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Weighted Merging Operators: Product, Utility-based Operators and Egalitarianism

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Abstract

We propose new operators for weighted propositional belief merging. We introduce distance-based operators that use the product as aggregation function. In social choice theory, the product, called the Nash welfare function, is known to be a more equitable social welfare function than the classical utilitarian welfare function (based on a sum). We study which properties are satisfied by the obtained corresponding weighted merging operators. In particular, we show that, unlike the Nash welfare function, distance-based operators using the product do not satisfy the Pigou-Dalton property. Then, we introduce a new family of weighted merging operators, which we call utility-based weighted merging operators, where the utility is roughly the converse of a distance for distance-based operators. For most well-known distance-based operators, it is easy to find the corresponding utility-based merging operators. But an interesting result is that the utility-based weighted merging operator based on the product does not correspond to any standard distance-based weighted merging operator, and this operator satisfies the Pigou-Dalton property.

1 Introduction

Belief merging aims at combining the beliefs from a set of agents to form a coherent belief base (Revesz 1997; Konieczny and Pino Pérez 2002; Konieczny, Lang, and Marquis 2004). This requires in particular to solve the logical conflicts (inconsistencies) generated when the different beliefs are combined (Baral, Kraus, and Minker 1991). The behaviour of belief merging operators has been investigated under several aspects: logical properties (Konieczny and Pino Pérez 2002; Haret and Woltran 2019), strategy-proofness (Everaere, Konieczny, and Marquis 2007), truth-tracking (Everaere, Konieczny, and Marquis 2010; Everaere, Konieczny, and Marquis 2020), egalitarianism (Everaere, Konieczny, and Marquis 2014), relations with judgement aggregation (Pigozzi 2021; Everaere, Konieczny, and Marquis 2015), applications in medical diagnosis (Kareem, Parra, and Wilson 2017) etc.

In all these works, all the agents/sources that participate in the merging are supposed to have the same weight, so they have the same importance in the process of defining the result of the merging. But in many situations there are differences between agents, for instance, some sources may be more reliable than others, and so we want to listen to them more than others. So, this requires to be able to define weighted merging operators that take these differences into account. This was done in (Everaere et al. 2023) where the authors propose a characterization of weighted merging operators.

In this work, we want to explore weighted merging operators that exhibit a more egalitarist behaviour. We start by introducing the product as aggregation function for distance-based merging operators. This product, known in social choice theory as the Nash welfare function (Sen 2005; Moulin 1988; Sen 2017), is a fairer social welfare function than the classical utilitarian welfare function (based on a sum). In particular, it satisfies the Pigou-Dalton property (Pigou 1920; Dalton 1920). This property states that reducing inequalities between two agents (without decreasing the global utility) is beneficial for the (welfare of the) group. In our setting, this means that if we have the choice between two interpretations, the one that is closest (most plausible) to the furthest agent is the best one. We will make this clearer in the formal definition.

We investigate the properties of the distance-based merging operators using the product as aggregation function, and illustrate their interesting behaviour. But we obtain also a quite surprising result: in this merging setting, these operators do not satisfy the (corresponding translation of the) Pigou-Dalton property.

Then we introduce a new family of belief merging operators: the utility-based merging operators. The rough idea is to reverse the evaluation of interpretations: for distance-based operators we look for the closest interpretations, i.e. the ones with the smallest distance. For utility-based operators we look for the interpretations with the biggest utility. At first sight, what is basically a reverse of the scale looks innocuous. But we found out interesting results. Obviously for some operators it is easy to find the utility-based operator that corresponds to a known distance-based operator. But for other operators the correspondence is not that simple. An interesting result is that if we use the product as aggregation function for utility-based operators, this leads to operators which are different from known distance-based operators. Moreover, these utility based operators based on the product satisfy the Pigou-Dalton property.

After some preliminaries in the next Section, we will recall some background on weighted belief merging in Section

3. In Section 4 we introduce and study distance-based merging operators based on the product aggregation function. In Section 5 we introduce weighted utility-based merging operators. We conclude in Section 6.

2 Background

We consider a propositional language $\mathcal{L}$ over a finite alphabet $\mathcal{P}$ of propositional letters. The set of consistent formulas is denoted $\mathcal{L}^*$. An interpretation is a function from $\mathcal{P}$ to $\{0,1\}$. The set of all interpretations is denoted Ω. An interpretation ω is a model of a formula if and only if it makes it true in the usual classical truth functional way. $[\![\varphi]\!]$ denotes the set of models of φ, i.e. $[\![\varphi]\!] = \{\omega \in \Omega : \omega \models \varphi\}$. We write $\varphi \equiv \varphi'$ when φ and φ' have exactly the same models.

An agent is characterized by some beliefs and by a reliability degree, that encodes how important/reliable/expert he is. Thus, an agent a is encoded by a couple of the form (φ, α) where φ is a consistent propositional formula, the beliefs of a, and where α is a strictly positive real number,[1] the degree of reliability associated to agent a.

The set of agents will be denoted by A. A finite tuple of agents is called a profile. We use capital Greek letters to denote profiles. We will denote $\sqcup$ the union (concatenation) of profiles. The set of profiles is denoted $\mathcal{E}$.

Let $\mathcal{B}$ and δ be functions giving the beliefs and the reliability of an agent respectively, that is $\mathcal{B} : A \to \mathcal{L}$ and $\delta : A \to \mathbb{R}^+_*$ are the functions such that for every agent $a = (\varphi, \alpha)$, $\mathcal{B}(a) = \varphi$ and $\delta(a) = \alpha$.

We say that two agents a and a' are equivalent (noted by $a \leftrightarrow a'$) if and only if $\mathcal{B}(a) \equiv \mathcal{B}(a')$ and $\delta(a) = \delta(a')$.

We say that two profiles Ψ and Ψ' are equivalent (noted by $\Psi \leftrightarrow \Psi'$) if and only if there is a bijection g from Ψ to Ψ' such that $a \leftrightarrow g(a)$.

The union of a profile Ψ and a profile $\{a\}$ such that $\mathcal{B}(a) = \varphi$ and $\delta(a) = \sigma$ is noted by $\Psi \sqcup a$ or $\Psi \sqcup (\varphi, \sigma)$. We define Ψ^n as $\sqcup_{i=1}^n \Psi_i$.

We denote the conjunction between the bases of a profile $\mathcal{B}(a_1) \wedge ... \wedge \mathcal{B}(a_n)$ by $\bigwedge \Psi$.

We say that the profile Ψ is consistent if and only if $\bigwedge \Psi$ is consistent, in this case we write $\omega \models \Psi$ instead of $\omega \models \bigwedge \Psi$.

3 Weighted Merging

Recently a characterization of weighted IC merging operators[2] has been given (Everaere et al. 2023), extending the classical unweighted case (Konieczny and Pino Pérez 2002). A set of syntactical postulates has been given, in order to characterize the behaviour of the merging of propositional bases when weights are considered. A merging operator Δ is a function that associates a formula $\Delta_\mu(\Psi)$ to any profile Ψ and to any formula μ.

Definition 1. *A merging operator Δ is called a weighted IC merging operator (WIC merging operator for short) if it satisfies the following postulates. Let Ψ, Ψ_1, Ψ_2 be profiles and μ a formula that represents the integrity constraints.*

(WIC0) $\Delta_\mu(\Psi) \vdash \mu$

(WIC1) If μ is consistent, then $\Delta_\mu(\Psi)$ is consistent

(WIC2) If Ψ is consistent with μ, then $\Delta_\mu(\Psi) = \bigwedge \Psi \wedge \mu$

(WIC3) If $\Psi_1 \leftrightarrow \Psi_2$ and $\mu_1 \equiv \mu_2$, then $\Delta_{\mu_1}(\Psi_1) \equiv \Delta_{\mu_2}(\Psi_2)$

(WIC4) If $\mathcal{B}(a_1) \vdash \mu$, $\mathcal{B}(a_2) \vdash \mu$ and $\delta(a_1) = \delta(a_2)$, then $\Delta_\mu(a_1 \sqcup a_2) \wedge \mathcal{B}(a_1) \nvdash \bot \Rightarrow \Delta_\mu(a_1 \sqcup a_2) \wedge \mathcal{B}(a_2) \nvdash \bot$

(WIC5) $\Delta_\mu(\Psi_1) \wedge \Delta_\mu(\Psi_2) \vdash \Delta_\mu(\Psi_1 \sqcup \Psi_2)$

(WIC6) If $\Delta_\mu(\Psi_1) \wedge \Delta_\mu(\Psi_2)$ is consistent, then $\Delta_\mu(\Psi_1 \sqcup \Psi_2) \vdash \Delta_\mu(\Psi_1) \wedge \Delta_\mu(\Psi_2)$

(WIC7) $\Delta_{\mu_1}(\Psi) \wedge \mu_2 \vdash \Delta_{\mu_1 \wedge \mu_2}(\Psi)$

(WIC8) If $\Delta_{\mu_1}(\Psi) \wedge \mu_2$ is consistent, then $\Delta_{\mu_1 \wedge \mu_2}(\Psi) \vdash \Delta_{\mu_1}(\Psi) \wedge \mu_2$

(WIC9) If $\beta > \alpha$, if $\Delta_\mu(\Psi \sqcup (\varphi, \alpha)) \vdash \varphi$, then $\Delta_\mu(\Psi \sqcup (\varphi, \beta)) \vdash \varphi$.

(WIC10) If $\varphi \wedge \Delta_\mu(\Psi \sqcup (\varphi, \alpha)) \nvdash \bot$ and $\varphi \wedge \Delta_\mu(\Psi \sqcup (\varphi, \beta)) \nvdash \bot$, then $\Delta_\mu(\Psi \sqcup (\varphi, \alpha)) \wedge \varphi \equiv \Delta_\mu(\Psi \sqcup (\varphi, \beta)) \wedge \varphi$

(WIC11) $\Delta_\mu((\varphi, \alpha)) \equiv \Delta_\mu((\varphi, \beta))$

(WIC12) If φ is consistent with μ, then $\exists \alpha$, $\Delta_\mu(\Psi \sqcup (\varphi, \alpha)) \vdash \varphi$

These postulates are a generalization of those proposed for merging propositional belief bases (Konieczny and Pino Pérez 2002). In particular, postulates **(WIC1)** to **(WIC8)** are identical to the original ones[3]. The postulates **(WIC9)** to **(WIC12)** address more precisely the weight-related behaviour. **(WIC9)** states that increasing the weight of an agent can only be beneficial for this agent. **(WIC10)** ensures that the weight associated to an agent is a penalty against conflicting formulas, but has no impact on formulas consistent with the agent. **(WIC11)** expresses that the plausibility relation associated to the beliefs of one agent alone does not depend on the weights, only on the beliefs of this agent. **(WIC12)** is a kind of success postulate for the weights: if the weight of an agent is sufficiently large, then this agent imposes its view for the result of the merging.

A semantic characterization has been given with the definition of a plausibility relation between interpretations.

Definition 2. *A function $\Psi \mapsto \preccurlyeq_\Psi$ that maps each profile Ψ to a total preorder over interpretations $\preccurlyeq_\Psi$ is called a* weighted syncretic assignment *if it satisfies the conditions 1-10 below:*

1. If $\omega \models \Psi$ and $\omega' \models \Psi$, then $\omega \simeq_\Psi \omega'$
2. If $\omega \models \Psi$ and $\omega' \not\models \Psi$, then $\omega \prec_\Psi \omega'$
3. If $\Psi_1 \leftrightarrow \Psi_2$, then $\preccurlyeq_{\Psi_1} = \preccurlyeq_{\Psi_2}$
4. For any a, a' with $\delta(a) = \delta(a'), \forall \omega \models \mathcal{B}(a)$, $\exists \omega' \models \mathcal{B}(a')$ such that $\omega' \preccurlyeq_{a \sqcup a'} \omega$
5. If $\omega \preccurlyeq_{\Psi_1} \omega'$ and $\omega \preccurlyeq_{\Psi_2} \omega'$, then $\omega \preccurlyeq_{\Psi_1 \sqcup \Psi_2} \omega'$

[1] We denote $\mathbb{R}^+$ the set $\{x \in \mathbb{R} : x \geq 0\}$, *i.e.*, the set of non negative real numbers. And we denote $\mathbb{R}^+_*$ the set $\{x \in \mathbb{R} : x > 0\}$, *i.e.*, the set of strictly positive real numbers.

[2] IC stands for integrity constraints, that are represented by a logical formula μ and allow to encode physical or legal constraints for the result of the merging.

[3] **(WIC4)** has simply been adapted with identical weights for the two bases.

6. If $\omega \preccurlyeq_{\Psi_1} \omega'$ and $\omega \prec_{\Psi_2} \omega'$, then $\omega \prec_{\Psi_1 \sqcup \Psi_2} \omega'$

7. If $\omega \models \varphi$, $\omega' \not\models \varphi$ and $\alpha < \beta$, then
$\omega \prec_{\Psi \sqcup (\varphi,\alpha)} \omega' \implies \omega \prec_{\Psi \sqcup (\varphi,\beta)} \omega'$

8. If $\omega, \omega' \models \varphi$ then $\forall \alpha, \beta$,
$\omega \preccurlyeq_{\Psi \sqcup (\varphi,\alpha)} \omega' \iff \omega \preccurlyeq_{\Psi \sqcup (\varphi,\beta)} \omega'$

9. $\omega \preccurlyeq_{(\varphi,\alpha)} \omega'$ iff $\omega \preccurlyeq_{(\varphi,\beta)} \omega'$

10. If $\omega \models \varphi$ and $\omega' \models \neg\varphi$, then $\exists \alpha$ s.t. $\omega \prec_{\Psi \sqcup (\varphi,\alpha)} \omega'$

A representation theorem for WIC merging operators gives an equivalence between the syntactical postulates and the semantic conditions:

Theorem 1 ((Everaere et al. 2023))**.** *An operator Δ is a WIC merging operator if and only if there exists a weighted syncretic assignment $\Psi \mapsto \preccurlyeq_\Psi$ such that for every formula μ and every profile Ψ the following equality holds:*

$$[\![\Delta_\mu(\Psi)]\!] = min([\![\mu]\!], \preccurlyeq_\Psi)$$

In (Everaere et al. 2023), a concrete way to obtain weighted merging operators has been given, based on a pseudo-distance between interpretations, a weight function and a weighted aggregation function. Next, we recall the construction.

Definition 3. *A pseudo-distance over interpretations is a function $d : \Omega \times \Omega \to \mathbb{R}^+$ such that $d(\omega, \omega') = d(\omega', \omega)$ and $d(\omega, \omega') = 0$ iff $\omega = \omega'$.*

Definition 4. *The distance between an interpretation ω and a formula φ is defined by $d(\omega, \varphi) = \min_{\omega' \models \varphi} d(\omega, \omega')$.*

Some examples of such pseudo-distances are the drastic distance d_D, with $d_D(\omega, \omega') = 0$ iff $\omega = \omega'$, $d_D(\omega, \omega') = 1$ otherwise; and the Hamming distance d_H, where $d_H(\omega, \omega')$ is the number of variables on which the two interpretations differ.

Definition 5. *A weight function is a function $\bullet : \mathbb{R}^+ \times \mathbb{R}^+_* \to \mathbb{R}^+$, which satisfies the following properties:*

- ***Increasing:*** *If $d \neq 0$ and $\alpha > \beta$, then $\bullet(d, \alpha) > \bullet(d, \beta)$, and if $d > d'$, then $\bullet(d, \beta) > \bullet(d', \beta)$*
- ***Invariance of 0:*** $\forall \alpha, \beta, \bullet(0, \alpha) = \bullet(0, \beta) \stackrel{def}{=} \bullet_0$
- ***Unbounded:*** $\forall d > 0$ *,* $\forall K > 0, \exists \alpha$ *s.t.* $\bullet(d, \alpha) > K$

Some examples of weight functions are the multiplication $\times(x, y) = x \times y$ and the power $pow(x, y) = (x + 1)^y$.

Definition 6. *A weighted aggregation function is a mapping $f : \bigcup_{n \in \mathbb{N}} {\mathbb{R}^+}^n \to (\mathcal{I}, \leq)$, where $\mathcal{I}$ is a totally ordered set*[4]*, which has the following properties:*

- ***Identity:*** *$\forall x$, $f(x) = x$.*
- ***Symmetry:*** *If σ is a permutation over $\{1, ..., n\}$, then $f(\alpha_1, ..., \alpha_n) = f(\alpha_{\sigma(1)}, ..., \alpha_{\sigma(n)})$*
- ***Composition:*** *If $f(\alpha_1, ..., \alpha_n) \geq f(\beta_1, ..., \beta_n)$, then $\forall \gamma \geq 0$, $f(\alpha_1, ..., \alpha_n, \gamma) \geq f(\beta_1, ..., \beta_n, \gamma)$*
- ***Decomposition:*** *$\forall \gamma \geq 0$, if $f(\alpha_1, ..., \alpha_n, \gamma) \geq f(\beta_1, ..., \beta_n, \gamma)$, then $f(\alpha_1, ..., \alpha_n) \geq f(\beta_1, ..., \beta_n)$*
- ***Unbounded:*** *$\forall (\alpha_1, ..., \alpha_n), \forall (\beta_1, ..., \beta_n), \forall \beta, \exists \alpha$ s.t. $f(\alpha_1, ..., \alpha_n, \alpha) > f(\beta_1, ..., \beta_n, \beta)$*

Most usual aggregation functions (Σ and Gmax for instance) for merging operators are also weighted aggregation functions. But it is not the case for all functions, for example Gmin (see definition below) is not a weighted aggregation function (the **Unbounded** condition is not satisfied).

Definition 7. *Let $\mathsf{Gmin}/\mathsf{Gmax} : \bigcup_n {\mathbb{R}^+}^n \to (\bigcup_n {\mathbb{R}^+}^n, \leq_{lex})$ be aggregation functions, such that:*

$$\mathsf{Gmin}(y_1, \ldots, y_n) = (y_{\rho_m(i_1)}, \ldots, y_{\rho_m(i_n)})$$

$$\mathsf{Gmax}(y_1, \ldots, y_n) = (y_{\rho_M(i_1)}, \ldots, y_{\rho_M(i_n)})$$

where ρ_m is a permutation of $\{1, \ldots, n\}$ such that the $y_{\rho_m(i)}$ are in increasing order, ρ_M is a permutation of $\{1, \ldots, n\}$ such that the $y_{\rho_M(i)}$ are in decreasing order, and $\leq_{lex}$ is the lexicographic order.

The weighted distance between an interpretation ω and an agent a is:

$$d^\bullet_d(\omega, a) = \bullet(d(\omega, \mathcal{B}(a)), \delta(a))$$

The distance between an interpretation ω and a profile Ψ is:[5]

$$d^\bullet_{d,f}(\omega, \Psi) = f_{a \in \Psi}\, d^\bullet_d(\omega, a)$$

An assignment $\Psi \mapsto \preccurlyeq^{d^\bullet_{d,f}}_\Psi$ is defined by :

$$\omega \preccurlyeq^{d^\bullet_{d,f}}_\Psi \omega' \text{ iff } d^\bullet_{d,f}(\omega, \Psi) \leq d^\bullet_{d,f}(\omega', \Psi)$$

Finally, the operator $\Delta^{d^\bullet_{d,f}}_\mu$ associated to d, $\bullet$ and f is defined semantically as:

$$[\![\Delta^{d^\bullet_{d,f}}_\mu(\Psi)]\!] = min([\![\mu]\!], \preccurlyeq^{d^\bullet_{d,f}}_\Psi)$$

Note that the assignment defined in this way is a weighted syncretic assignment, hence:

Theorem 2 ((Everaere et al. 2023))**.** *Let d, $\bullet$ and f be a pseudo-distance, a weight function and a weighted aggregation function respectively. Then, the operator $\Delta^{d^\bullet_{d,f}}$ is a WIC merging operator.*

Note that in the definition of weighted aggregation functions in (Everaere et al. 2023), the **Increasing** condition was required. But it can be shown that **Identity** plus **Decomposition** imply this condition. Note also that all the conditions except for **Unbounded** are exactly the usual conditions used in the unweighted setting (for IC merging operators).

[4] Usually, for example for the sum, $\mathcal{I}$ is simply $\mathcal{R}^+$ and $\leq$ the natural order between real numbers. For Gmax or Gmin, $\mathcal{I}$ is $\bigcup_{n \in \mathbb{N}} {\mathbb{R}^+}^n$ and $\leq$ the lexicographic order between vectors of real numbers.

[5] If $\Psi = \{a_1, \ldots, a_n\}$, $f_{a \in \Psi}\, d^\bullet_d(\omega, a)$ is the shorthand notation for $f(d^\bullet_d(\omega, a_1), \ldots, d^\bullet_d(\omega, a_n))$.

4 The Aggregation Function Product

The product as aggregation function, known as the Nash welfare function in social choice theory, is reputed as computing the welfare of a society in a more egalitarian way than that used by the traditional (utilitarian) social function. In particular, the Nash welfare function satisfies the Pigou-Dalton property, which can be viewed as a minimal requirement for an egalitarian way of choosing a collective allocation.

In this section, we introduce operators that use the product as aggregation function for distanced-base merging. We show that the obtained operators are WIC merging operators and examine the supplementary properties of these operators.

4.1 Definition and Examples

First of all, as distances can be equal to 0, the direct use of product for distances may induce a problem, as 0 is an absorbant element for product, and this is not an expected property for aggregation operators for merging. We then have to avoid this behaviour, so we shift all distances by 1. This gives rise to the definition of the $*$ function as follows:

Definition 8. *Let the function* $* : \bigcup_n {\mathbb{R}^+}^n \to (\mathbb{R}^+, \leq)$*, be defined as:*

$$*(y_1, \ldots, y_n) = \left[\prod_{i=1}^{n}(y_i + 1)\right] - 1$$

Proposition 1. $*$ *is a weighted aggregation function.*

The operator $\Delta^{d^{\bullet}_{d,*}}$ is then defined as follows:

Definition 9. *Let* Ψ *be a profile and* ω *an interpretation, the weighted distance* $d^{\bullet}_{d,*}$ *between* ω *and* Ψ *is defined by:*

$$d^{\bullet}_{d,*}(\omega, \Psi) \quad = \quad *_{a \in \Psi}(d^{\bullet}_{d}(\omega, a))$$

Associated to $d^{\bullet}_{d,*}$*, the following preorder is defined:*

$$\omega \preccurlyeq_{\Psi}^{d^{\bullet}_{d,*}} \omega' \;\Leftrightarrow\; d^{\bullet}_{d,*}(\omega, \Psi) \;\leqslant\; d^{\bullet}_{d,*}(\omega', \Psi)$$

Finally the operator $\Delta^{d^{\bullet}_{d,*}}$ *is defined by:*

$$[\![\Delta_{\mu}^{d^{\bullet}_{d,*}}(\Psi)]\!] \;=\; min([\![\mu]\!], \preccurlyeq_{\Psi}^{d^{\bullet}_{d,*}})$$

From Theorem 2 and Proposition 1 we get the following result:

Corollary 1. *For every pseudo-distance* d *and every weight function* $\bullet$*, we have that* $\Delta^{d^{\bullet}_{d,*}}$ *is a WIC merging operator.*

This result gives a family of merging operators: each choice of a distance and of a weight function gives a WIC merging operator. We give an example of these operators:

Example 1. *Take* $\bullet = \times$ *(the multiplication between two numbers) and the profile* $\Psi = \{a_1, a_2, a_3, a_4\}$ *with:*
$\varphi_1 = \mathcal{B}(a_1) = \{000, 110\}$ $\quad \alpha_1 = \delta(a_1) = 10$
$\varphi_2 = \mathcal{B}(a_2) = \{010\}$ $\quad \alpha_2 = \delta(a_2) = 4$
$\varphi_3 = \mathcal{B}(a_3) = \{101, 110\}$ $\quad \alpha_3 = \delta(a_3) = 8$
$\varphi_4 = \mathcal{B}(a_4) = \{010, 100\}$ $\quad \alpha_3 = \delta(a_3) = 4$
$\mu = \{001, 011, 101, 111\}$ *(only models of* μ *can be considered as possible results of the merging – they correspond to unshaded lines in Table 1). The first columns in Table 1 contain the Hamming distance between each interpretation and the beliefs of the agents. Then, these distances are combined with the weights (using multiplication*[6] *(*$\bullet = \times$*) here), and then aggregated to obtain the weighted distance to the profile. For example,* $d^{\times}_{d_H,\Sigma}(101, \Psi) \;= 10 + 8 + 8 + 8 = 34$ *and* $d^{\times}_{d_H,*}(101, \Psi) \;= 11 * 9 * 9 * 9 - 1 = 8018.$

	a_1	a_2	a_3	a_4	$\Delta^{d^{\times}_{d_H,\Sigma}}$	$\Delta^{d^{\times}_{d_H,*}}$
000	0	1	2	1	24	424
001	1	2	1	2	**34**	8018
010	1	0	1	0	18	98
011	2	1	2	1	44	8924
100	1	2	1	0	26	890
101	2	3	0	1	36	**1364**
110	0	1	0	1	8	24
111	1	2	2	2	42	15146

Table 1: Example of weigthed merging

The result of the merging for $[\![\Delta_{\mu}^{d^{\times}_{d_H,\Sigma}}]\!]$ *is* $\{001\}$*, for the product* $[\![\Delta_{\mu}^{d^{\times}_{d_H,*}}]\!] = \{101\}$.

4.2 Behaviour of $\Delta^{d^{\bullet}_{d,*}}$

(WIC0) to **(WIC12)** are the basic postulates for the whole family of weighted merging operators. But some additional properties can characterize interesting subclasses. Two important classes of merging operators are the majority operators and the arbitration operators (Konieczny and Pino Pérez 2002). Roughly speaking, majority operators take into account the beliefs of the majority in the profile whereas arbitration[7] operators aim at being more egalitarian.

The postulate of Majority is the condition:
(Maj) $\exists n$ s.t. $\Delta_{\mu}(\Psi_1 \sqcup \Psi_2^n) \vdash \Delta_{\mu}(\Psi_2)$

The Arbitration postulate is expressed as:
(Arb)

$$\left.\begin{array}{l} \Delta_{\mu_1}(a_1) \equiv \Delta_{\mu_2}(a_2) \\ \Delta_{\mu_1 \leftrightarrow \neg\mu_2}(a_1 \sqcup a_2) \equiv (\mu_1 \leftrightarrow \neg\mu_2) \\ \mu_1 \not\vdash \mu_2 \\ \mu_2 \not\vdash \mu_1 \end{array}\right\} \Rightarrow \begin{array}{c} \Delta_{\mu_1 \vee \mu_2}(a_1 \sqcup a_2) \\ \equiv \\ \Delta_{\mu_1}(a_1) \end{array}$$

Proposition 2. *For every pseudo-distance* d *and weight function* $\bullet$*,* $\Delta^{d^{\bullet}_{d,*}}$ *is a majority WIC merging operator.*

Before the proof let us state this useful Lemma the proof of which is easily done by induction on n:

Lemma 1. *Let* n *be an integer,* d *a pseudo-distance and* $\bullet$ *a weight function. Then:*

$$\forall n, d^{\bullet}_{d,*}(\omega, \Psi^n) = (d^{\bullet}_{d,*}(\omega, \Psi) + 1)^n - 1$$

[6] Note that in the following we will use $\times$ for the multiplication and $*$ for the product aggregation function of definition 8.

[7] Note that "arbitration" here denotes operators defined in (Konieczny and Pino Pérez 2002), but that this term was also used in (Revesz 1997) and (Liberatore and Schaerf 1995) to denote two other families of operators.

Proof of Proposition 2. We want to show that $\exists n, \Delta_{\mu}^{d^{\bullet}_{d,*}}(\Psi_1 \sqcup \Psi_2^n) \vdash \Delta_{\mu}^{d^{\bullet}_{d,*}}(\Psi_2)$.
Suppose $\forall n, \Delta_{\mu}^{d^{\bullet}_{d,*}}(\Psi_1 \sqcup \Psi_2^n) \nvdash \Delta_{\mu}^{d^{\bullet}_{d,*}}(\Psi_2)$. This means that for each n, $\exists \omega_n \in \Omega$ s.t. $\omega_n \models \Delta_{\mu}^{d^{\bullet}_{d,*}}(\Psi_1 \sqcup \Psi_2^n)$ and $\omega_n \not\models \Delta_{\mu}^{d^{\bullet}_{d,*}}(\Psi_2)$. As Ω is a finite set, according to the pigeonhole principle, there is at least one interpretation $\omega \in \Omega$ for which there exist infinite numbers n s.t. $\omega \models \Delta_{\mu}^{d^{\bullet}_{d,*}}(\Psi_1 \sqcup \Psi_2^n)$ and $\omega \not\models \Delta_{\mu}^{d^{\bullet}_{d,*}}(\Psi_2)$. As a consequence, for this ω, there exist infinite numbers n s.t. for each n, there is an interpretation ω'_n with:

$$\omega \preccurlyeq^{d^{\bullet}_{d,*}}_{\Psi_1 \sqcup \Psi_2^n} \omega'_n \text{ and } \omega'_n \prec^{d^{\bullet}_{d,*}}_{\Psi_2} \omega \quad (1)$$

Again, according to the pigeonhole principle, as there are infinite numbers n satisfying 1 but there are only a finite number of ω'_n, there exists ω' for which there is an infinite number of integers n s.t. the following holds:

$$\omega \preccurlyeq^{d^{\bullet}_{d,*}}_{\Psi_1 \sqcup \Psi_2^n} \omega' \text{ and } \omega' \prec^{d^{\bullet}_{d,*}}_{\Psi_2} \omega \quad (2)$$

We get $d^{\bullet}_{d,*}(\omega', \Psi_2) < d^{\bullet}_{d,*}(\omega, \Psi_2)$ and
$(d^{\bullet}_{d,*}(\omega, \Psi_1)+1) \cdot (d^{\bullet}_{d,*}(\omega, \Psi_2^n)+1) - 1 \leqslant (d^{\bullet}_{d,*}(\omega', \Psi_1)+1) \cdot (d^{\bullet}_{d,*}(\omega', \Psi_2^n)+1) - 1$.
By Lemma 1, we have

$$(d^{\bullet}_{d,*}(\omega, \Psi_1)+1) \cdot (d^{\bullet}_{d,*}(\omega, \Psi_2)+1)^n - 1 \leqslant$$

$$(d^{\bullet}_{d,*}(\omega', \Psi_1)+1) \cdot (d^{\bullet}_{d,*}(\omega', \Psi_2)+1)^n - 1 \quad (3)$$

and we know that this holds for infinite integers n. So, we can choose n s.t.:

$$n > \frac{ln\left(d^{\bullet}_{d,*}(\omega', \Psi_1)+1\right) - ln\left(d^{\bullet}_{d,*}(\omega, \Psi_1)+1\right)}{ln\left(d^{\bullet}_{d,*}(\omega, \Psi_2)+1\right) - ln\left(d^{\bullet}_{d,*}(\omega', \Psi_2)+1\right)}$$

Then we get:

$$(d^{\bullet}_{d,*}(\omega, \Psi_1)+1) \cdot (d^{\bullet}_{d,*}(\omega, \Psi_2)+1)^n >$$

$$(d^{\bullet}_{d,*}(\omega', \Psi_1)+1) \cdot (d^{\bullet}_{d,*}(\omega', \Psi_2)+1)^n$$

which contradicts inequality 3. Therefore, by *reductio ad absurdum*, we obtain $\exists n, \Delta_{\mu}^{d^{\bullet}_{d,*}}(\Psi_1 \sqcup \Psi_2^n) \vdash \Delta_{\mu}^{d^{\bullet}_{d,*}}(\Psi_2)$. ☐

The next question is if operators based on such a product are also arbitration ones. The answer is no, in general:

Proposition 3. *$\Delta^{d^{pow}_{d_H,*}}$ and $\Delta^{d^{\times}_{d_H,*}}$ are not Arbitration WIC merging operators.*

Note that it is not the case for all distances. If the drastic distance d_D is considered, we get:

Proposition 4. *$\Delta^{d^{\bullet}_{d_D,\Sigma}}$ and $\Delta^{d^{\bullet}_{d_D,*}}$ are WIC merging operators which are majority and arbitration operators.*

Remark that $\Delta^{d^{\bullet}_{d_D,\Sigma}}$ and $\Delta^{d^{\bullet}_{d_D,*}}$ are different operators (contrary to the unweighted case where $\Delta^{d_D,f}$ leads to the same operator for all f (Everaere et al. 2021)).

The arbitration condition seems too demanding for operators based on product. But a classical egalitarian property, often considered in social choice theory, is the Pigou-Dalton principle, which can be stated as follows, for weighted distance-based operators (Everaere, Konieczny, and Marquis 2014):

Definition 10. *(Pigou-Dalton for distances) A WIC merging operator $\Delta^{\bullet}_{d,f}$ satisfies the D-Pigou-Dalton property if for any $\Psi = \{a_1, \ldots, a_n\}$, if $\exists$ k and l such that $d^{\bullet}_{d}(\omega, a_k) < d^{\bullet}_{d}(\omega', a_k) \leqslant d^{\bullet}_{d}(\omega', a_l) < d^{\bullet}_{d}(\omega, a_l)$, $d^{\bullet}_{d}(\omega', a_k) - d^{\bullet}_{d}(\omega, a_k) = d^{\bullet}_{d}(\omega, a_l) - d^{\bullet}_{d}(\omega', a_l)$ and $\forall i \neq k$ and $i \neq l, d^{\bullet}_{d}(\omega, a_i) = d^{\bullet}_{d}(\omega', a_i)$, then $\omega' \prec^{d^{\bullet}_{d,f}}_{\Psi} \omega$.*

Pigou-Dalton property is an important fairness property. For belief merging it can be seen as some kind of strengthening of the arbitration property, where we want to choose a result that is as satisfactory as possible for each agent (whereas majority operators just want to be globally satisfactory). In particular, operators that satisfy D-Pigou-Dalton will choose, between two interpretations that are identical up to a transfer from an agent to another one, the one that satisfies the most the least satisfied agent. For instance, if the distances of interpretation ω to the profile are (10, 8, 5, 12, 15), and the distances of interpretation ω' to the profile are (10, 8, 8, 12, 12), then we should prefer ω' since the last agent, that is the least satisfied one for ω prefers ω' whereas for the other agents it changes nothing, except for agent 3 that was the most satisfied agent for ω and that is just a bit less satisfied with ω'.

In particular, the class of problems where the Pigou-Dalton property is useful concerns those where we want to ensure the best satisfaction of each agent. For instance for goal merging, there can be cases where if one agent is not satisfied enough by the global decision, he may decide to quit the group and it may ruin the whole consensus obtained through merging (look at Example 22 of (Konieczny and Pino Pérez 2002) for instance).

Note that the previous definition encodes that property in our framework: when a fair transfer is realized from ω to ω', the world ω' is better than ω. However, this principle is not satisfied in general for weighted merging operators using product:

Proposition 5. *$\Delta^{d^{\times}_{d_H,*}}$ and $\Delta^{d^{pow}_{d_H,*}}$ do not satisfy the Pigou-Dalton property.*

Proof. The following example shows that *D-Pigou-Dalton* is not satisfied. Consider the profile $\Psi = \{a_1, a_2, a_3, a_4\}$ with:
$\varphi_1 = \mathcal{B}(a_1) = \{11\}$ $\quad$ $\alpha_1 = \delta(a_1) = 1$
$\varphi_2 = \mathcal{B}(a_2) = \{00\}$ $\quad$ $\alpha_2 = \delta(a_2) = 1$
$\varphi_2 = \mathcal{B}(a_3) = \{01\}$ $\quad$ $\alpha_2 = \delta(a_3) = 2$
$\varphi_2 = \mathcal{B}(a_4) = \{01\}$ $\quad$ $\alpha_2 = \delta(a_4) = 2$
In Table 2, the columns 2 to 5 contain the $d^{\times}_{d_H}$ distance of each interpretation to each base:

	a_1	a_2	a_3	a_4	$d^{\times}_{d_H,*}$
$\omega_1 = 00$	2	0	2	2	26
$\omega_2 = 01$	1	1	2	2	35

Table 2: $\Delta^{d^{\times}_{d_H,*}}$ does not satisfy *(D-Pigou-Dalton)*

We have:

$$d^{\times}_{d_H}(\omega_1, a_2) < d^{\times}_{d_H}(\omega_2, a_2) \leqslant d^{\times}_{d_H}(\omega_2, a_1) < d^{\times}_{d_H}(\omega_1, a_1)$$

$$d^{\times}_{d_H}(\omega_2, a_2) - d^{\times}_{d_H}(\omega_1, a_2) = d^{\times}_{d_H}(\omega_1, a_1) - d^{\times}_{d_H}(\omega_2, a_1)$$

And $d^{\times}_{d_H}(\omega_1, a_3) = d^{\times}_{d_H}(\omega_2, a_3)$, $d^{\times}_{d_H}(\omega_1, a_4) = d^{\times}_{d_H}(\omega_2, a_4)$.

In accordance with Definition 10, we should obtain $\omega_2 \prec_{\Psi}^{d^{\times}_{d_H,*}} \omega_1$. Nevertheless (see table 2), we get $d^{\times}_{d_H,*}(\omega_1, \Psi) < d^{\times}_{d_H,*}(\omega_2, \Psi)$, i.e. $\omega_1 \prec_{\Psi}^{d^{\times}_{d_H,*}} \omega_2$. Contradiction.

The same profile also leads to a counter-example for $\Delta^{d^{pow}_{d_H,*}}$. □

This result is quite surprising, as the product is well-known as satisfying the Pigou-Dalton principle when utilities, instead of distances, are considered (Moulin 1988). It is natural then to investigate the following question. Is it possible to define WIC merging operators starting from utilities? In this case, is the Pigou-Dalton principle satisfied for the product? This is the aim of the following Section.

5 Weighted Utility Merging

Usually, concrete merging operators are based on distances. In social choice theory, utility is generally used to measure the satisfaction of an individual or a society (Social Welfare) (Barbera, Hammond, and Seidl 1998; Sen 2017). To our knowledge, using utilities to merge belief bases has not been studied yet. Utility is a sort of dual of distance: the greater the utility, the more satisfied an agent, whereas for distances, the closer an agent is to a base (and so, the less the distance), the more it is satisfied. First, we have to define weighted utility for an agent, when an agent is represented with a propositional formula and a weight.

Definition 11. *Let $K > 1$ be a real number. A function $u_K : \Omega \times \Omega \to [1, K]$ is called a utility function if the following conditions are satisfied for all interpretations ω and ω':*
1. $u_K(\omega, \omega') = u_K(\omega', \omega)$,
2. $u_K(\omega, \omega') = K$ iff $\omega = \omega'$.

Note that $u_K(\omega, \omega') \in [1, K[\iff \omega \neq \omega'$: the greater the utility, the better the satisfaction. The best value is K, corresponding to the case where $\omega = \omega'$.

We now give a very simple way to construct a utility function from a pseudo-distance d.

Definition 12. *Let $d : \Omega^2 \to \mathbb{R}^+$ be a pseudo-distance. Take[8] $K = 1 + \max\{d(\omega, \omega') : \omega, \omega' \in \Omega\}$. Define $u_d : \Omega^2 \to [1, K]$ by $u_d(\omega, \omega') = K - d(\omega, \omega')$.*

It is straightforward to see that such a function u_d is a utility function. We will call it the utility function associated to a pseudo-distance d.

Definition 13. *The utility of a formula φ for an interpretation ω, denoted u_K, is:*

$$u_K(\omega, \varphi) = \max_{\omega' \models \varphi} u_K(\omega, \omega')$$

The satisfaction of a formula for an interpretation ω is maximal if and only if ω is a model of the formula.

Definition 14. *The utilitarian weight function is a function $\circ : [1, K] \times [1, +\infty] \mapsto [0, K]$, which satisfies:*

- ***Weight decreasing:** If $u \neq K$ and $\alpha > \beta$, then $\circ(u, \alpha) < \circ(u, \beta)$*
- ***Utility increasing:** If $u > u'$, then $\circ(u, \beta) > \circ(u', \beta)$*
- ***Invariance of K:** $\forall \alpha, \beta, \circ(K, \alpha) = \circ(K, \beta) = K$*
- ***Infinitely reducible** $\forall u \in [1, K[$, $\forall \epsilon > 0, \exists \alpha$ s.t. $\circ(u, \alpha) < \epsilon$*

The differences between the weight function (for the distance) and the utilitarian weight function (for utility) rely mainly on the Weight decreasing and Infinitely reducible conditions. These conditions impose that when the weights increase, the utilities decrease. The invariance of K fixes K as the maximal possible utility, whatever the weight.

To define a utilitarian aggregation function, there is only one condition which has to be adapted from the distance case. More precisely, the **Unbounded** condition is replaced by the following **Fully reducible** condition:

Definition 15. *Consider a function $f : \bigcup_n \mathbb{R}^{+n}_* \mapsto \mathbb{R}^+$ satisfying identity, symmetry, composition and decomposition. f is a utilitarian aggregation function if it satisfies:*

- ***Fully reducible:** $\forall(\alpha_1, ..., \alpha_n), \forall(\beta_1, ..., \beta_n), \forall \beta, \exists \alpha$ s.t. $f(\alpha_1, ..., \alpha_n, \alpha) < f(\beta_1, ..., \beta_n, \beta)$*

We have now all the ingredients to define a plausibility preorder from a utility, a utilitarian weight function and a utilitarian aggregation function. The weighted utility between an interpretation and an agent is defined by:

$$u^{\circ}_{u_K}(\omega, a) = \circ(u_K(\omega, \mathcal{B}(a)), \delta(a))$$

And now we define the weighted utility between an interpretation and a profile as follows:

$$u^{\circ}_{u_K,f}(\omega, \Psi) = f_{a \in \Psi}\, u^{\circ}_{u_K}(\omega, a)$$

Finally we obtain a plausibility preorder profile $\Psi \mapsto \preccurlyeq_{\Psi}^{u^{\circ}_{u_K,f}}$ by setting:

$$\omega \preccurlyeq_{\Psi}^{u^{\circ}_{u_K,f}} \omega' \text{ iff } u^{\circ}_{u_K,f}(\omega, \Psi) \geqslant u^{\circ}_{u_K,f}(\omega', \Psi)$$

[8] We add 1 here to shift utilities and avoid 0 values, that would cause problems for multiplications and divisions.

and the operator $\Delta^{u^{\circ}_{u_K},f}$ associated to u_K, $\circ$ and f by putting:

$$[\![\Delta_{\mu}^{u^{\circ}_{u_K},f}(\Psi)]\!] = min([\![\mu]\!], \preccurlyeq_{\Psi}^{u^{\circ}_{u_K},f})$$

Let us now state that these operators satisfy the expected logical properties:

Theorem 3. *Let u_K, $\circ$ and f be a utility, a utilitarian weight function and a utilitarian aggregation function respectively. Then, the operator $\Delta^{u^{\circ}_{u_K},f}$ is a weighted merging operator IC (it satisfies the postulates (WIC0-WIC12).*

Proof. The proof is based on showing that the assignment obtained with u_K, $\circ$ and f is a weighted syncretic assignment. Then using Theorem 1, the result follows. We focus here on the conditions linked to the weights (conditions 7 to 10), as the first conditions are clear (similar to the unweighted case).

7. Suppose that $\omega \models \varphi, \omega' \not\models \varphi, \omega \prec^{u^{\circ}_{u_K},f}_{\Psi\sqcup(\varphi,\alpha)} \omega'$ and $\beta > \alpha$. We want to show that $\omega \prec^{u^{\circ}_{u_K},f}_{\Psi\sqcup(\varphi,\beta)} \omega'$.
By definition, we have:

$$f(\{u^{\circ}_{u_K}(\omega, a_i) \mid a_i \in \Psi \sqcup (\varphi,\alpha)\}) >$$
$$f(\{u^{\circ}_{u_K}(\omega', a_i) \mid a_i \in \Psi \sqcup (\varphi,\alpha)\}) \quad (4)$$

As $\omega \models \varphi$, we have $u_u(\omega,\varphi) = K$. From invariance of K, we obtain $u^{\circ}_{u_K}(\omega,(\varphi,\alpha)) = u^{\circ}_{u_K}(\omega,(\varphi,\beta)) = K$. Then:

$$f(\{u^{\circ}_{u_K}(\omega, a_i) \mid a_i \in \Psi \sqcup (\varphi,\alpha)\}) =$$
$$f(\{u^{\circ}_{u_K}(\omega, a_i) \mid a_i \in \Psi \sqcup (\varphi,\beta)\}) \quad (5)$$

Note that, as $\omega' \not\models \varphi$, $u_u(\omega',\varphi) < K$. $\circ$ is weight decreasing, so $\circ(u_u(\omega',\varphi),\beta) < \circ(u_u(\omega',\varphi),\alpha)$, that is $u^{\circ}_{u_K}(\omega',(\varphi,\beta)) < u^{\circ}_{u_K}(\omega',(\varphi,\alpha))$.
Then, using increasing and composition of f, we get

$$f(\{u^{\circ}_{u_K}(\omega', a_i) \mid a_i \in \Psi \sqcup (\varphi,\alpha)\}) >$$
$$f(\{u^{\circ}_{u_K}(\omega', a_i) \mid a_i \in \Psi \sqcup (\varphi,\beta)\}) \quad (6)$$

From equations (4), (5) and (6) we obtain $f(\{u^{\circ}_{u_K}(\omega, a_i) \mid a_i \in \Psi \sqcup (\varphi,\beta)\}) > f(\{u^{\circ}_{u_K}(\omega', a_i) \mid a_i \in \Psi \sqcup (\varphi,\beta)\})$.
Then $\omega \prec^{u^{\circ}_{u_K},f}_{\Psi\sqcup(\varphi,\beta)} \omega'$.

8. Suppose that $\omega, \omega' \models \varphi$. We want to show that $\forall \alpha, \beta, \omega \preccurlyeq^{u^{\circ}_{u_K},f}_{\Psi\sqcup(\varphi,\alpha)} \omega' \Leftrightarrow \omega \preccurlyeq^{u^{\circ}_{u_K},f}_{\Psi\sqcup(\varphi,\beta)} \omega'$. We show ($\Rightarrow$) ($\Leftarrow$ is symmetrical). Suppose that $\omega \preccurlyeq^{u^{\circ}_{u_K},f}_{\Psi\sqcup(\varphi,\alpha)} \omega'$. By definition, we have $f(\{u^{\circ}_{u_K}(\omega, a_i) \mid a_i \in \Psi \sqcup (\varphi,\alpha)\}) \geqslant f(\{u^{\circ}_{u_K}(\omega', a_i) \mid a_i \in \Psi \sqcup (\varphi,\alpha)\})$. Then:

$$f(\{u^{\circ}_{u_K}(\omega, a_i) \mid a_i \in \Psi, u^{\circ}_{u_K}(\omega,(\varphi,\alpha))\}) \geqslant$$
$$f(\{u^{\circ}_{u_K}(\omega', a_i) \mid a_i \in \Psi, u^{\circ}_{u_K}(\omega',(\varphi,\alpha))\}) \quad (7)$$

As $\omega, \omega' \models \varphi$, $u_u(\omega,\varphi) = u_u(\omega',\varphi) = K$. From invariance of K, we get $u^{\circ}_{u_K}(\omega,(\varphi,\alpha)) = u^{\circ}_{u_K}(\omega,(\varphi,\beta)) = K$ and $u^{\circ}_{u_K}(\omega',(\varphi,\alpha)) = u^{\circ}_{u_K}(\omega',(\varphi,\beta)) = K$.
Then, $f(\{u^{\circ}_{u_K}(\omega, a_i) \mid a_i \in \Psi, u^{\circ}_{u_K}(\omega,(\varphi,\alpha))\}) = f(\{u^{\circ}_{u_K}(\omega, a_i) \mid a_i \in \Psi, u^{\circ}_{u_K}(\omega,(\varphi,\beta))\})$ and $f(\{u^{\circ}_{u_K}(\omega', a_i) \mid a_i \in \Psi, u^{\circ}_{u_K}(\omega',(\varphi,\alpha))\}) = f(\{u^{\circ}_{u_K}(\omega', a_i) \mid a_i \in \Psi, u^{\circ}_{u_K}(\omega',(\varphi,\beta))\})$. Therefore, using (7), we obtain $f(\{u^{\circ}_{u_K}(\omega, a_i) \mid a_i \in \Psi, u^{\circ}_{u_K}(\omega,(\varphi,\beta))\}) \geqslant f(\{u^{\circ}_{u_K}(\omega', a_i) \mid a_i \in \Psi, u^{\circ}_{u_K}(\omega',(\varphi,\beta))\})$. So $f(\{u^{\circ}_{u_K}(\omega, a_i) \mid a_i \in \Psi \sqcup (\varphi,\beta)\}) \geqslant f(\{u^{\circ}_{u_K}(\omega', a_i) \mid a_i \in \Psi \sqcup (\varphi,\beta)\})$. As a consequence, $\omega \preccurlyeq^{u^{\circ}_{u_K},f}_{\Psi\sqcup(\varphi,\beta)} \omega'$.

9. Suppose that $\omega \preccurlyeq^{u^{\circ}_{u_K},f}_{(\varphi,\alpha)} \omega'$. We want to show $\omega \preccurlyeq^{u^{\circ}_{u_K},f}_{(\varphi,\beta)} \omega'$ (it is sufficient as α and β play symmetrical role). By definition we have $f(u^{\circ}_{u_K}(\omega,(\varphi,\alpha))) \geqslant f(u^{\circ}_{u_K}(\omega',(\varphi,\alpha)))$. We can deduce $u^{\circ}_{u_K}(\omega,(\varphi,\alpha)) \geqslant u^{\circ}_{u_K}(\omega',(\varphi,\alpha))$ (otherwise, the increasing property of f would be violated). Then $\circ(u_u(\omega,\varphi),\alpha) \geqslant \circ(u_u(\omega',\varphi),\alpha)$. Suppose that $u_u(\omega,\varphi) < u_u(\omega',\varphi)$. As $\circ$ is utility increasing, we get $\circ(u_u(\omega,\varphi),\alpha) > \circ(u_u(\omega',\varphi),\alpha)$: contradiction. So $u_u(\omega,\varphi) \geqslant u_u(\omega',\varphi)$. As $\circ$ is utility increasing, we obtain $\circ(u_u(\omega,\varphi),\beta) \geqslant \circ(u_u(\omega',\varphi),\beta)$. As a consequence, $u^{\circ}_{u_K}(\omega,(\varphi,\beta)) \geqslant u^{\circ}_{u_K}(\omega',(\varphi,\beta))$.
As f is increasing, $f(u^{\circ}_{u_K}(\omega,(\varphi,\beta))) \geqslant f(u^{\circ}_{u_K}(\omega',(\varphi,\beta)))$. So, $\omega \preccurlyeq^{u^{\circ}_{u_K},f}_{(\varphi,\beta)} \omega'$.

10. Suppose that $\omega \models \varphi$ and $\omega' \not\models \varphi$. We want to show that $\exists \alpha'$, s.t. $\omega \prec^{u^{\circ}_{u_K},f}_{\Psi\sqcup(\varphi,\alpha')} \omega'$. From the fully reducible property of f, we know that $\forall(\alpha_1,...,\alpha_n), \forall(\beta_1,...,\beta_n), \forall\beta, \exists\alpha$ such that $f(\alpha_1,...,\alpha_n,\alpha) < f(\beta_1,...,\beta_n,\beta)$.
We state $\alpha_i = u^{\circ}_{u_K}(\omega', a_i), \beta_i = u^{\circ}_{u_K}(\omega, a_i)$. As $\omega \models \varphi$, $\forall \alpha', u^{\circ}_{u_K}(\omega,(\varphi,\alpha')) = K$. We state $\beta = K$.
We know that $\exists\alpha$ s.t. $f(\alpha_1,...,\alpha_n,\alpha) < f(\beta_1,...,\beta_n,\beta)$.
We state $u = u(\omega',\varphi)$ and $\epsilon = \alpha$. As $\omega' \not\models \varphi$ we have $u < K$, thus from the infinitely reducible property for $\circ$, $\exists\alpha'$ s.t. $\circ(u,\alpha') < \alpha$.
Then, $f(\alpha_1,...,\alpha_n,\circ(u_u(\omega',\varphi),\alpha')) < f(\beta_1,...,\beta_n,\beta)$.
This gives $f(u^{\circ}_{u_K}(\omega', a_i) \mid a_i \in \Psi \sqcup (\varphi,\alpha')) < f(u^{\circ}_{u_K}(\omega, a_i) \mid a_i \in \Psi \sqcup (\varphi,\alpha'))$. So $\omega \prec^{u^{\circ}_{u_K},f}_{\Psi\sqcup(\varphi,\alpha')} \omega'$. □

5.1 Concrete Utilitarian IC Merging Operators

An example of a utilitarian weight function is the function div defined as follows:

Definition 16. *Let div be the function $div : [1,K] \times [1,+\infty] \mapsto]0,K]$, defined by:*

$$div(x,y) = \begin{cases} K \text{ if } x = K \\ x \cdot \frac{1}{y} \text{ otherwise.} \end{cases}$$

A first example of utilitarian aggregation function is the leximin function (Moulin 1988), denoted here the Gmin function.

The product is also a utilitarian aggregation function.

Definition 17. *The function* $\pi : \bigcup_{n\geq 0} (\mathbb{R}^+)^n \longrightarrow \mathbb{R}^+$ *is defined in the following way:*

$$\pi(x_1, \ldots, x_n) = (\prod_{i=1}^{n}(x_i))$$

It is easy to see that:

Proposition 6. *The functions* Gmin *and* π *are utilitarian aggregation functions.*

It is important to stress that not all the aggregation functions are utilitarian aggregation functions. For example, sum and Gmax are not utilitarian aggregation functions (they do not satisfy the fully reducible property). The following example illustrates the behaviour of these operators.

Example 2. *We consider again the profile of the Example 1, with* div *and* π *or* Gmin*. The integrity constraint is* $\mu = \{001, 011, 101, 111\}$. *We obtain:*

	a_1	a_2	a_3	a_4	$u^{div}_{u_{d_H},\pi}$	$u^{div}_{u_{d_H},\mathsf{Gmin}}$
000	4	3	2	3	0.56	(0.25,0.75,0.75,4)
001	3	2	3	2	0.028	**(0.3,0.375,0.5,0.5)**
010	3	4	3	4	1.8	(0.3,0.375,4,4)
011	2	3	2	3	0.028	(0.2,0.25,0.75,0.75)
100	3	2	3	4	0.225	(0.3,0.375,0.5,4)
101	2	1	4	3	**0.15**	(0.2,0.25,0.75,4)
110	4	3	4	3	9	(0.75,0.75,4,4)
111	3	2	2	2	0.019	(0.25,0.3,0.5,0.5)

The operator $\Delta^{u^{div}_{u_{d_H},\pi}}$ *gives* $\{101\}$ *as result, and* $\Delta^{u^{div}_{u_{d_H},\mathsf{Gmin}}}$ *gives* $\{001\}$.

A first question to address is the majoritarian behaviour of these operators.

Proposition 7. $\Delta^{u^{\circ}_{u_K},\pi}$ *is a majority operator, whereas* $\Delta^{u^{\circ}_{u_K},\mathsf{Gmin}}$ *does not satisfy* **(Maj)**.

Proof. Note that Postulate **(Maj)** is indeed equivalent, modulo the other postulates, to the following semantical property:

(SemMaj) For any Ψ_1, Ψ_2, ω and ω'

$$\omega \prec_{\Psi_2} \omega' \quad \Rightarrow \quad \exists n \quad \text{such that} \quad \omega \prec_{\Psi_1 \sqcup \Psi_2^n} \omega'$$

where $\Psi \mapsto \preceq_\Psi$ is the weighted syncretic assignment of Theorem 1. This is Property 7 in (Konieczny and Pino Pérez 2002), and the proof of the equivalence between (SemMaj) and **(Maj)** follows exactly the same argument as in that work. Thus, in order to prove that the operator $\Delta^{u^{\circ}_{u_K},\pi}$ satisfies **(Maj)**, we are going to prove (SemMaj) for the the syncretic weighted assignment representing it. Then, assume $\omega \prec_{\Psi_2} \omega'$, that is $\prod_{a\in\Psi_2} u^{\circ}_{u_K}(\omega, a) > \prod_{a\in\Psi_2} u^{\circ}_{u_K}(\omega', a)$. Then

$$\frac{\prod_{a\in\Psi_2} u^{\circ}_{u_K}(\omega, a)}{\prod_{a\in\Psi_2} u^{\circ}_{u_K}(\omega', a)} > 1 \tag{8}$$

We want to see that there exists an n such that $\prod_{a\in\Psi_1\sqcup\Psi_2^n} u^{\circ}_{u_K}(\omega, a) > \prod_{a\in\Psi_1\sqcup\Psi_2^n} u^{\circ}_{u_K}(\omega', a)$. But this is equivalent to

$$\prod_{a\in\Psi_1} u^{\circ}_{u_K}(\omega, a) \cdot \left(\prod_{a\in\Psi_2} u^{\circ}_{u_K}(\omega, a)\right)^n >$$
$$\prod_{a\in\Psi_1} u^{\circ}_{u_K}(\omega', a) \cdot \left(\prod_{a\in\Psi_2} u^{\circ}_{u_K}(\omega', a)\right)^n$$

for a certain n. That is

$$\left(\frac{\prod_{a\in\Psi_2} u^{\circ}_{u_K}(\omega, a)}{\prod_{a\in\Psi_2} u^{\circ}_{u_K}(\omega', a)}\right)^n > \frac{\prod_{a\in\Psi_1} u^{\circ}_{u_K}(\omega', a)}{\prod_{a\in\Psi_1} u^{\circ}_{u_K}(\omega, a)}$$

for a certain n. An the existence of this n is guaranteed by the fact that the exponential function is unbounded when the base is strictly greater than 1, which is the case because of Equation 8. Thus, the first part of the Proposition is proved.

Now we prove that $\Delta^{u^{\circ}_{u_K},\mathsf{Gmin}}$ does not satisfy **(Maj)**. For this it is enough to find a situation in which (SemMaj) is not satisfied. Suppose that $\Psi_2 = \{a_2\}$, $u^{\circ}_{u_K}(\omega, a_2) = 4$, $u^{\circ}_{u_K}(\omega', a_2) = 2$. Thus we have $\omega \prec_{\Psi_2} \omega'$. Suppose now that $\Psi_1 = \{a_1\}$, $u^{\circ}_{u_K}(\omega, a_1) = 1$, $u^{\circ}_{u_K}(\omega', a_1) = 2$. With this data is easy to see that for any n we have $\omega' \prec_{\Psi_1\sqcup\Psi_2^n} \omega$ because $(2, 2, \ldots, 2) >_{lex} (1, 4, \ldots, 4)$. Thus, (SemMaj) doesn't hold. □

Let us now investigate the behaviour of these new operators regarding the egalitarian side. We first look at the Pigou-Dalton property, which can be expressed as follow when utilities are considered:

Definition 18. *(Pigou-Dalton for utilities) A WIC merging operator* $\Delta^{u^{\circ}_{u_K},f}$*, satisfies the U-Pigou-Dalton property if for any* $\Psi = \{a_1, \ldots, a_n\}$*, if* $\exists$ k *and* l *such that* $u^{\circ}_{u_K}(\omega, a_l) < u^{\circ}_{u_K}(\omega', a_l) \leqslant u^{\circ}_{u_K}(\omega', a_k) < u^{\circ}_{u_K}(\omega, a_k)$, $u^{\circ}_{u_K}(\omega', a_k) - u^{\circ}_{u_K}(\omega, a_k) = u^{\circ}_{u_K}(\omega, a_l) - u^{\circ}_{u_K}(\omega', a_l)$ *and* $\forall i \neq k$ *and* $i \neq l, u^{\circ}_{u_K}(\omega, a_i) = u^{\circ}_{u_K}(\omega', a_i)$*, then* $\omega' \prec^{u^{\circ}_{u_K},f}_{\Psi} \omega$

Proposition 8. *For every utility* u_K *and utilitarian weight function* $\circ$*,* $\Delta^{u^{\circ}_{u_K},\pi}$ *and* $\Delta^{u^{\circ}_{u_K},\mathsf{Gmin}}$ *satisfy the U-Pigou-Dalton property.*

Proof. As the proof for Gmin is quite straightforward, we only give the proof for π. Suppose that $u^{\circ}_{u_K}(\omega, a_k) > u^{\circ}_{u_K}(\omega', a_k) \geqslant u^{\circ}_{u_K}(\omega', a_l) > u^{\circ}_{u_K}(\omega, a_l)$ and $u^{\circ}_{u_K}(\omega', a_k) - u^{\circ}_{u_K}(\omega, a_k) = u^{\circ}_{u_K}(\omega, a_l) - u^{\circ}_{u_K}(\omega', a_l)$. We have to show that $\omega' \prec^{u^{\circ}_{u_K},f}_{\Psi} \omega$, so we want to show that $u^{\circ}_{u_K,\pi}(\omega', \Psi) > u^{\circ}_{u_K,\pi}(\omega, \Psi)$.

Consider $\Psi = \{a_k, a_l, a_{l+1}, \ldots, a_n\}$, $\epsilon \in \mathbb{R}^+_*$, s.t.

$$u^{\circ}_{u_K}(\omega', a_l) = u^{\circ}_{u_K}(\omega, a_l) + \epsilon \tag{9}$$

$$u^{\circ}_{u_K}(\omega', a_k) = u^{\circ}_{u_K}(\omega, a_k) - \epsilon \tag{10}$$

Compute first $u^{\circ}_{u_K,\pi}(\omega', \Psi)$:
$u^{\circ}_{u_K,\pi}(\omega', \Psi) = u^{\circ}_{u_K}(\omega', a_l) \cdot u^{\circ}_{u_K}(\omega', a_k)$

From Equations 9 and 10, we get: $u^{\circ}_{u_K,\pi}(\omega', \Psi) = (u^{\circ}_{u_K}(\omega, a_l) + \epsilon) \cdot (u^{\circ}_{u_K}(\omega, a_k) - \epsilon)$.
We obtain:

$$u^{\circ}_{u_K,\pi}(\omega', \Psi) = u^{\circ}_{u_K}(\omega, a_k) \cdot \epsilon - u^{\circ}_{u_K}(\omega, a_l) \cdot \epsilon - \epsilon^2 + u^{\circ}_{u_K}(\omega, a_k) \cdot u^{\circ}_{u_K}(\omega, a_l) \tag{11}$$

From assumption: $u^{\circ}_{u_K}(\omega, a_k) > u^{\circ}_{u_K}(\omega', a_l)$
As $\epsilon > 0$, we have: $u^{\circ}_{u_K}(\omega, a_k) \cdot \epsilon > u^{\circ}_{u_K}(\omega', a_l) \cdot \epsilon$
With Equation 9, we obtain $u^{\circ}_{u_K}(\omega, a_k) \cdot \epsilon > (u^{\circ}_{u_K}(\omega, a_l) + \epsilon) \cdot \epsilon$

And then $u^{\circ}_{u_K}(\omega, a_k) \cdot \epsilon - u^{\circ}_{u_K}(\omega, a_l) \cdot \epsilon - \epsilon^2 > 0$

Finally we get $u^{\circ}_{u_K}(\omega, a_k) \cdot \epsilon - u^{\circ}_{u_K}(\omega, a_l) \cdot \epsilon - \epsilon^2 + u^{\circ}_{u_K}(\omega, a_k) \cdot u^{\circ}_{u_K}(\omega, a_l) > u^{\circ}_{u_K}(\omega, a_k) \cdot u^{\circ}_{u_K}(\omega, a_l)$

With Equation 11 and definition of π, we obtain $u^{\circ}_{u_K,\pi}(\omega', \Psi) > u^{\circ}_{u_K,\pi}(\omega, \Psi)$. □

The satisfaction of U-Pigou-Dalton by product with utility-based operators is intuitively explained by the fact that this framework is really close to the classical Nash function in social choice. The reason why D-Pigou-Dalton is not satisfied by product with distance-based is a consequence of the behaviour of the product. The product in the framework of distances will penalise interpretations with small gaps between the satisfaction of the agents, and that goes against D-Pigou-Dalton property. For instance, if the distances of interpretation ω to the profile are (10, 8, 5, 12, 15), and the distances of interpretation ω' to the profile are (10, 8, 8, 12, 12), we get $\pi(10, 8, 5, 12, 15) = 72000$ and $\pi(10, 8, 8, 12, 12) = 92160$ and the interpretation with the smallest distance, which is selected, is the bad one with respect to Pigou-Dalton.

At first glance, it may seem that there is a correspondence between a distance d and the utility u_d defined from d. In fact, it is not the case, as π satisfies the Pigou-Dalton property when utilities are considered and not when distances are considered.

Corollary 2. *Let u_K be a utility function and $\circ$ a utilitarian weight function, then*

$$\Delta^{u^{\circ}_{u_K,\pi}} \neq \Delta^{d^{\bullet}_{d,*}}$$

for any pseudo-distance d and any weight function $\bullet$.

As $\Delta^{u^{\circ}_{u_K,\pi}}$ and $\Delta^{u^{\circ}_{u_K,\mathsf{Gmin}}}$ satisfy the U-Pigou-Dalton property, a natural question is to investigate if the other egalitarian property, **Arb** is also satisfied or not. In fact, it is not the case:

Proposition 9. *$\Delta^{u^{\circ}_{u_K,\mathsf{Gmin}}}$ is an Arbitration operator but $\Delta^{u^{\circ}_{u_K,\pi}}$ does not satisfy (Arb).*

Proof. Note that Postulate **(Arb)** is indeed equivalent, modulo the other postulates, to the following semantical property for any a_1, a_2, ω, ω' and ω'':

$$\text{(SemArb)} \qquad \left.\begin{array}{l} \omega \prec_{a_1} \omega' \\ \omega \prec_{a_2} \omega'' \\ \omega' \simeq_{a_1 \sqcup a_2} \omega'' \end{array}\right\} \Rightarrow \omega \prec_{a_1 \sqcup a_2} \omega'$$

where $\Psi \mapsto \preceq_\Psi$ is the weighted syncretic assignment of Theorem 1. This is Property 8 in (Konieczny and Pino Pérez 2002), and the proof of the equivalence between (SemArb) and **(Arb)** follows exactly the same arguments as in that work. Thus, in order to prove the satisfaction of **(Arb)** for the operator $\Delta^{u^{\circ}_{u_K,\mathsf{Gmin}}}$, we are going to prove (SemArb) for the the syncretic weighted assignment representing it. Then, assume $\omega \prec_{a_1} \omega'$, $\omega \prec_{a_2} \omega''$ and $\omega' \simeq_{a_1 \sqcup a_2} \omega''$. Suppose that $u^{\circ}_{u_K}(\omega, a_1) = a$, $u^{\circ}_{u_K}(\omega', a_1) = a'$, $u^{\circ}_{u_K}(\omega, a_2) = b$, $u^{\circ}_{u_K}(\omega'', a_2) = b'$, $u^{\circ}_{u_K}(\omega'', a_1) = x$ and $u^{\circ}_{u_K}(\omega', a_2) = y$. Then, the assumptions means $a > a'$, $b > b'$ and $(a', y) \simeq_{\mathsf{Gmin}} (b', x)$. Without loss of generality, we can suppose $a \leq b$. We consider three cases. First, suppose that $a' < b'$. Then, as $(a', y) \simeq_{\mathsf{Gmin}} (b', x)$, $x = a'$ and $y = b'$. Thus, $u^{\circ}_{u_K,\mathsf{Gmin}}(\omega, a_1 \sqcup a_2) = (a, b) >_{lex} (a', b') = u^{\circ}_{u_K,\mathsf{Gmin}}(\omega', a_1 \sqcup a_2)$. The second case, $b' < a'$, is analogous to the first case. The third case is when $a' = b'$. Then, as $(a', y) \simeq_{\mathsf{Gmin}} (b', x)$, $x = y$. If $a' \leq x$, we have $u^{\circ}_{u_K,\mathsf{Gmin}}(\omega, a_1 \sqcup a_2) = (a, b) >_{lex} (a', x) = u^{\circ}_{u_K,\mathsf{Gmin}}(\omega', a_1 \sqcup a_2)$. If $x < a'$ then $u^{\circ}_{u_K,\mathsf{Gmin}}(\omega, a_1 \sqcup a_2) = (a, b) >_{lex} (x, a') = u^{\circ}_{u_K,\mathsf{Gmin}}(\omega', a_1 \sqcup a_2)$. Therefore, in any case, (SemArb) holds.

Now we prove that $\Delta^{u^{\circ}_{u_K,\pi}}$ does not satisfy **(Arb)**. For this it is enough to find a situation in which (SemArb) is not satisfied. Suppose that $u^{\circ}_{u_K}(\omega, a_1) = 5$, $u^{\circ}_{u_K}(\omega', a_1) = 1$, $u^{\circ}_{u_K}(\omega, a_2) = 5$, $u^{\circ}_{u_K}(\omega'', a_2) = 1$, $u^{\circ}_{u_K}(\omega'', a_1) = 30$ and $u^{\circ}_{u_K}(\omega', a_2) = 30$. Then, $u^{\circ}_{u_K,\pi}(\omega, a_1 \sqcup a_2) = 5 \times 5 < 1 \times 30 = u^{\circ}_{u_K,\pi}(\omega', a_1 \sqcup a_2)$. Therefore, (SemArb) is violated. □

This result helps to have a picture of the links between **Arb** and the Pigou-Dalton property: some operators satisfy both **Arb** and Pigou-Dalton (those using Gmax and distances); some operators satisfy neither **Arb** nor Pigou-Dalton ($\Delta^{d^{\times}_{d_H,*}}$ or $\Delta^{d^{pow}_{d_H,*}}$); some operators satisfy Pigou-Dalton and not **Arb**, for instance $\Delta^{u^{\circ}_{u_K,\pi}}$. An open question is whether it is possible to have a WIC operator satisfying **Arb** and not Pigou-Dalton. These two conditions seems logically independent. More precisely, there is no apparent reason why **Arb** implies Pigou-Dalton. But in fact, there is only one known Arbitration operator built with distances: Gmax. As this operator also satisfies Pigou-Dalton, the question is still open. It leads to another interesting question: is the **Arb** postulate a characterization for Gmax with distances?

6 Conclusion

In this paper we aimed at studying weighted merging operators that exhibit some egalitarian behaviour. We introduce the product aggregation function for distance-based merging operators. These operators, surprisingly, do not satisfy the expected Pigou-Dalton property, one of the basic egalitarian properties.

In order to be able to satisfy this Pigou-Dalton property we introduce a new family of merging operators: Utility-based merging operators. While they have close connections with distance-based merging operators, and while some operators can be easily defined in both families, for some operators it is far less direct. And the utility-based operators using the product as aggregation function, that satisfy the Pigou-Dalton property, do not correspond to any known distance-based merging operators.

As future work, a more in depth investigation of utility-based merging operators seems interesting. In particular, it would be interesting to know if these two families are distinct, or if they are just two representations of the same families of operators, and that the choice of one over the other has to be made just to obtain easier definitions of operators.

Another interesting aspect to explore in relation to Nash welfare is whether there are natural concepts in belief merging related to the envy-free notion with respect to which the Nash function has very good properties (Caragiannis et al. 2019).

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