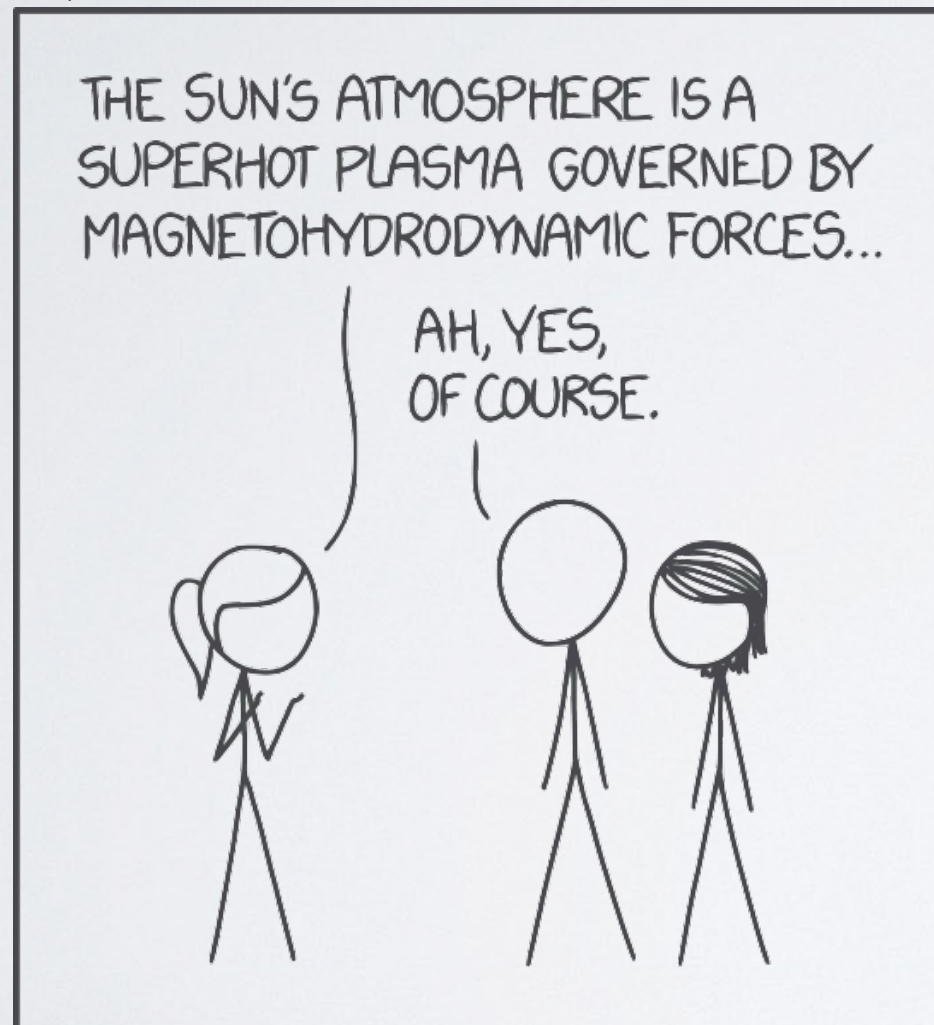


Magnetohydrodynamics and the Geometry of Conservation Laws in Physics

<https://xkcd.com/1851/>



WHENEVER I HEAR THE WORD "MAGNETOHYDRODYNAMIC" MY BRAIN JUST REPLACES IT WITH "MAGIC."

Mark Gillespie



Why?

Variational Integrators for Ideal Magnetohydrodynamics

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March 13, 2018

Abstract

A variational integrator for ideal magnetohydrodynamics is derived by applying a discrete action principle to a formal Lagrangian. Discrete exterior calculus is used for the discretisation of the field variables in order to preserve their geometrical character. The resulting numerical method is free of numerical resistivity, thus the magnetic field line topology is preserved and unphysical reconnection is absent. In 2D numerical examples we find that important conservation laws like total energy, magnetic helicity and cross helicity are satisfied within machine accuracy.

Keywords: Conservation Laws, Discrete Exterior Calculus, Geometric Discretization, Lagrangian Field Theory, Magnetohydrodynamics, Variational Integrators,

arXiv:1707.03227v2 [math.NA] 12 Mar 2018

VARIATIONAL INTEGRATION FOR IDEAL MAGNETOHYDRODYNAMICS AND FORMATION OF CURRENT SINGULARITIES

YAO ZHOU

A DISSERTATION
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OF DOCTOR OF PHILOSOPHY

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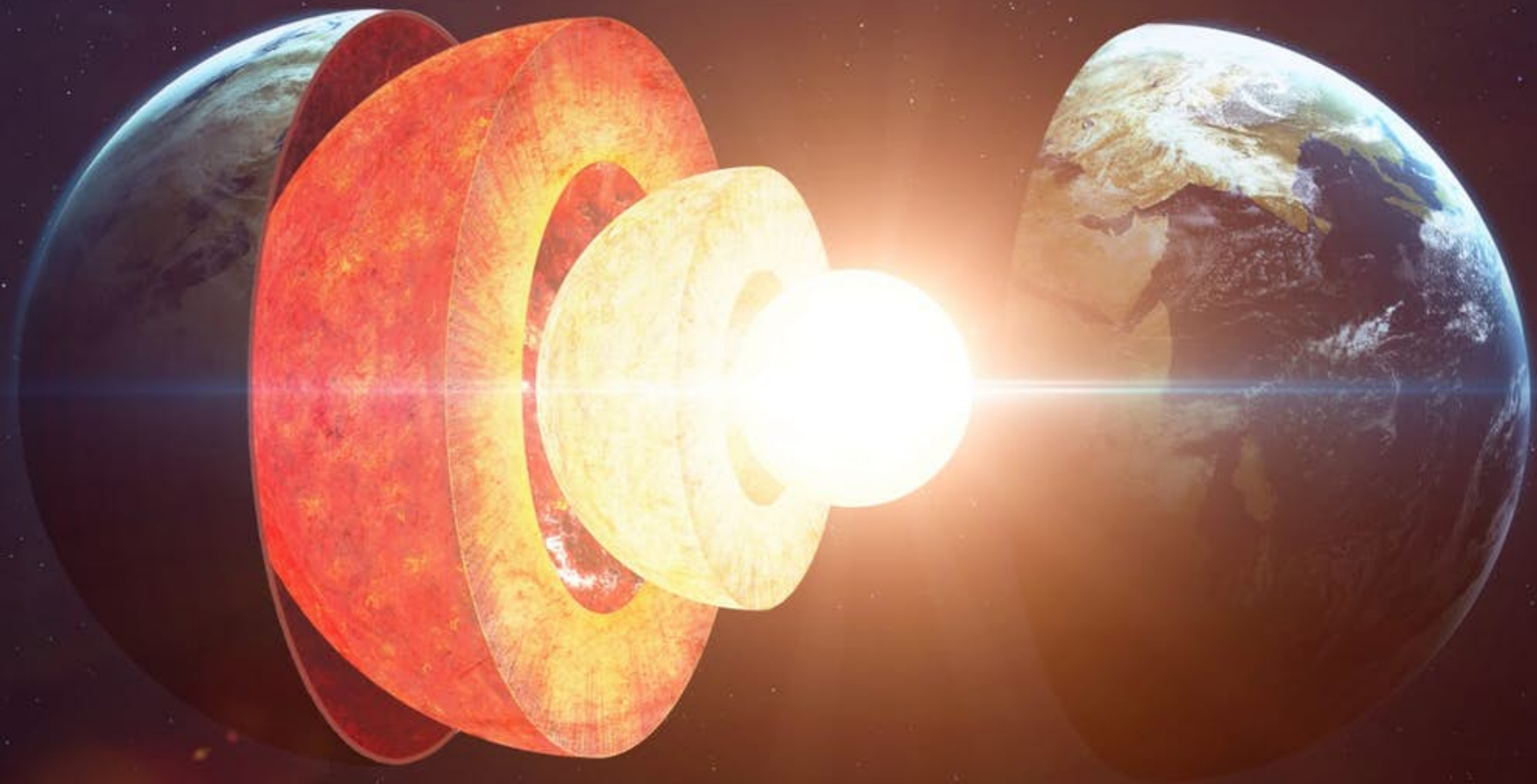
Why?

https://upload.wikimedia.org/wikipedia/commons/e/e3/Magnificent_CME_Erupts_on_the_Sun_-_August_31.jpg



Why?

<https://newatlas.com/earth-inner-core-solid-soft/56882/>



Structure-Preserving Integrators

Eurographics/ACM SIGGRAPH Symposium on Computer Animation (2006)
M.-P. Cani, J. O'Brien (Editors)

Geometric, Variational Integrators for Computer Animation

L. Kharevych Weiwei Y. Tong E. Kanso† J. E. Marsden P. Schröder M. Desbrun

Abstract

We present a particular time integration scheme for Hamiltonian systems, such as linear behavior, even simply; finally properties set an update step factor of two in implementation called the applicability.

1. Introduction

Mathematical models of physical systems (whether in animation) generally involve solving a physical system forward in time. Although allowing the computation of a ball (i.e., its position in the air). Although analytically, direct solving a system are to resort to numerical description of a significant amount how to deal with solutions, leading to a plethora of properties, order of implementation of these integrators are of most physics-based methods (such as finite difference and more recently the choice in practice is better (i.e., faster and very little attention is dedicated to this goal).

In this paper, we focus on numerical-analytical integration. Motivated by approaches in geometric mechanics, we will take a different point of view. The vorticity is concentrated within the green region.

© The Eurographics Association

Schrödinger's Smoke

Albert Chern Caltech Felix Knöppel TU Berlin Ulrich Pinkall TU Berlin Peter Schröder Caltech Steffen Weißmann Google Inc.

Abstract

We describe a new approach for the purely Eulerian simulation of incompressible fluids. In it, the fluid state is represented by a \mathbb{C}^2 -valued wave function evolving under the Schrödinger equation subject to incompressibility constraints. The underlying dynamical system is Hamiltonian and governed by the kinetic energy of the fluid together with an energy of Landau-Lifshitz type. The latter ensures that dynamics due to thin vortical structures, all important for visual simulation, are faithfully reproduced. This enables robust simulation of intricate phenomena such as vortical wakes and interacting vortex filaments, even on modestly sized grids. Our implementation uses a simple splitting method for time integration, employing the FFT for Schrödinger evolution as well as constraint projection. Using a standard penalty method we also allow arbitrary obstacles. The resulting algorithm is simple, unconditionally stable, and efficient. In particular it does not require any Lagrangian techniques for advection or to counteract the loss of vorticity. We demonstrate its use in a variety of scenarios, compare it with experiments, and evaluate it against benchmark tests. A full implementation is included in the ancillary materials.

Keywords: discrete differential geometry, fluid simulation, Schrödinger operator

Concepts: Mathematics of computing → Partial differential equations; Computing methodologies → Physical simulation; Applied computing → Physics;

1 Introduction

We introduce *incompressible Schrödinger flow* (ISF), a new method to simulate incompressible fluids (Fig. 1, middle). Instead of describing the fluid evolution in terms of the velocity or vorticity field, ISF evolves a two-component wave function $\psi = (\psi_1, \psi_2)^T: M \rightarrow \mathbb{C}^2$, which encodes the fluid state on a 3D domain M . The classical fluid density ρ and fluid velocity $v = (v_1, v_2, v_3)^T$ are extracted from ψ as

$$\rho = |\psi|^2 = \langle \psi, \psi \rangle_{\mathbb{R}} \quad \text{and} \quad \rho v_\alpha = \hbar \left(\frac{\partial \psi}{\partial x_\alpha}, i\psi \right)_{\mathbb{R}} \quad \alpha = 1, 2, 3$$

where $\langle \phi, \psi \rangle_{\mathbb{R}} = \text{Re}(\langle \phi, \psi \rangle_{\mathbb{C}}) = \text{Re}(\int \bar{\phi}_1 \psi_1 + \bar{\phi}_2 \psi_2)$. The time evolution of these wave functions is governed by the Schrödinger equation

$$i\hbar \psi_t = -\frac{\hbar^2}{2} \Delta \psi + p \psi \quad \frac{\partial \psi}{\partial x_\alpha} \Big|_{\partial M} = 0 \quad (1)$$

subject to the constraints

$$\langle \Delta \psi, i\psi \rangle_{\mathbb{R}} = 0 \quad \text{and} \quad |\psi|^2 = 1, \quad (2)$$

which correspond to $\text{div}(v) = 0$ and $\rho = 1$ in the classical variables (Sec. 4.1). The scalar potential $p: M \rightarrow \mathbb{R}$ in Eq. (1) is the Lagrange multiplier for the divergence constraint (App. A), and we will refer to it as *pressure* in analogy to the Euler equation. The reduced Planck constant \hbar of quantum Physics becomes the only parameter for our fluid and controls the quantization of vorticity. For a large range of initial conditions ISF tends to concentrate vorticity in filaments of strength $2\pi\hbar$ (Fig. 1, bottom).

We call Eqs. (1) and (2) the *incompressible Schrödinger equations* and the corresponding flow the *incompressible Schrödinger flow*.

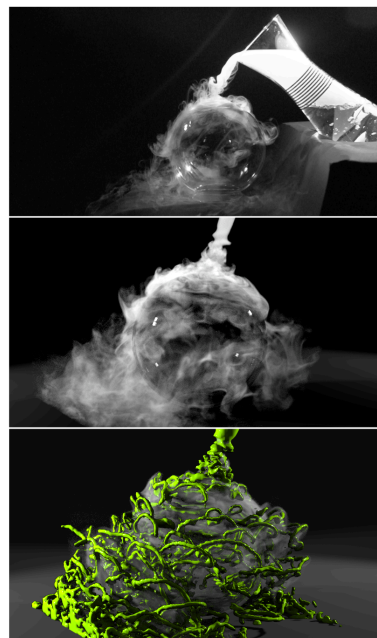


Figure 1: Comparing experiment (dry ice vapor, top) with ISF simulation (middle), followed by a visualization of the underlying wave function ψ . Vorticity is concentrated within the green region.

Energy-Preserving Integrators for Fluid Animation

Patrick Mullen Caltech Keenan Crane Caltech Dmitry Pavlov Caltech Yiying Tong MSU Mathieu Desbrun Caltech

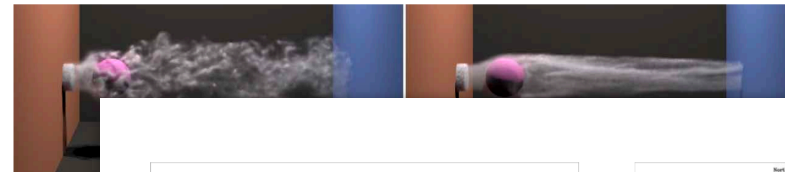


Figure 1: By viscosity in fluid

Abstract

Numerical visc Existing metho ten apply comp effects. Conse or modeled exp complicating th ping metho. fully Eulerian are capable of recourse to con ciently and sim cial grids, and in the case of in extension of the Nicolson sche

CR Categories: and Realism—A

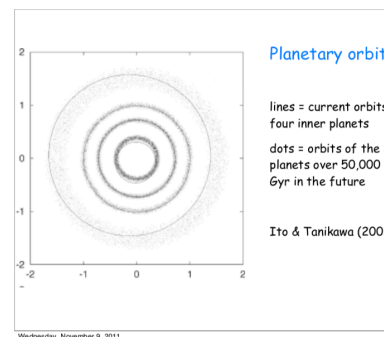
Keywords: Euler

1 Introduction

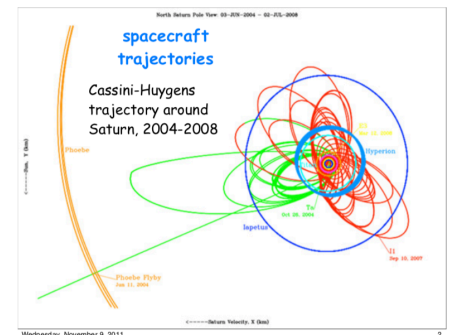
Physically-base incompressible these equations and has been a the past thirty y orate CFD meth Eulerian discre advection have for the last few

Geometric methods for orbit integration

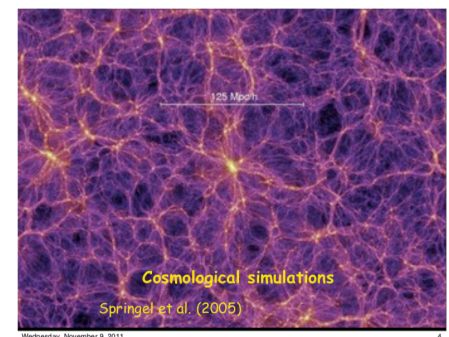
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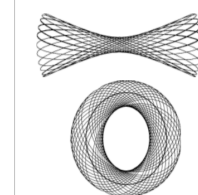


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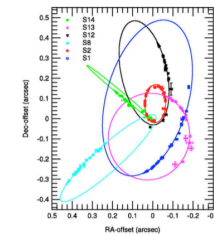


Wednesday, November 9, 2011

Galactic dynamics



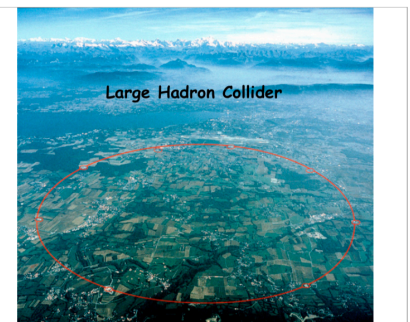
box and tube orbits in a galactic potential



orbits of stars near the Galactic center Eisenhauer et al. (2005)

Wednesday, November 9, 2011

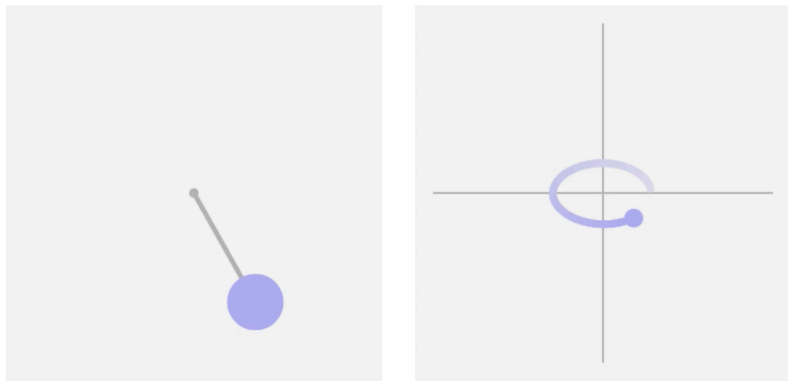
Large Hadron Collider



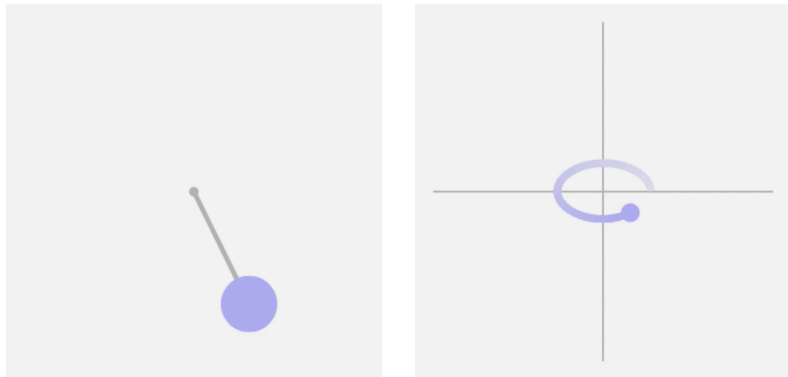
Wednesday, November 9, 2011

Symplectic Integrators

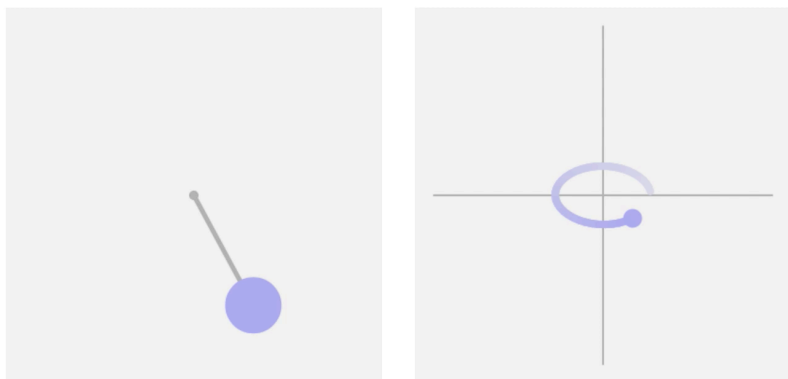
Explicit Euler



Implicit Euler

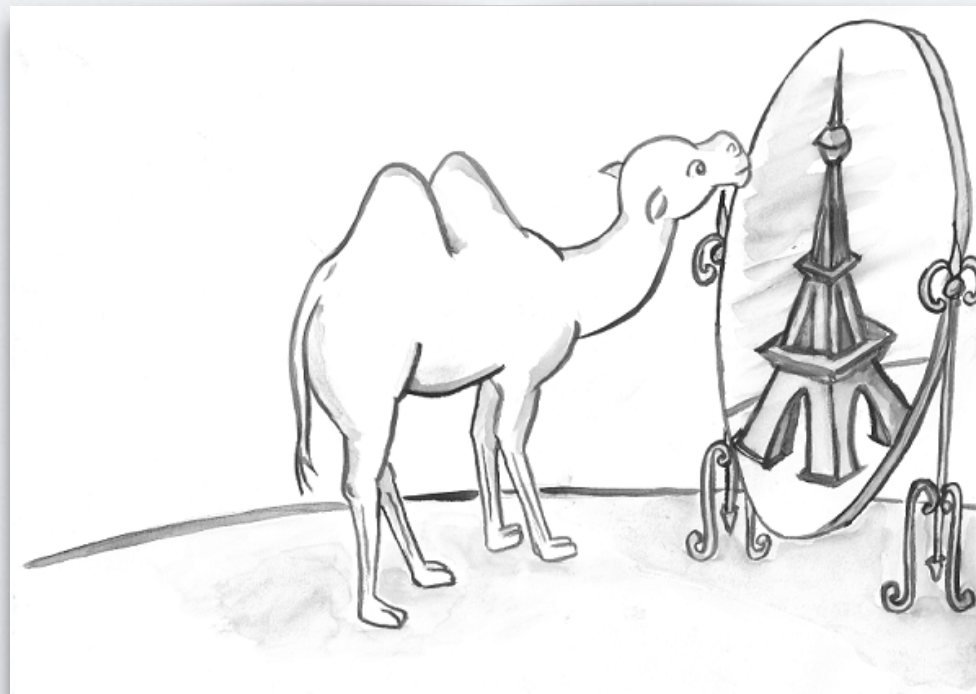
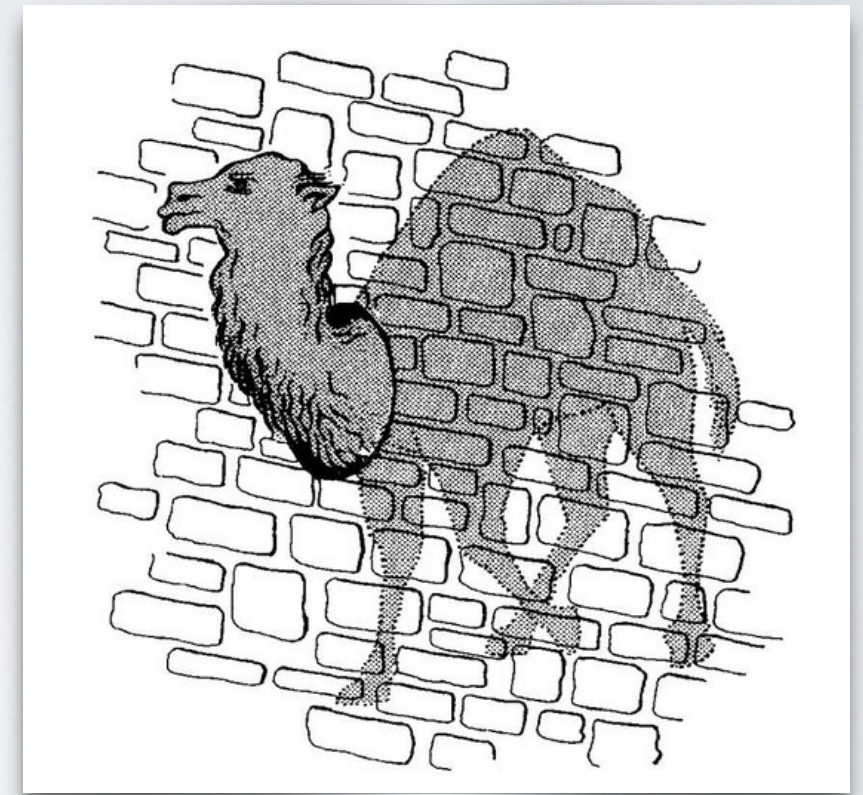
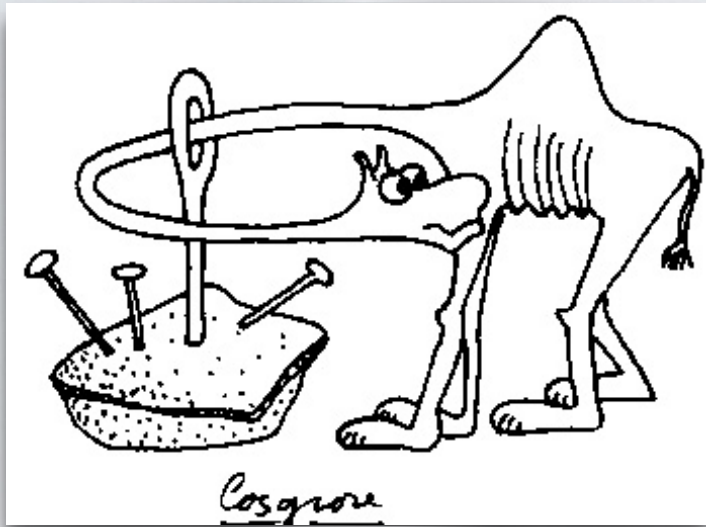


Symplectic Euler

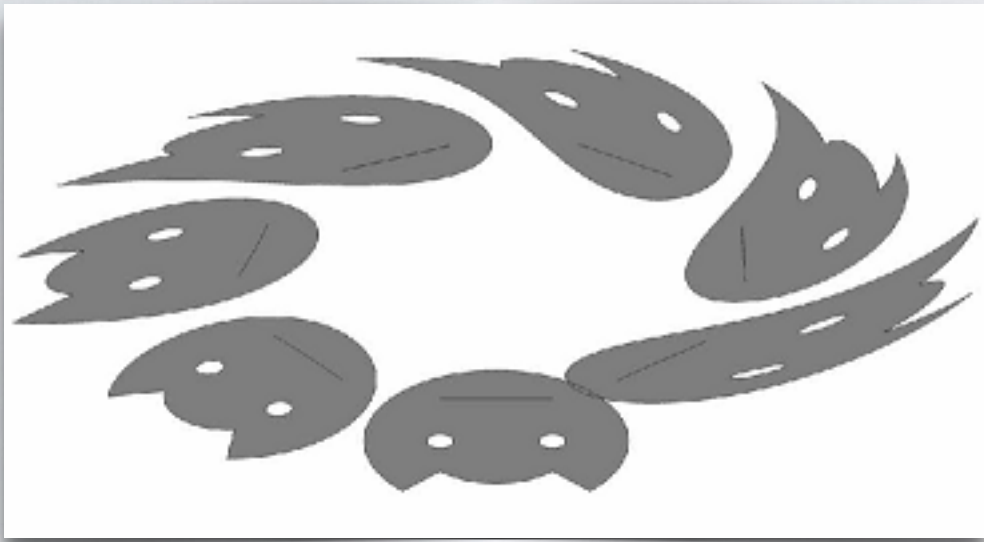


- How you update velocity makes a big difference
- One option makes your simulation *symplectic*
- It captures important features

Symplectic?



Symplectic?



<http://people.bath.ac.uk/tjs42/BNA/bna-res.html>

- In 2D, *symplectic* just means *area-preserving*
- Really, symplectic maps generalize area-preserving maps

Symplectic Euler is Symplectic

- Luckily for us, the pendulum's *phase space* is 2D.
- How do we measure areas in 2D?
 - Determinants
- Useful fact:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = \begin{pmatrix} a \\ c \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix}$$

Symplectic Euler is Symplectic

- So the matrix $\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ measures (infinitesimal) areas
- Our simulation preserves area (and is thus symplectic) if it “preserves” this matrix

Definitions

$$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{“Area matrix”}$$

Symplectic Euler is Symplectic

- Given a force function $F(q)$, and a time step h , symplectic Euler updates positions (q) and momenta (p) by

$$\begin{pmatrix} q_{n+1} \\ p_{n+1} \end{pmatrix} = \begin{pmatrix} q_n + hp_n + h^2 F(q_n) \\ p_n + hF(q_n) \end{pmatrix} =: T \begin{pmatrix} q_n \\ p_n \end{pmatrix}$$

- What does this do to Ω ?

Definitions

$$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{“Area matrix”}$$

$$T \quad \text{Update rule}$$

Symplectic Euler is Symplectic

- How is the area of a parallelogram related to the area of $T(\text{parallelogram})$?
- If the parallelogram is tiny, T looks like a linear map, its *linearization* (or *Jacobian*) $L = dT$
- We only need to check that L preserves area

Definitions

$$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{“Area matrix”}$$

$$T \quad \text{Update rule}$$

$$L = dT \quad \text{Linearization of } T$$

Symplectic Euler is Symplectic

- Note that $\text{Area}(Lv, Lw) = (Lv)^T \Omega (Lw)$
 $= v^T L^T \Omega L w$
 $= v^T [L^T \Omega L] w$
- So our transformation preserves area (and is symplectic) as long as

$$\Omega = L^T \Omega L$$

Definitions

$$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{“Area matrix”}$$

$$T \quad \text{Update rule}$$

$$L = dT \quad \text{Linearization of } T$$

Symplectic Euler is Symplectic

- Now, we can just do this computation

$$T\left(\begin{pmatrix} q_n \\ p_n \end{pmatrix}\right) = \begin{pmatrix} q_n + hp_n + h^2F(q_n) \\ p_n + hF(q_n) \end{pmatrix} \quad L = \begin{pmatrix} 1 + h^2\partial_q F & h \\ h\partial_q F & 1 \end{pmatrix}$$

...

$$\Omega = L^T \Omega L$$

(You can also observe that $\det(L) = 1$)

Definitions

$$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{“Area matrix”}$$

$$T \quad \text{Update rule}$$

$$L = dT \quad \text{Linearization of } T$$

Symplectic?

- In n dimensions, we can define* $\Omega = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}$
- Symplectic maps are still the maps which satisfy

$$\Omega = L^T \Omega L$$

Definitions

$$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{“Area matrix”}$$

T Update rule

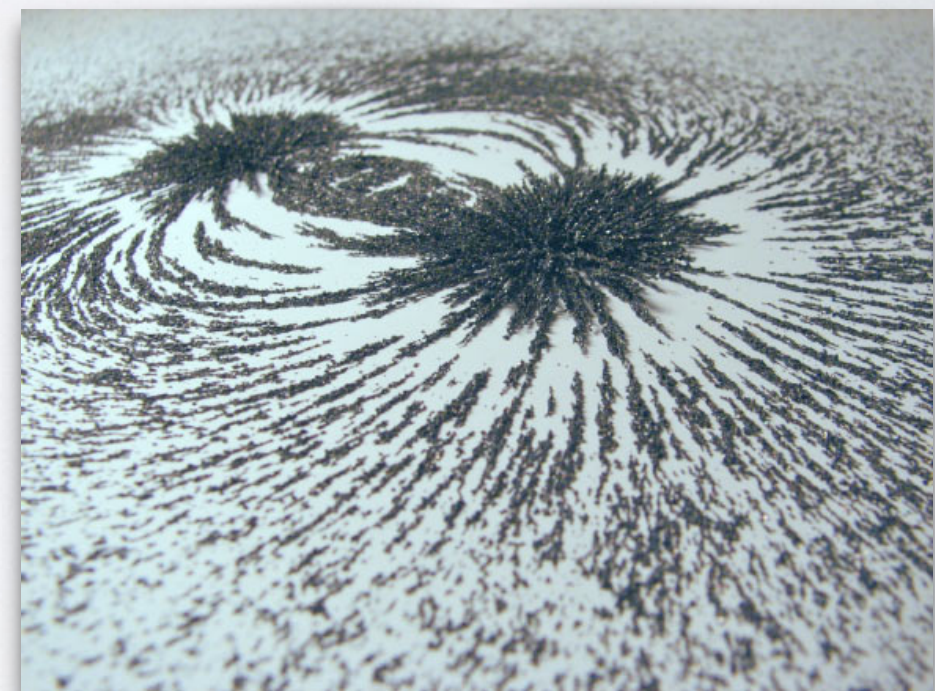
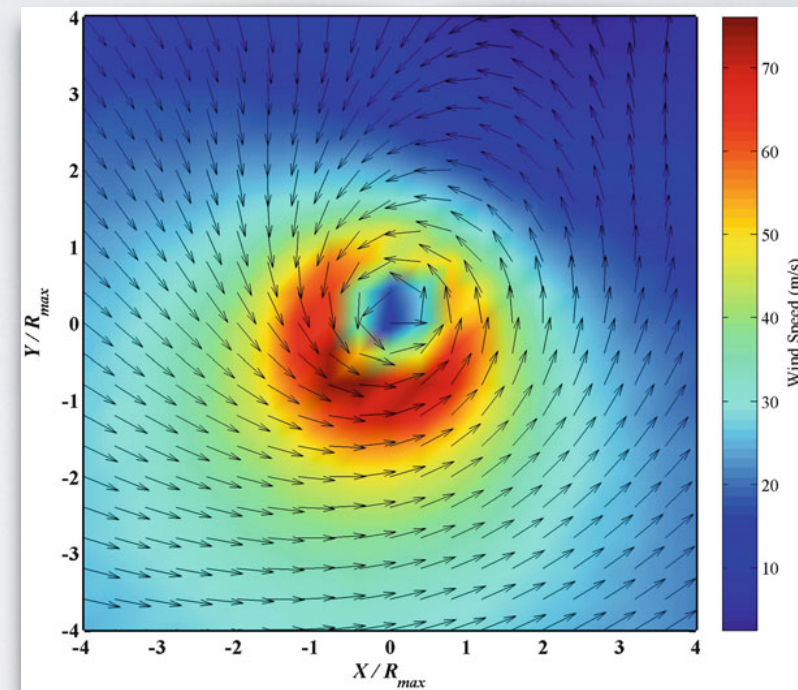
$L = dT$ Linearization of T

How does this relate to camels?

- Symplectic geometry was confusing, even to mathematicians at first
- It's "clear" that symplectic maps preserve volume
- Gromov's non-squeezing theorem

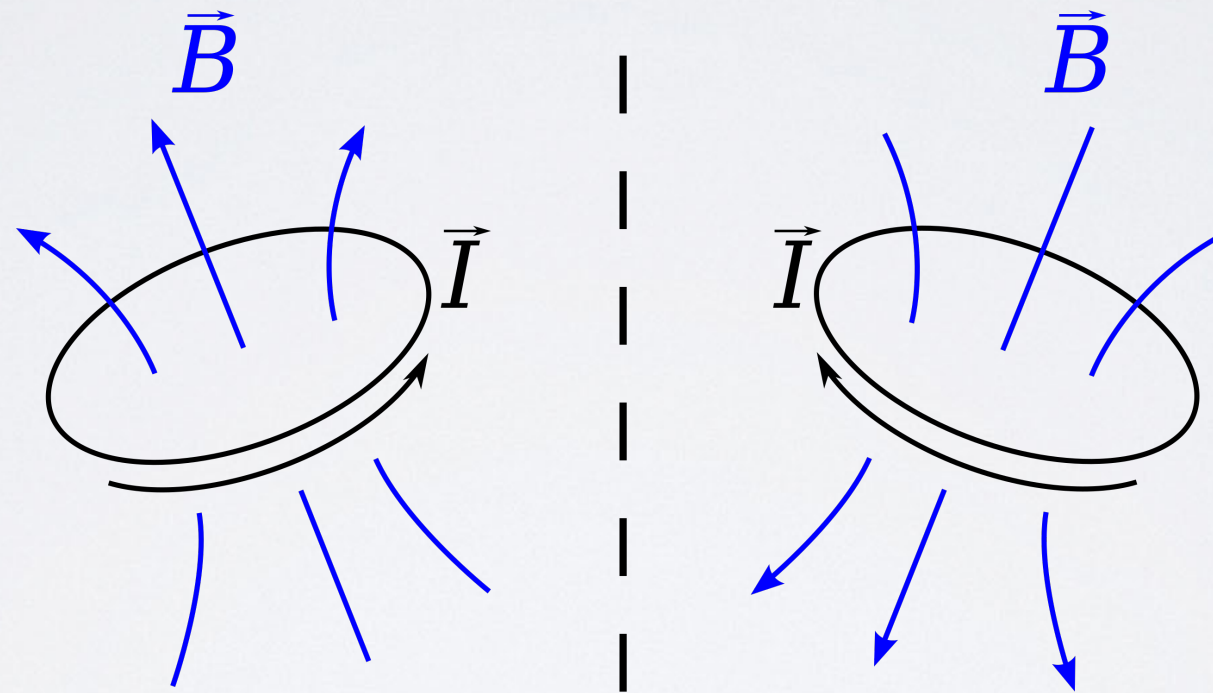
Magnetohydrodynamics

- Physics of conducting fluids
- Key ingredients
 - Velocity field (1-form) η
 - Magnetic field (2-form) β



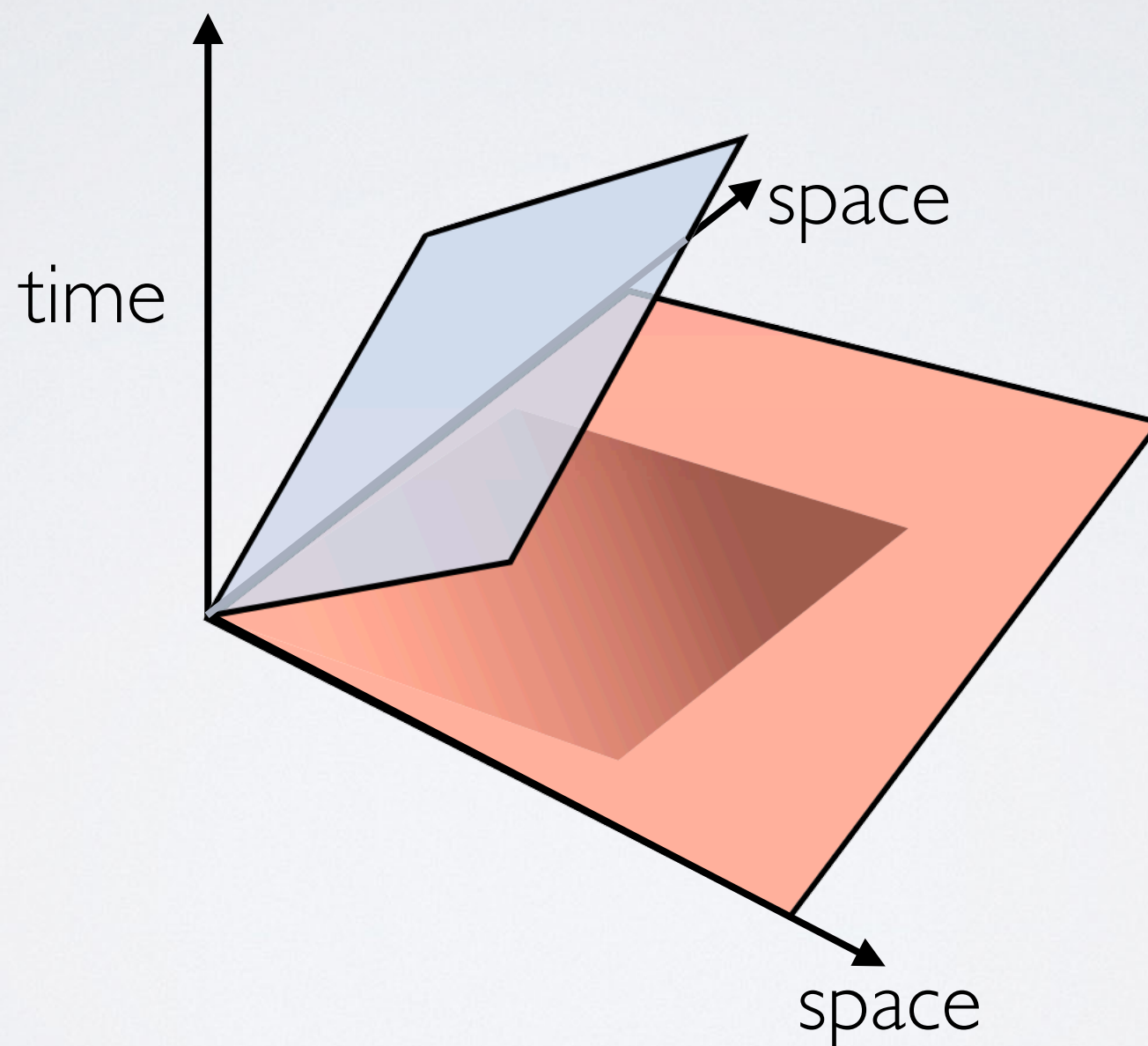
Aside: The Magnetic Field is not a Vector Field

- Wrong symmetries under reflection



- More like an oriented plane than a vector
 - 2D!

Even More Aside: Faraday 2-Form



Fluids

- Fluids are difficult
- Simplifying assumptions:
 - Incompressible
 - No viscosity
 - Unit density

Fluids

- How does a fluid's velocity change over time?
- Fluid is made up of small particles which each have a velocity
- The fluid drags the velocity field along

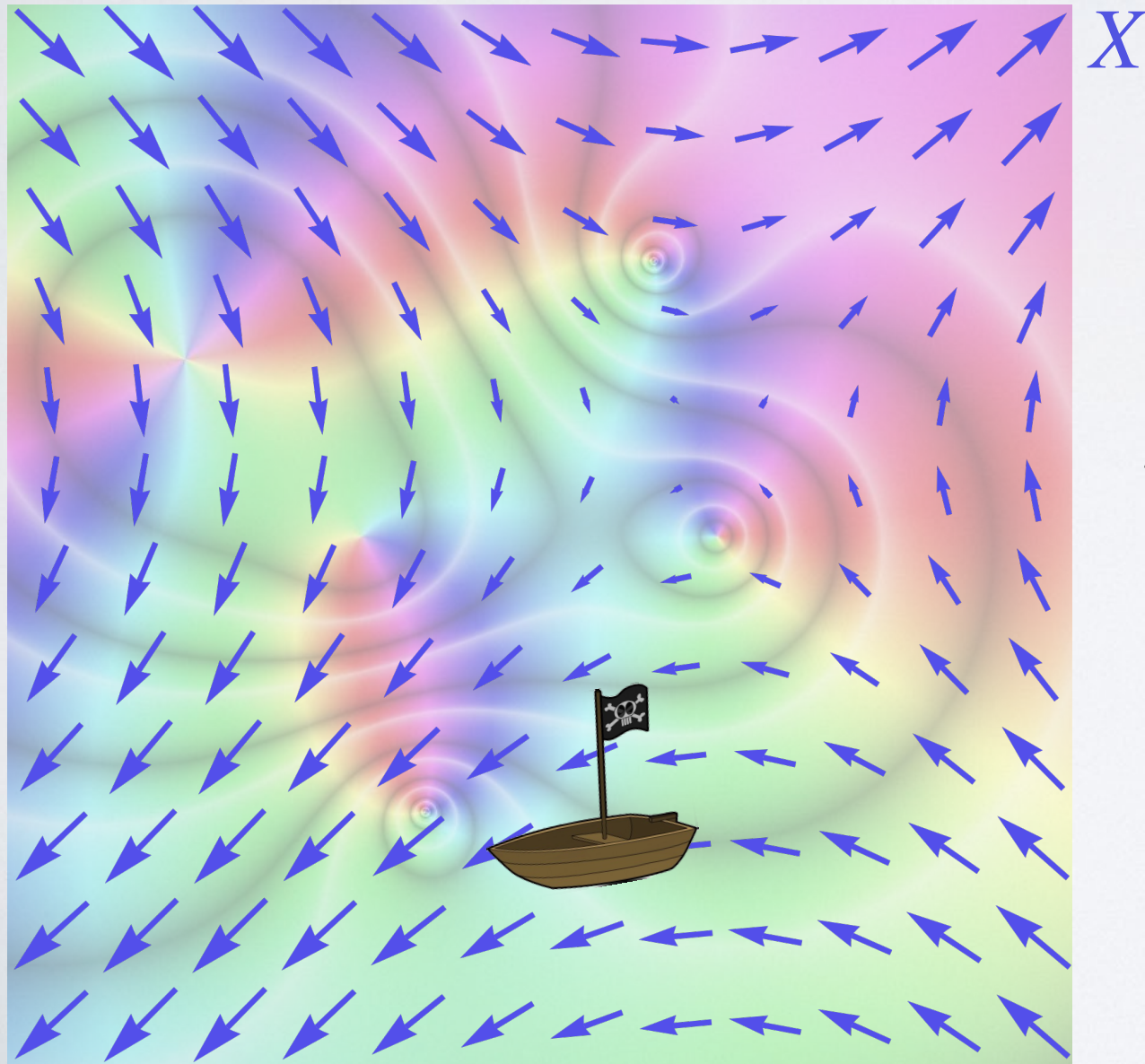
$$\frac{d\eta}{dt} = - \mathcal{L}_{\eta} \eta$$

Definitions

η Velocity

Aside: Lie Derivative

- What is that weird \mathcal{L} ?

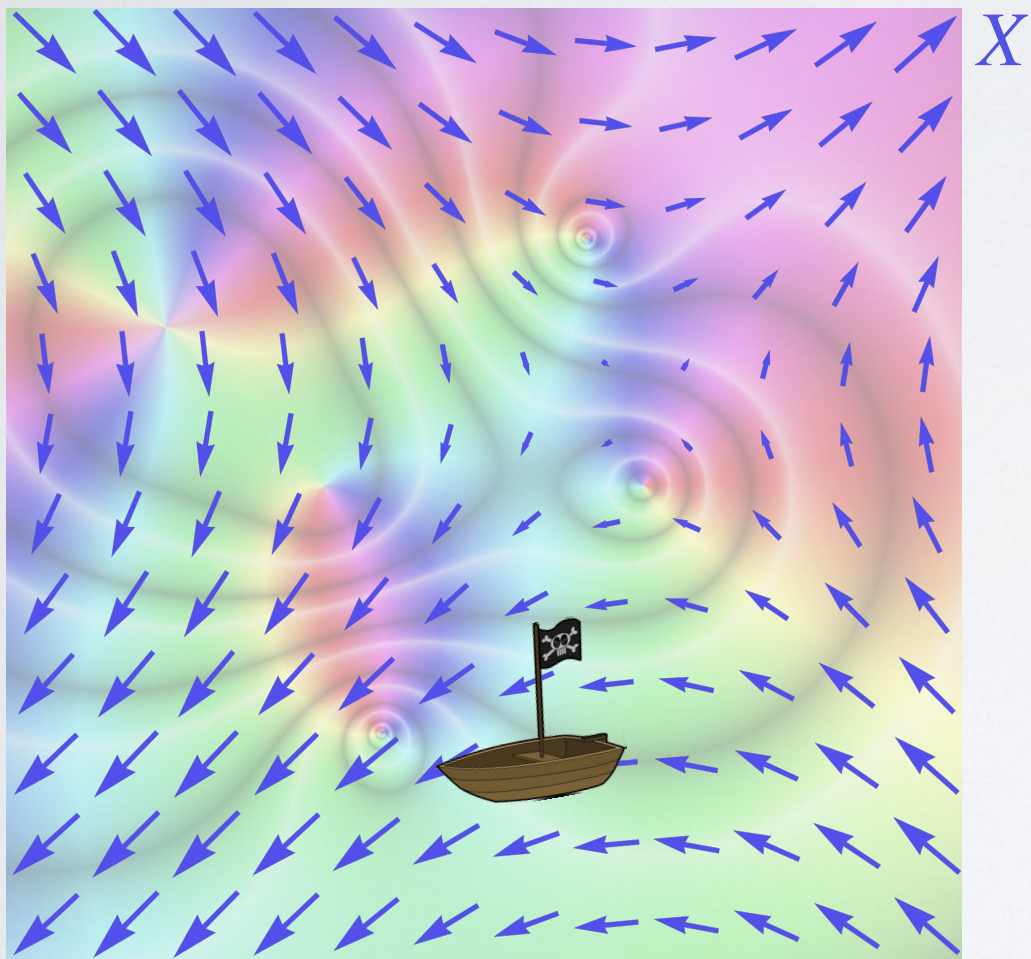


$\mathcal{L}_X Y$ “How much does
Y change as I flow
along X?”

Aside: Lie Derivative

φ_t Where do I wind up after time t ?

$\varphi_t^* \omega$ What does ω look like at my position after time t ?



$$\mathcal{L}_X \omega = \left. \frac{d}{dt} \right|_{t=0} \varphi_t^* \omega$$

Fluids

- Not incompressible yet
- Incompressible \leftrightarrow Divergence free

$$\frac{d\eta}{dt} = -\mathcal{L}_{\eta}\eta + dp$$

$$\delta\eta = 0$$

“Euler equation”

Definitions

η Velocity

p Pressure

dp Gradient of pressure

$\delta\eta$ Divergence of velocity

Magnetism

- Lorentz force law

$$F = J \times B$$

- Maxwell's fourth equation*

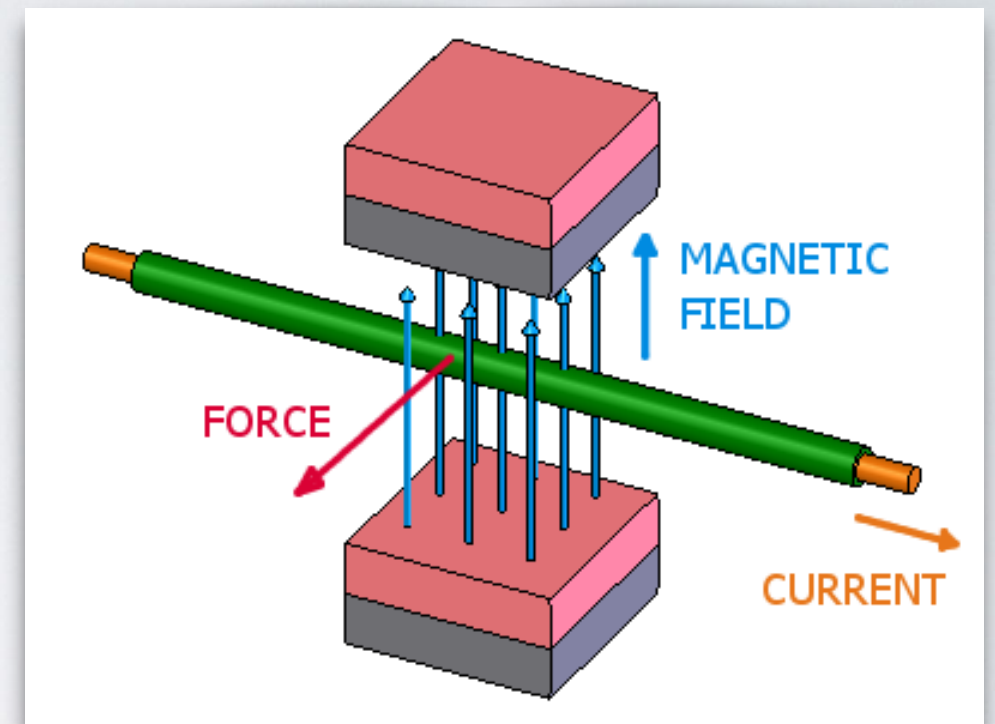
$$J = \nabla \times B$$

- Combining,

$$F = (\nabla \times B) \times B$$

$$= -\iota_{(\delta\beta)\#}\beta$$

* in the case of low E-field oscillation, and with some constants dropped



<https://www.kjmagnetics.com/images/blog/forcediagram1.png>

Definitions

β	Magnetic field
$\delta\beta$	“Curl of β ”
$-\iota_{(\delta\beta)\#}\beta$	Magnetic force from induced current

Magnetism

- Magnetic field caused by little ions
- Carried by flow

$$\frac{d\beta}{dt} = - \mathfrak{L}_{\eta\#}\beta$$

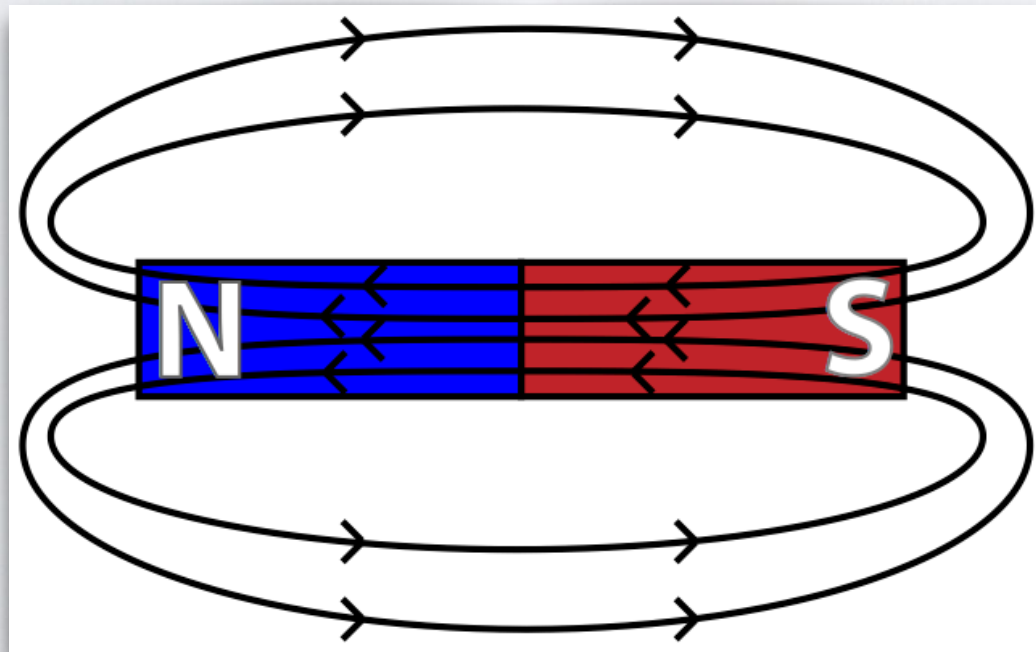
Definitions

β	Magnetic field
η	Velocity
$\mathfrak{L}_{\eta\#}$	Lie derivative along velocity field

Magnetism

- One last constraint: no magnetic monopoles

$$d\beta = 0$$



Definitions

β	Magnetic field
$d\beta$	Divergence of magnetic field

MHD Equations

$$\frac{d\eta}{dt} = -\mathcal{L}_{\eta\#}\eta + dp - \iota_{(\delta\beta)\#}\beta$$

$$\frac{d\beta}{dt} = -\mathcal{L}_{\eta\#}\beta$$

$$\delta\eta = 0$$

$$d\beta = 0$$

Advection along velocity field

Force from pressure

Force from magnetic field

Constraints

Definitions

η Velocity

β Magnetic field

p Pressure

dp Gradient of pressure

$\delta\eta$ Divergence of η

$d\beta$ Divergence of β

$-\iota_{(\delta\beta)\#}\beta$ Magnetic force from induced current

$\mathcal{L}_{\eta\#}$ Lie derivative along velocity field

MHD Equations

$$\frac{d\eta}{dt} = -\mathcal{L}_{\eta\#}\eta + dp - \iota_{(\delta\beta)\#}\beta$$

$$\frac{d\beta}{dt} = -\mathcal{L}_{\eta\#}\beta$$

$$\delta\eta = 0$$

~~$$d\beta = 0$$~~

Advection along velocity field

Force from pressure

Force from magnetic field

Constraints

Definitions

η Velocity

β Magnetic field

p Pressure

dp Gradient of pressure

$\delta\eta$ Divergence of η

$d\beta$ Divergence of β

$-\iota_{(\delta\beta)\#}\beta$ Magnetic force from induced current

$\mathcal{L}_{\eta\#}$ Lie derivative along velocity field

Conservation Laws: Energy

$$E = \frac{1}{2} \int \|\eta\|^2 + \|\beta\|^2 dV$$

Kinetic Energy

Potential Energy (Magnetic field strength)

Definitions

η Velocity

β Magnetic field

Conservation Laws: Magnetic Helicity

- Suppose we have a vector-potential such that

$$\beta = d\alpha$$

- Then we define

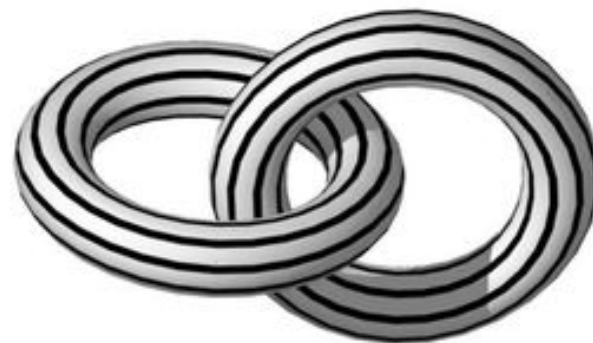
$$H_M := \int \alpha \wedge \beta$$



$$H=0$$



$$H=T\Phi^2$$



$$H=\pm 2\Phi_1\Phi_2$$

Definitions

α Vector potential

β Magnetic field

Conservation Laws: Cross Helicity

- Similarly

$$H_\chi := \int \eta \wedge \beta$$

“How linked are the velocity and magnetic fields?”

Definitions

η Velocity field

β Magnetic field

Discretization

- Standard Discrete Exterior Calculus gives us d, δ
- Now, we just need ι, \mathfrak{L}
- In fact, thanks to *Cartan's Magic Formula*, we only need ι

$$\mathfrak{L}_X \omega = \iota_X d\omega + d\iota_X \omega$$

MHD Equations

$$\frac{d\eta}{dt} = -\mathfrak{L}_{\eta\#}\eta + dp - \iota_{(\delta\beta)\#}\beta$$

$$\frac{d\beta}{dt} = -\mathfrak{L}_{\eta\#}\beta$$

$$\delta\eta = 0$$

Discretization

- The proofs of conservation of energy and cross helicity rely on the fact that ι and Λ are adjoint
- We can use the standard discrete wedge product
- Also gives us behavior on boundaries

MHD Equations

$$\frac{d\eta}{dt} = -\iota_{\eta\#}d\eta + dp - \iota_{(\delta\beta)\#}\beta$$

$$\frac{d\beta}{dt} = -d\iota_{\eta\#}\beta$$

$$\delta\eta = 0$$

Discretization

- This conserves energy and cross helicity for free!
- Magnetic helicity doesn't work out so well

MHD Equations

$$\frac{d\eta}{dt} = -\iota_{\eta\#}d\eta + dp - \iota_{(\delta\beta)\#}\beta$$

$$\frac{d\beta}{dt} = -d\iota_{\eta\#}\beta$$

$$\delta\eta = 0$$

Complication: 2D MHD

- Limit of MHD equations to a thin layer of fluid in strong background magnetic field
- Equations look a bit different

2D MHD Equations

$$\frac{d\eta}{dt} = -\iota_{\eta\#}d\eta + dp + \iota_{b\#}db$$

$$\frac{db}{dt} = -\delta(b \wedge \eta)$$

$$\delta\eta = 0$$

3D MHD Equations

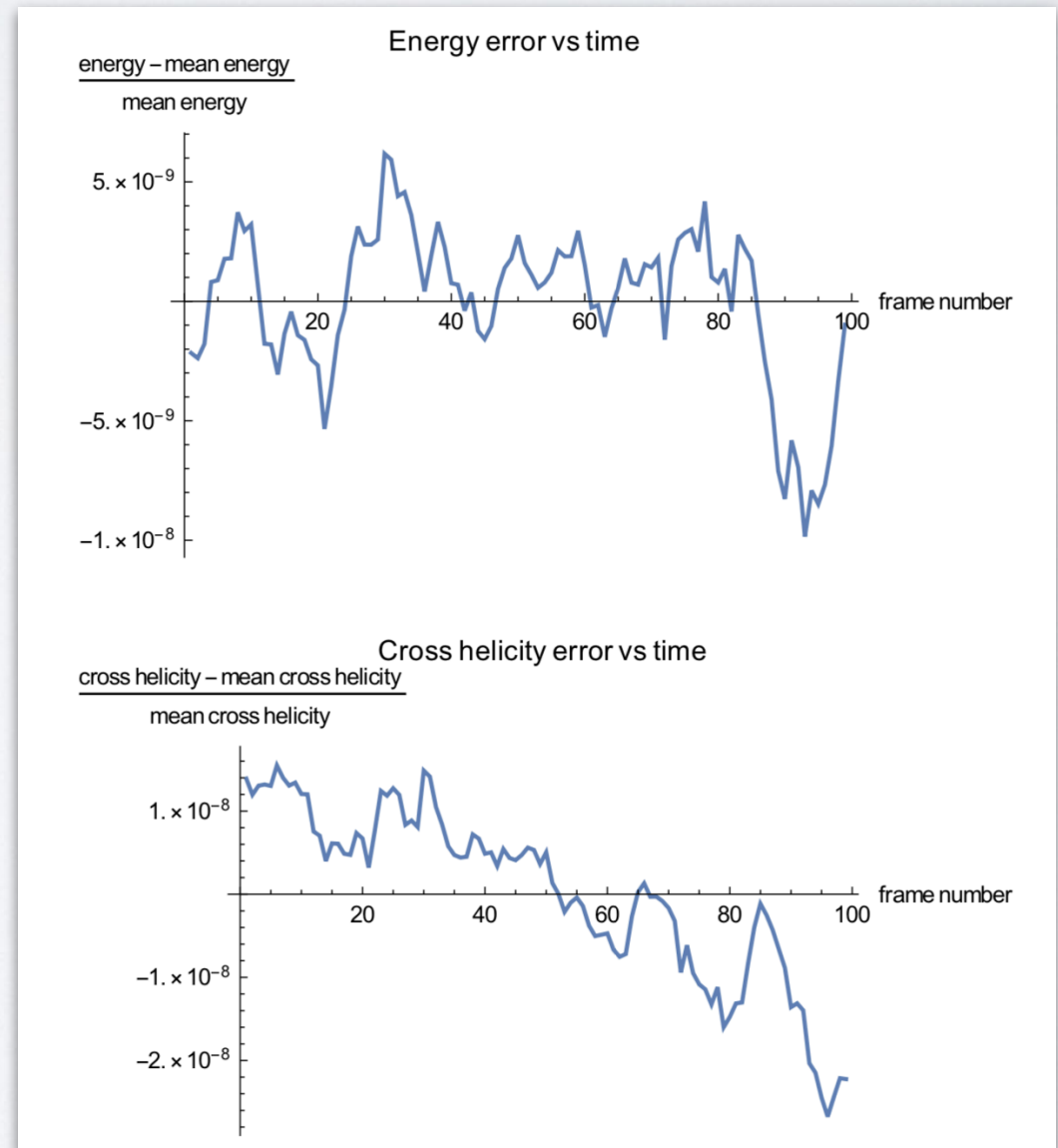
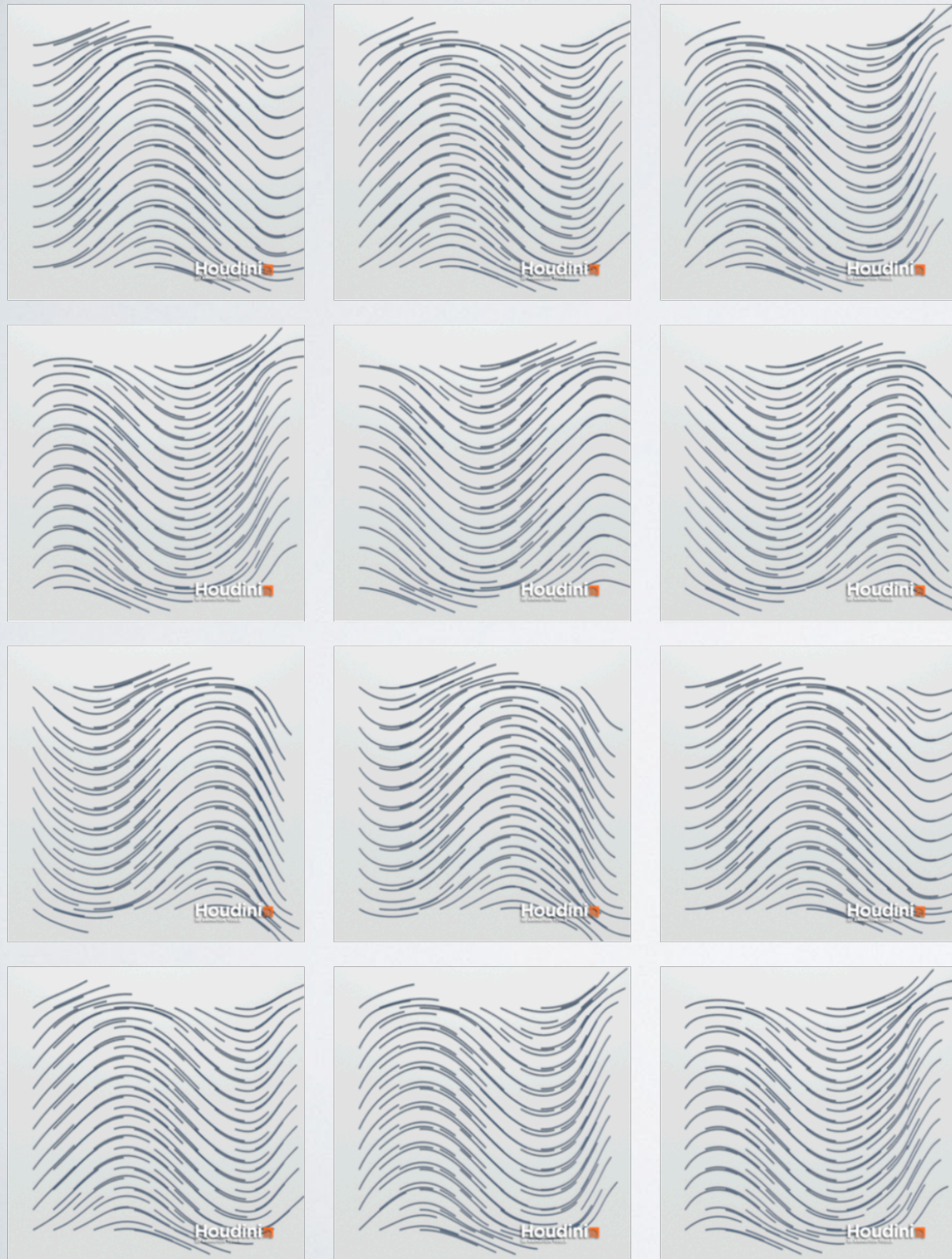
$$\frac{d\eta}{dt} = -\iota_{\eta\#}d\eta + dp - \iota_{(\delta\beta)\#}\beta$$

$$\frac{d\beta}{dt} = -d\iota_{\eta\#}\beta$$

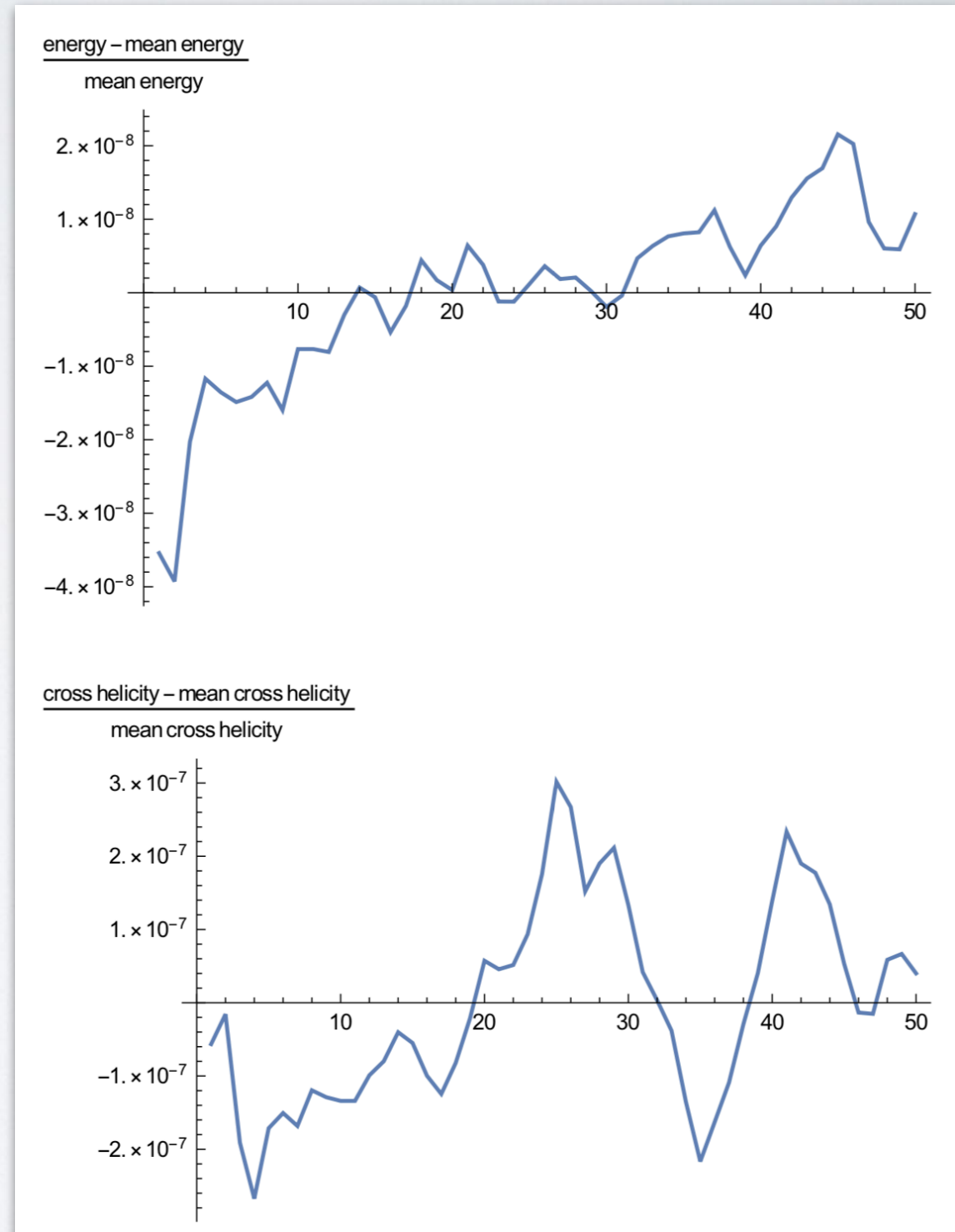
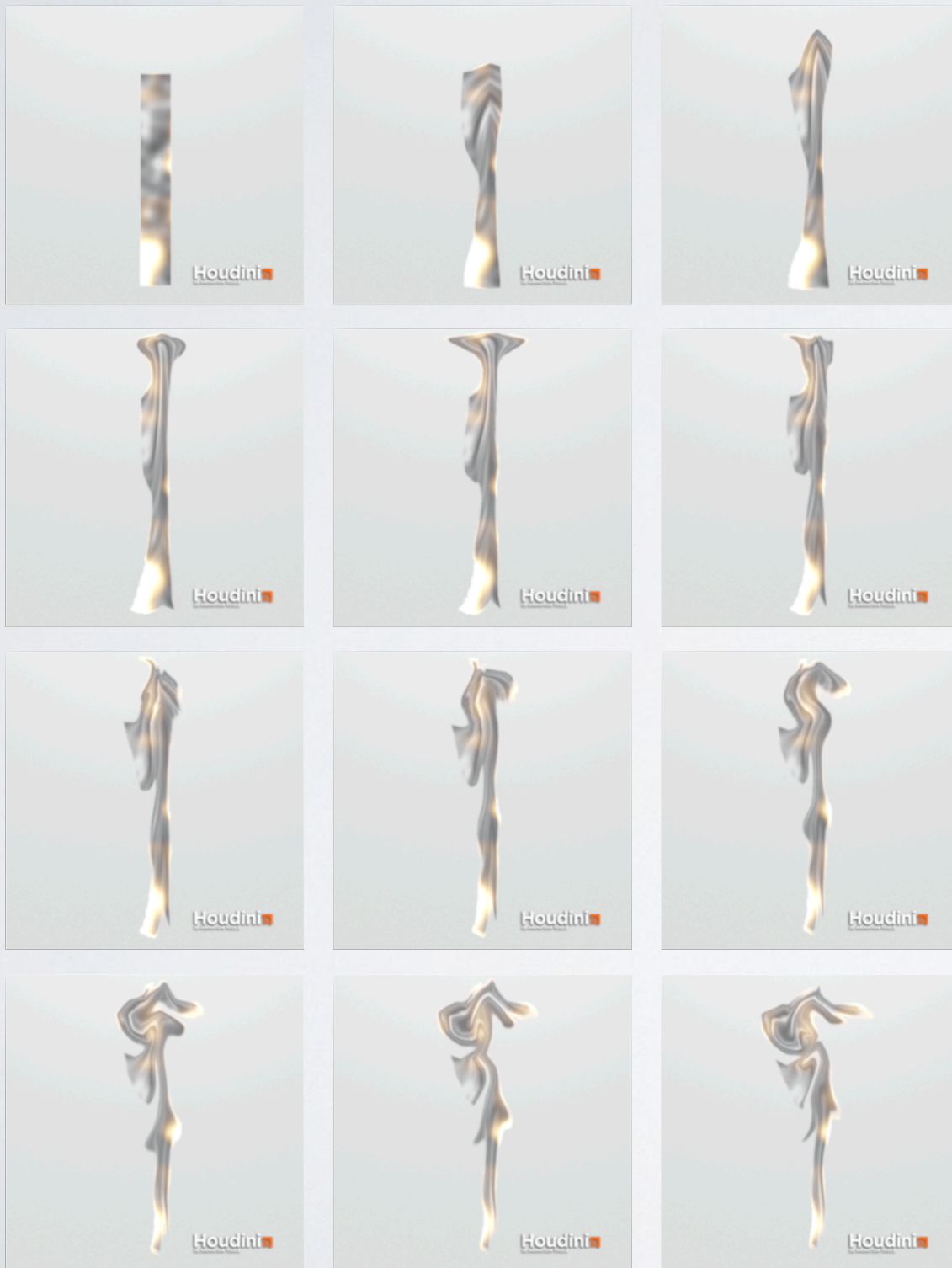
$$\delta\eta = 0$$

Test Case: Alfvén Wave

- Waves in the magnetic field lines



Test Case: Plume in a Box



Further questions?

- Is this integrator (multi)symplectic?
- Topological properties of magnetic field
- 3D simulation
- Boundaries between fluids

Quick Aside: Hamiltonian Mechanics is Symplectic

- Hamilton's equations of motion

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

$$X := \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \Omega \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}$$

Definitions

$$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ Symplectic Form}$$

Quick Aside: Hamiltonian Mechanics is Symplectic

$$X := \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \Omega \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} \quad \begin{aligned} X^T \Omega Y &= \begin{pmatrix} \frac{\partial H}{\partial q} & \frac{\partial H}{\partial p} \end{pmatrix} \Omega^T \Omega Y \\ &= \begin{pmatrix} \frac{\partial H}{\partial q} & \frac{\partial H}{\partial p} \end{pmatrix} Y \\ &= dH(Y) \end{aligned}$$

$$dH = X^T \Omega =: \iota_X \Omega$$

Definitions

$$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ Symplectic Form}$$

Quick Aside: Hamiltonian Mechanics is Symplectic

- Change in Ω along time evolution: $\mathfrak{L}_X\Omega$
- Cartan: $\mathfrak{L}_X\Omega = d\iota_X\Omega + \iota_X d\Omega$

$$\mathfrak{L}_X\Omega = d(d\Omega) = 0$$

Definitions / Facts

$$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ Symplectic Form}$$

$$dH = \iota_X\Omega$$

Thanks!

