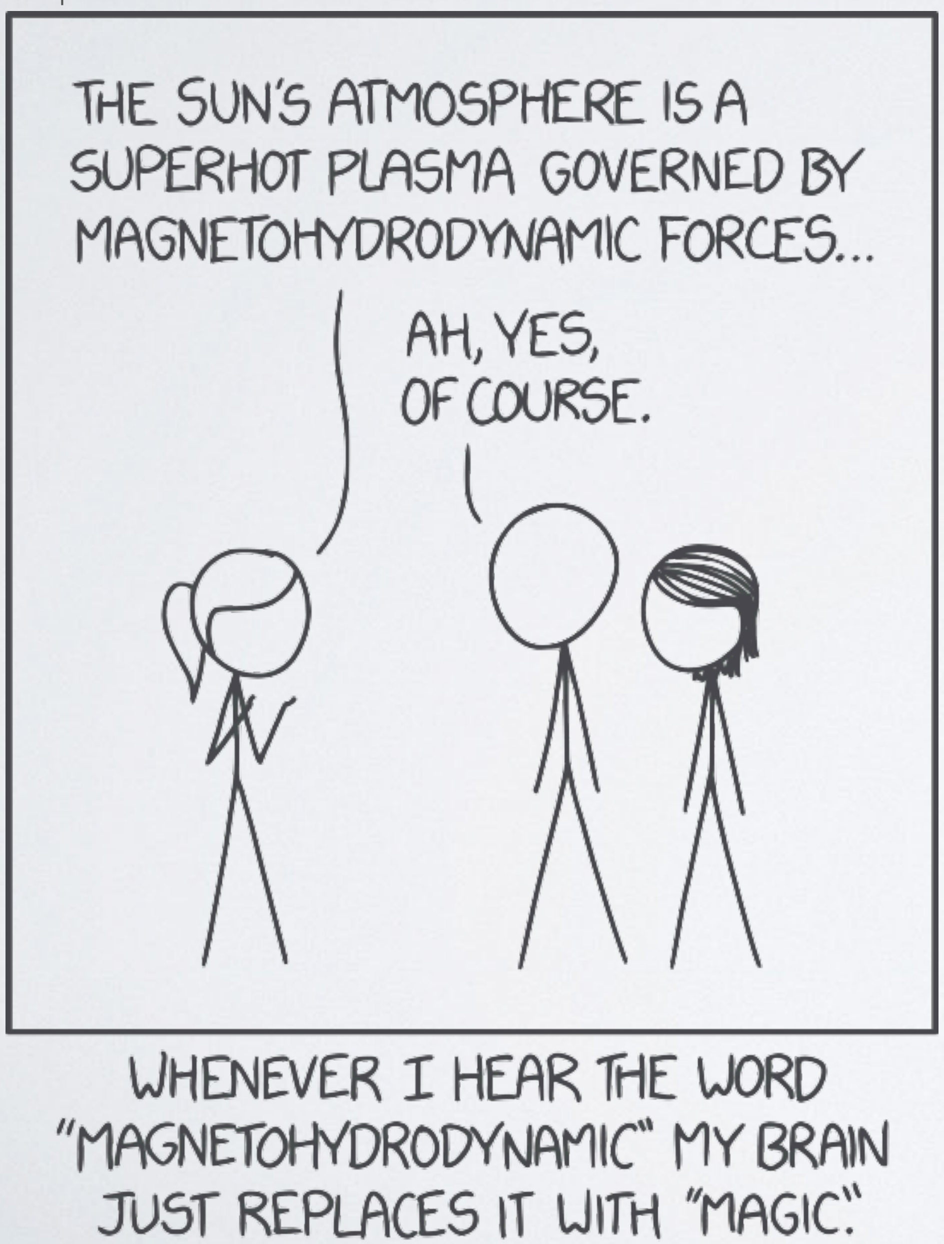
Magnetohydrodynamics

and the Geometry of Conservation Laws in Physics

https://xkcd.com/1851/



Mark Gillespie

Why?

arXiv:1707.03227v2 [math.NA] 12 Mar 2018

Variational Integrators for
Ideal Magnetohydrodynamics

Michael Kraus
(michael.kraus@ipp.mpg.de)

Max-Planck-Institut für Plasmaphysik
Boltzmannstraße 2, 85748 Garching, Deutschland

Technische Universität München, Zentrum Mathematik
Boltzmannstraße 3, 85748 Garching, Deutschland

Omar Maj
(omar.maj@ipp.mpg.de)

Max-Planck-Institut für Plasmaphysik
Boltzmannstraße 2, 85748 Garching, Deutschland

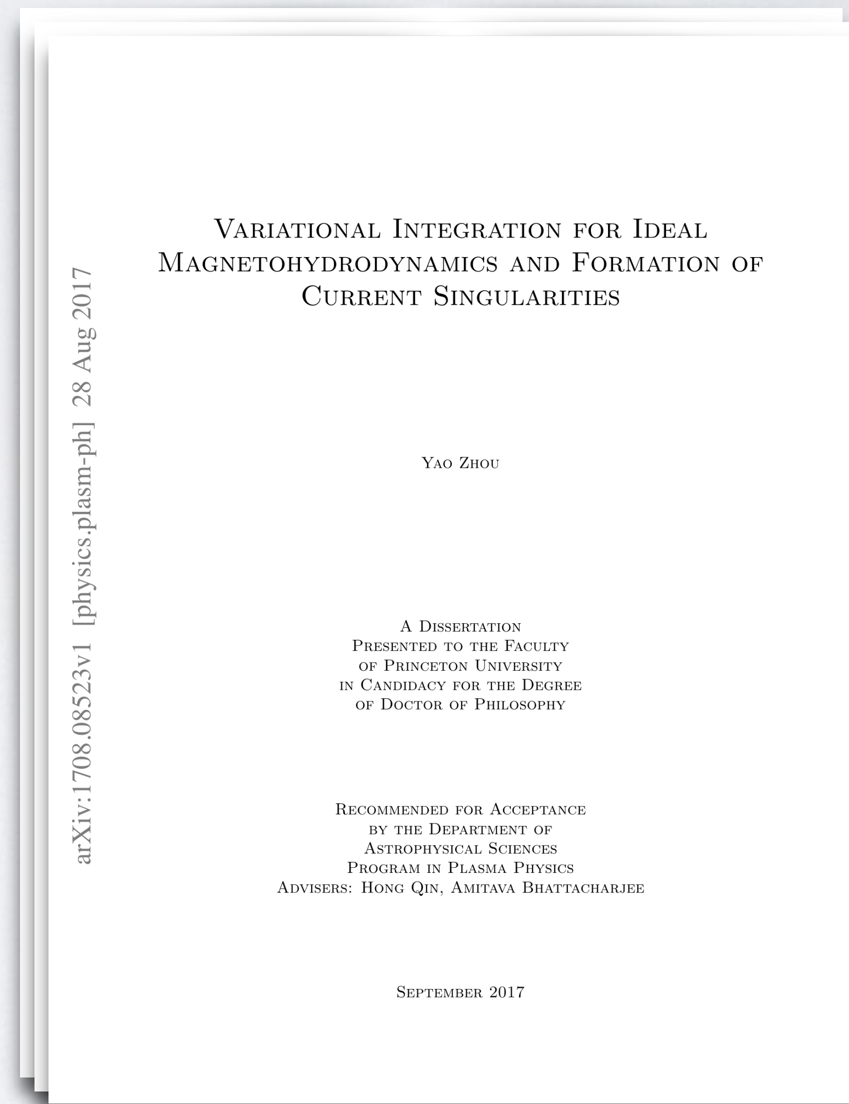
Technische Universität München, Zentrum Mathematik
Boltzmannstraße 3, 85748 Garching, Deutschland

March 13, 2018

Abstract

A variational integrator for ideal magnetohydrodynamics is derived by applying a discrete action principle to a formal Lagrangian. Discrete exterior calculus is used for the discretisation of the field variables in order to preserve their geometrical character. The resulting numerical method is free of numerical resistivity, thus the magnetic field line topology is preserved and unphysical reconnection is absent. In 2D numerical examples we find that important conservation laws like total energy, magnetic helicity and cross helicity are satisfied within machine accuracy.

Keywords: Conservation Laws, Discrete Exterior Calculus, Geometric Discretization, Lagrangian Field Theory, Magnetohydrodynamics, Variational Integrators,



arXiv:1708.08523v1 [physics.plasm-ph] 28 Aug 2017

Variational Integration for Ideal
Magnetohydrodynamics and Formation of
Current Singularities

Yao Zhou

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
Astrophysical Sciences
Program in Plasma Physics
Advisers: Hong Qin, Amitava Bhattacharjee

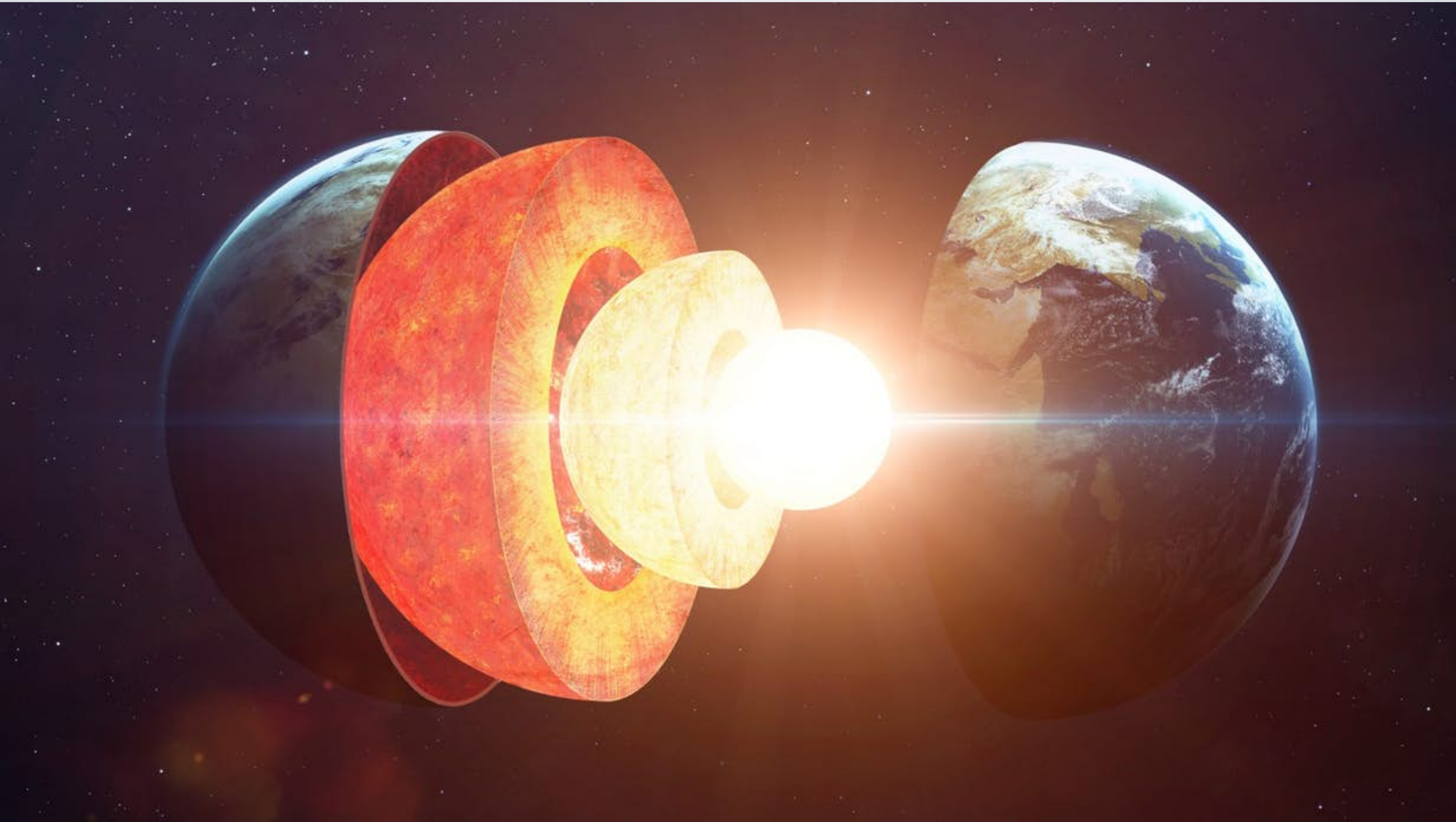
September 2017

Why?

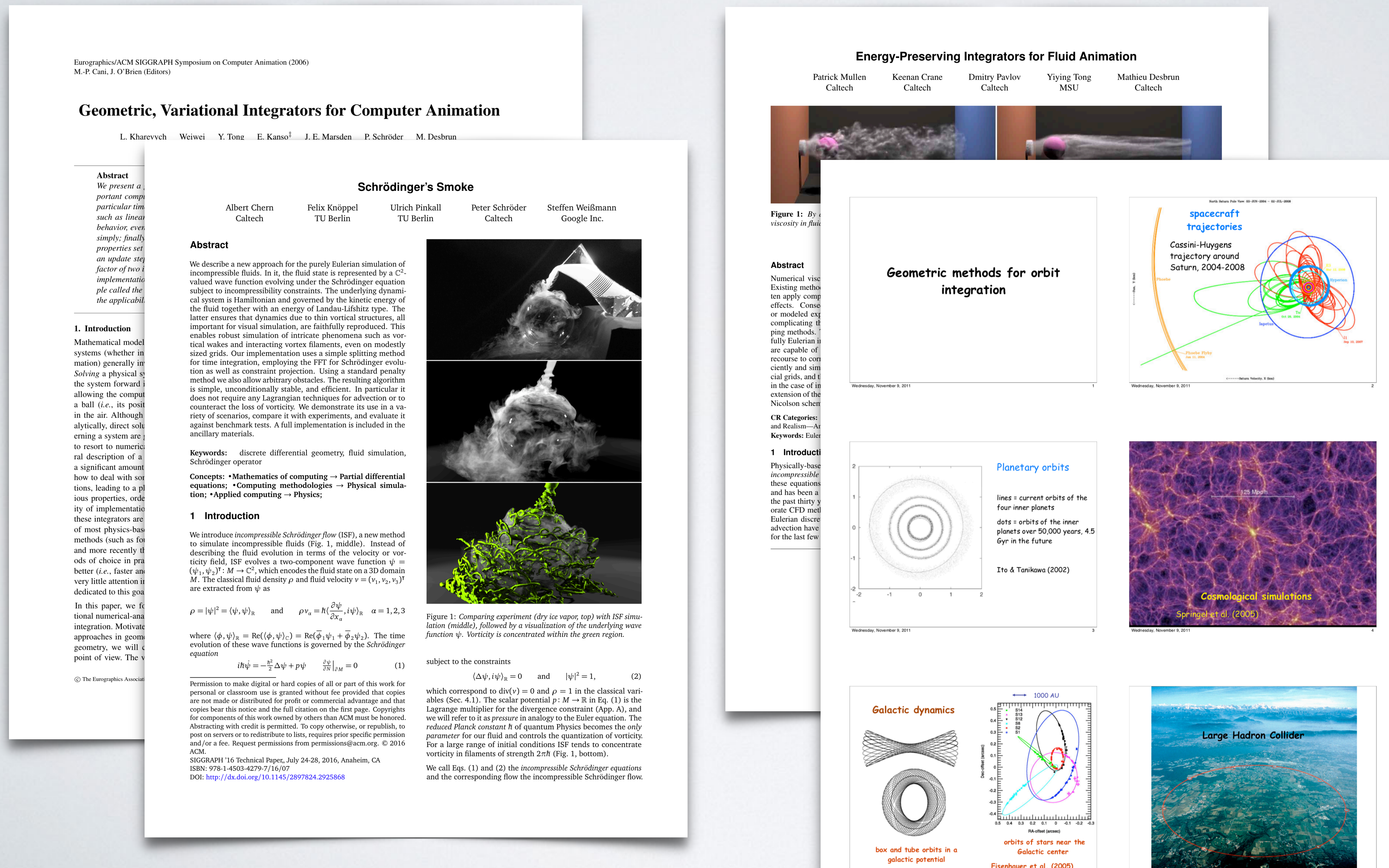
https://upload.wikimedia.org/wikipedia/commons/e/e3/Magnificent_CME_Erupts_on_the_Sun_-_August_31.jpg

Why?

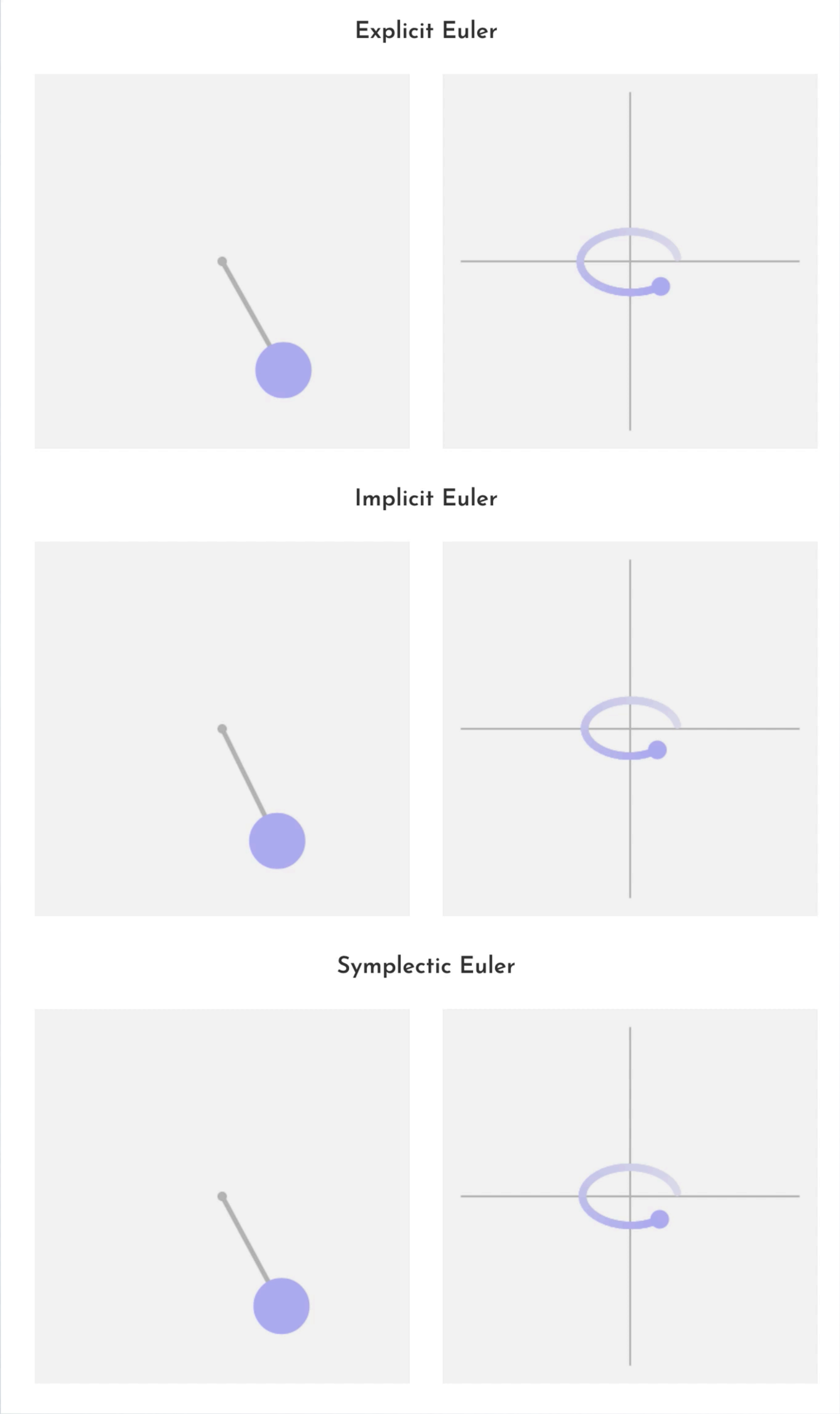
https://newatlas.com/earth-inner-core-solid-soft/56882/



Structure-Preserving Integrators

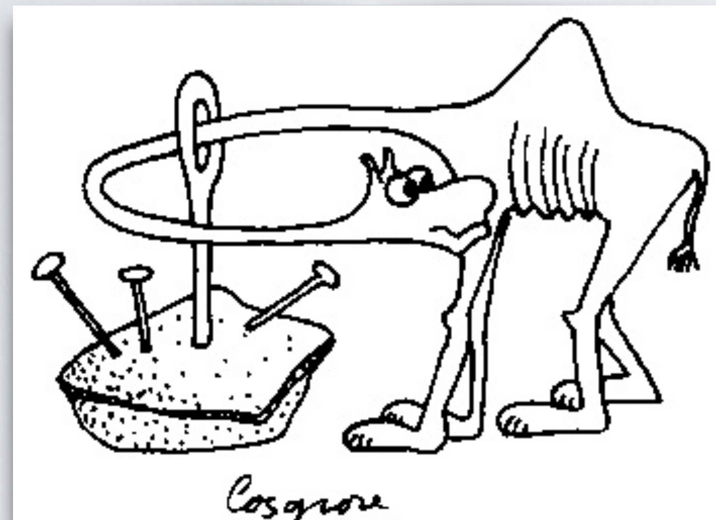


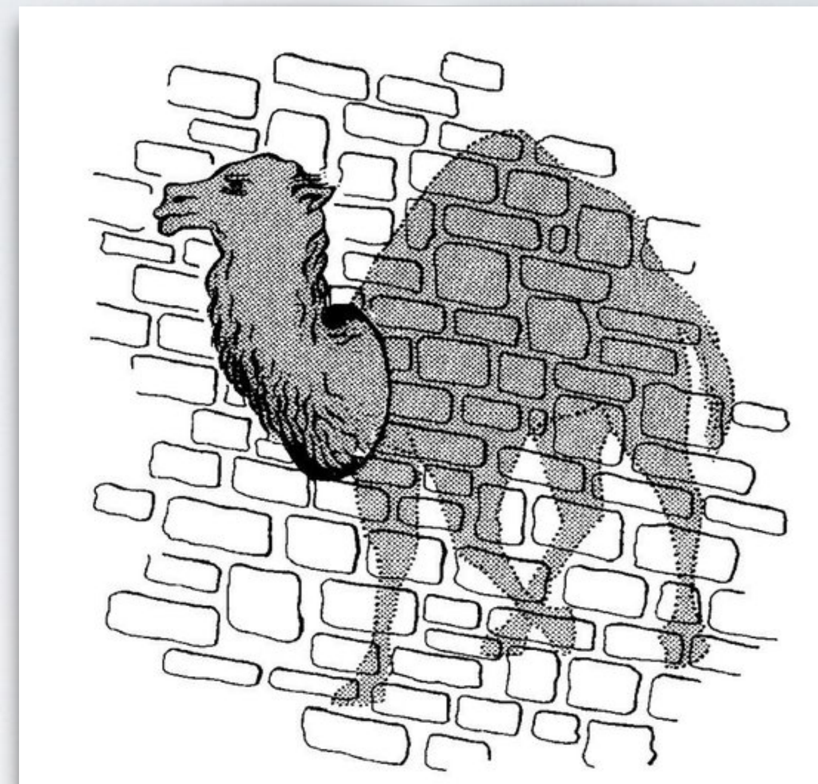
Symplectic Integrators

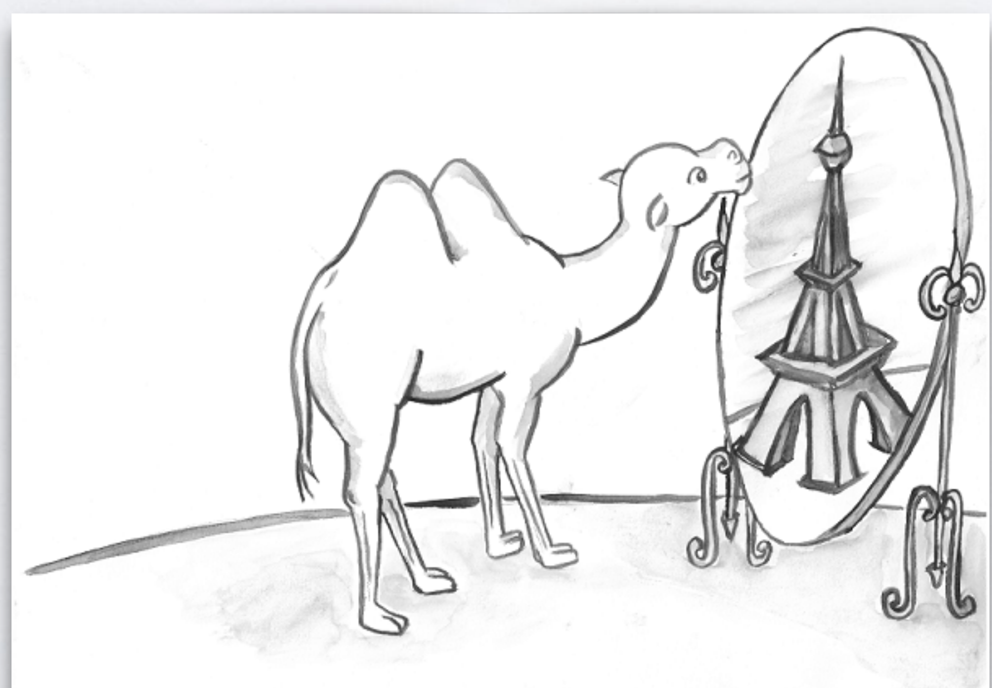


- How you update velocity makes a big difference
- One option makes your simulation *symplectic*
- It captures important features

Symplectic?

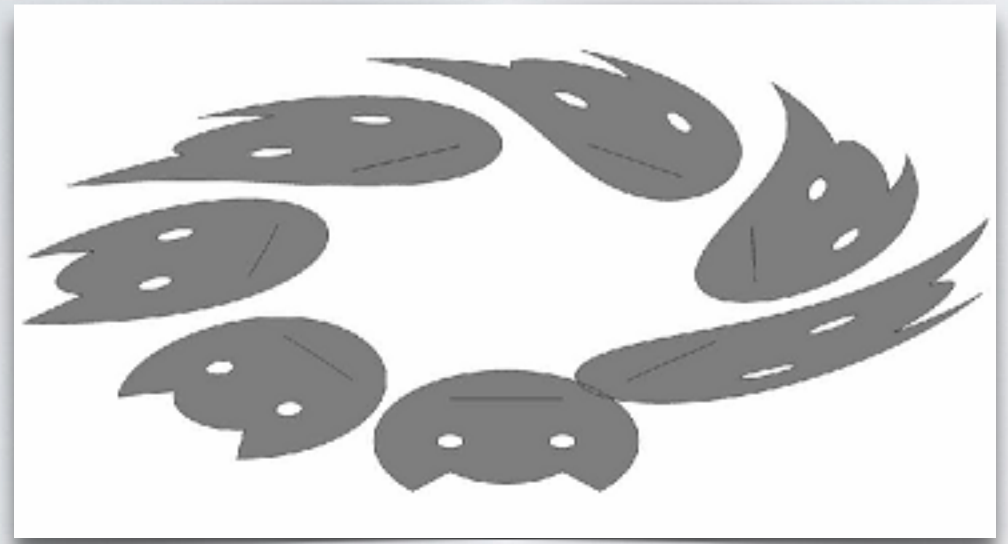






http://www.math.ucla.edu/~cm/symcon.html, https://math.berkeley.edu/~jchaidez/math290f16.html, http://school2016.imj-prg.fr/

Symplectic?



http://people.bath.ac.uk/tjs42/BNA/bna-res.html

- In 2D, *symplectic* just means *area-preserving*
- Really, symplectic maps generalize area-preserving maps

Symplectic Euler is Symplectic

- Luckily for us, the pendulum's *phase space* is 2D.
- How do we measure areas in 2D?
    - Determinants
- Useful fact:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = \begin{pmatrix} a \\ c \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix}$$

Symplectic Euler is Symplectic

- So the matrix $\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ measures (infinitesimal) areas

- Our simulation preserves area (and is thus symplectic) if it "preserves" this matrix

Definitions

$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ "Area matrix"

Symplectic Euler is Symplectic

- Given a force function *F(q)*, and a time step *h*, symplectic Euler updates positions (*q*) and momenta (*p*) by

$$\begin{pmatrix} q_{n+1} \\ p_{n+1} \end{pmatrix} = \begin{pmatrix} q_n + hp_n + h^2F(q_n) \\ p_n + hF(q_n) \end{pmatrix} =: T\left(\begin{pmatrix} q_n \\ p_n \end{pmatrix}\right)$$

- What does this do to $\mathbf{\Omega}$?

Definitions

$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$		"Area matrix"
T		Update rule

Symplectic Euler is Symplectic

- How is the area of a parallelogram related to the area of *T(parallelogram)* ?

- If the parallelogram is tiny, *T* looks like a linear map, its *linearization* (or *Jacobian*) $\boldsymbol{L = dT}$

- We only need to check that *L* preserves area

Definitions

$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	"Area matrix"
T	Update rule
$L = dT$	Linearization of T

Symplectic Euler is Symplectic

- Note that $\text{Area}(Lv, Lw) = (Lv)^T \Omega (Lw)$
  $= v^T L^T \Omega L w$
  $= v^T [L^T \Omega L] w$

- So our transformation preserves area (and is symplectic) as long as

$$\Omega = L^T \Omega L$$

Definitions	
$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	"Area matrix"
T	Update rule
$L = dT$	Linearization of T

Symplectic Euler is Symplectic

- Now, we can just do this computation

$$T\left(\begin{pmatrix} q_n \\ p_n \end{pmatrix}\right) = \begin{pmatrix} q_n + hp_n + h^2F(q_n) \\ p_n + hF(q_n) \end{pmatrix} \qquad L = \begin{pmatrix} 1 + h^2\partial_q F & h \\ h\partial_q F & 1 \end{pmatrix}$$

$$\cdots$$

$$\Omega = L^T \Omega L$$

(You can also observe that det(L) = 1)

Definitions

$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	"Area matrix"
T	Update rule
$L = dT$	Linearization of T

Symplectic?

- In n dimensions, we can define* $\Omega = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}$

- Symplectic maps are still the maps which satisfy

$$\Omega = L^T \Omega L$$

Definitions

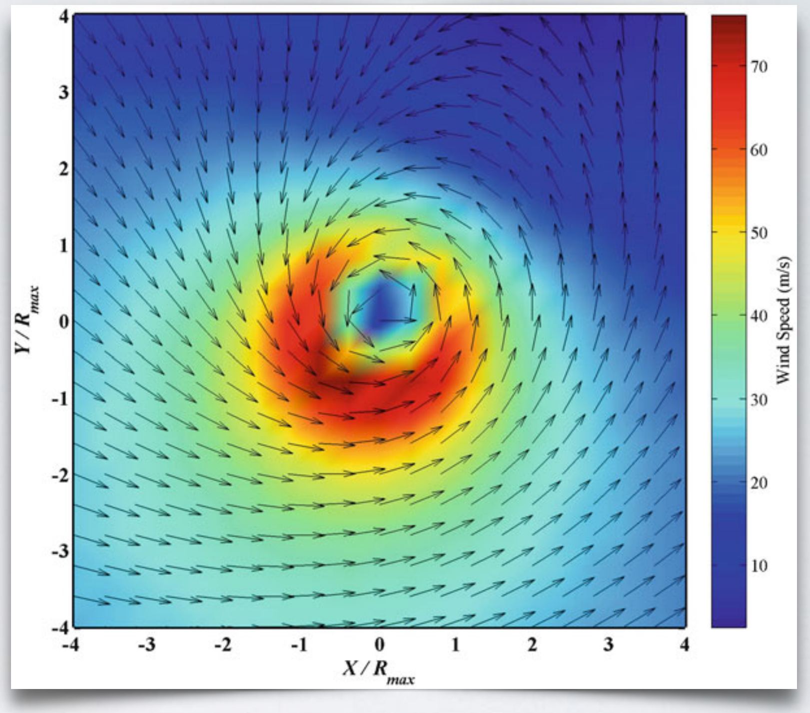
$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	"Area matrix"
T	Update rule
$L = dT$	Linearization of T

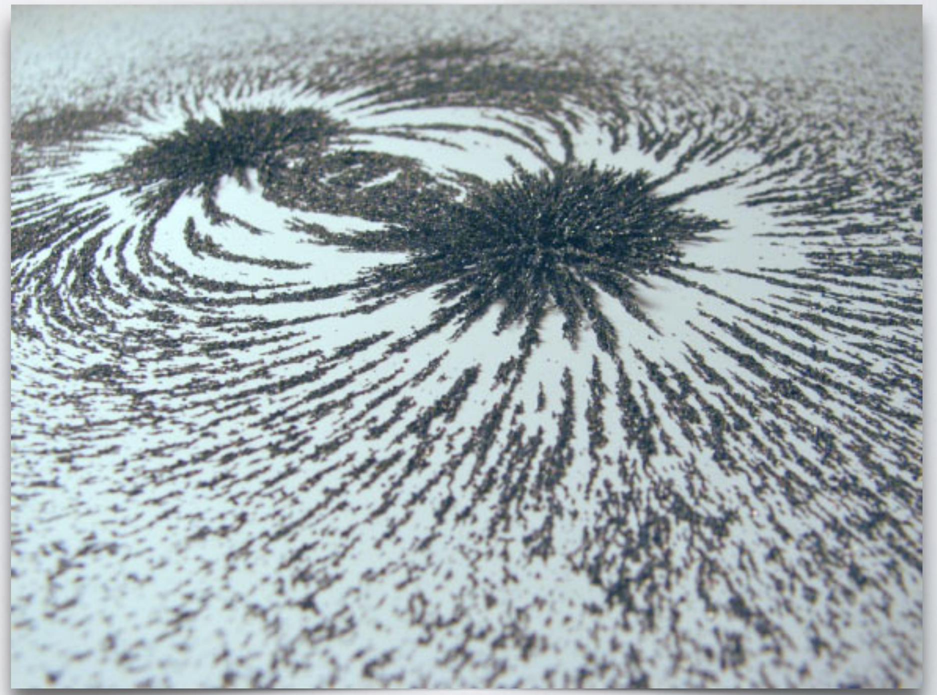
How does this relate to camels?

- Symplectic geometry was confusing, even to mathematicians at first
- It's "clear" that symplectic maps preserve volume
- Gromov's non-squeezing theorem

Magnetohydrodynamics

- Physics of conducting fluids
- Key ingredients
  - Velocity field (1-form) η
  - Magnetic field (2-form) β

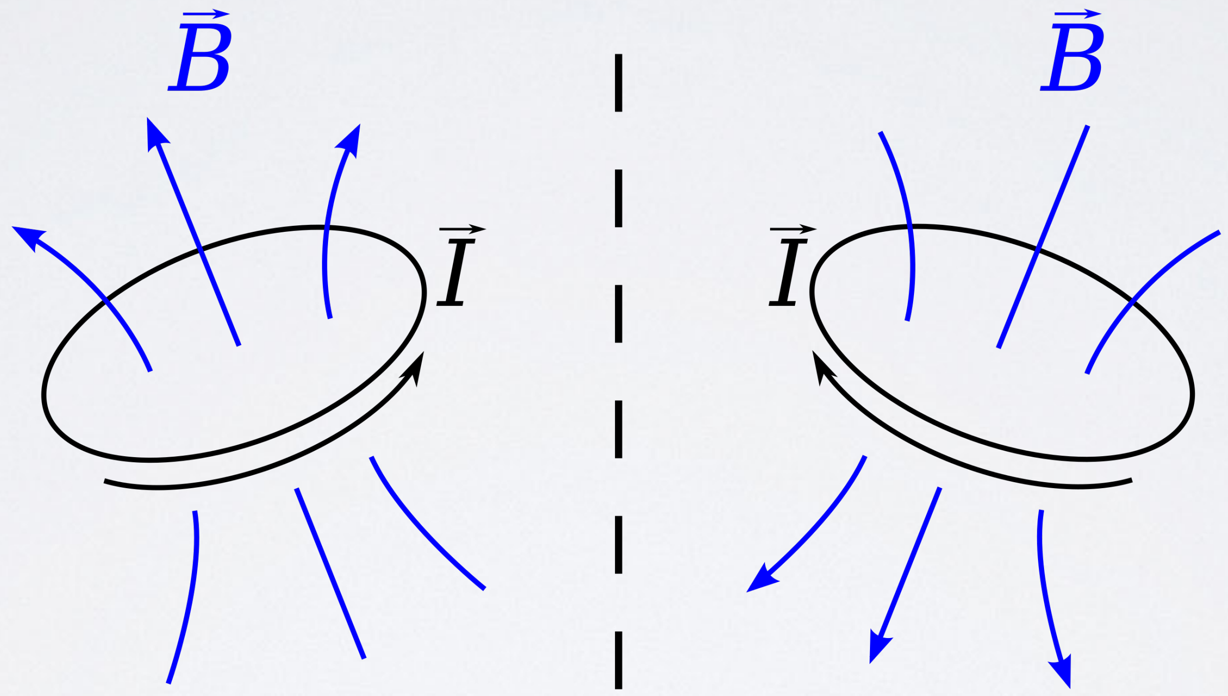




https://www.researchgate.net/figure/Wind-vector-and-isotachs-representing-a-wind-velocity-field-of-2011-Joplin-MO-tornado_fig5_304360364,
http://www.sci-news.com/physics/strongest-magnetic-field-achieved-indoors-06420.html,

Aside: The Magnetic Field is not a Vector Field

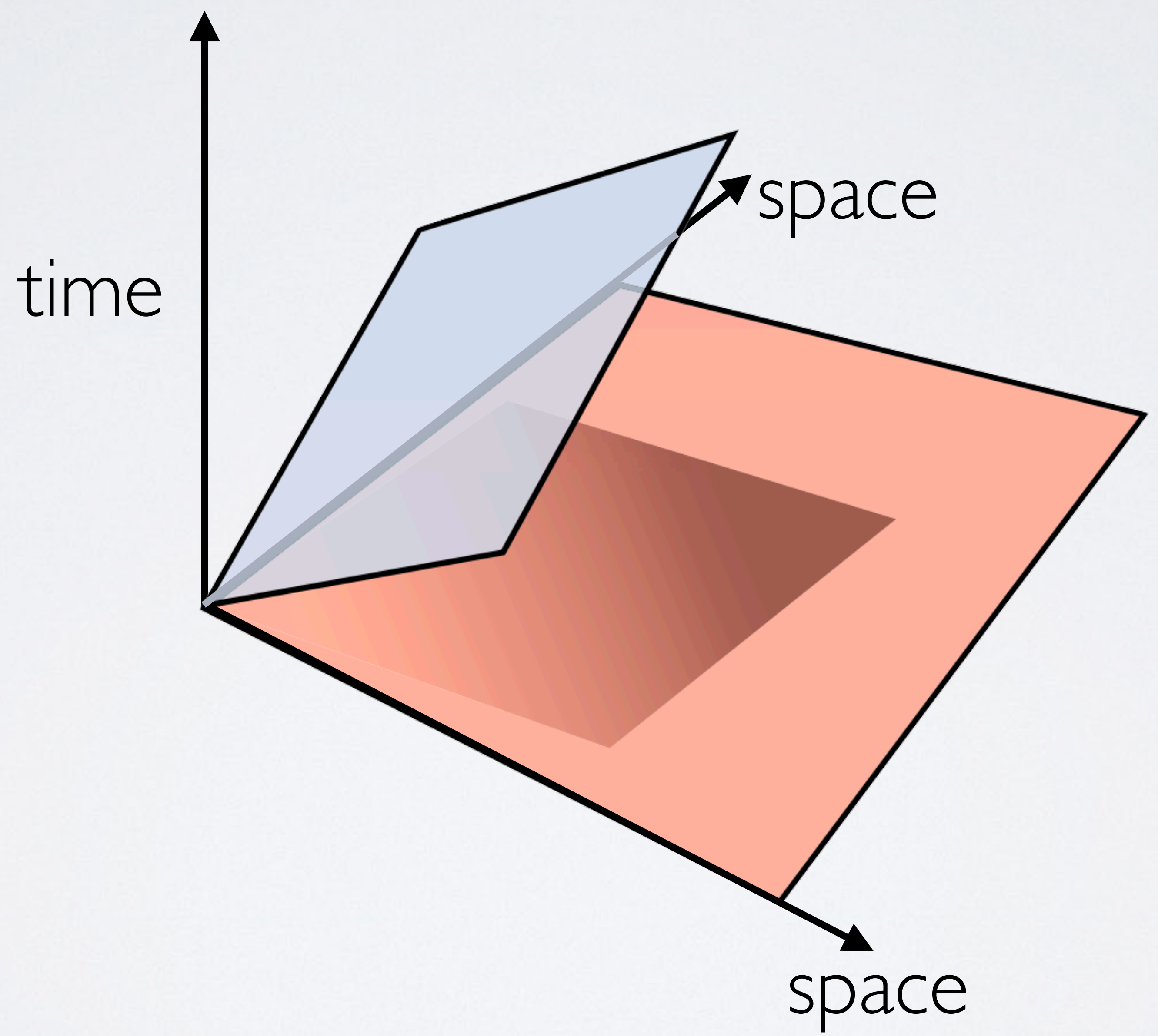
- Wrong symmetries under reflection



- More like an oriented plane than a vector
  - 2D!

https://en.wikipedia.org/wiki/Pseudovector#/media/File:BIsAPseudovector.svg

Even More Aside: Faraday 2-Form



Fluids

- Fluids are difficult
- Simplifying assumptions:
  - Incompressible
  - No viscosity
  - Unit density

Fluids

- How does a fluid's velocity change over time?
- Fluid is made up of small particles which each have a velocity
- The fluid drags the velocity field along

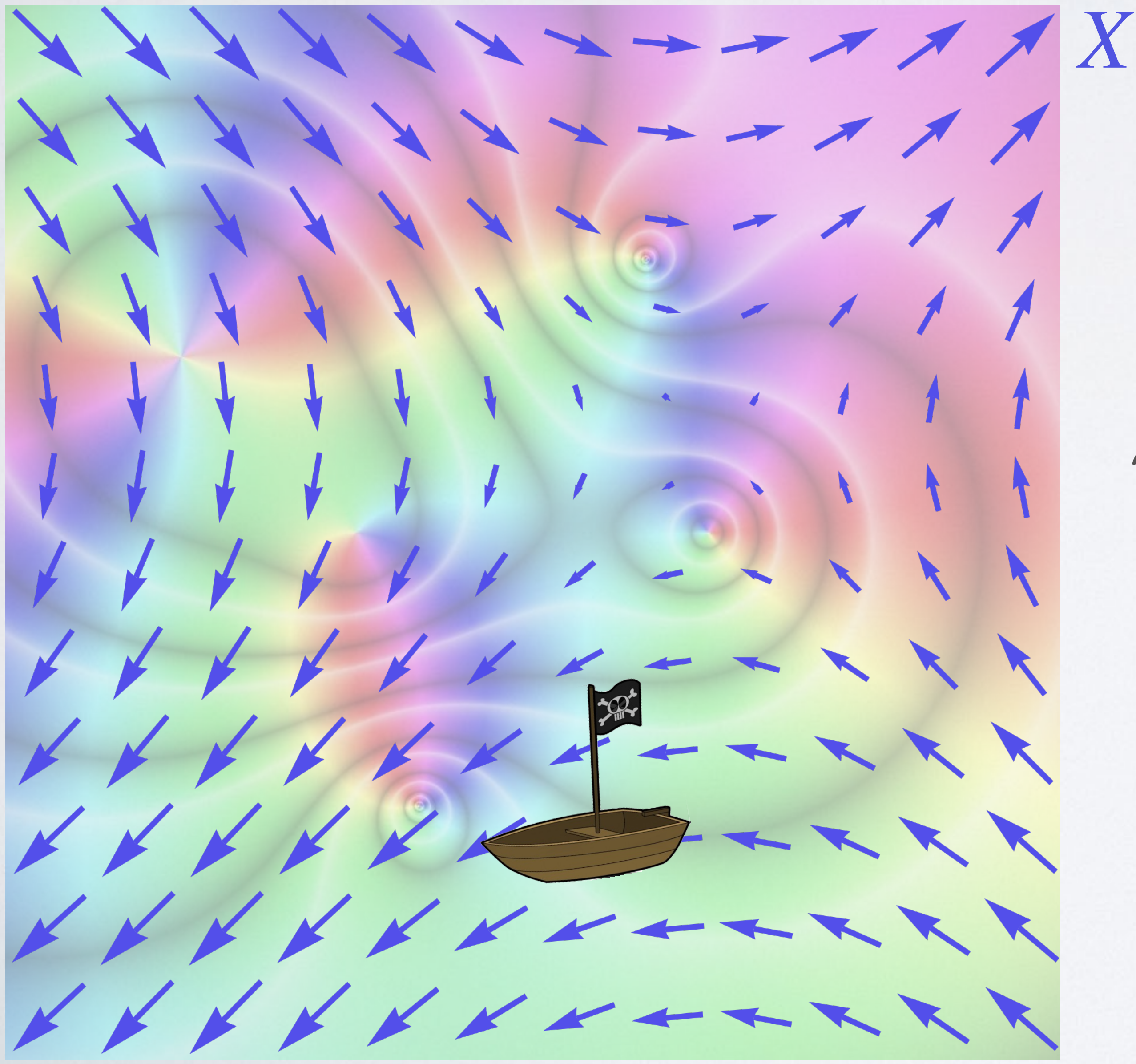
$$\frac{d\eta}{dt} = -\mathfrak{L}_{\eta^\sharp}\eta$$

Definitions

η Velocity

Aside: Lie Derivative

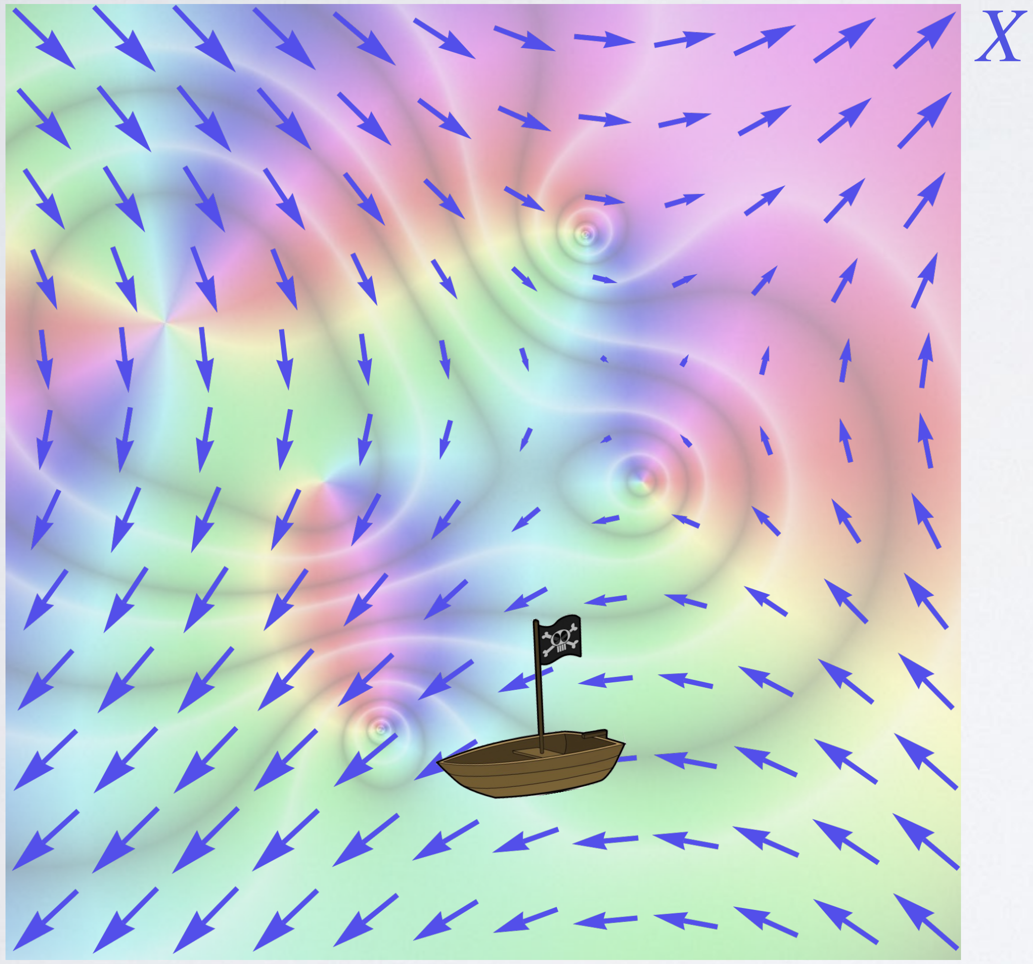
- What is that weird $\mathfrak{L}$?



$$\mathfrak{L}_X Y$$

"How much does Y change as I flow along X?"

Aside: Lie Derivative



φ_t Where do I wind up after time t?

$\varphi_t^*\omega$ What does ω look like at my position after time t?

$$\mathfrak{L}_X\omega = \frac{d}{dt}\bigg|_{t=0} \varphi_t^*\omega$$

Fluids

- Not incompressible yet
- Incompressible ↔ Divergence free

$$\frac{d\eta}{dt} = -\mathfrak{L}_{\eta^\sharp}\eta + dp$$

$$\delta\eta = 0$$

"Euler equation"

Definitions

- η Velocity
- p Pressure
- dp Gradient of pressure
- $\delta\eta$ Divergence of velocity

Magnetism

- Lorentz force law

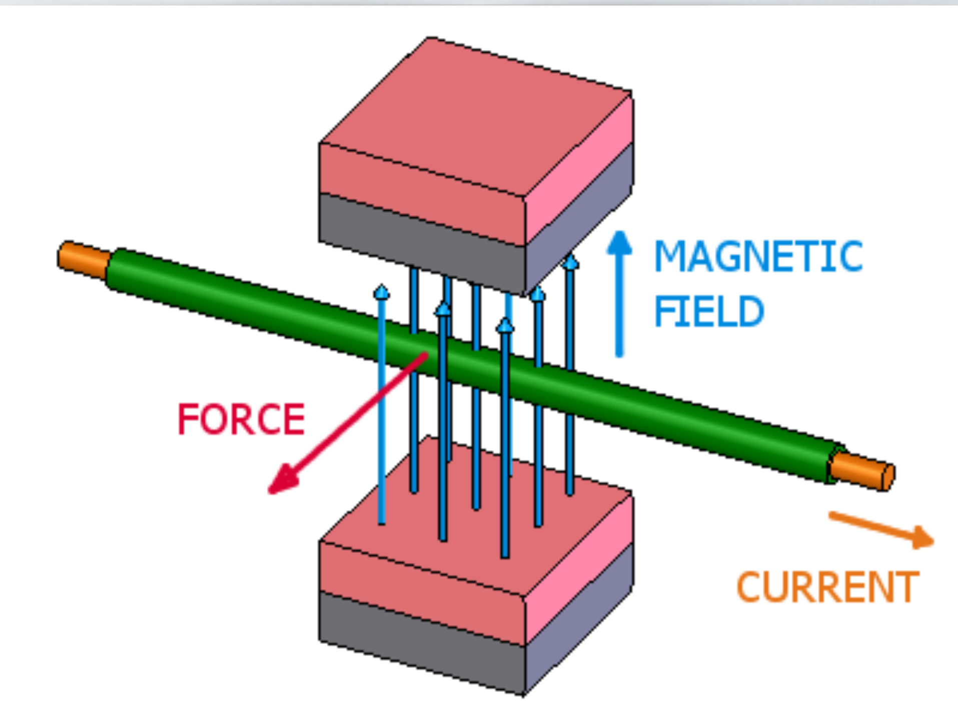
$$F = J \times B$$

- Maxwell's fourth equation*

$$J = \nabla \times B$$

- Combining,

$$F = (\nabla \times B) \times B$$
$$= -\iota_{(\delta\beta)^\sharp}\beta$$



https://www.kjmagnetics.com/images/blog/forcediagram1.png

Definitions	
β	Magnetic field
$\delta\beta$	"Curl of β"
$-\iota_{(\delta\beta)^\sharp}\beta$	Magnetic force from induced current

* in the case of low E-field oscillation, and with some constants dropped

Magnetism

- Magnetic field caused by little ions
- Carried by flow

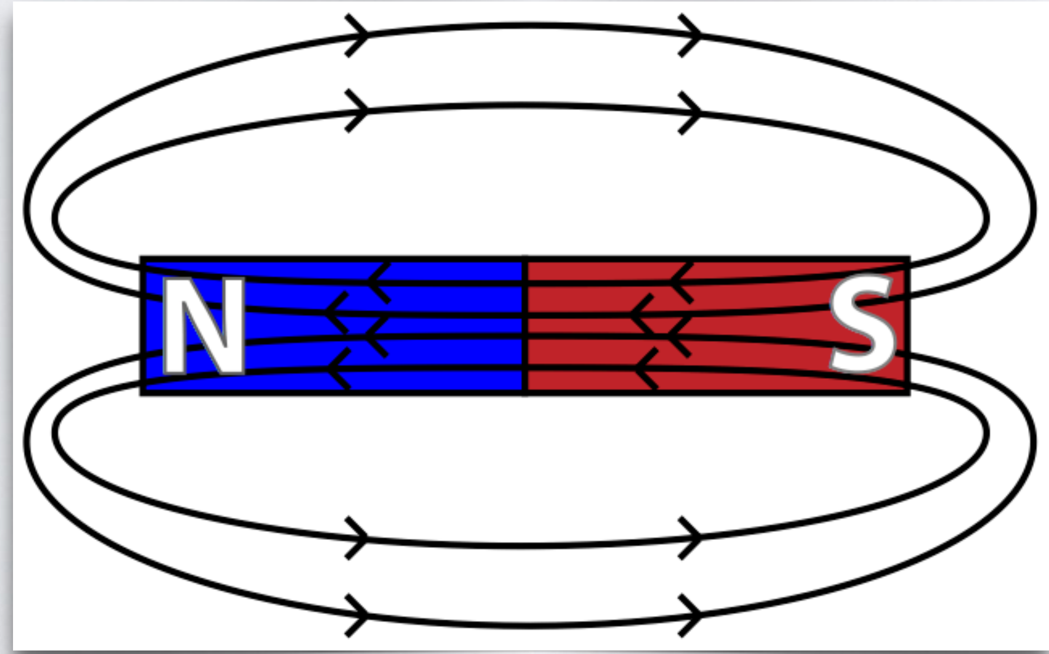
$$\frac{d\beta}{dt} = -\mathfrak{L}_{\eta^\sharp}\beta$$

Definitions	
β	Magnetic field
η	Velocity
$\mathfrak{L}_{\eta^\sharp}$	Lie derivative along velocity field

Magnetism

- One last constraint: no magnetic monopoles

$$d\beta = 0$$



http://aplusphysics.com/courses/honors/magnets/images/Magnetic%20Field%20Lines.png

Definitions	
β	Magnetic field
$d\beta$	Divergence of magnetic field

MHD Equations

$$\frac{d\eta}{dt} = -\mathfrak{L}_{\eta^\sharp}\eta \quad + dp \quad - \iota_{(\delta\beta)^\sharp}\beta$$

$$\frac{d\beta}{dt} = -\mathfrak{L}_{\eta^\sharp}\beta$$

$$\delta\eta = 0$$

$$d\beta = 0$$

Advection along velocity field

Force from pressure

Force from magnetic field

Constraints

Definitions

η	Velocity
β	Magnetic field
p	Pressure
dp	Gradient of pressure
$\delta\eta$	Divergence of η
$d\beta$	Divergence of β
$-\iota_{(\delta\beta)^\sharp}\beta$	Magnetic force from induced current
$\mathfrak{L}_{\eta^\sharp}$	Lie derivative along velocity field

MHD Equations

$$\frac{d\eta}{dt} = -\mathfrak{L}_{\eta^\sharp}\eta \quad + dp \quad - \iota_{(\delta\beta)^\sharp}\beta$$

$$\frac{d\beta}{dt} = -\mathfrak{L}_{\eta^\sharp}\beta$$

$$\delta\eta = 0$$

~~$d\beta = 0$~~

Definitions

η	Velocity
β	Magnetic field
p	Pressure
dp	Gradient of pressure
$\delta\eta$	Divergence of η
$d\beta$	Divergence of β
$-\iota_{(\delta\beta)^\sharp}\beta$	Magnetic force from induced current
$\mathfrak{L}_{\eta^\sharp}$	Lie derivative along velocity field

Advection along velocity field

Force from pressure

Force from magnetic field

Constraints

Conservation Laws: Energy

$$E = \frac{1}{2} \int \|\eta\|^2 + \|\beta\|^2 \, dV$$

Kinetic Energy

Potential Energy (Magnetic field strength)

Definitions

η Velocity

β Magnetic field

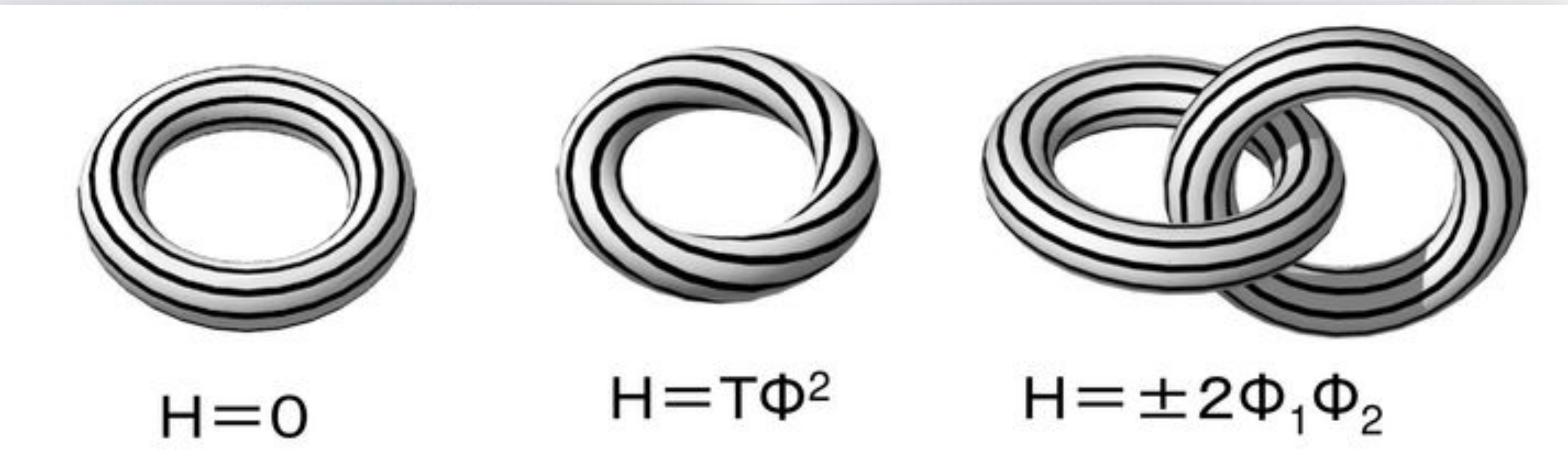
Conservation Laws: Magnetic Helicity

- Suppose we have a vector-potential such that

$$\beta = d\alpha$$

- Then we define

$$H_M := \int \alpha \wedge \beta$$



Definitions

α Vector potential

β Magnetic field

Wiegelmann & Sakurai (2012).

Conservation Laws: Cross Helicity

- Similarly

$$H_\chi := \int \eta \wedge \beta$$

"How linked are the velocity and magnetic fields?"

Definitions	
η	Velocity field
β	Magnetic field

Discretization

- Standard Discrete Exterior Calculus gives us d, δ
- Now, we just need $\iota, \mathfrak{L}$
- In fact, thanks to *Cartan's Magic Formula*, we only need ι

$$\mathfrak{L}_X \omega = \iota_X d\omega + d\iota_X \omega$$

MHD Equations

$$\frac{d\eta}{dt} = -\mathfrak{L}_{\eta^\sharp}\eta + dp - \iota_{(\delta\beta)^\sharp}\beta$$

$$\frac{d\beta}{dt} = -\mathfrak{L}_{\eta^\sharp}\beta$$

$$\delta\eta = 0$$

Discretization

- The proofs of conservation of energy and cross helicity rely on the fact that ι and $\wedge$ are adjoint
- We can use the standard discrete wedge product
- Also gives us behavior on boundaries

MHD Equations

$$\frac{d\eta}{dt} = -\iota_{\eta^\sharp} d\eta + dp - \iota_{(\delta\beta)^\sharp}\beta$$

$$\frac{d\beta}{dt} = -d\iota_{\eta^\sharp}\beta$$

$$\delta\eta = 0$$

Discretization

- This conserves energy and cross helicity for free!
- Magnetic helicity doesn't work out so well

MHD Equations

$$\frac{d\eta}{dt} = -\iota_{\eta^\sharp} d\eta + dp - \iota_{(\delta\beta)^\sharp}\beta$$

$$\frac{d\beta}{dt} = -d\iota_{\eta^\sharp}\beta$$

$$\delta\eta = 0$$

Complication: 2D MHD

- Limit of MHD equations to a thin layer of fluid in strong background magnetic field
- Equations look a bit different

2D MHD Equations

$$\frac{d\eta}{dt} = -\iota_{\eta^\sharp} d\eta + dp + \iota_{b^\sharp} db$$

$$\frac{db}{dt} = -\delta\,(b \wedge \eta)$$

$$\delta\eta = 0$$

3D MHD Equations

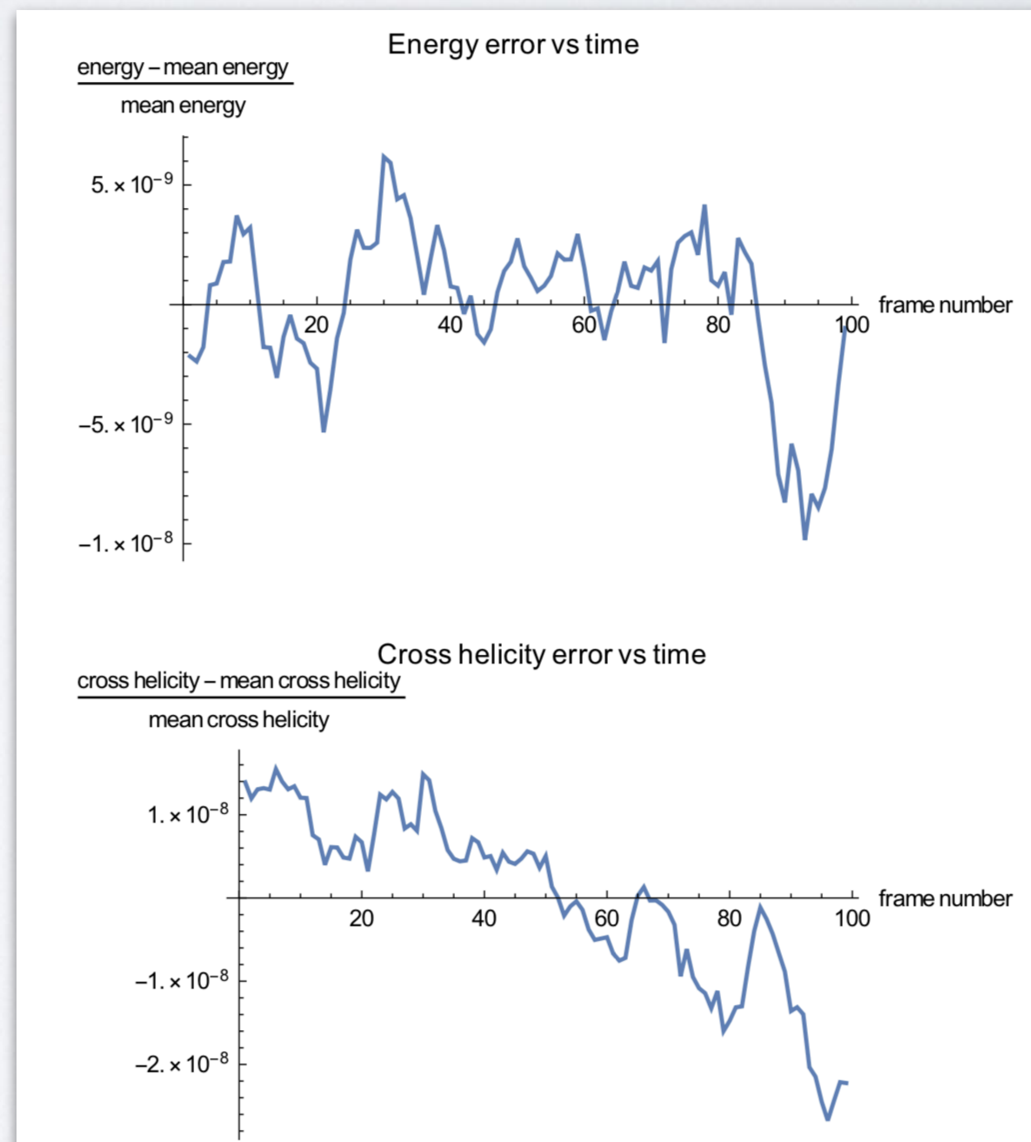
$$\frac{d\eta}{dt} = -\iota_{\eta^\sharp} d\eta + dp - \iota_{(\delta\beta)^\sharp}\beta$$

$$\frac{d\beta}{dt} = -d\iota_{\eta^\sharp}\beta$$

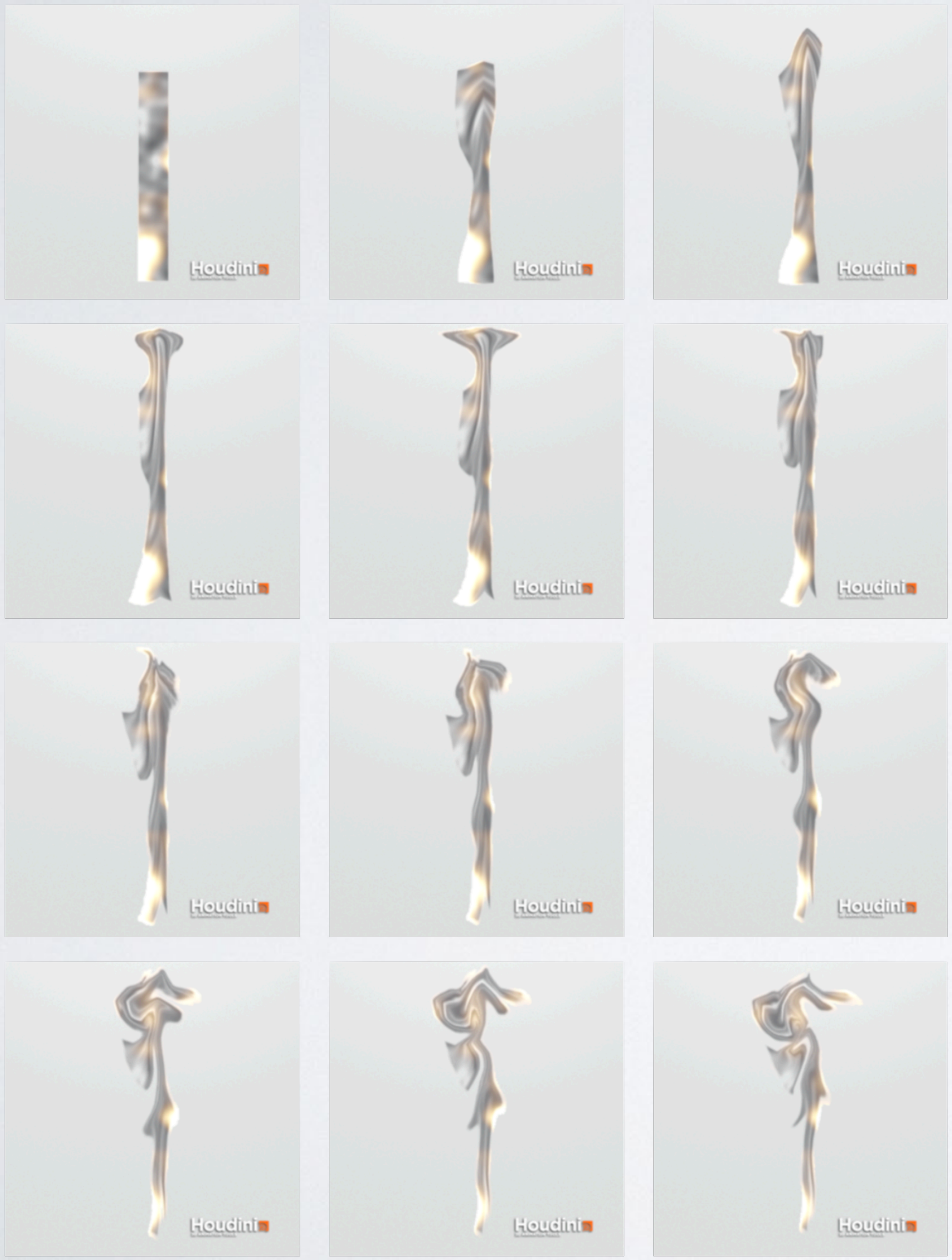
$$\delta\eta = 0$$

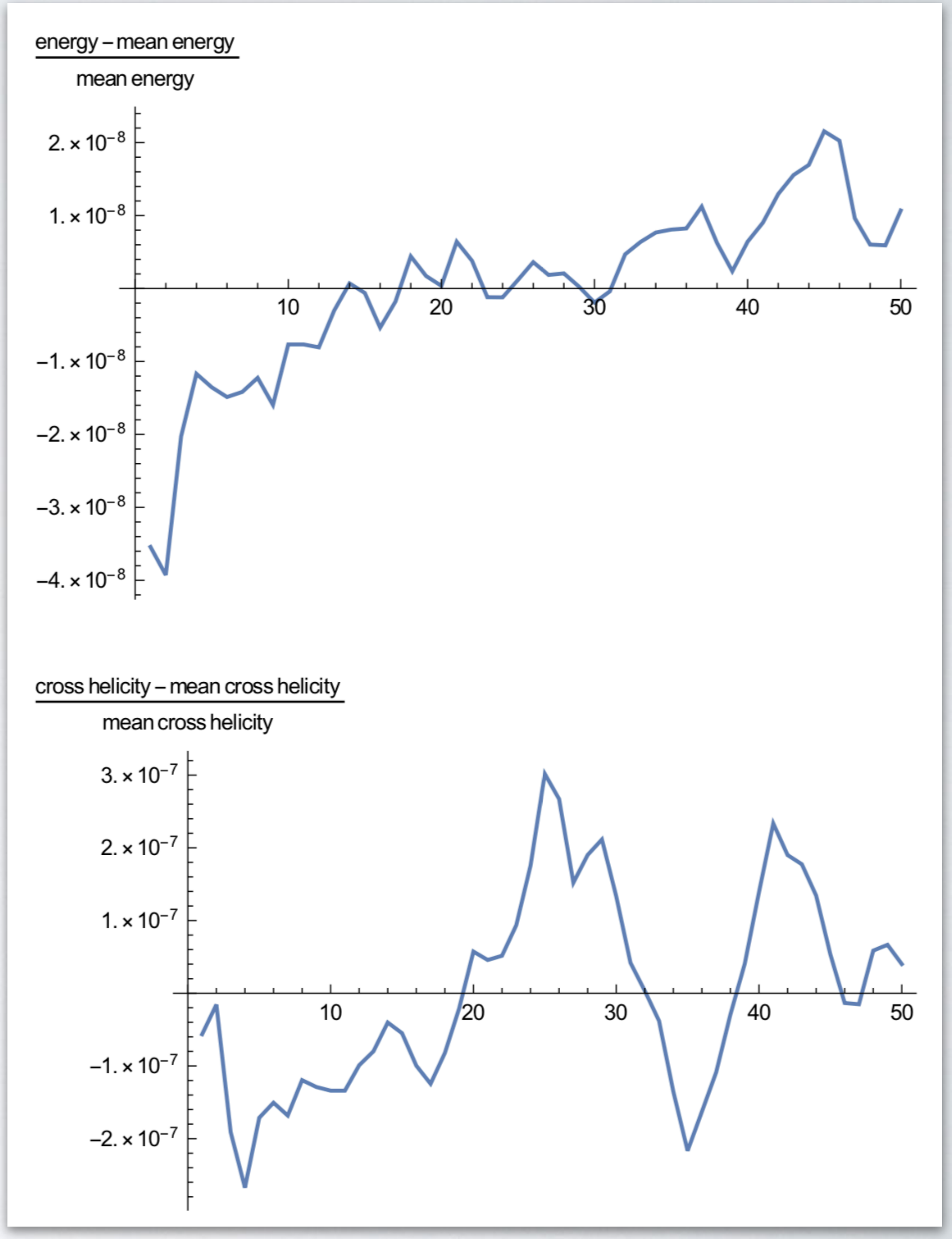
Test Case: Alfvén Wave

- Waves in the magnetic field lines



Test Case: Plume in a Box





Further questions?

- Is this integrator (multi)symplectic?
- Topological properties of magnetic field
- 3D simulation
- Boundaries between fluids

Quick Aside: Hamiltonian Mechanics is Symplectic

- Hamilton's equations of motion

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

$$X := \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \Omega \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}$$

Definitions

$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ Symplectic Form

Quick Aside: Hamiltonian Mechanics is Symplectic

$$X := \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \Omega \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}$$

$$\begin{aligned} X^T \Omega Y &= \begin{pmatrix} \frac{\partial H}{\partial q} & \frac{\partial H}{\partial p} \end{pmatrix} \Omega^T \Omega Y \\ &= \begin{pmatrix} \frac{\partial H}{\partial q} & \frac{\partial H}{\partial p} \end{pmatrix} Y \\ &= dH(Y) \end{aligned}$$

$$dH = X^T \Omega =: \iota_X \Omega$$

Definitions

$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ Symplectic Form

Quick Aside: Hamiltonian Mechanics is Symplectic

- Change in Ω along time evolution: $\mathfrak{L}_X \Omega$
- Cartan: $\mathfrak{L}_X \Omega = d\iota_X \Omega + \iota_X d\Omega$

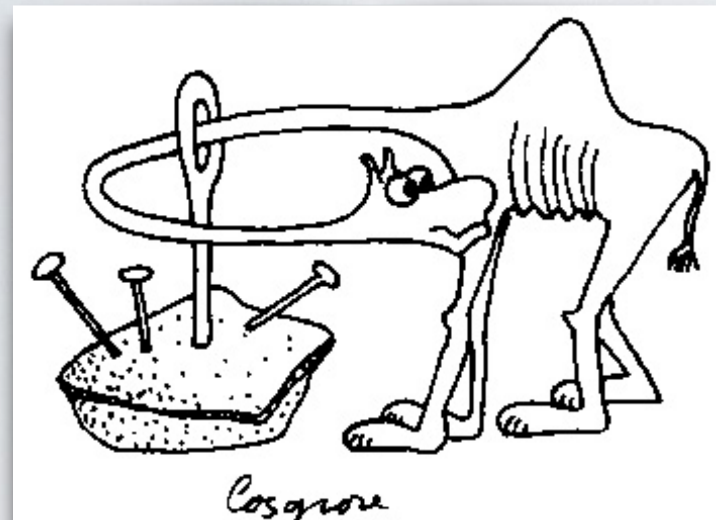
$$\mathfrak{L}_X \Omega = d(d\Omega) = 0$$

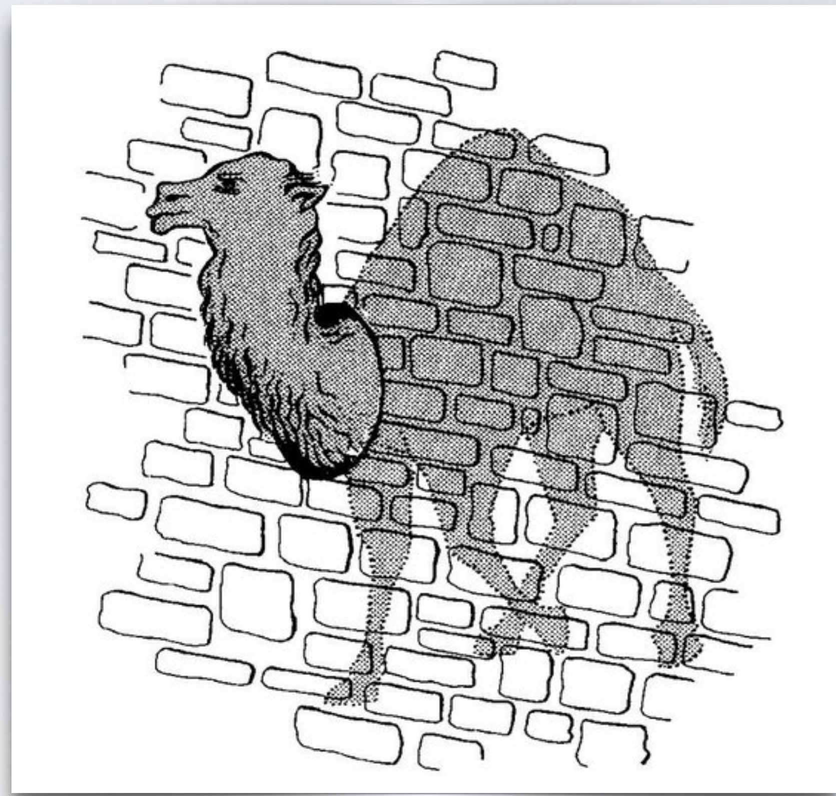
Definitions / Facts

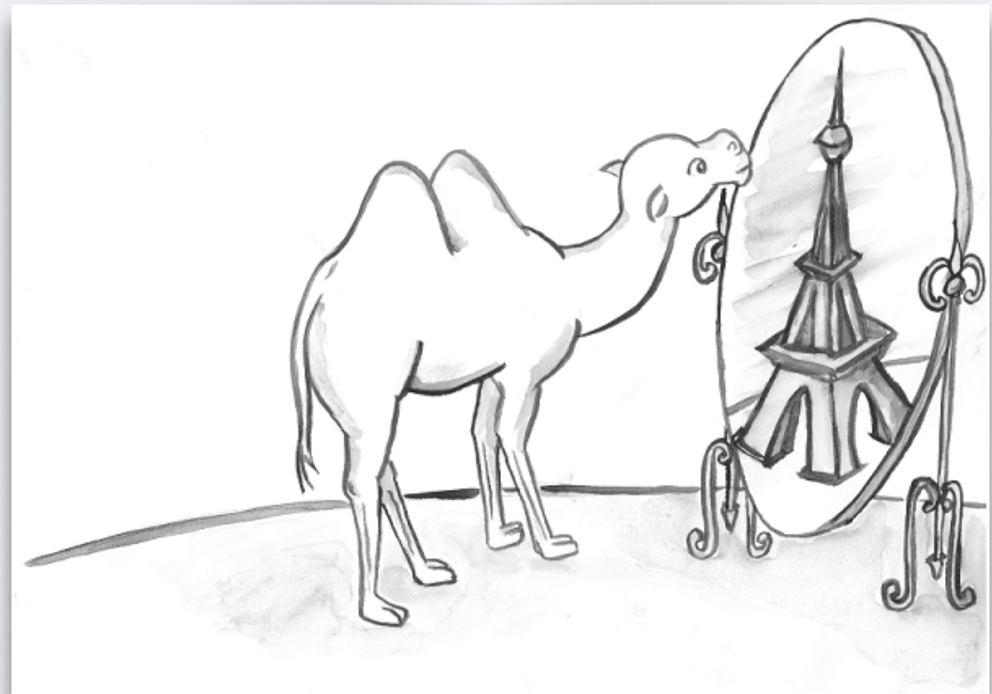
$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ Symplectic Form

$$dH = \iota_X \Omega$$

Thanks!







http://www.math.ucla.edu/~cm/symcon.html, https://math.berkeley.edu/~jchaidez/math290f16.html, http://school2016.imj-prg.fr/