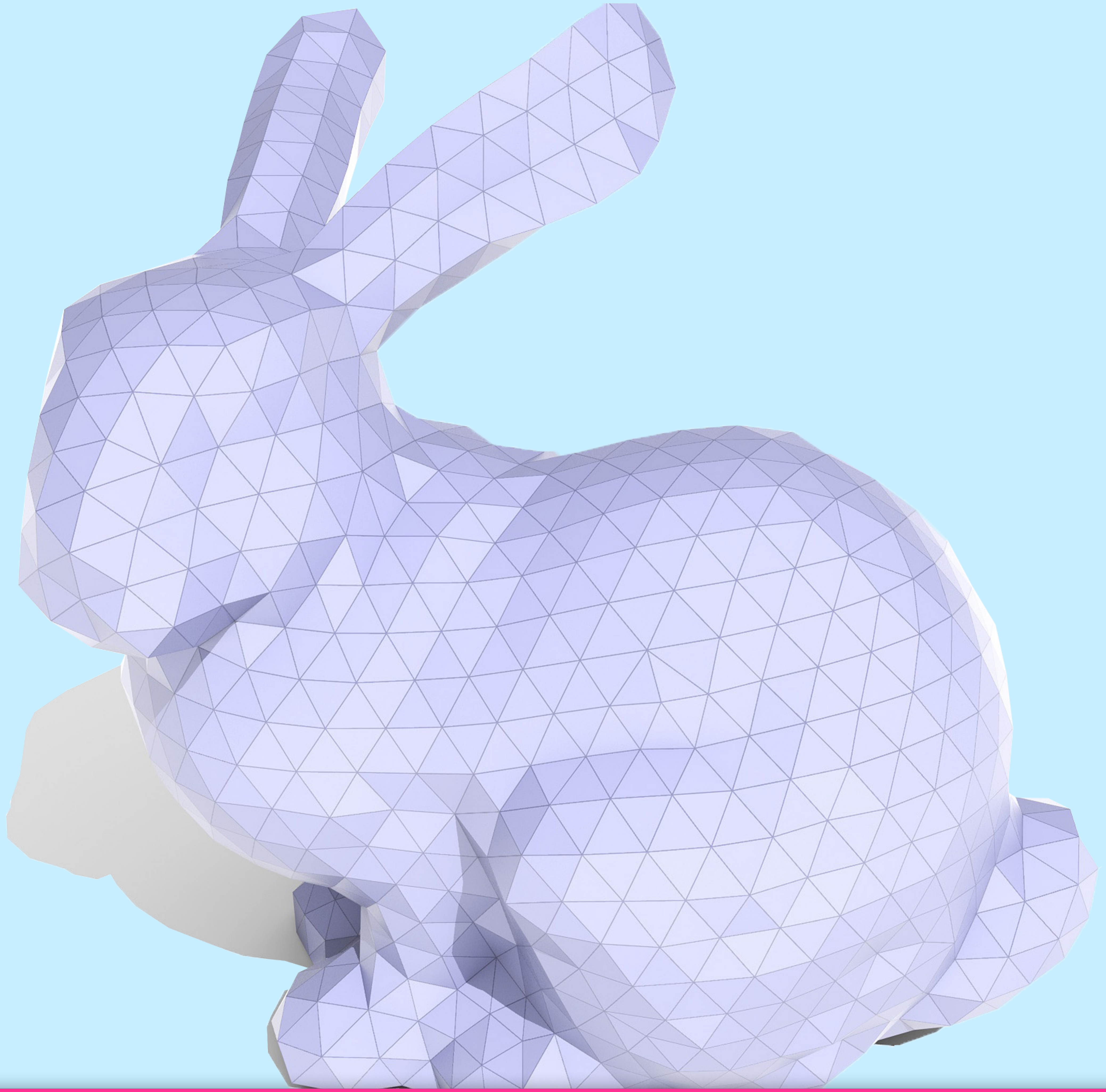
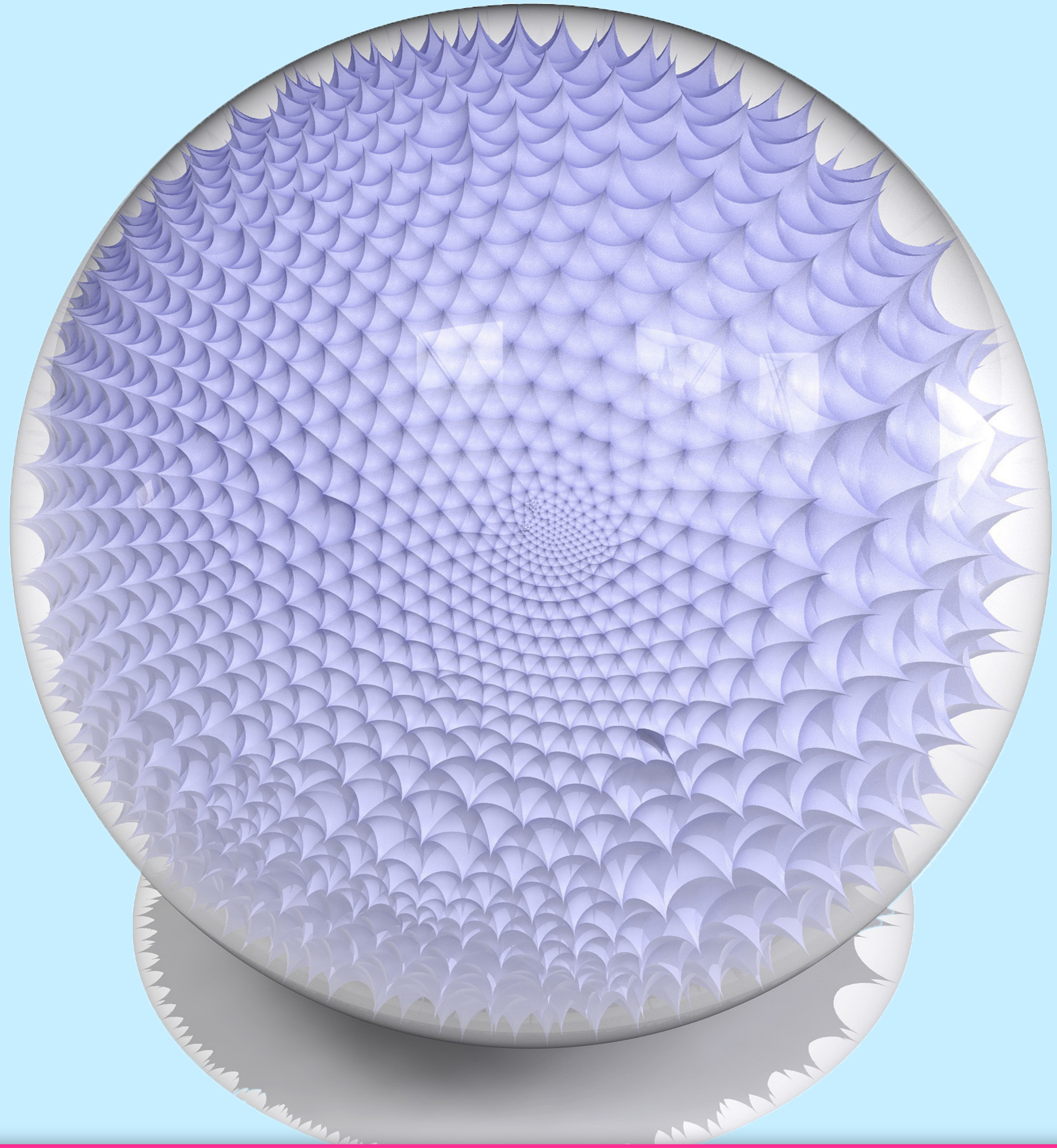




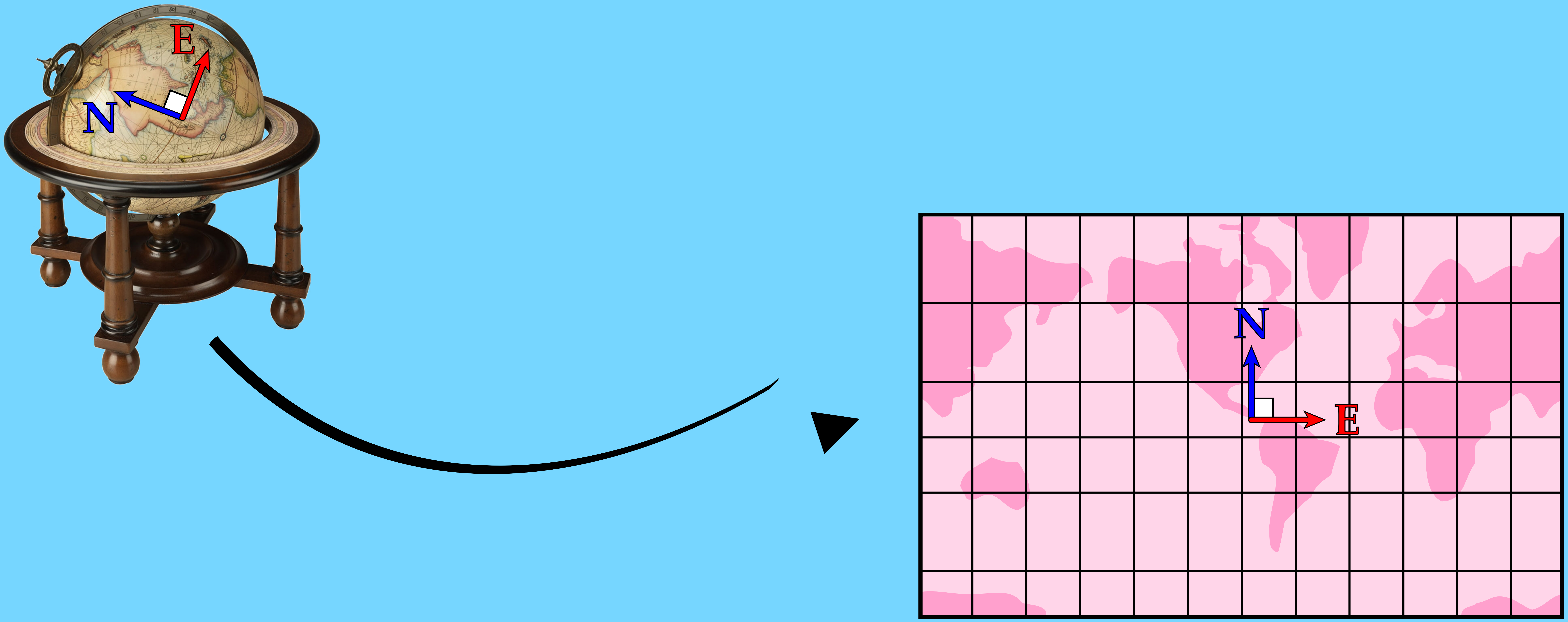
Discrete Conformal Structures and Hyperbolic Geometry

Mark Gillespie

Conformal Maps

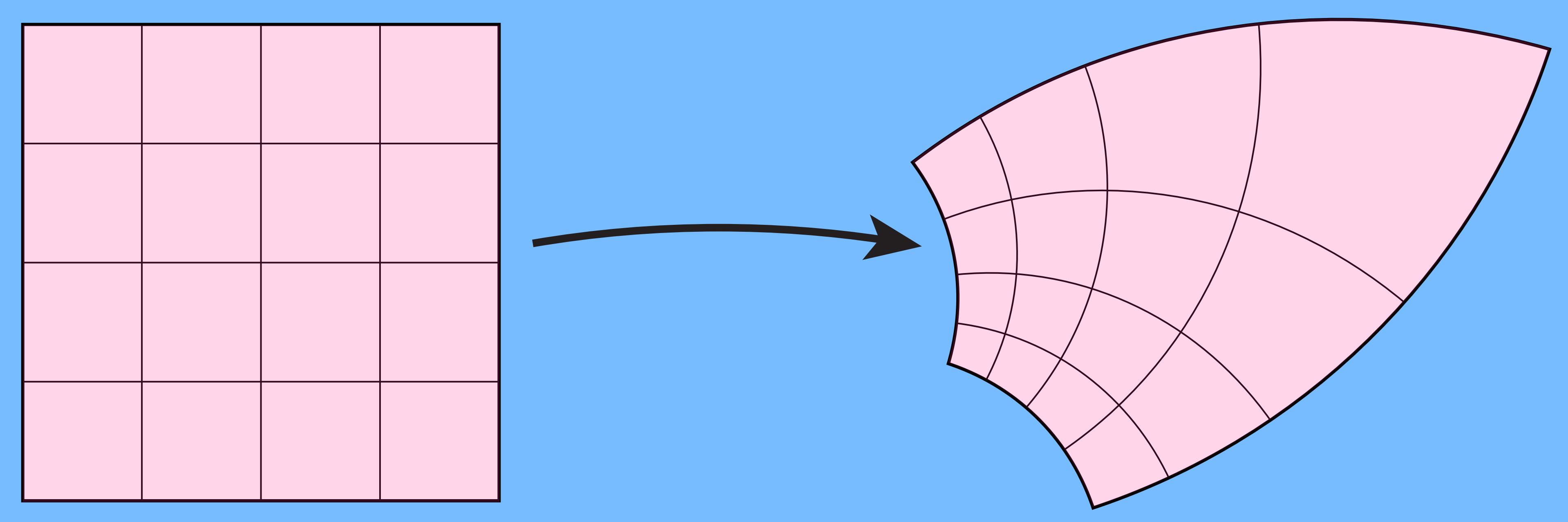
Conformal Maps

- Conformal maps preserve angles

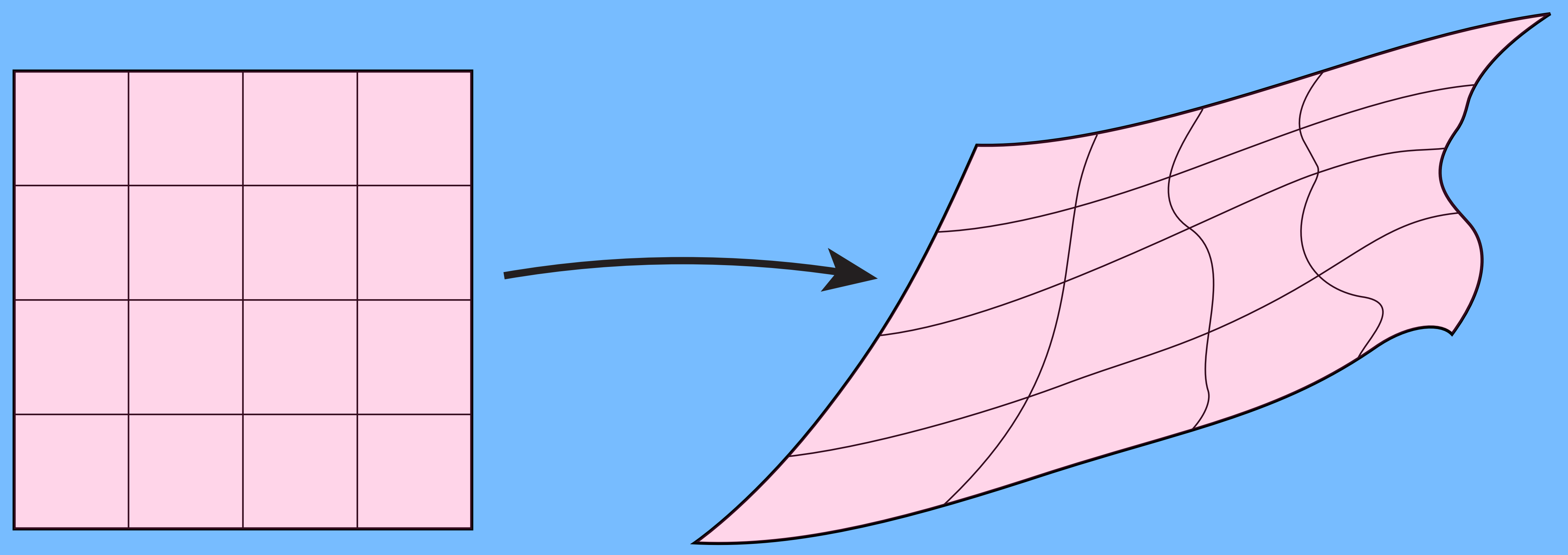


Conformal Maps

- Conformal maps preserve angles



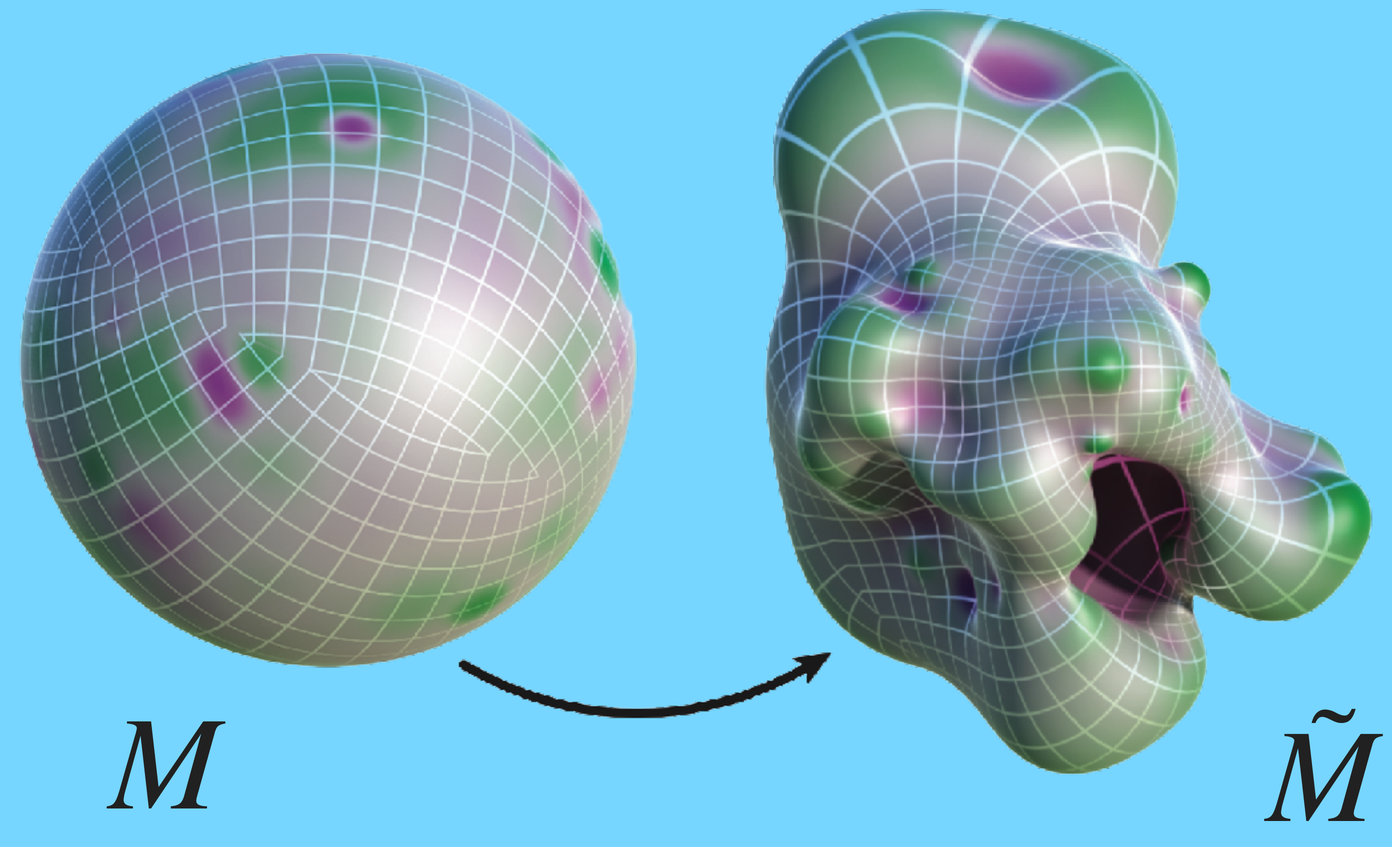
Conformal

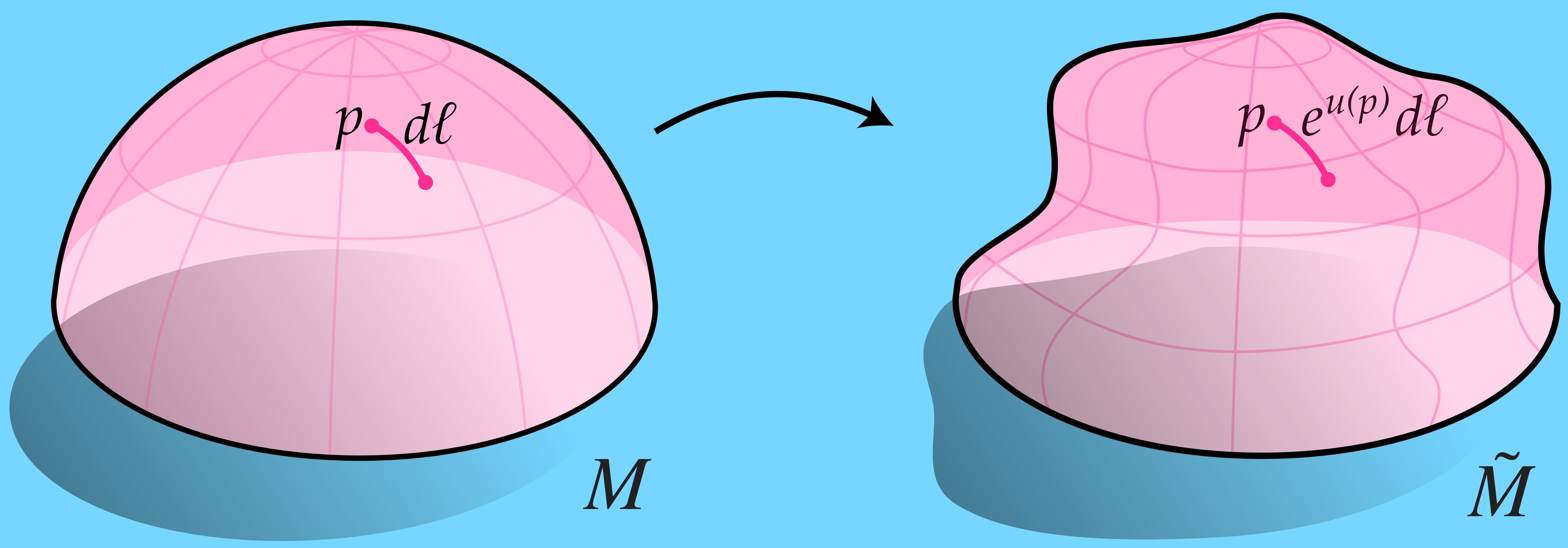


Not Conformal

Conformal Maps

- Conformal maps are specified by *conformal scale factors* $u : M \to \mathbb{R}$

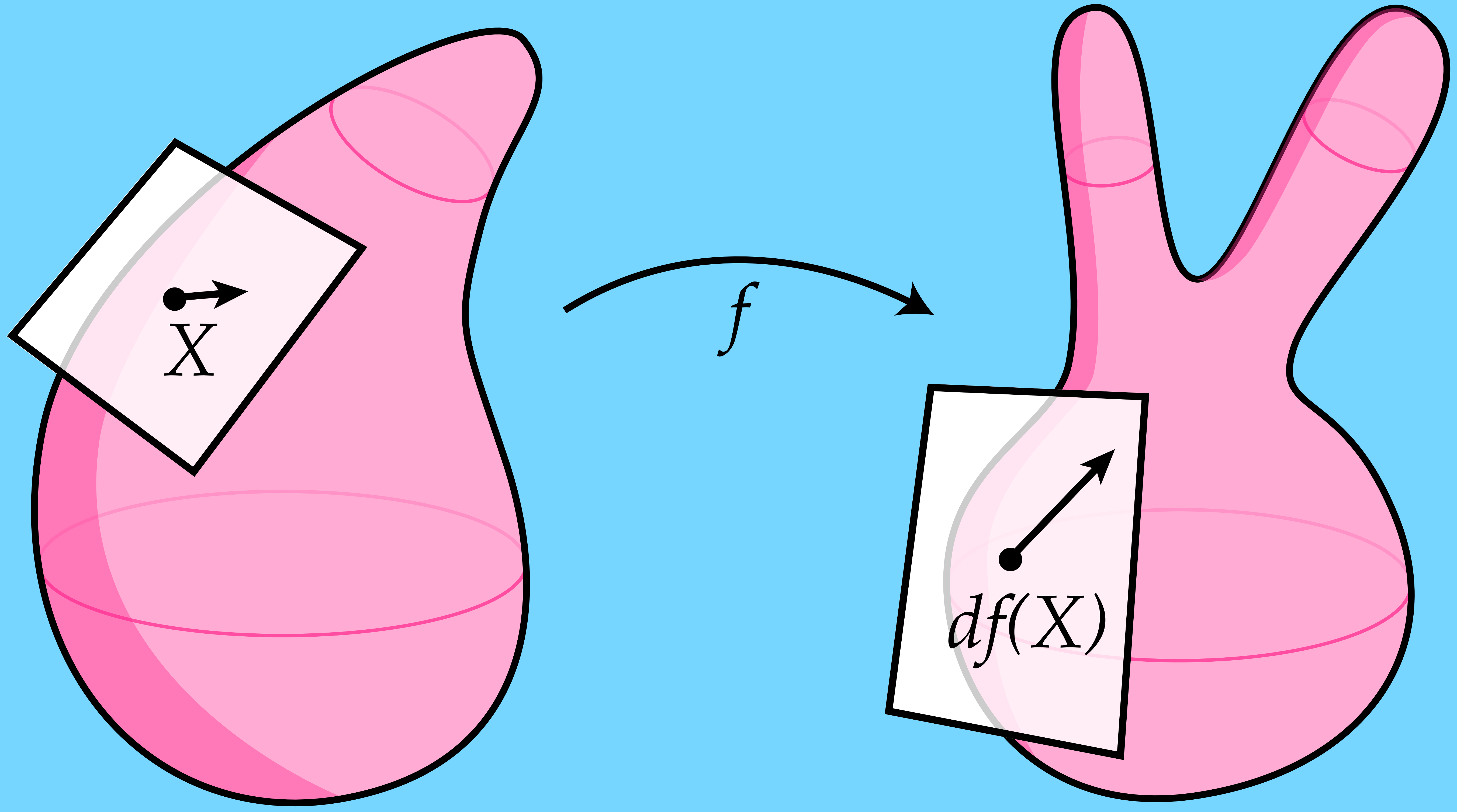




$$e^{2u(p)}g_p = \tilde{g}_p$$

Conformal Maps

- Infinitesimally, conformal maps look like *rotations* and *isotropic scalings*

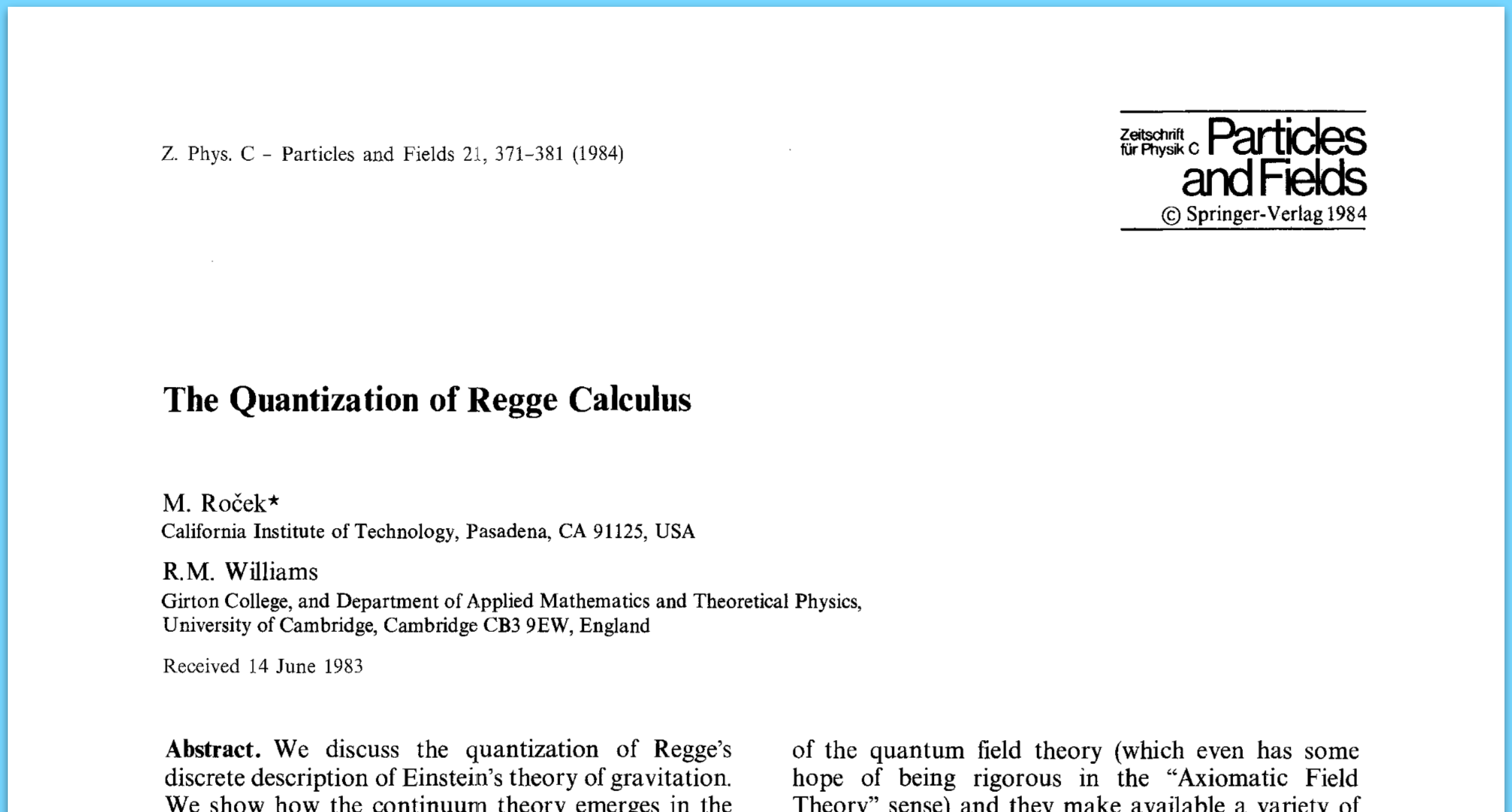


$df(X) = sRX$ for some rotation matrix R and scalar s

Discrete Conformal Maps (Definition 1)

- Specify a scale factor at each vertex
- Rescale edge lengths by

$$\tilde{\ell}_{ij} = e^{(u_i+u_j)/2}\ell_{ij}$$



Z. Phys. C – Particles and Fields 21, 371–381 (1984)

Zeitschrift für Physik C Particles and Fields

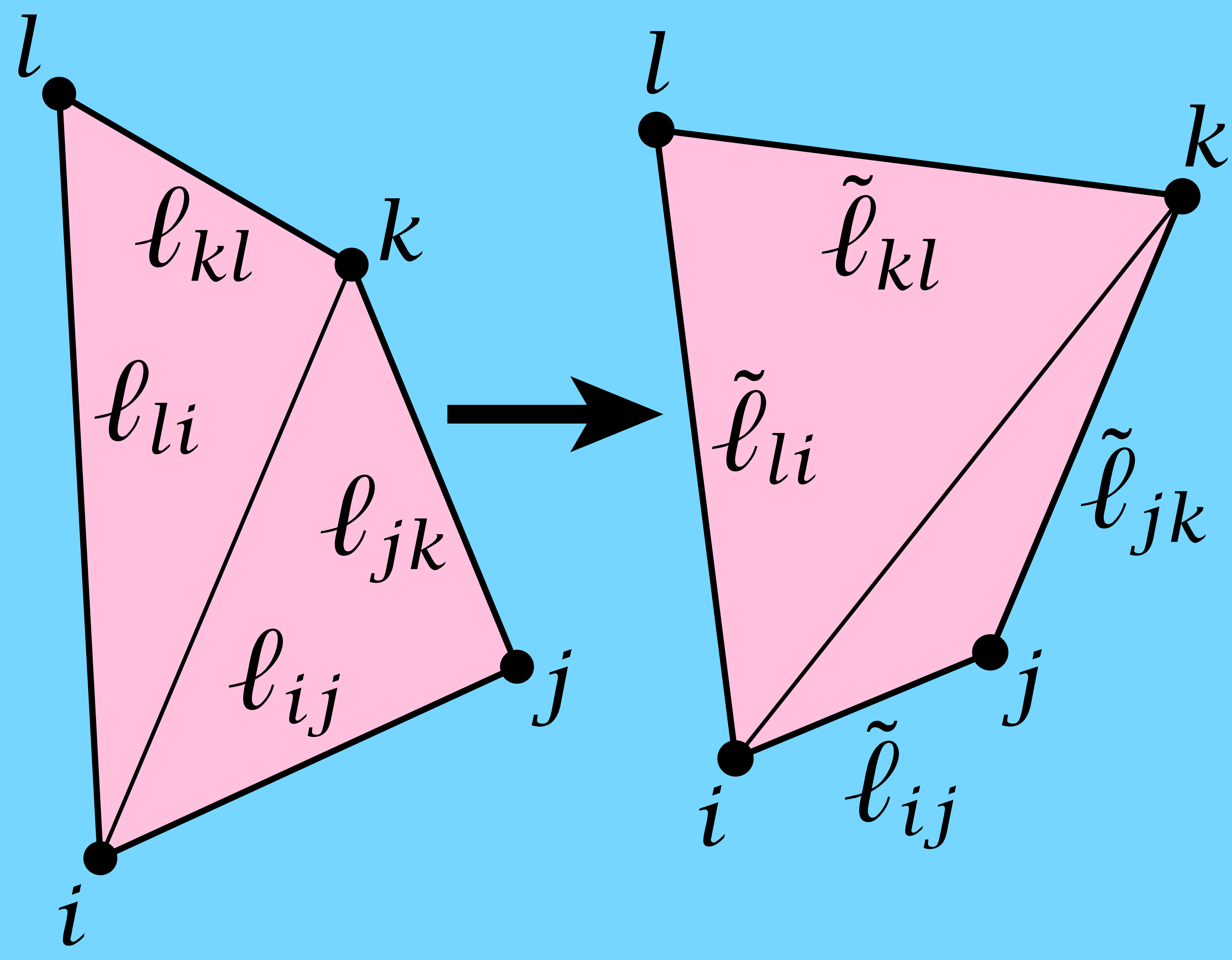
The Quantization of Regge Calculus

M. Roček*
California Institute of Technology, Pasadena, CA 91125, USA

R.M. Williams
Girton College, and Department of Applied Mathematics and Theoretical Physics,
University of Cambridge, Cambridge CB3 9EW, England

Received 14 June 1983

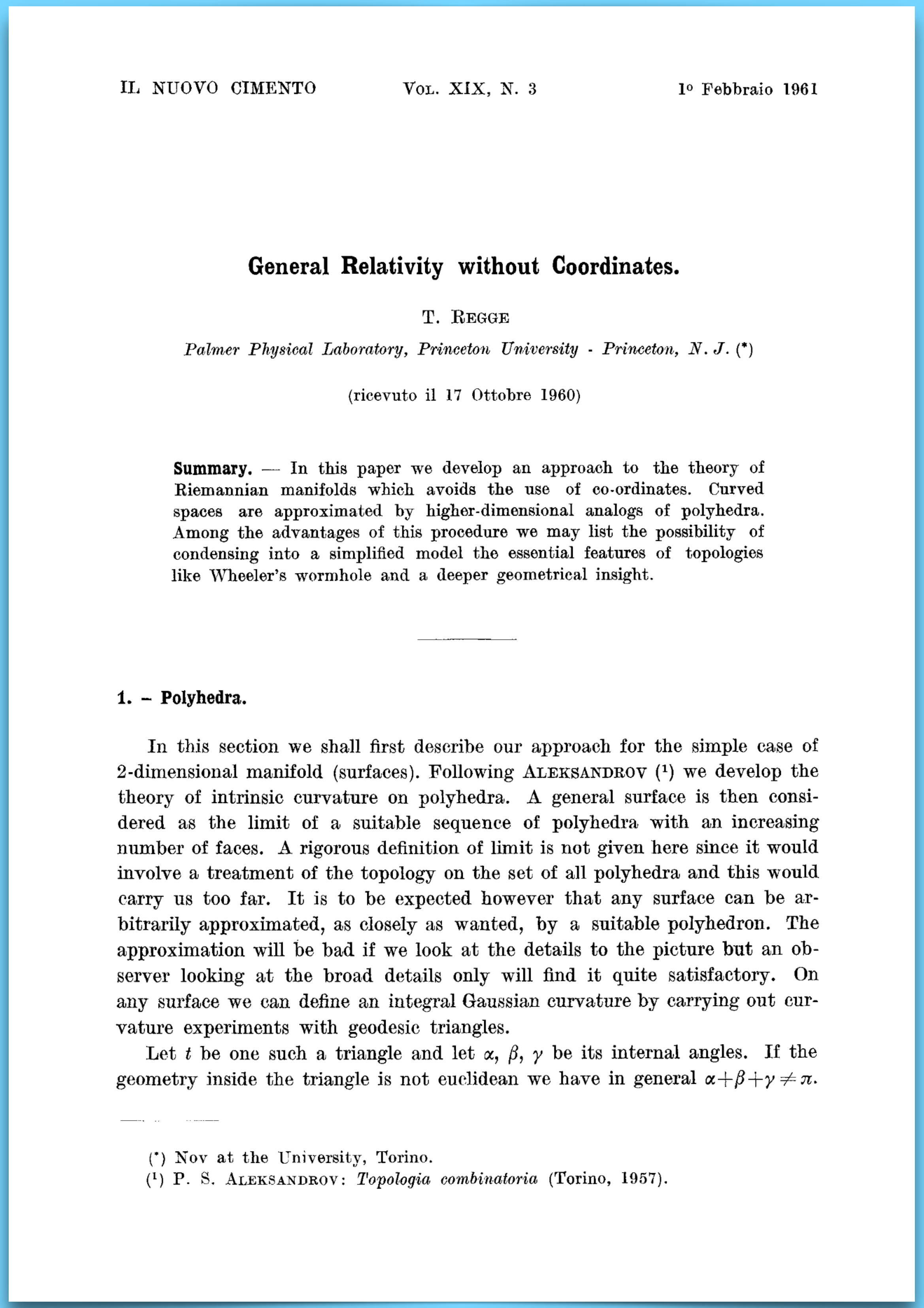
Abstract. We discuss the quantization of Regge's discrete description of Einstein's theory of gravitation. We show how the continuum theory emerges in the of the quantum field theory (which even has some hope of being rigorous in the "Axiomatic Field Theory" sense) and they make available a variety of



M. Roček, R.M. Williams, "*The Quantization of Regge Calculus*" (1984)

Aside: Regge Calculus

- Lays out a lot of discrete differential geometry
  - Gaussian curvature as angle defect
  - Gauss-Bonnet
  - Cone metrics



IL NUOVO CIMENTO   VOL. XIX, N. 3   1° Febbraio 1961

General Relativity without Coordinates.

T. REGGE

Palmer Physical Laboratory, Princeton University - Princeton, N. J. (*)

(ricevuto il 17 Ottobre 1960)

Summary. — In this paper we develop an approach to the theory of Riemannian manifolds which avoids the use of co-ordinates. Curved spaces are approximated by higher-dimensional analogs of polyhedra. Among the advantages of this procedure we may list the possibility of condensing into a simplified model the essential features of topologies like Wheeler's wormhole and a deeper geometrical insight.

1. – Polyhedra.

In this section we shall first describe our approach for the simple case of 2-dimensional manifold (surfaces). Following ALEKSANDROV (1) we develop the theory of intrinsic curvature on polyhedra. A general surface is then considered as the limit of a suitable sequence of polyhedra with an increasing number of faces. A rigorous definition of limit is not given here since it would involve a treatment of the topology on the set of all polyhedra and this would carry us too far. It is to be expected however that any surface can be arbitrarily approximated, as closely as wanted, by a suitable polyhedron. The approximation will be bad if we look at the details to the picture but an observer looking at the broad details only will find it quite satisfactory. On any surface we can define an integral Gaussian curvature by carrying out curvature experiments with geodesic triangles.

Let t be one such a triangle and let α, β, γ be its internal angles. If the geometry inside the triangle is not euclidean we have in general $\alpha+\beta+\gamma \neq \pi$.

(*) Now at the University, Torino.
(1) P. S. ALEKSANDROV: *Topologia combinatoria* (Torino, 1957).

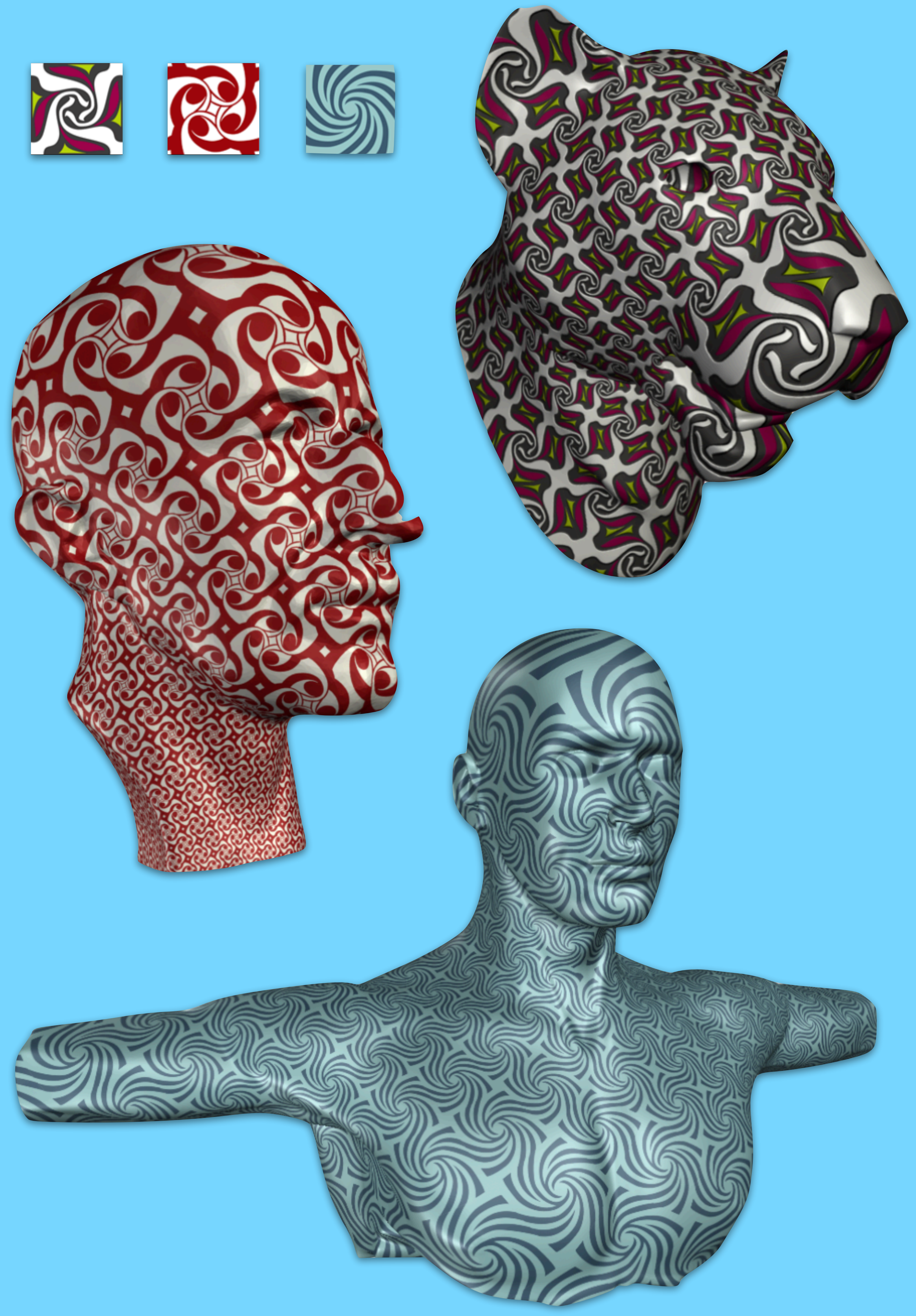
T. Regge, "*General Relativity without Coordinates*" (1961)

A Conformal Flattening Algorithm

1. Specify a target curvature $\tilde{K}_i$ at each vertex

2. Iteratively update the lengths using conformal scale factors

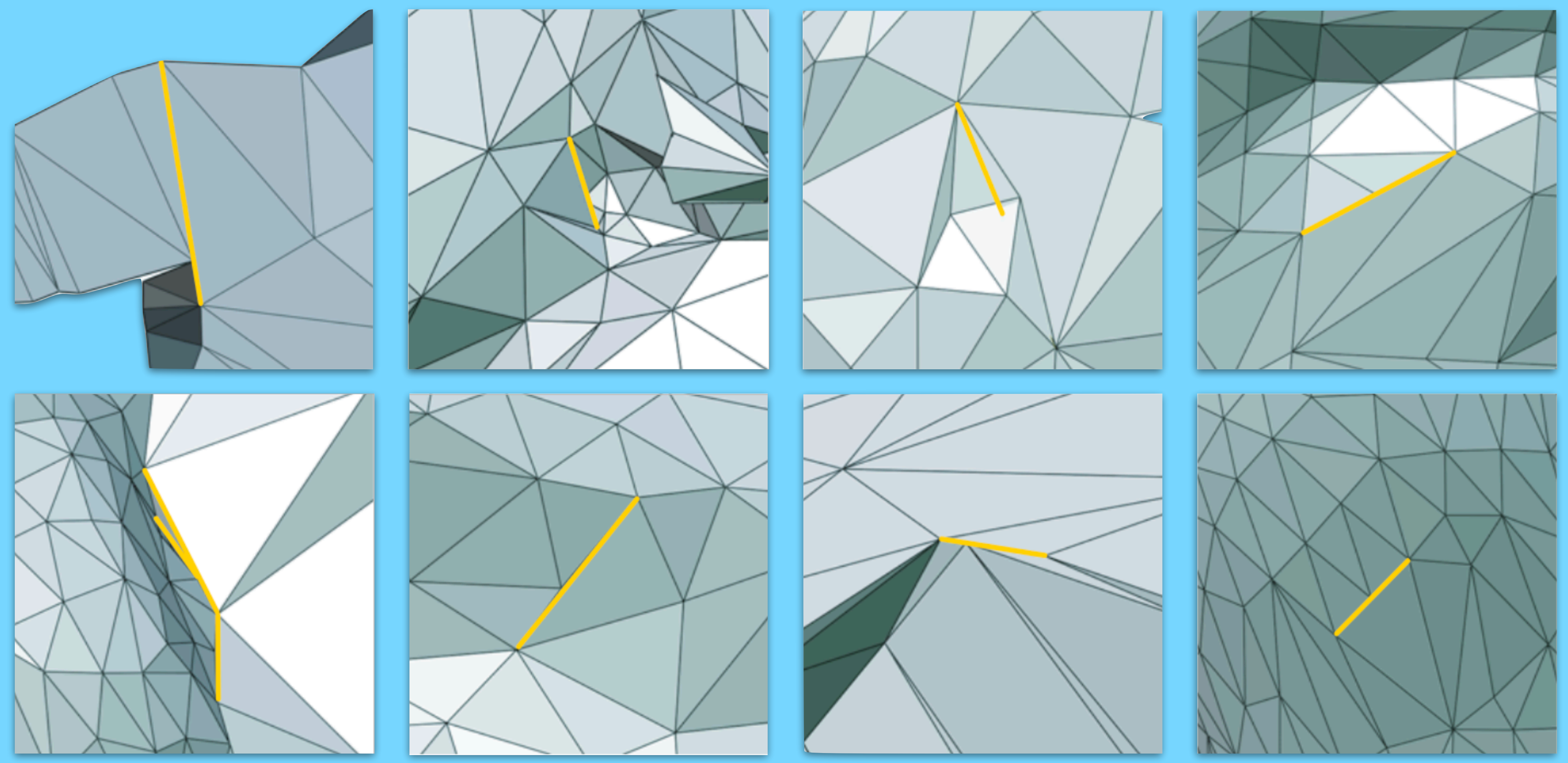
$$u_i = \tilde{K}_i - K_i$$



Springborn, Schröder, Pinkall, "*Conformal Equivalence of Triangle Meshes*" (2008)

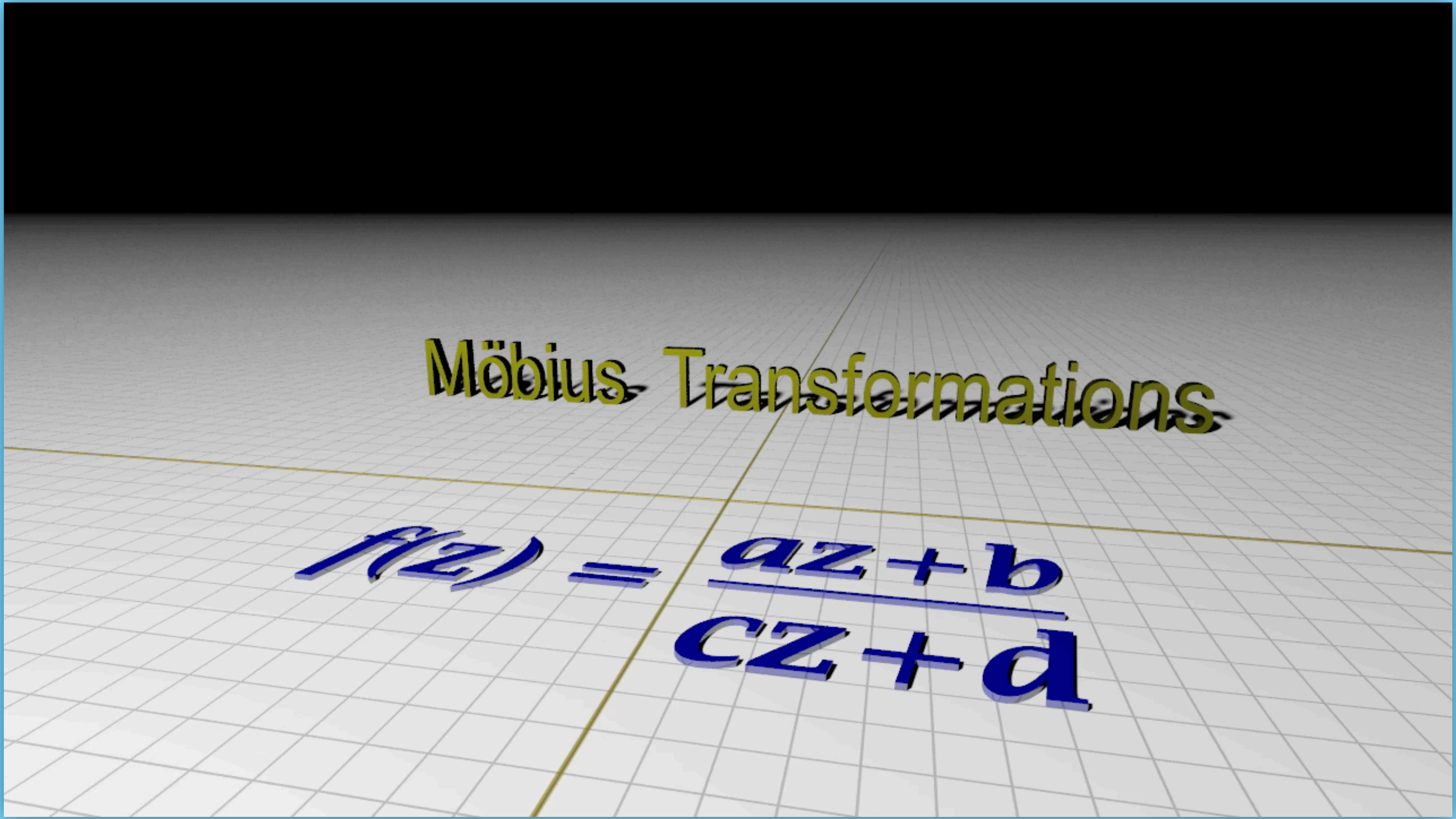
A Conformal Flattening Algorithm

- Problem: sometimes this rescaling breaks your mesh



> *“One can show that the product of two conformal transformations (40) such that each separately preserves these constraints is a transformation which in general will violate the constraints. Therefore, globally the group property is violated. Furthermore, no' subset of the transformations (40) forms a group”.*
>
> M. Roček, R.M. Williams,
> “*The Quantization of Regge Calculus*” (1984)

Möbius Transformations



Möbius Transformations

- Complex functions of the form $f(z) = \frac{az+b}{cz+d}$
- Note that if $ad = bc$, then

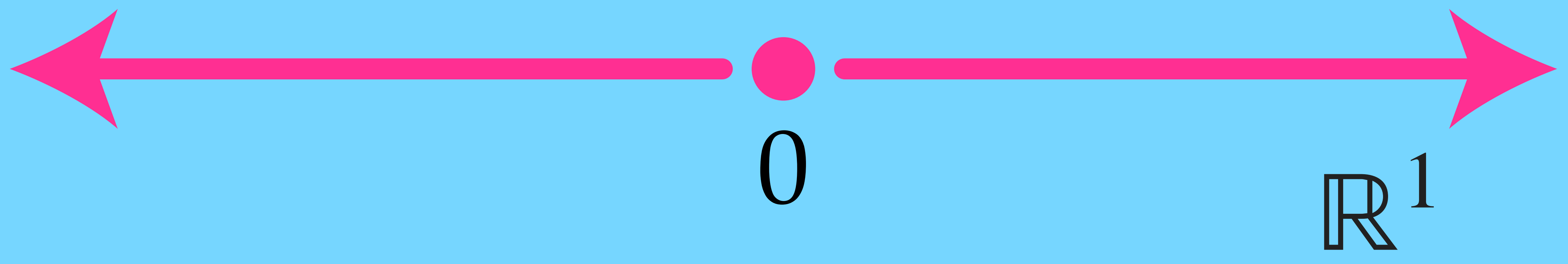
$$f(z) = \frac{az+b}{cz+d} = \frac{c(az+b)}{c(cz+d)} = \frac{caz+ad}{ccz+cd} = \frac{a}{c}$$

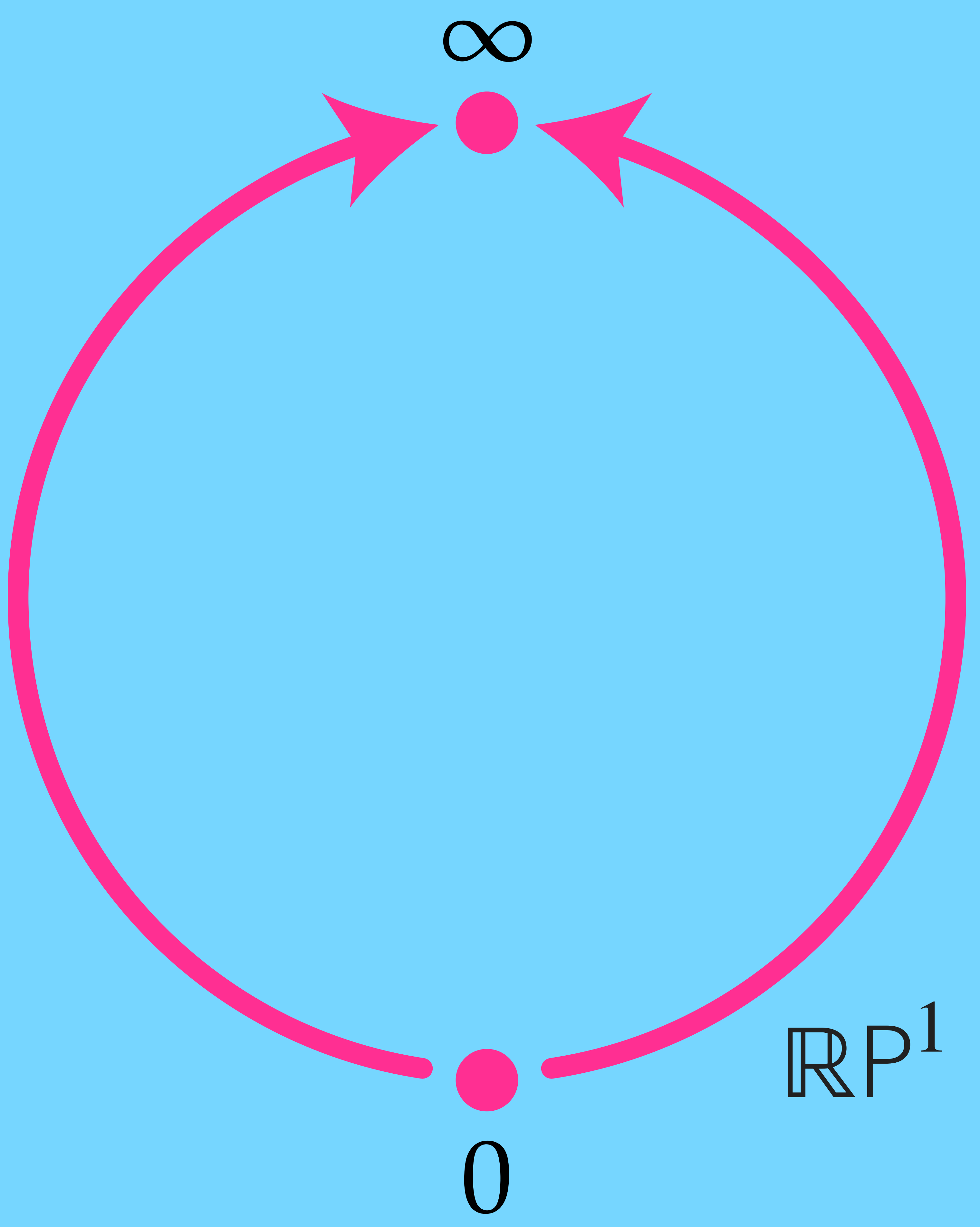
- We disallow this
- Remark: we can scale all coefficients

Möbius Transformations are Projective

$$\begin{pmatrix} az+b \\ cz+d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}$$

- Homogeneous coordinates



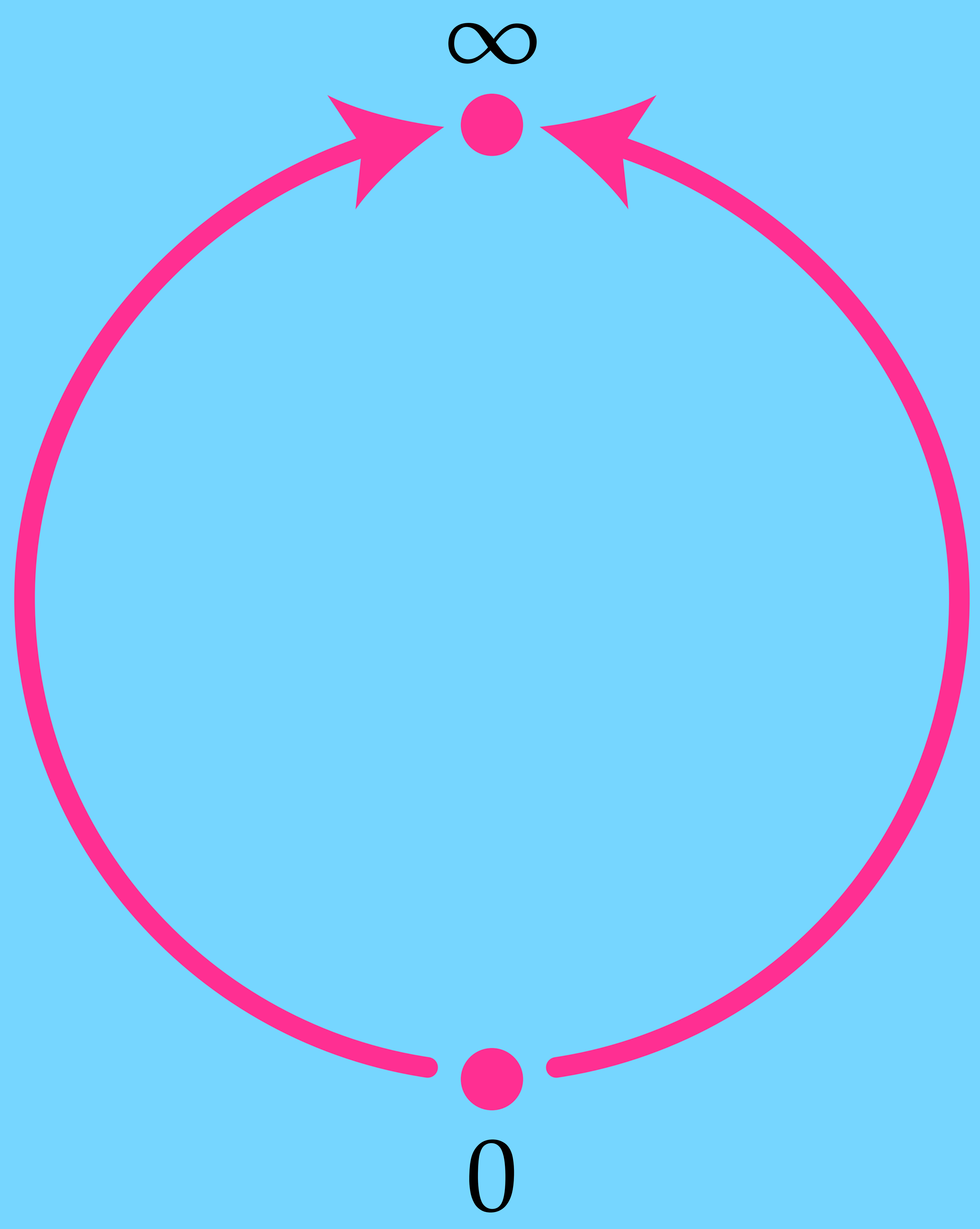


Complex Projective Space

- Just like real projective space

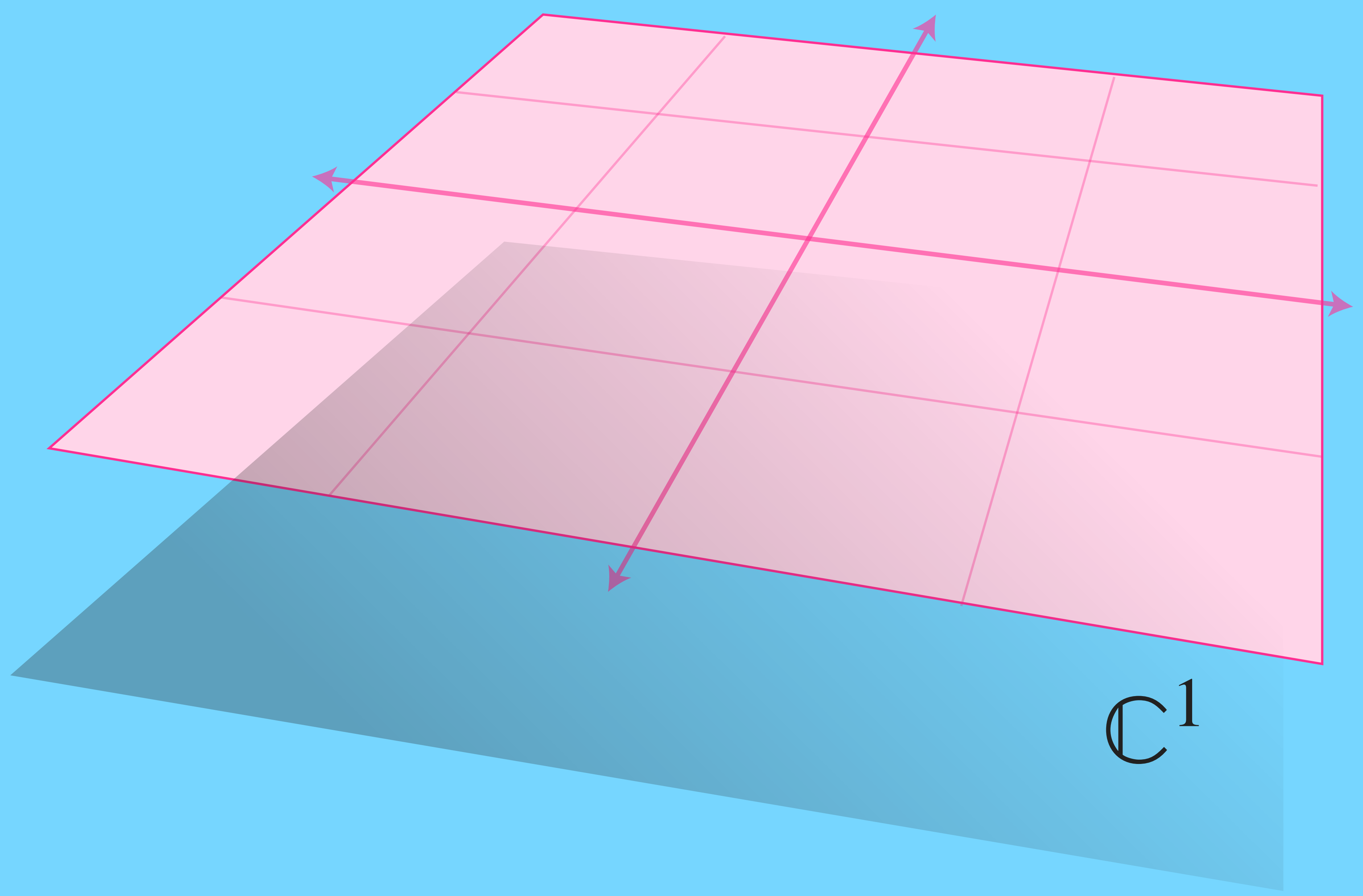
$$[z_1, z_2] \sim [\lambda z_1, \lambda z_2]$$

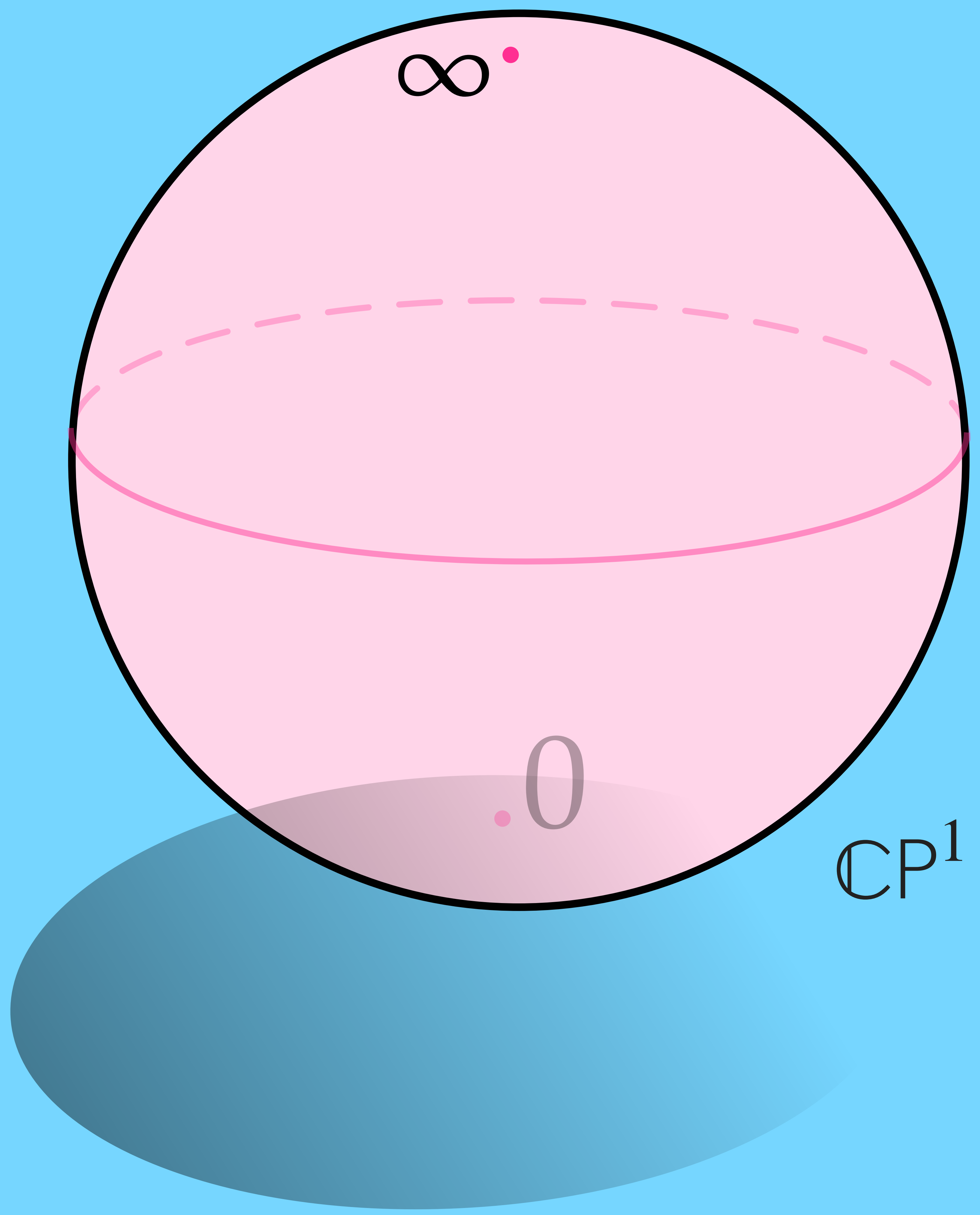
- What shape is it?
- Let's look at $\mathbb{R}P^1$ first: $[x_1, x_2] \sim [sx_1, sx_2]$
- Every vector has a canonical form $[x, 1]$
    - Except $[1, 0]$ - point at infinity



Complex Projective Space

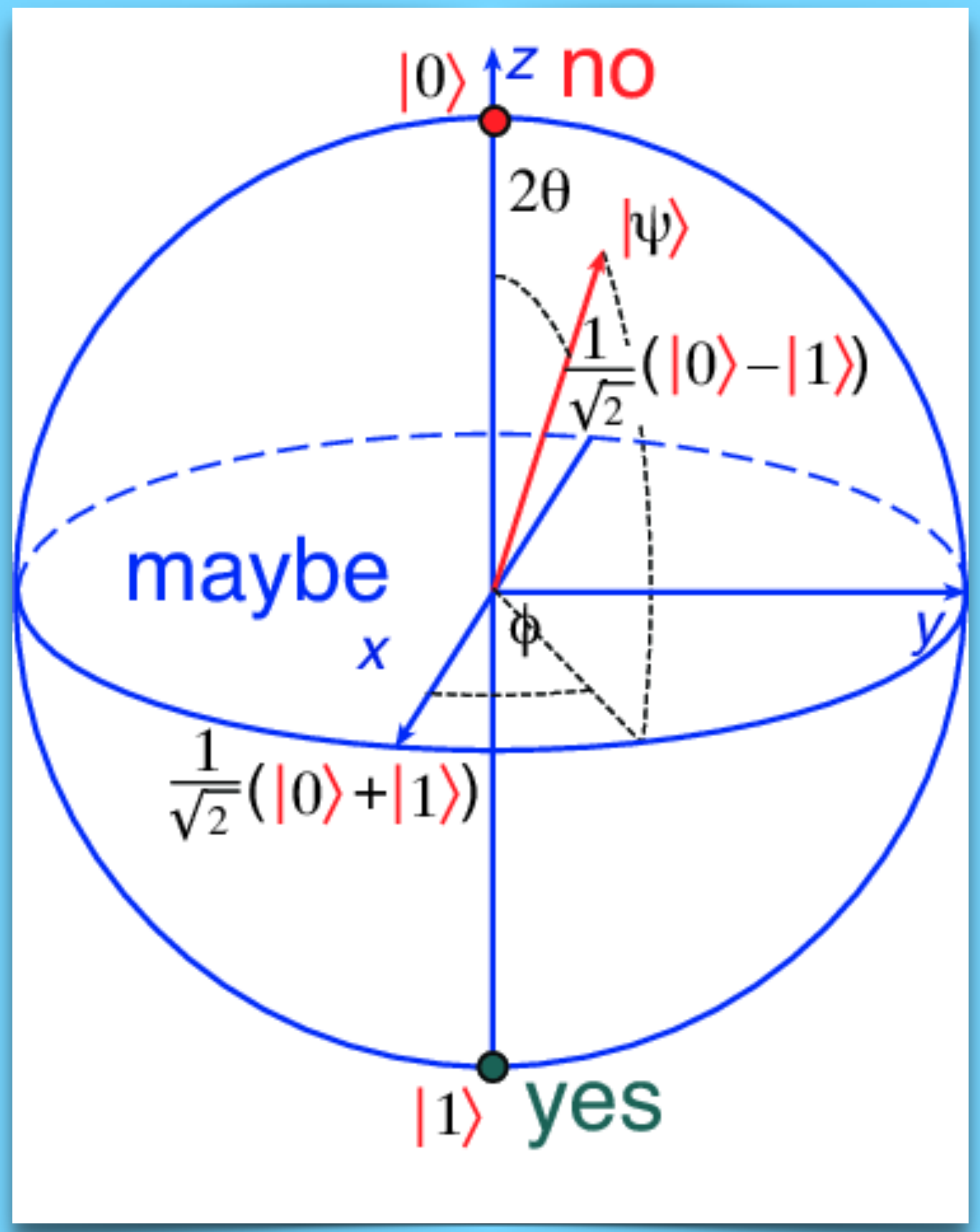
- Similarly, complex vectors look like $[z, 1]$





Aside: Bloch Sphere

- $\mathbb{C}P^1$ is also the state space of a qbit
- qbits live in 2-state quantum systems, i.e. $\mathbb{C}^2$
    - But we normalize and ignore phase
- qbits evolve in time by Möbius transformations!



Möbius Transformations

$$f(z) = \frac{az + b}{cz + d} \qquad ad \neq bc$$

- 4 complex degrees of freedom, 1 complex constraint
- Determined by 3 points

Complex Cross Ratios

- Consider 4 points $a, b, c, d, \in \mathbb{C}$

- Pick φ so that

$$\varphi(a) = \infty, \;\; \varphi(b) = 1, \;\; \varphi(c) = 0$$

- We define $[a, b; c, d]_{\mathbb{C}} := \varphi(d)$

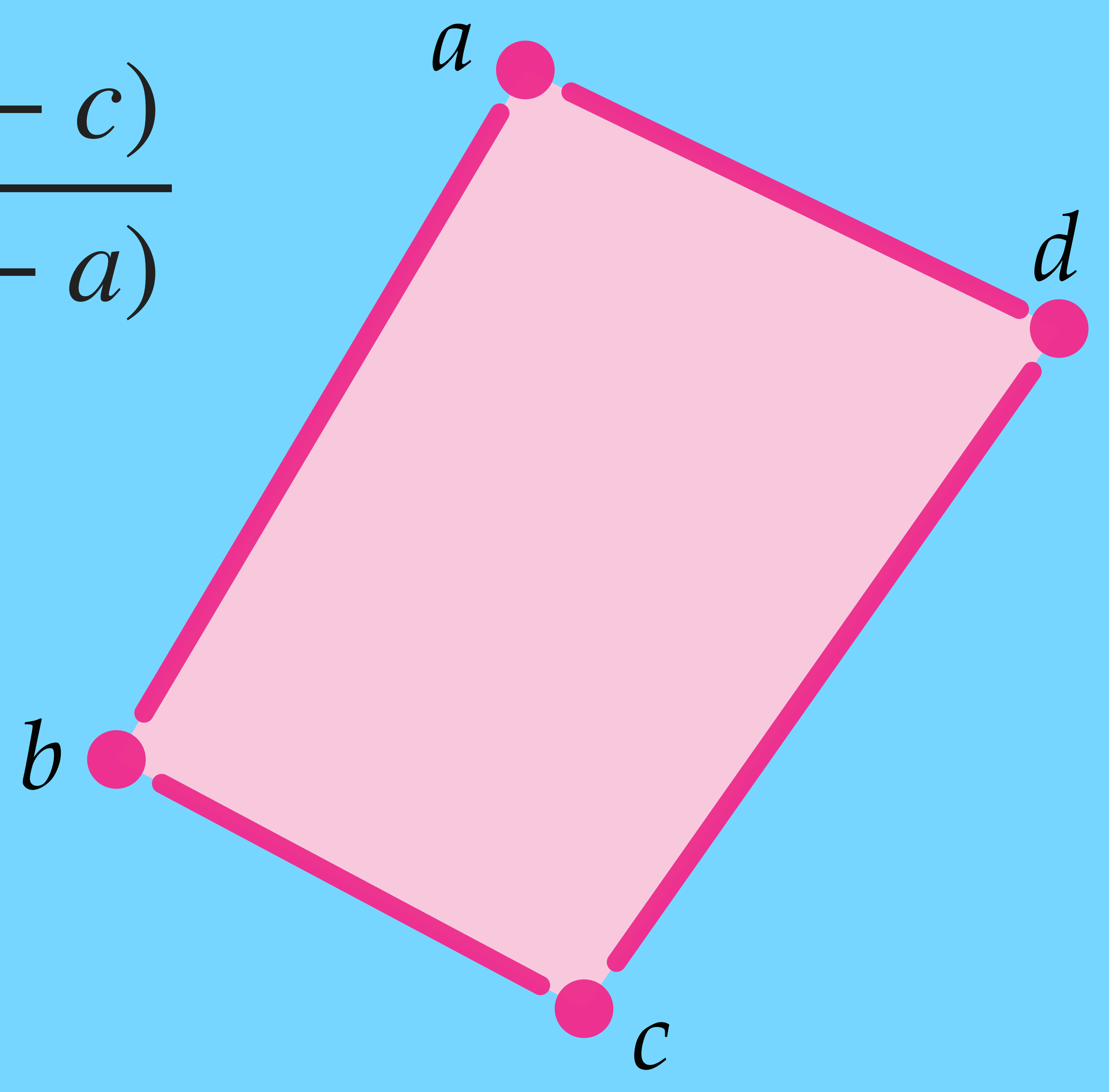
Length Cross Ratios

- Some computation reveals that

$$[a, b; c, d]_{\mathbb{C}} = \frac{(b-a)(d-c)}{(b-c)(d-a)}$$

- We define the *length cross ratio* by

$$[a, b; c, d] = \left| \frac{(b-a)(d-c)}{(b-c)(d-a)} \right|$$



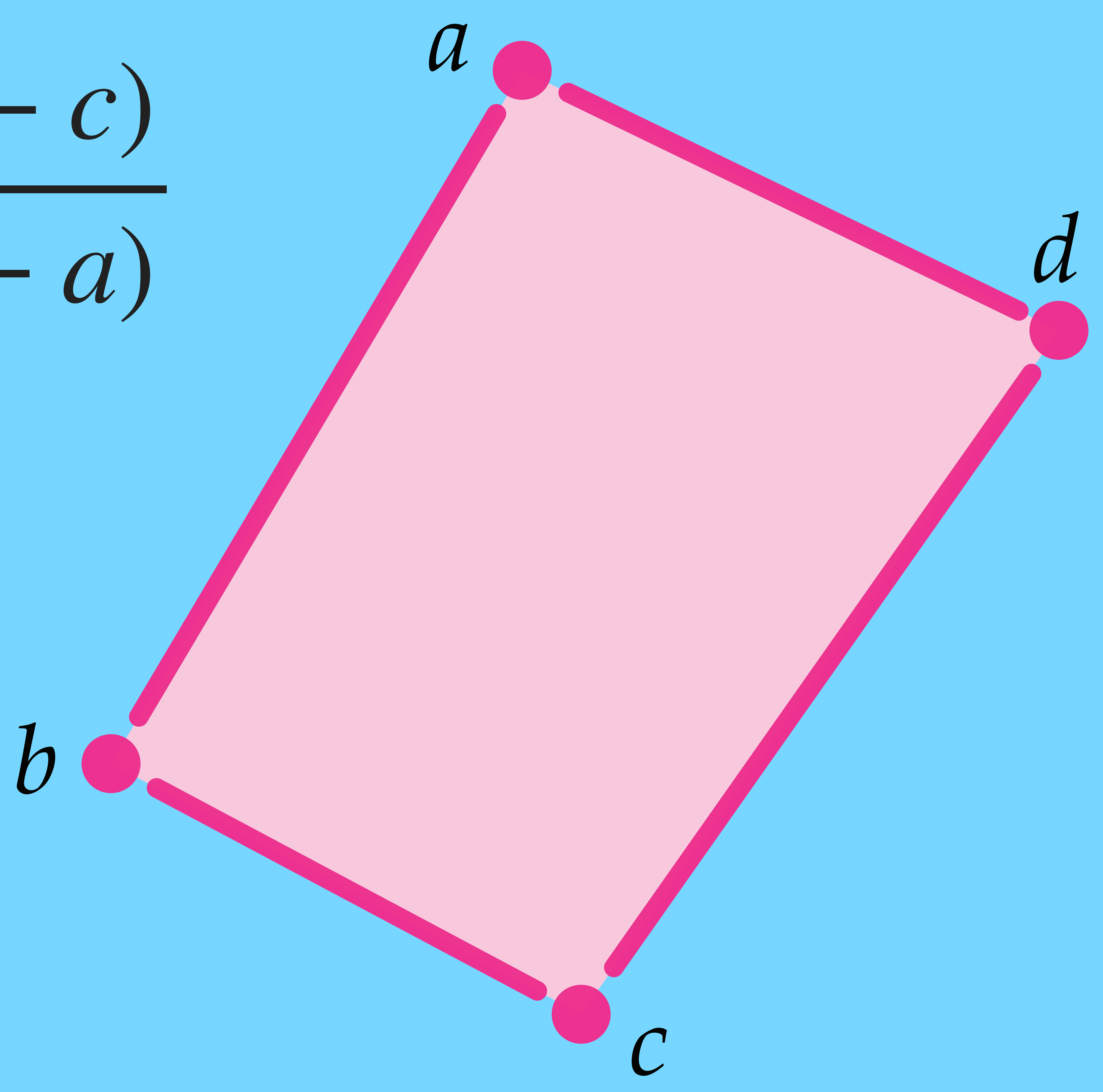
Length Cross Ratios

- Some computation reveals that

$$[a, b; c, d]_{\mathbb{C}} = \frac{(b-a)(d-c)}{(b-c)(d-a)}$$

- We define the *length cross ratio* by

$$[a, b; c, d] = \left| \frac{(b-a)(d-c)}{(b-c)(d-a)} \right|$$



Length Cross Ratios

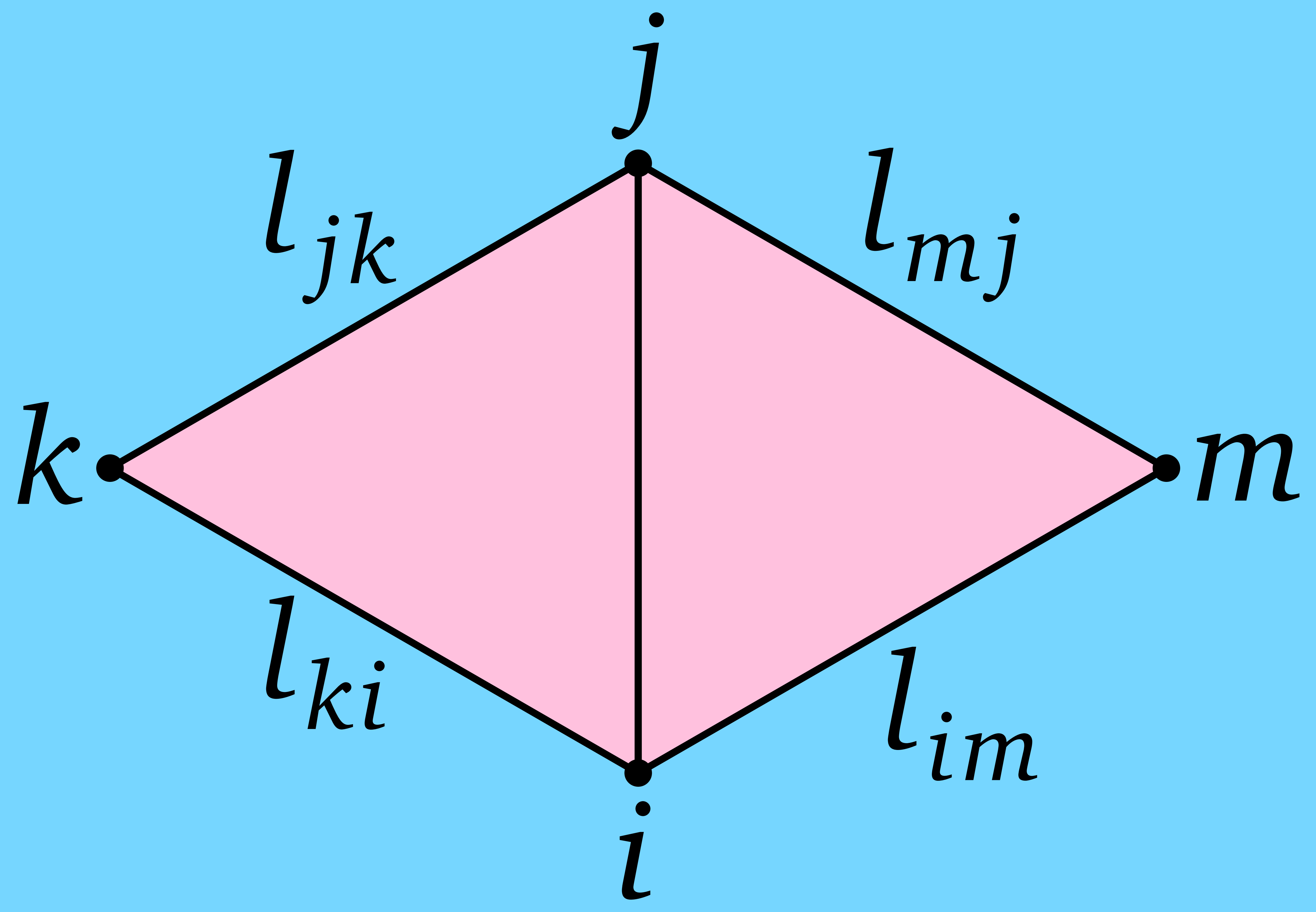
- A map is conformal if and only if its derivative preserves length cross ratios
- Easy direction: the derivative of a conformal map is a rotation and scaling - these preserve length cross ratios

Discrete Conformal Maps (Definition 2)

- We can associate length cross ratios with the edges of a triangle mesh

$$\mathfrak{c}_{ij} := \frac{l_{im}}{l_{mj}} \frac{l_{jk}}{l_{ki}}$$

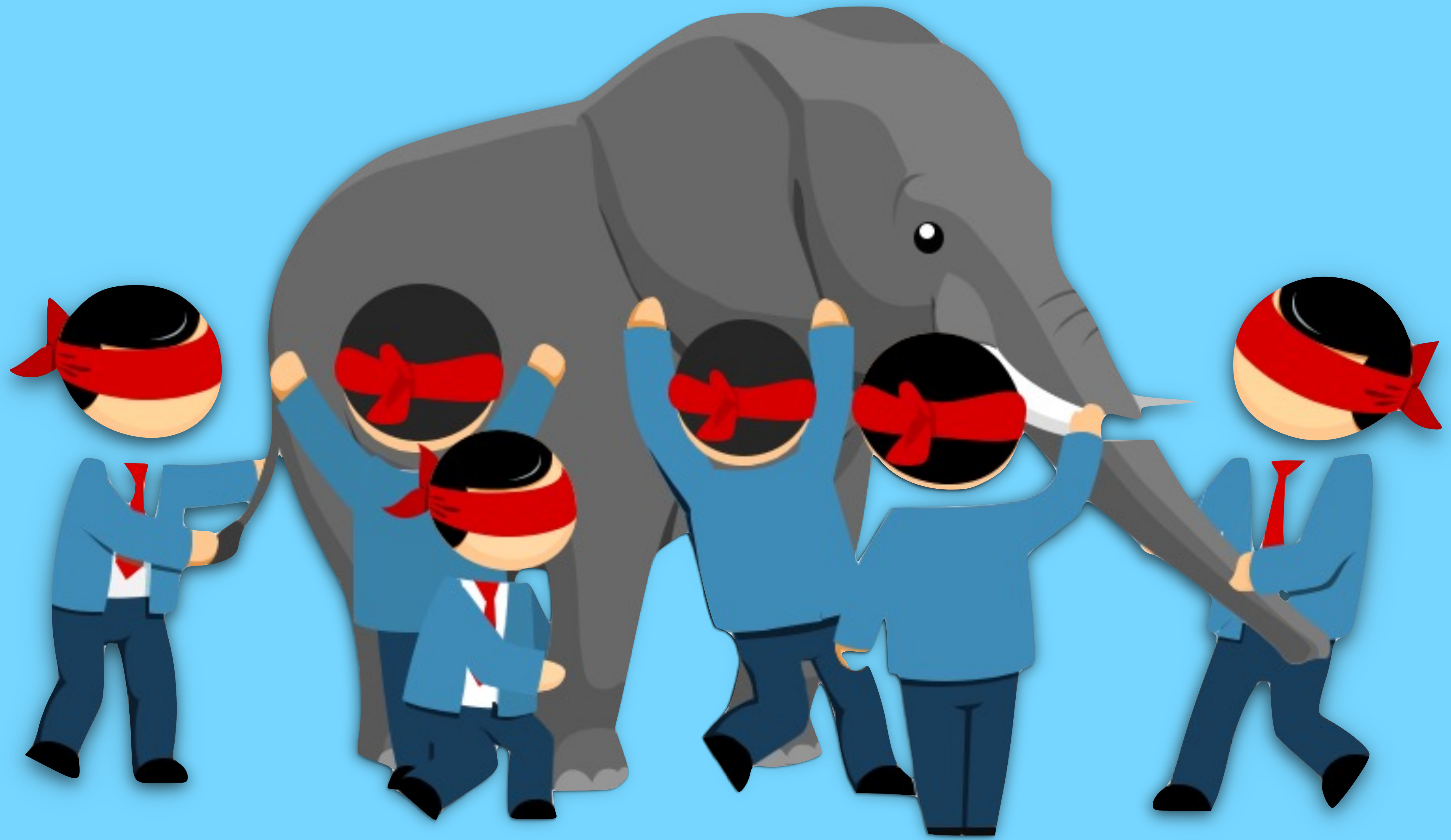
- Discrete conformal equivalence means having the same cross ratios



Hyperbolic Geometry

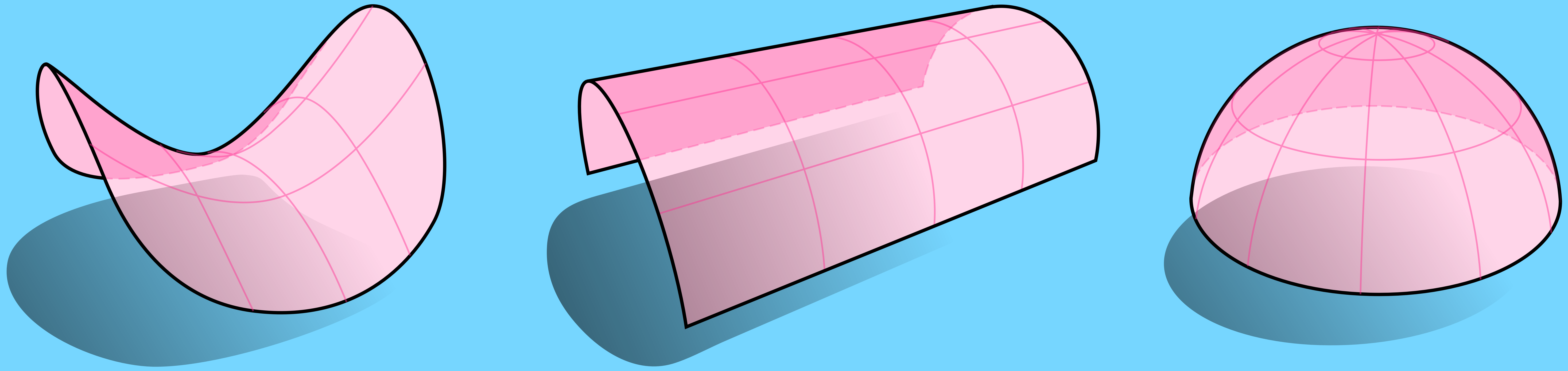
The Hyperbolic Plane

- The hyperbolic plane is a 2D surface, but it is so big that you can't fit it into $\mathbb{R}^3$!
- We study it through "models"



The Hyperbolic Plane

- Characterization: Gaussian curvature -1 everywhere
- What is Gaussian curvature?

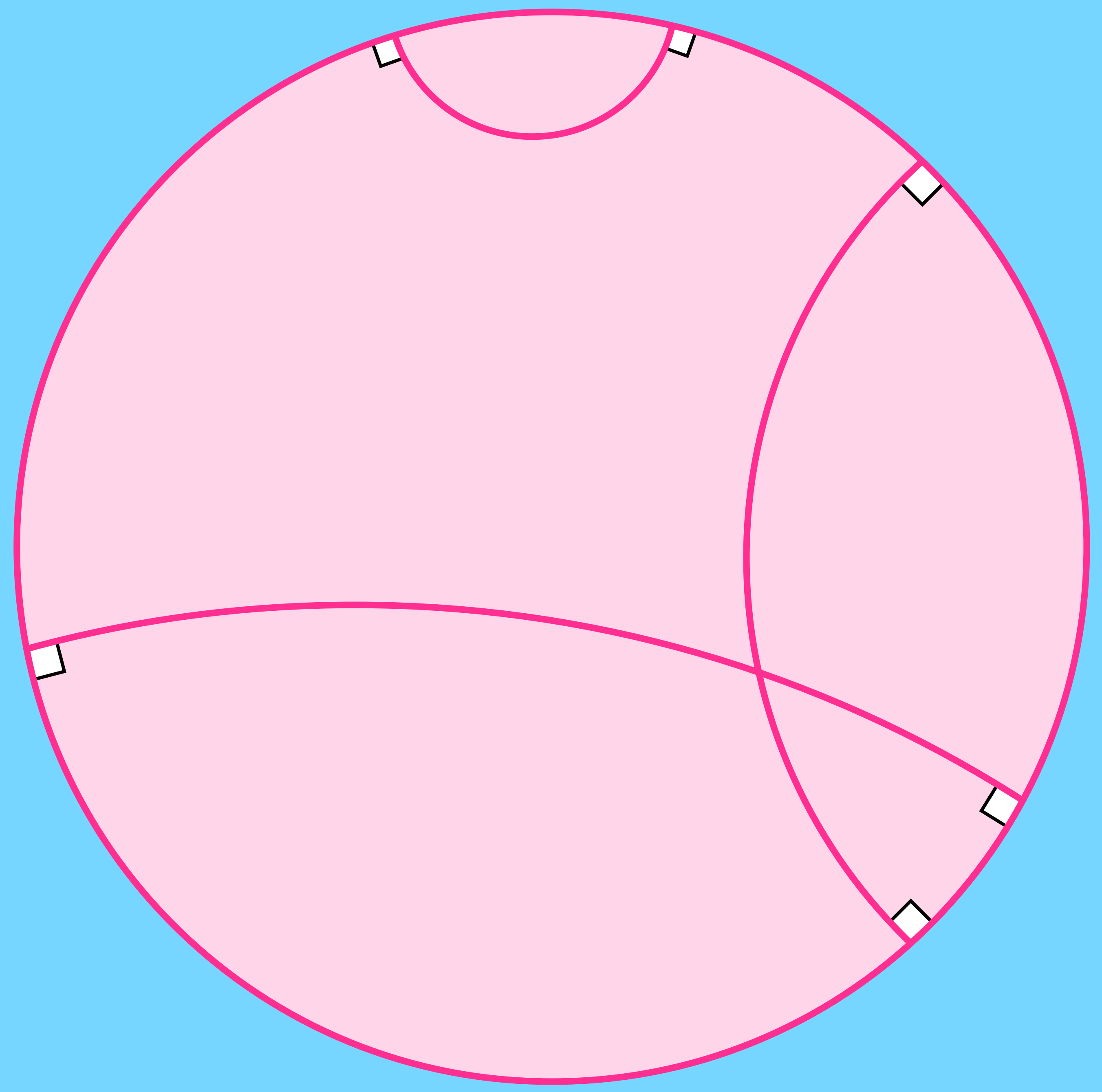


The Hyperbolic Plane

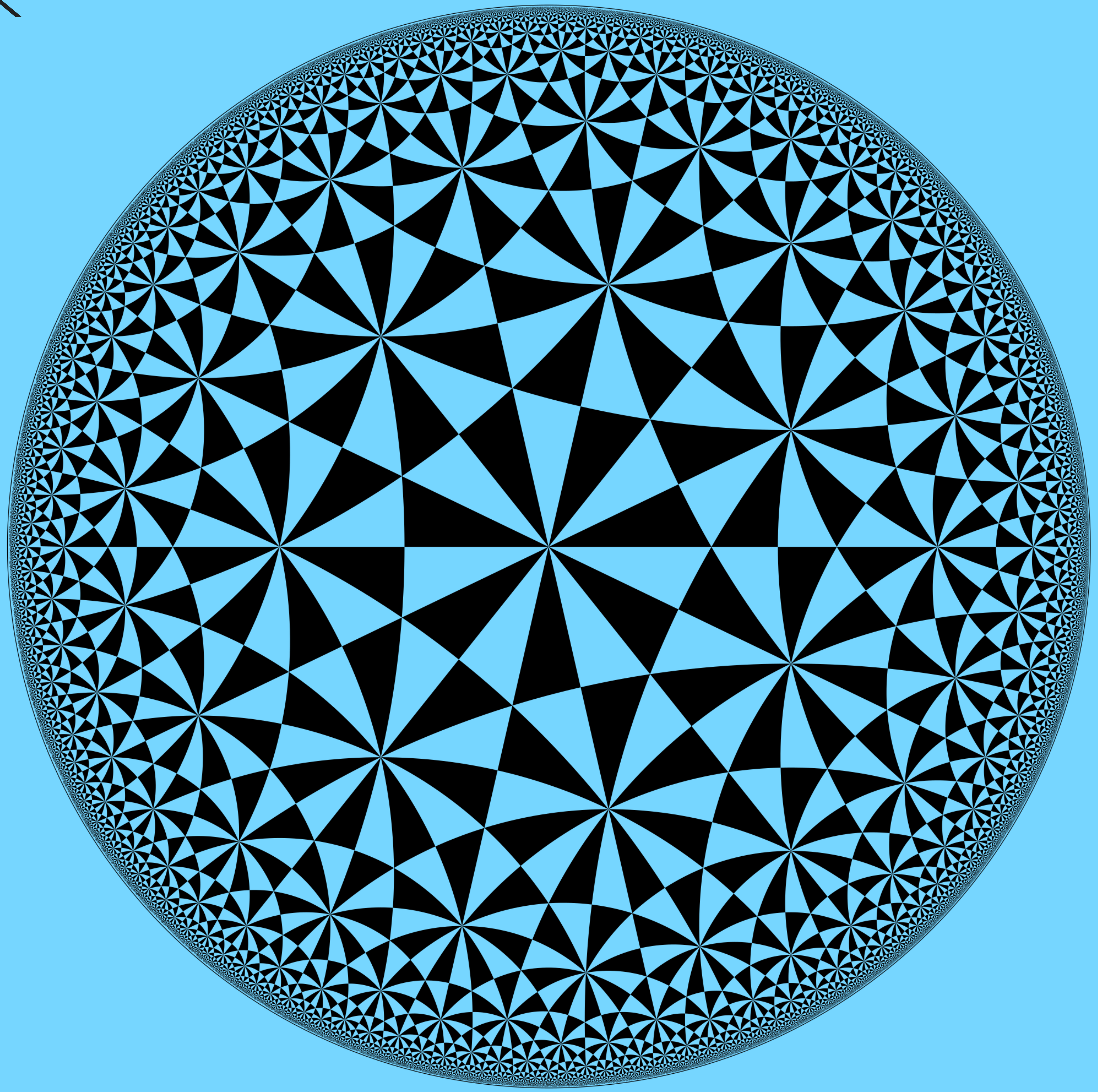
- Curvature -1 => wrinkly

Poincaré Disk

- Hyperbolic plane squished into unit disk



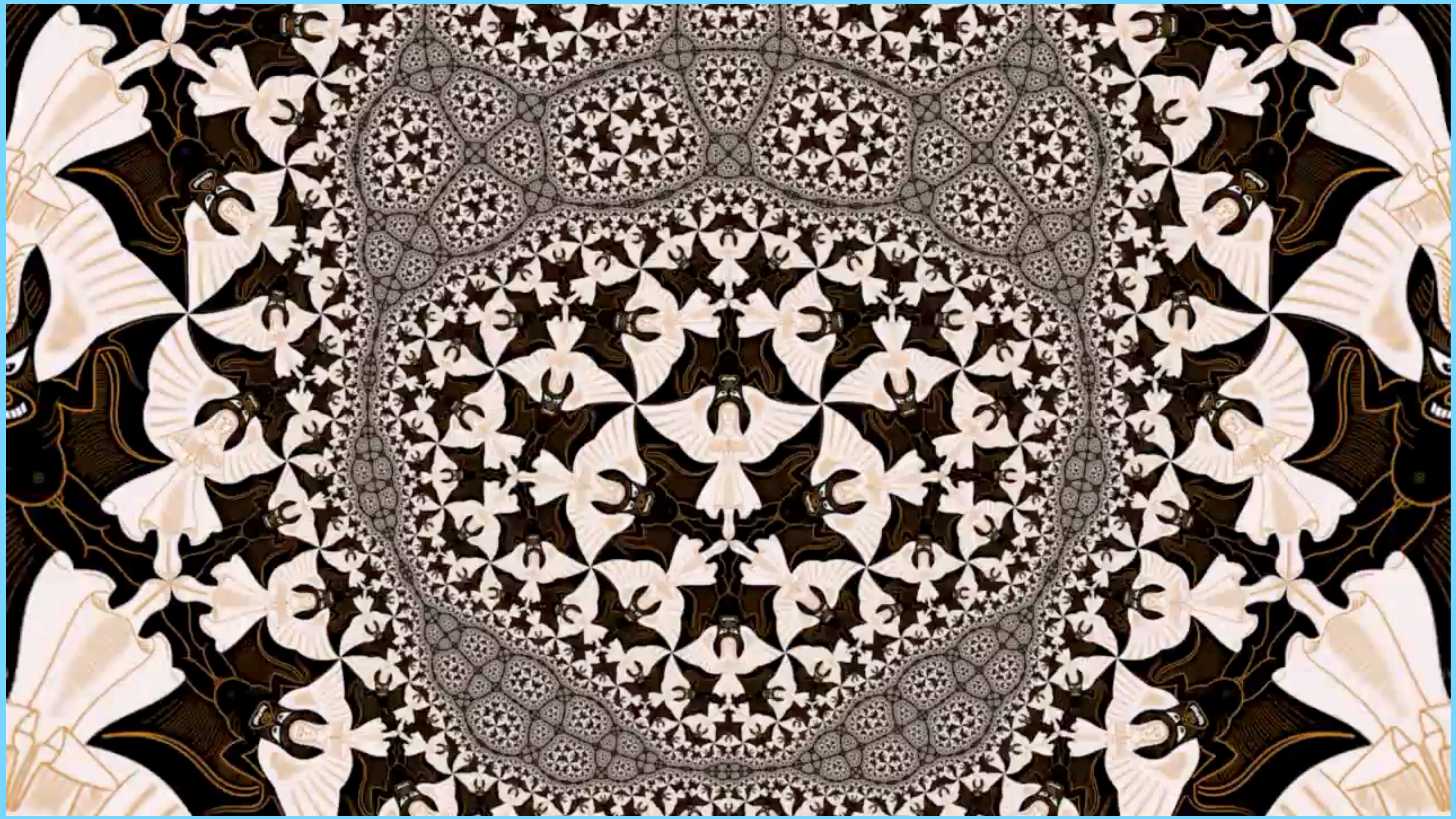
$$ds^2 = \frac{4\|d\mathbf{x}\|^2}{(1 - \|x\|^2)^2}$$

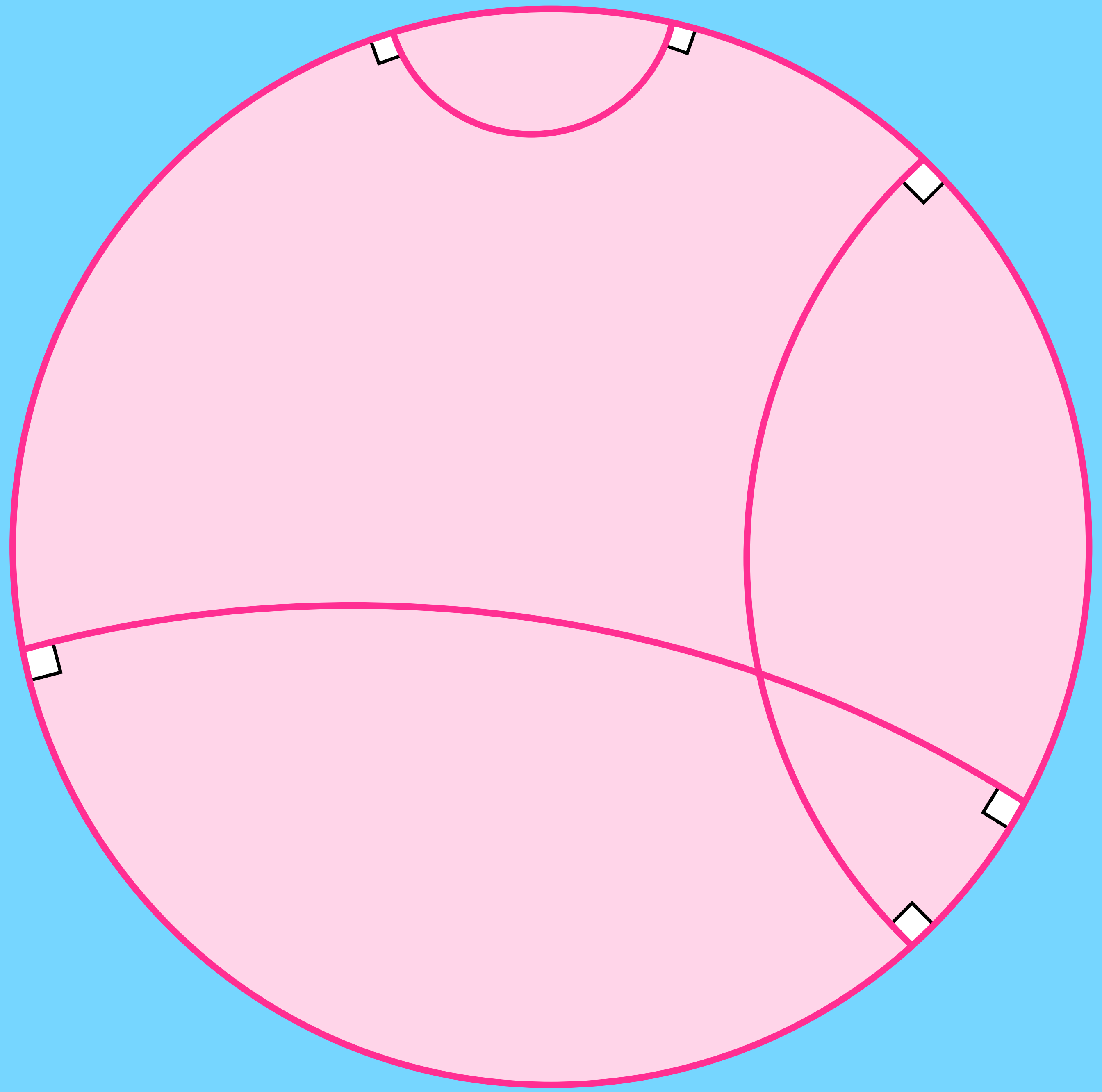


Poincaré Disk

- Rigid transformations - Möbius transformations which take the disk to itself!

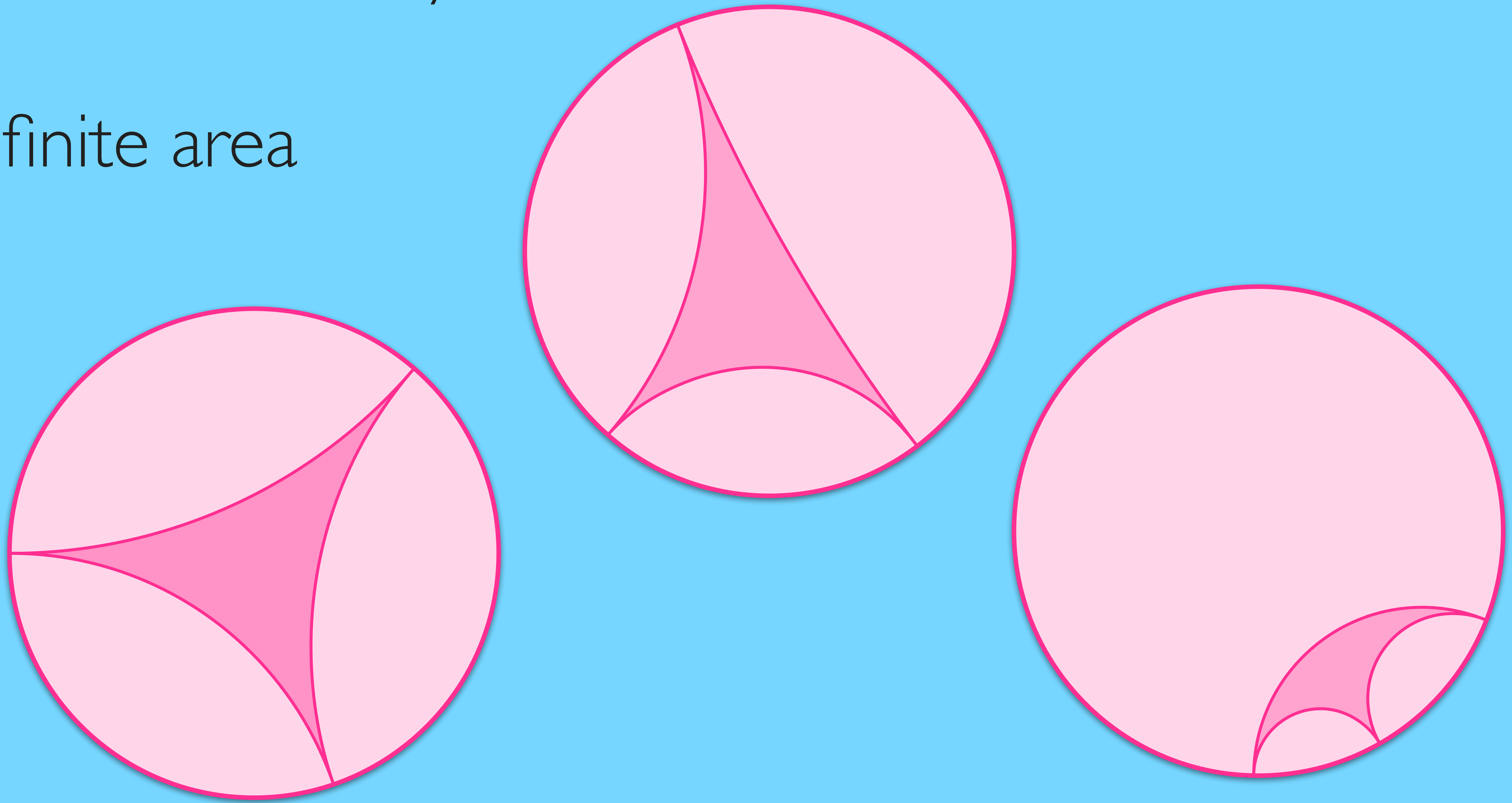
$$f(z) = \lambda \frac{z - a}{\bar{a}z - 1}$$





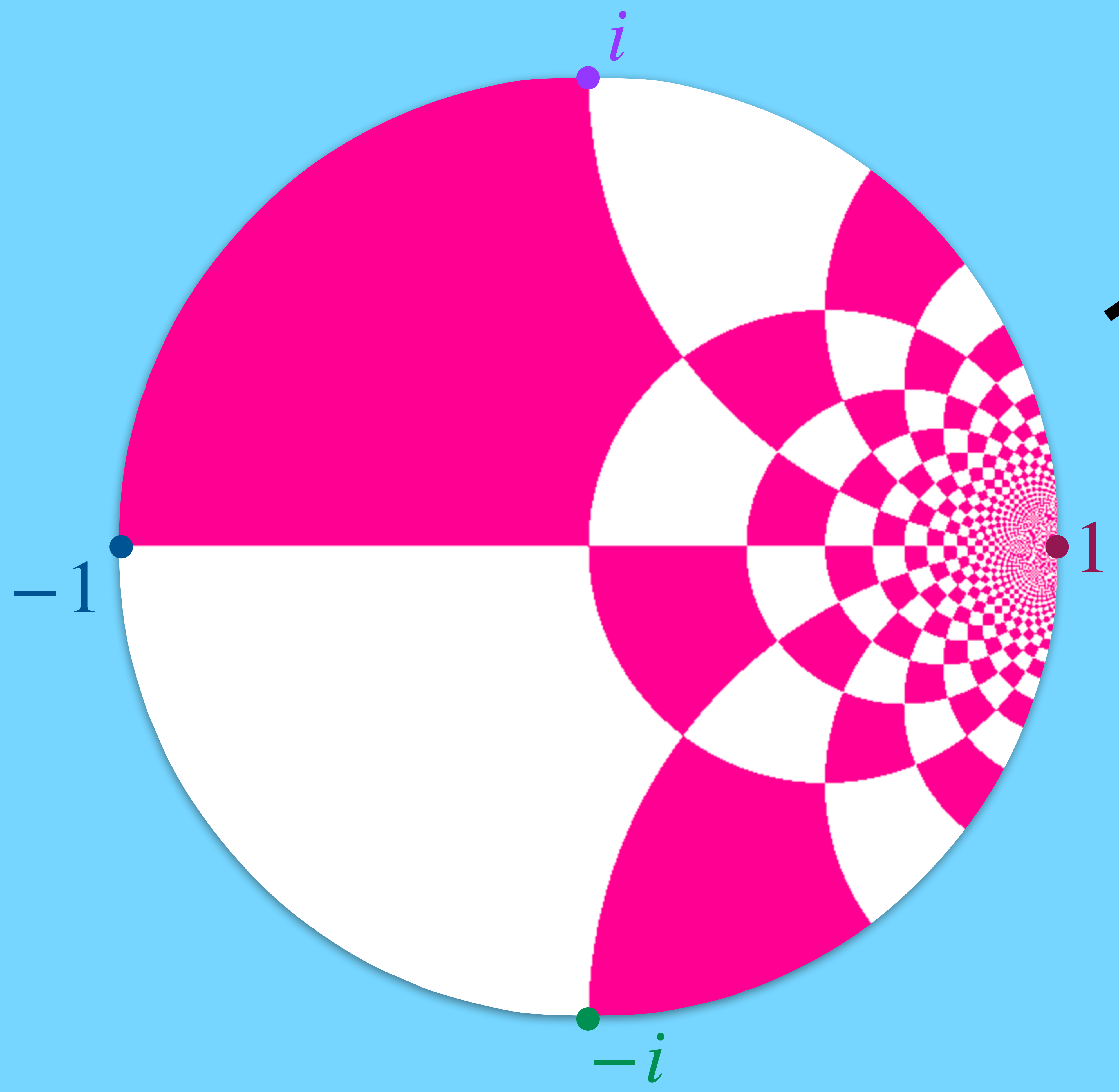
Ideal Hyperbolic Triangles

- Ideal points - points on boundary
- Infinite perimeter, finite area
- All congruent!

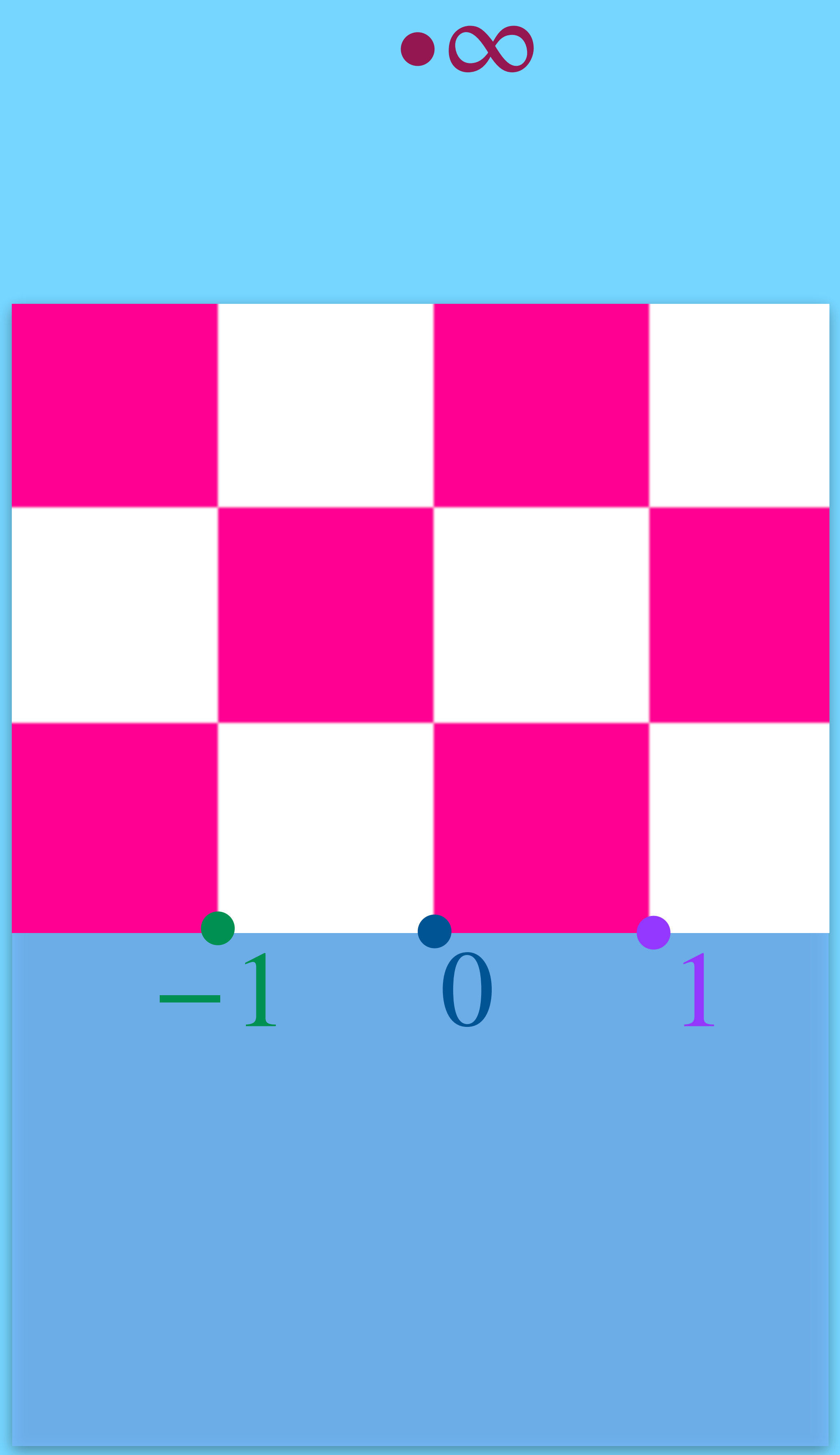


The Halfspace Model

- There is a conformal map from the disk to the upper half-plane

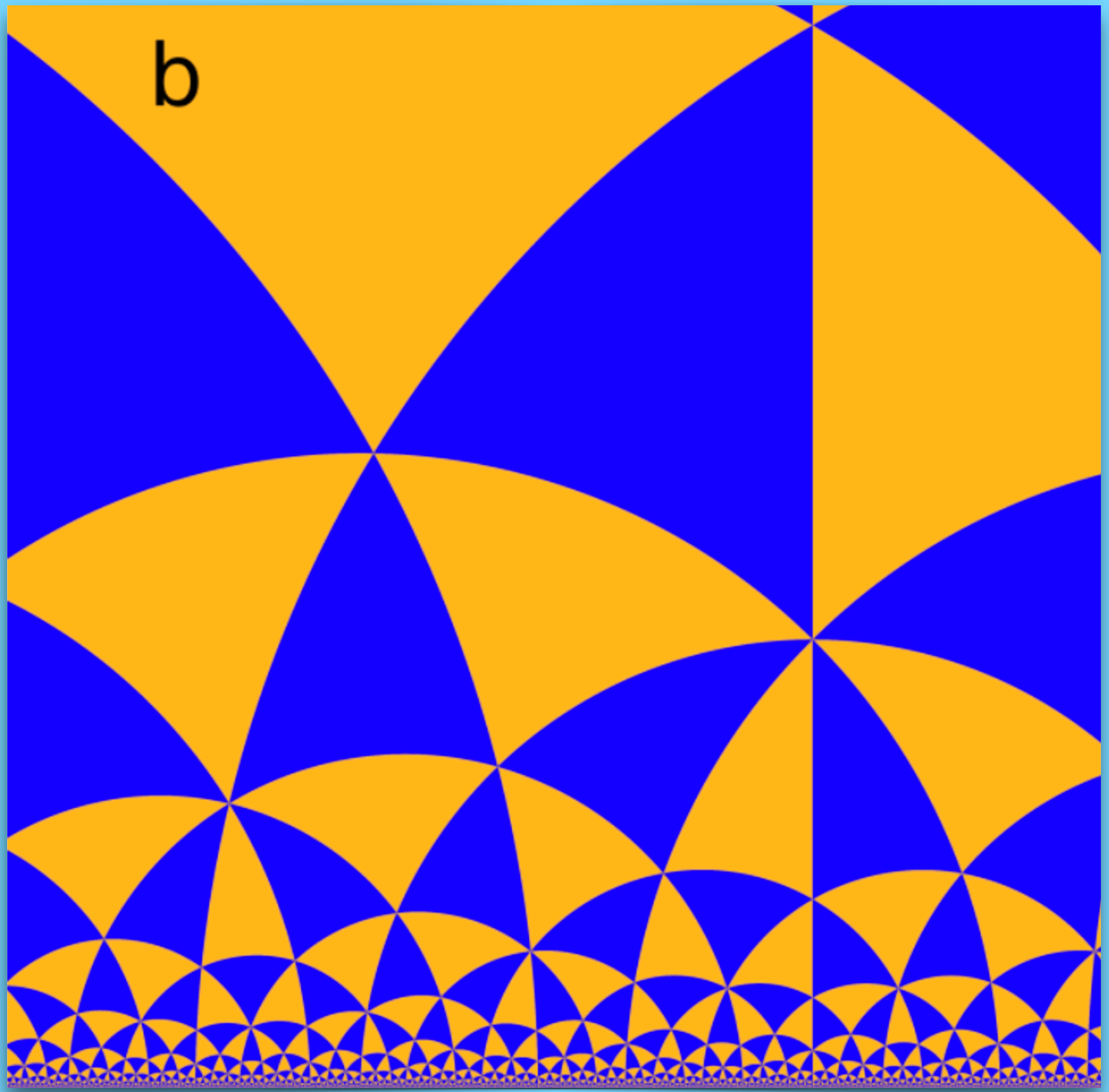


$$\frac{-iz - i}{z - 1}$$

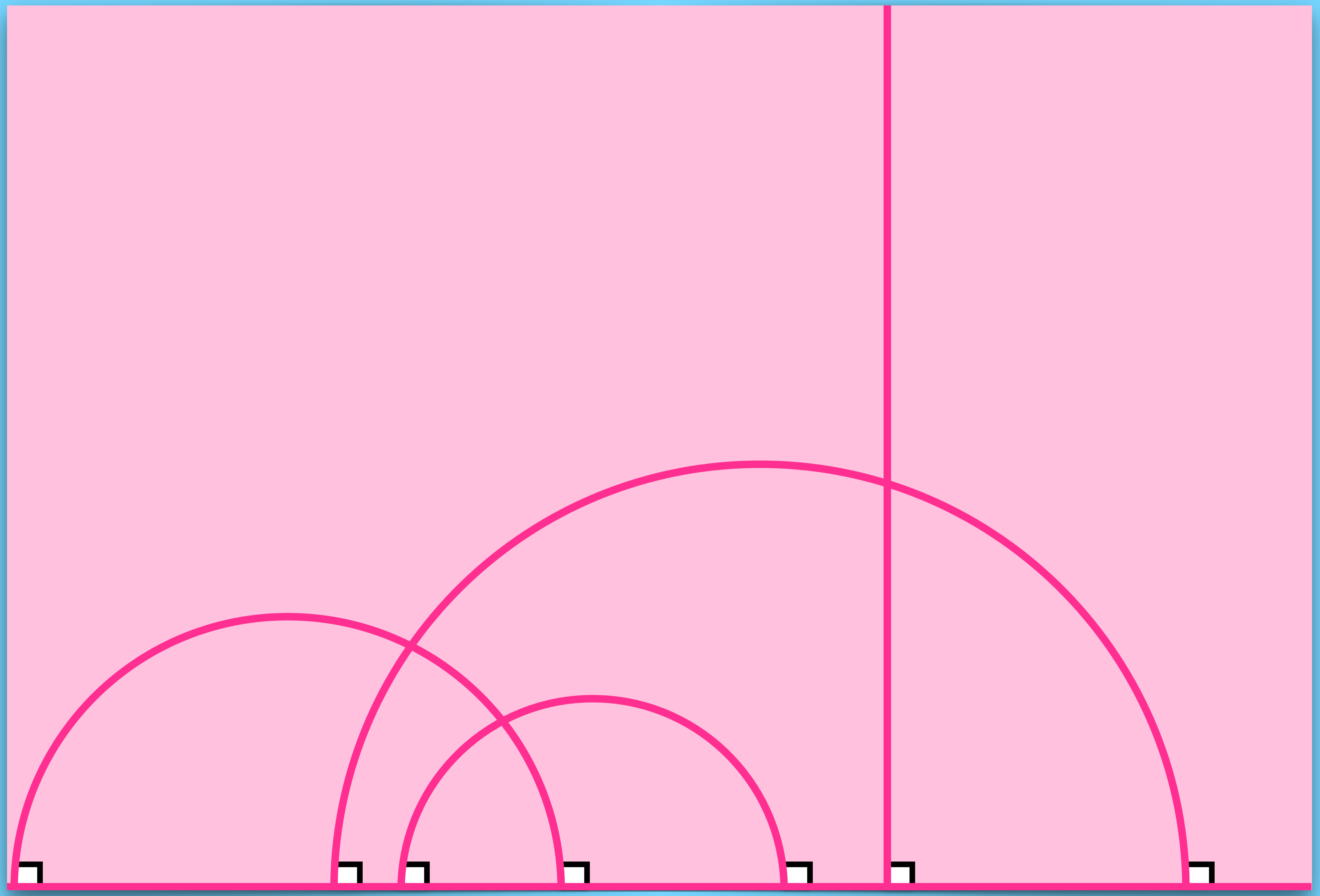


The Halfspace Model

- There is a conformal map from the disk to the upper half-plane



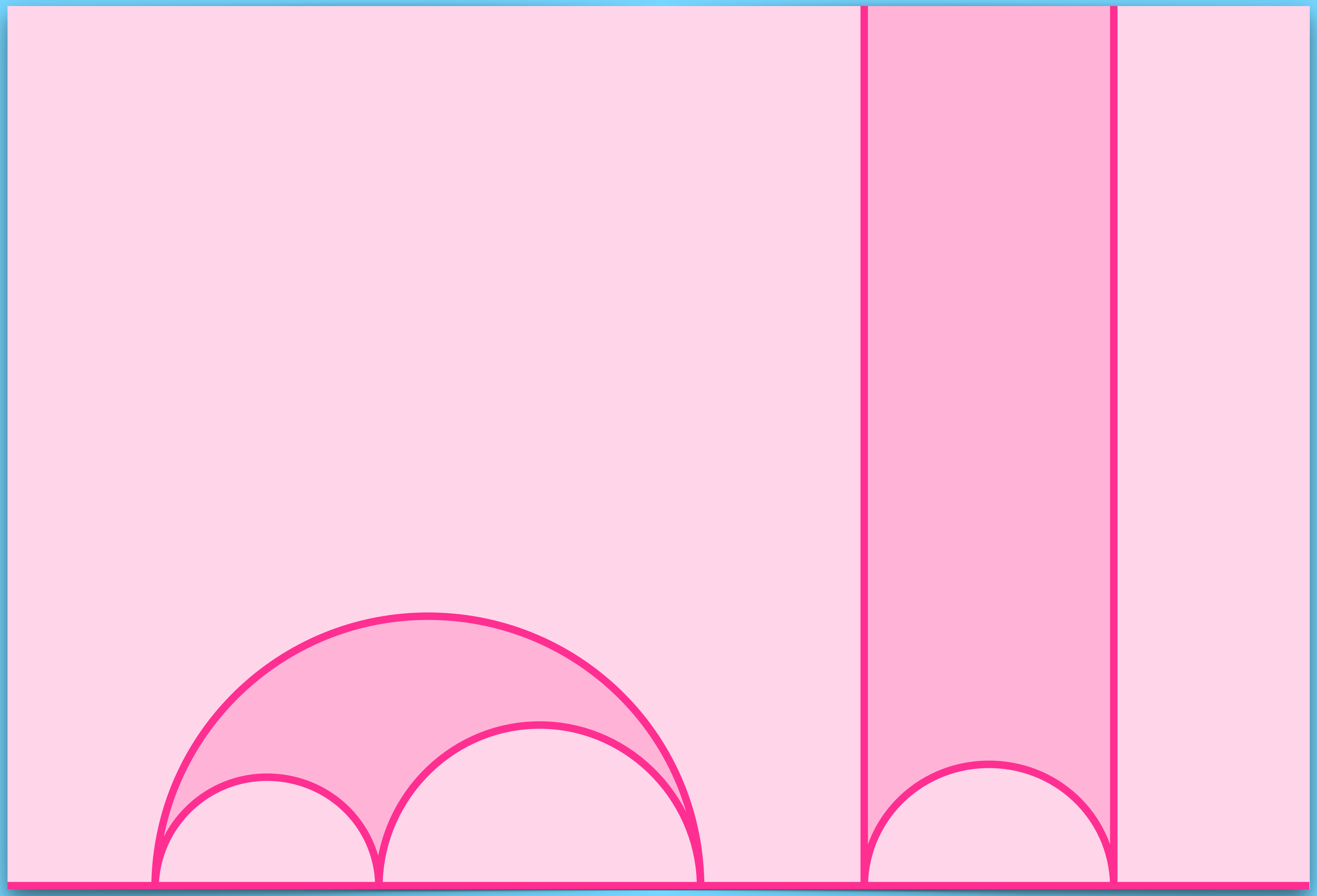
$$ds^2 = \frac{\|d\mathbf{x}\|^2}{y^2}$$



Horizontal slices look Euclidean

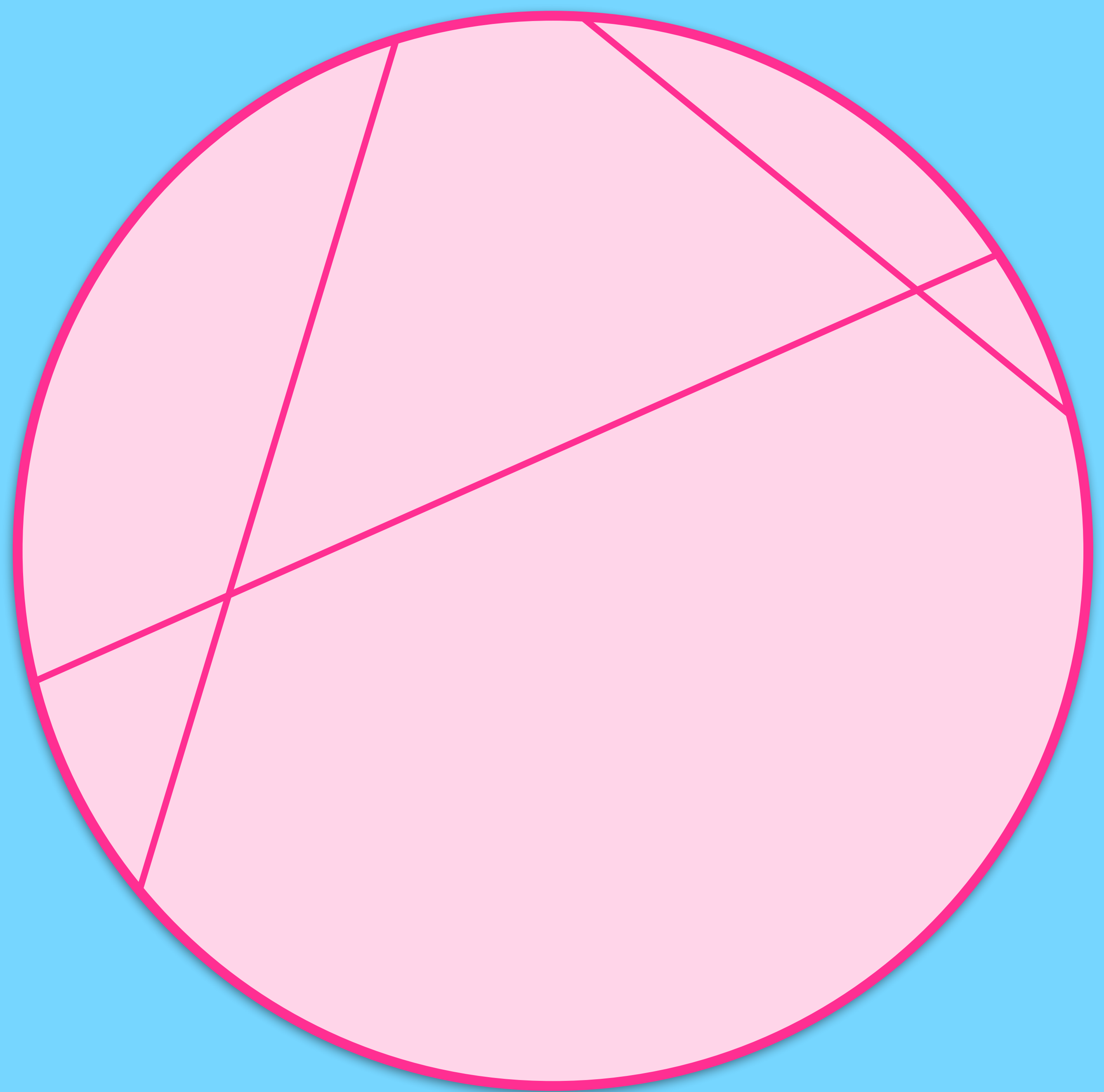
Ideal Triangles in the Halfspace Model

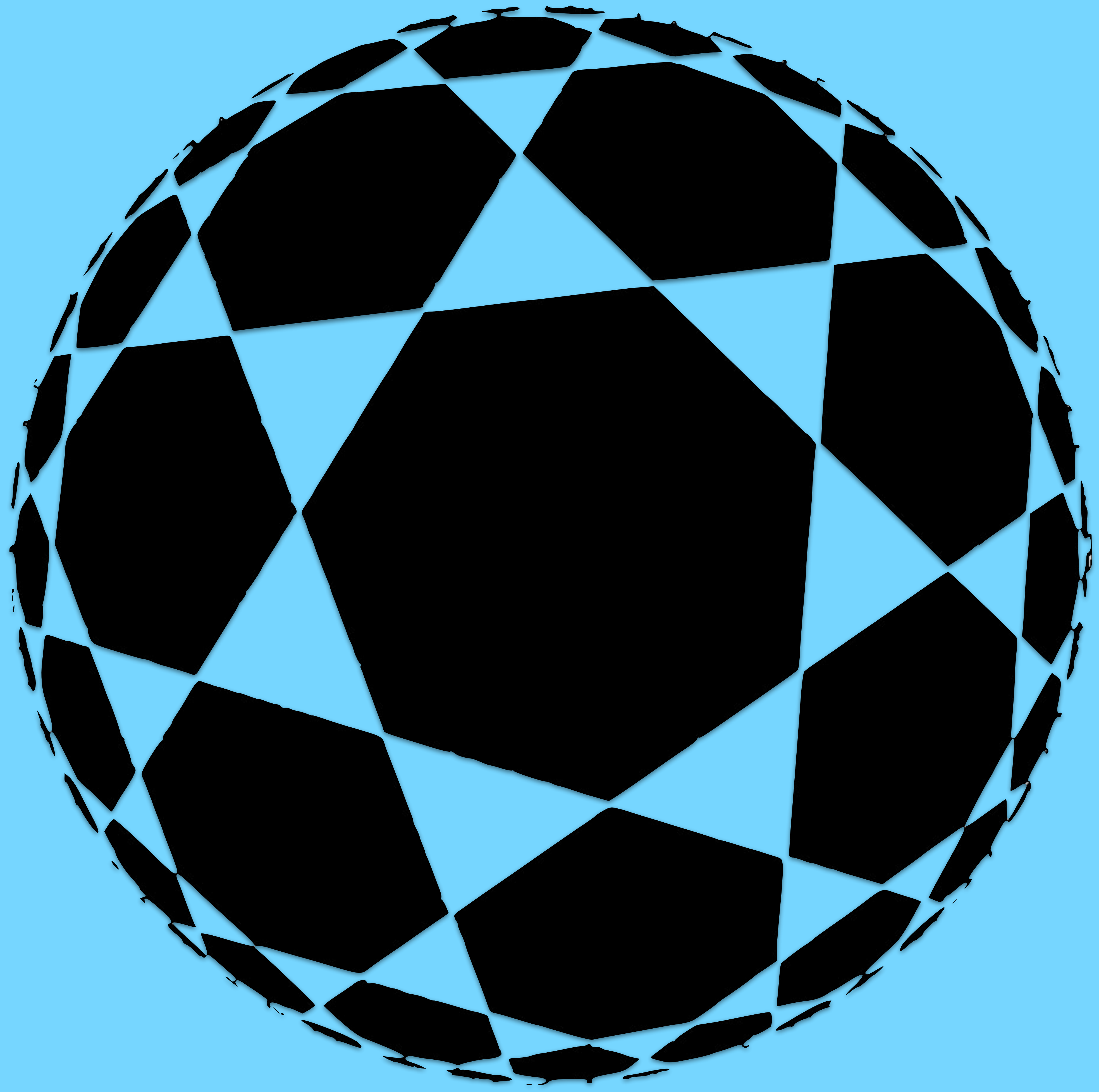
- Ideal points - points on boundary
- Infinite perimeter, finite area
- All congruent!



The Klein Model

- Straight lines are straight lines
- Angles are wonky





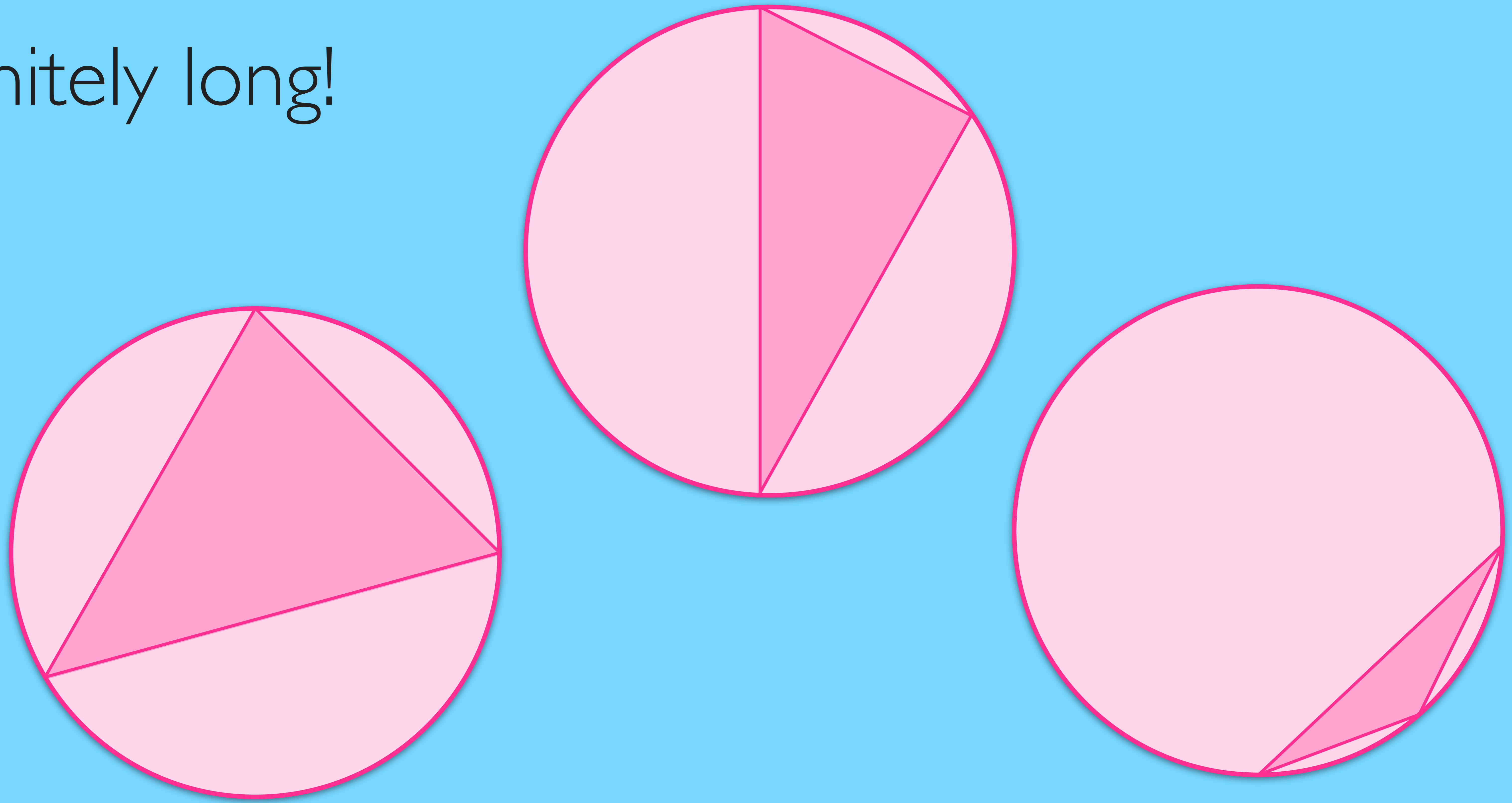
The Klein Model

- What are the rigid transformations of the Klein model?
- They must map straight lines to straight lines
    - (Real) projective transformations
- They must preserve the unit circle
    - Circle-preserving projective maps

The Klein Model

- Any Euclidean triangle is also a triangle in the Klein model

- But their sides are infinitely long!

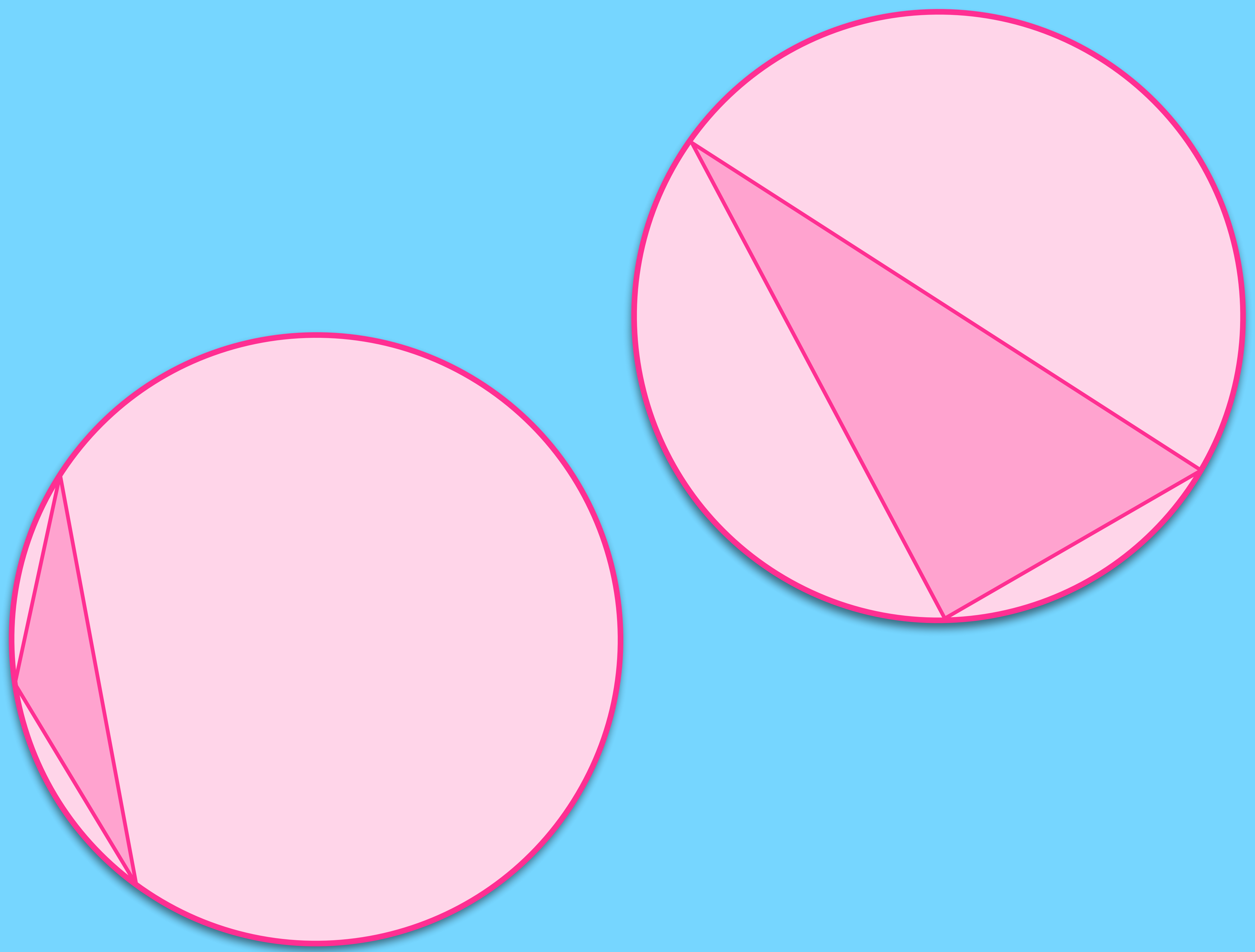


The Klein Model

- There's a unique rigid motion between any 2 Klein triangles

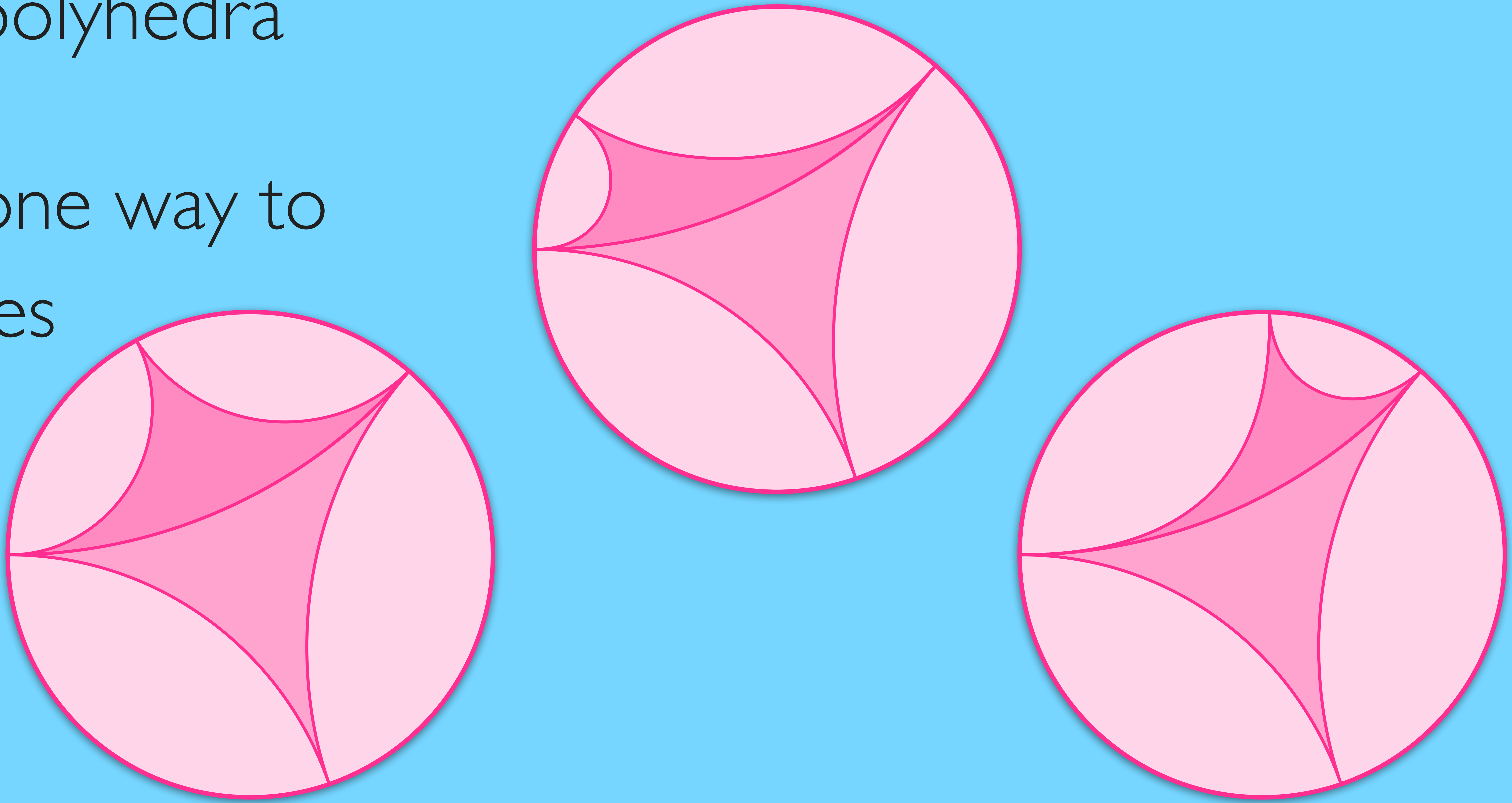
- It must be a projective map

- The coefficients are the conformal scale factors!



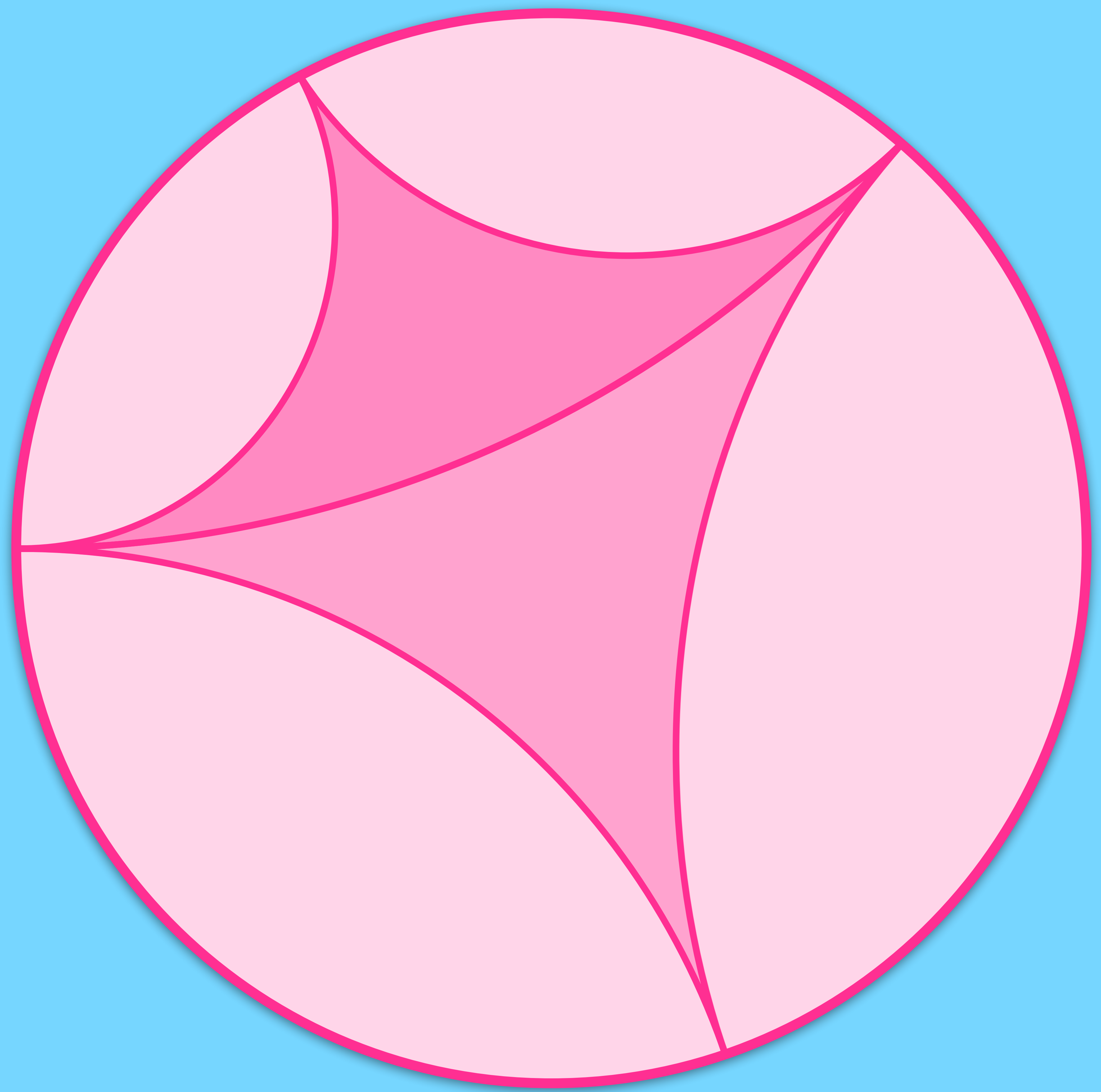
Ideal Hyperbolic Polyhedra

- We can glue ideal triangles together into ideal polyhedra
- There's more than one way to glue a pair of triangles



Ideal Hyperbolic Polyhedra

- 4 points cocircular: real cross ratio
    - Equals length cross ratio (up to sign)
- 4th point determined by cross ratio

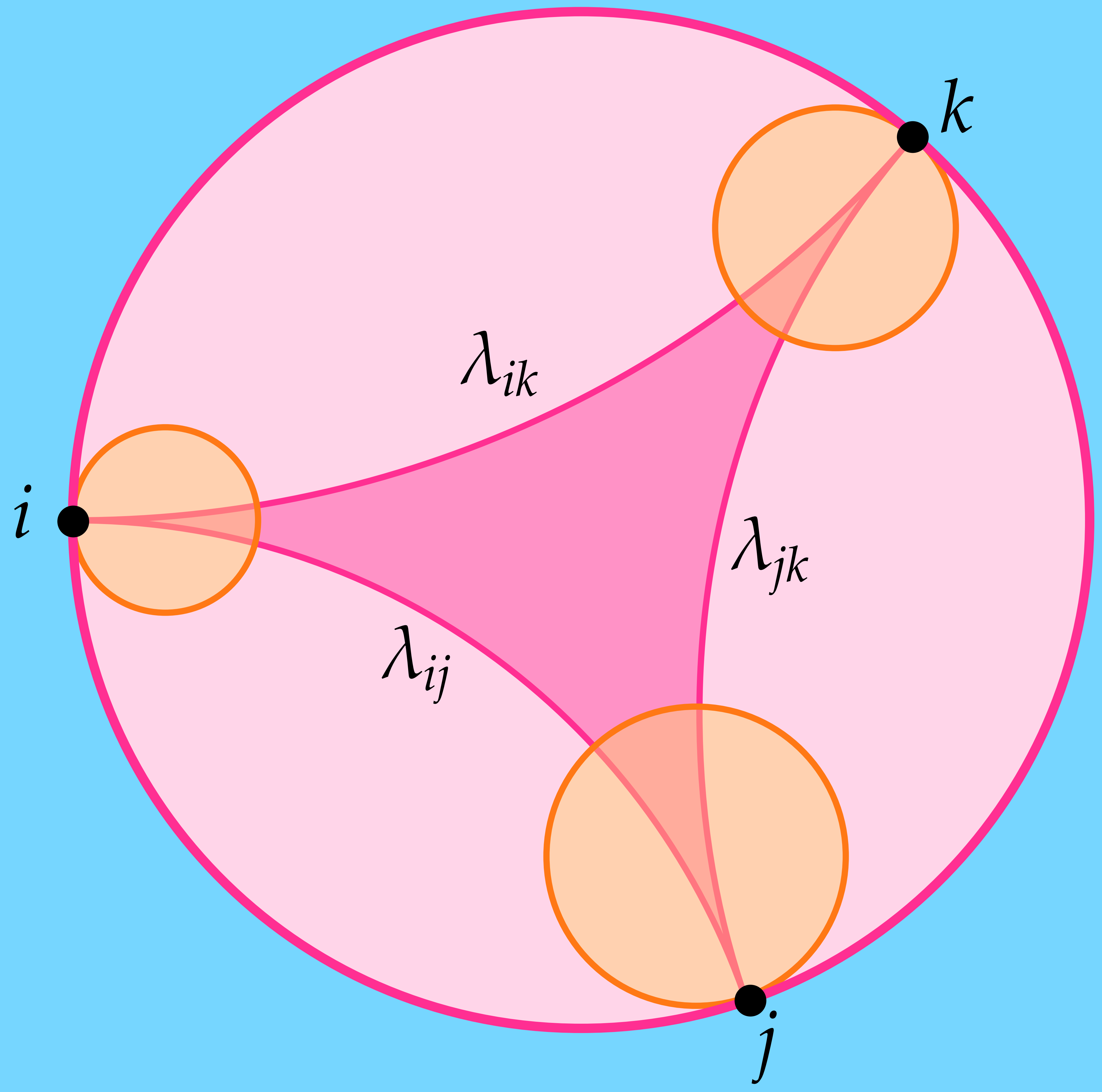


Ideal Hyperbolic Polyhedra

- An ideal hyperbolic polyhedron is specified by a length cross ratio per edge
- Rigid transformations of hyperbolic polyhedra preserve the length cross ratios at edges

Hyperbolic Edge Lengths

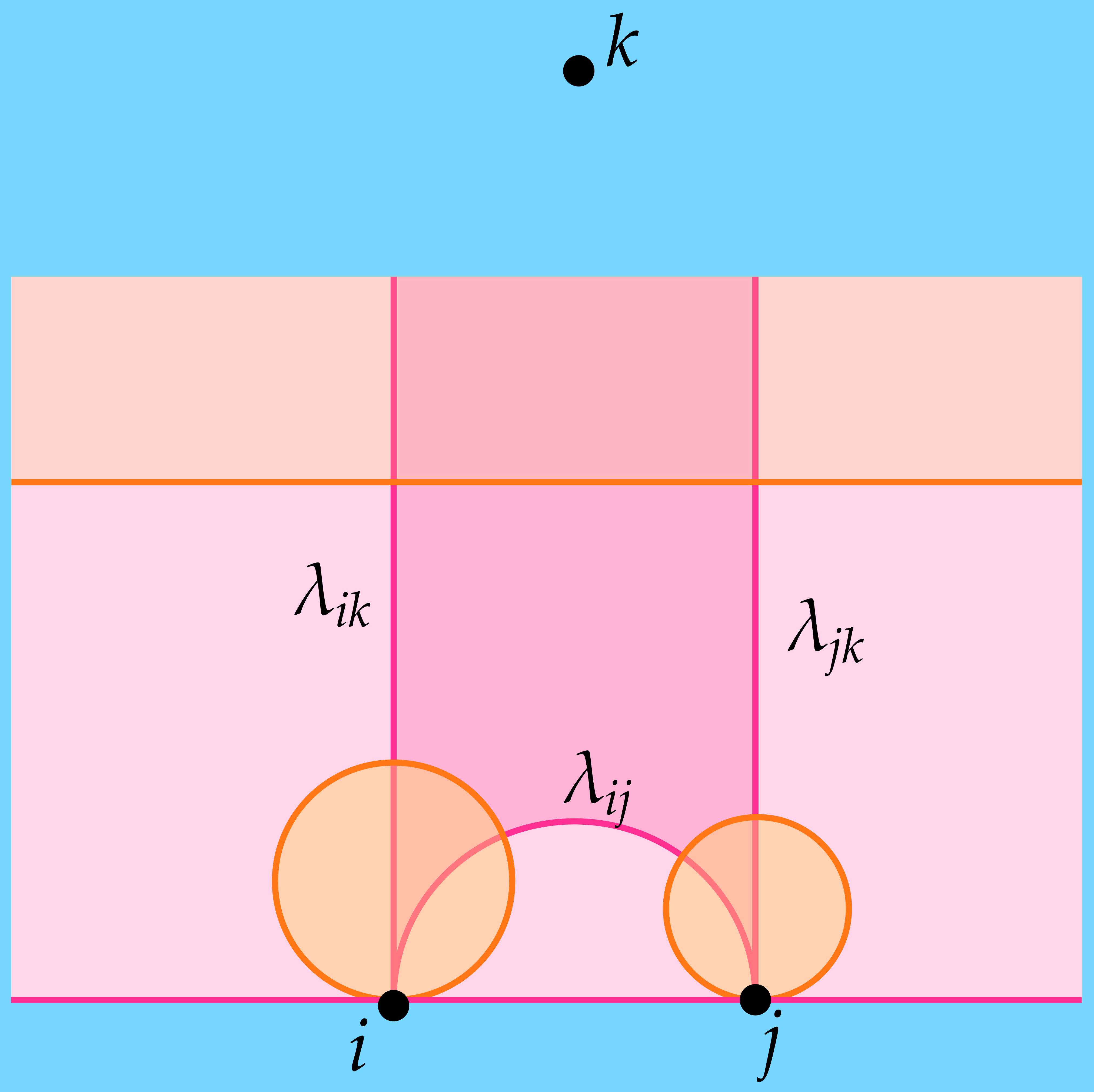
- Edge lengths are convenient
- By cutting off the infinite ends of the lines, we obtain finite lengths
- "Decorated" ideal triangle
- What happens if we pick a different horocycle?



Hyperbolic Edge Lengths

- Horocycles around infinity are horizontal (Euclidean) planes

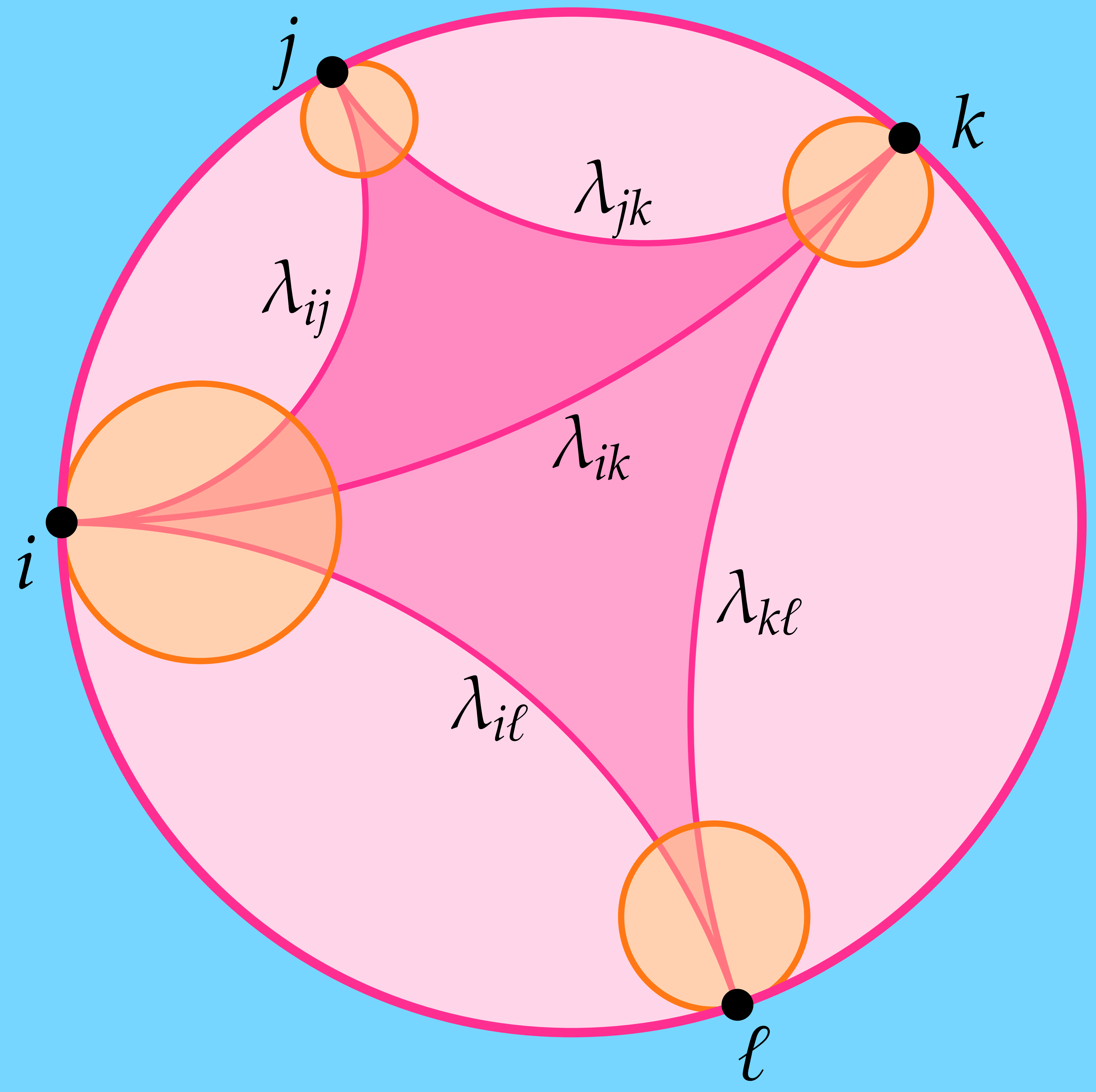
- Picking a different horocycle shifts the plane - changes lengths by a constant



Hyperbolic Edge Lengths

- Changing horocycles doesn't change $\lambda_{ij} - \lambda_{jk} + \lambda_{k\ell} - \lambda_{i\ell}$
- This is twice the (log of the) length cross ratio!

$$\mathfrak{cr} = \frac{e^{\lambda_{ij}/2}\, e^{\lambda_{k\ell}/2}}{e^{\lambda_{jk}/2}\, e^{\lambda_{i\ell}/2}}$$



Hyperbolic Edge Lengths

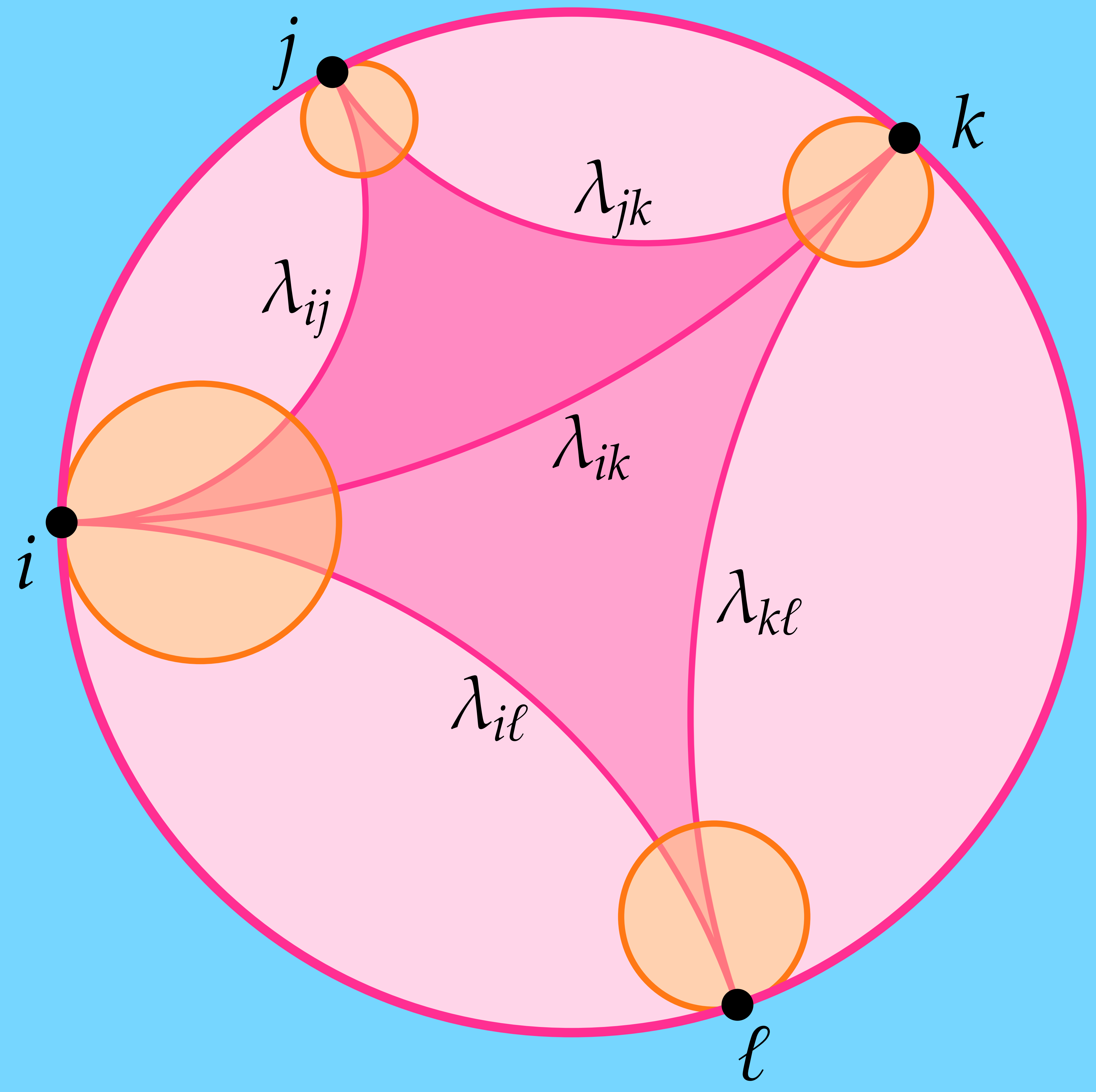
- Given a mesh, set hyperbolic lengths

$$\lambda_{ij} = 2 \log \ell_{ij}$$

- Then a conformal rescaling
- $\tilde{\ell}_{ij} = e^{(u_i + u_j)/2} \ell_{ij}$ looks like

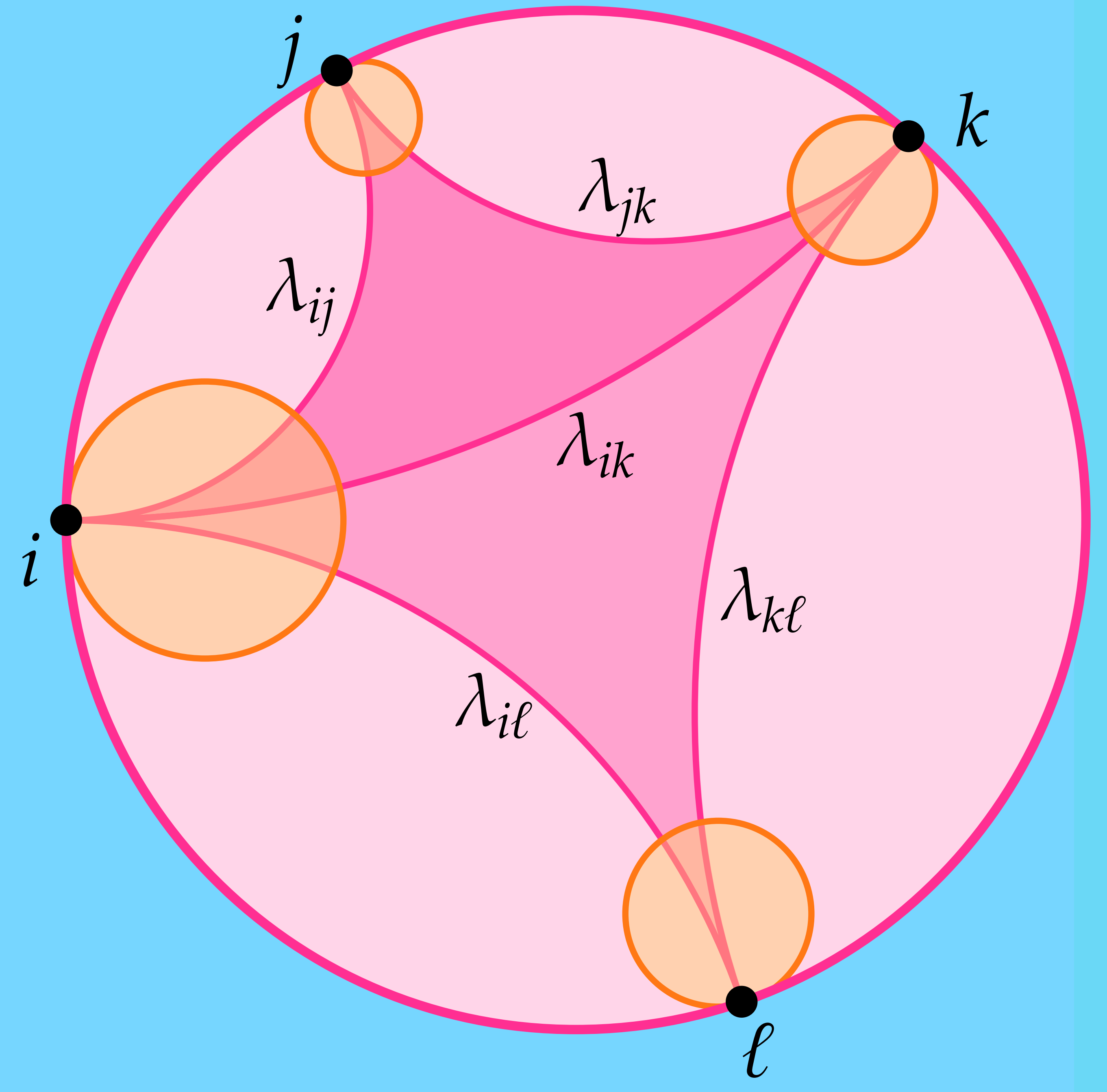
$$\tilde{\lambda}_{ij} = \lambda_{ij} + u_i + u_j$$

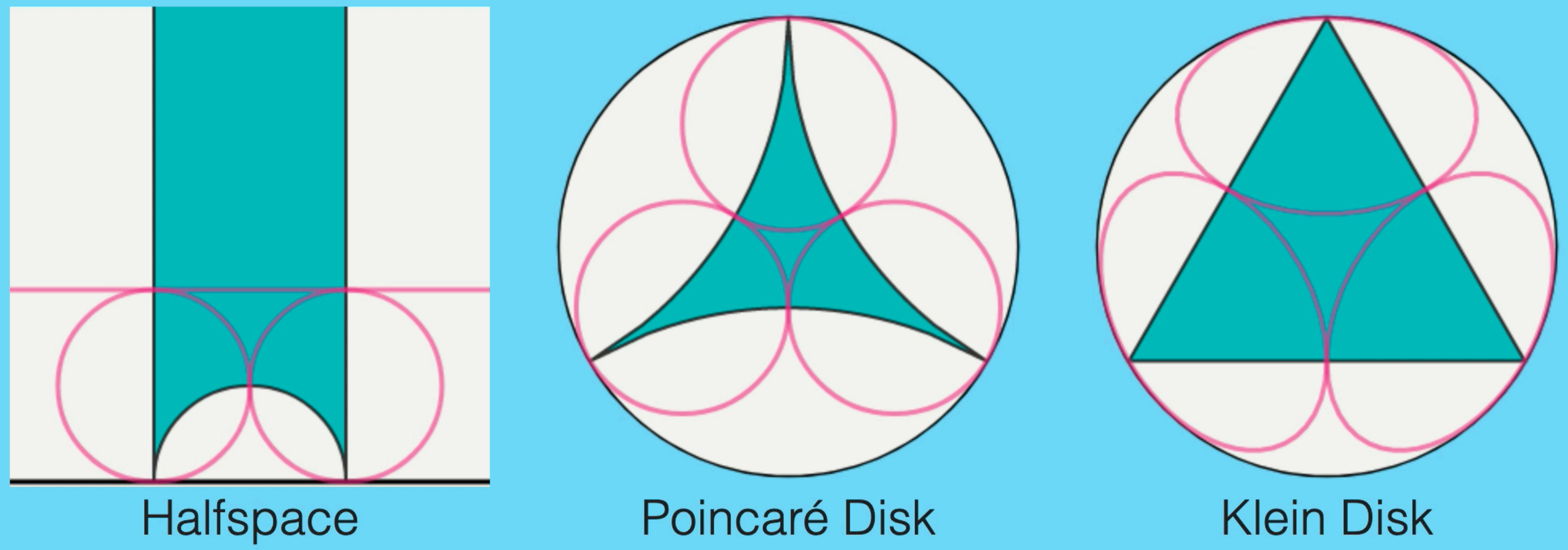
- This is just changing your horocycles!

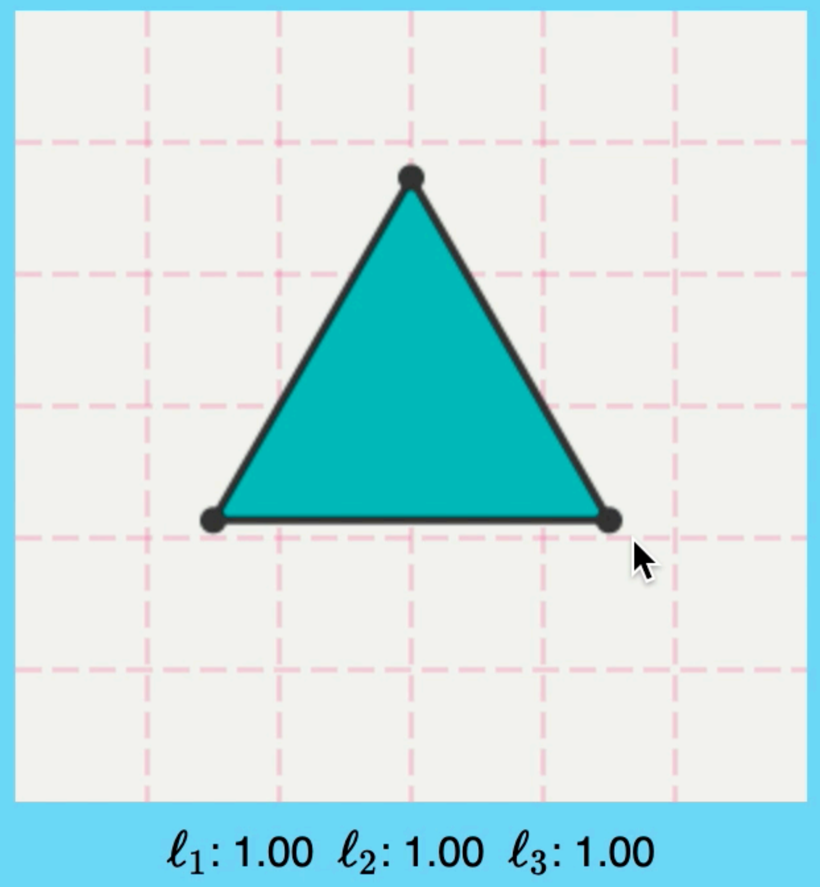


Hyperbolic Edge Lengths

$$\lambda_{ij} = 2 \log \ell_{ij}$$

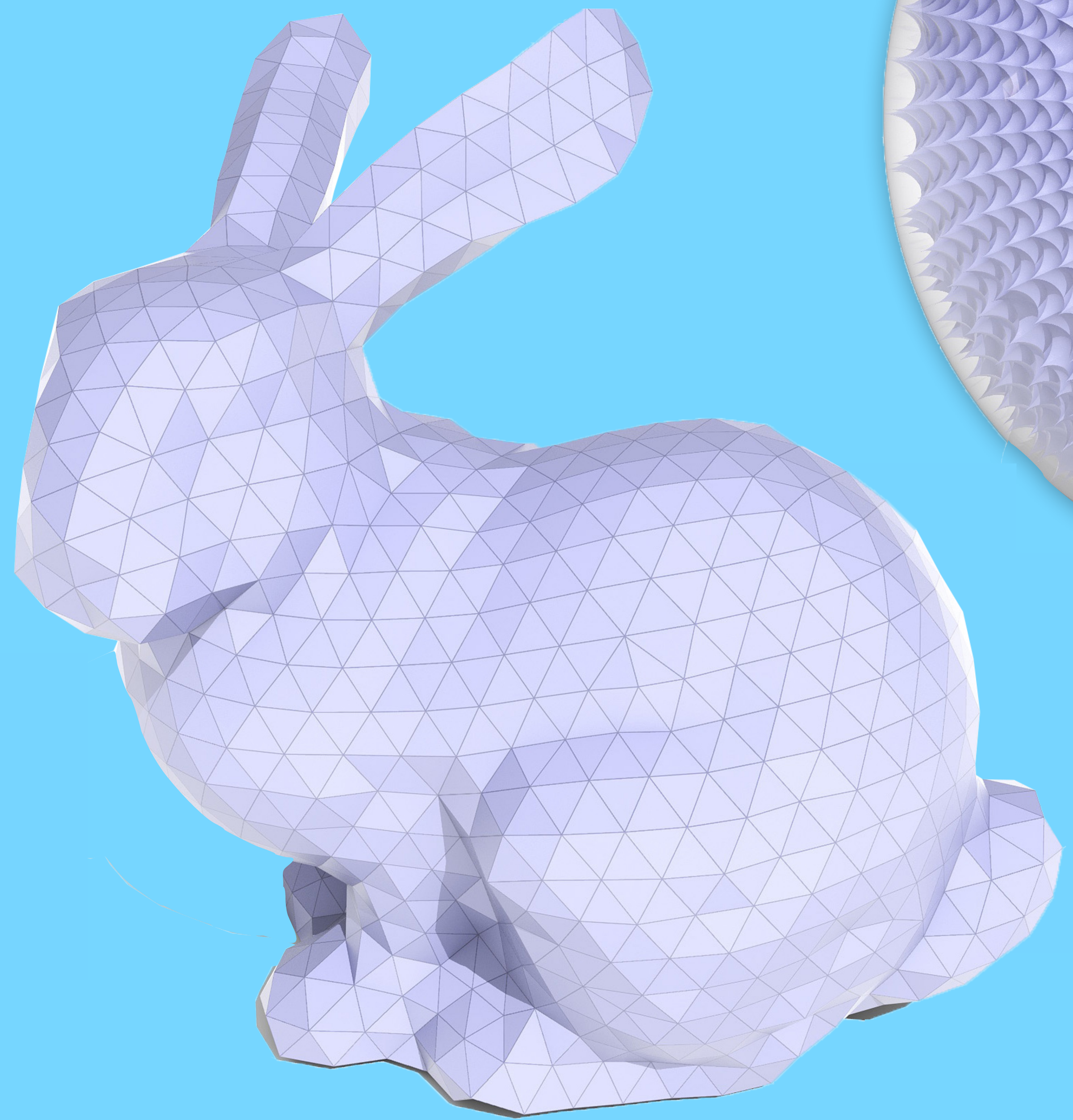


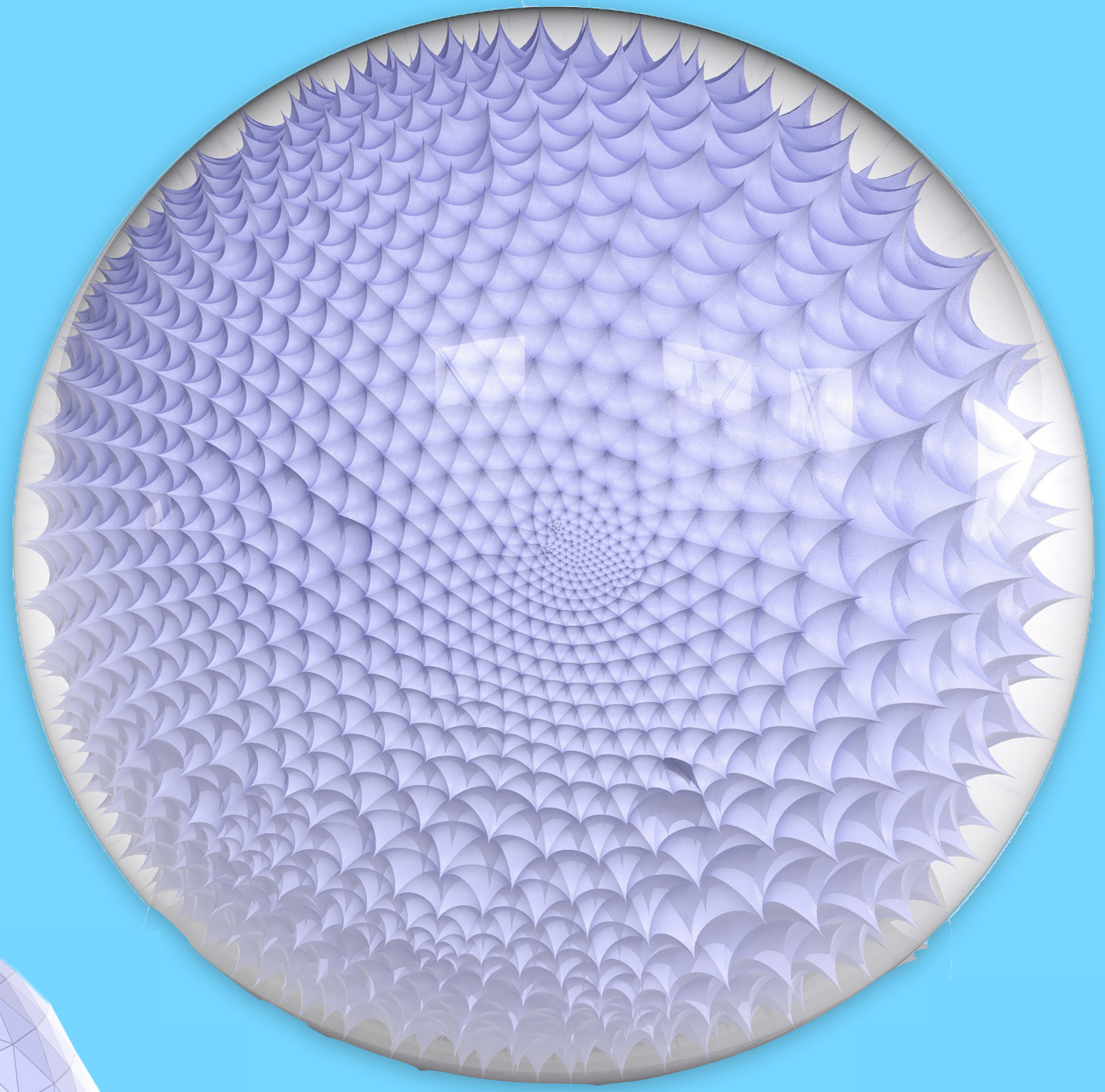




Discrete Conformal Equivalence

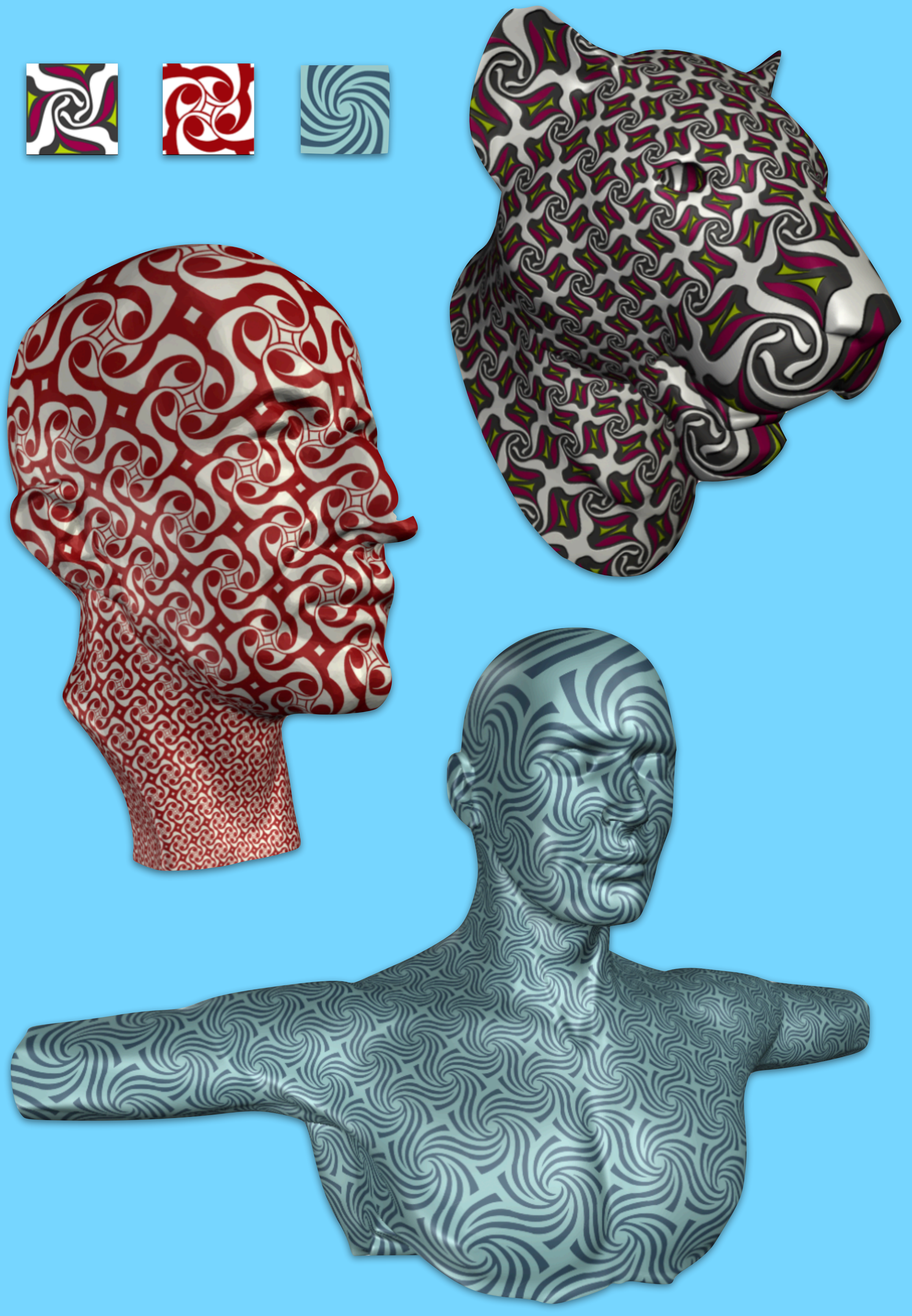
- Two triangle meshes are discretely conformally equivalent if they have the same hyperbolic metric
    - This is equivalent to both earlier definitions!

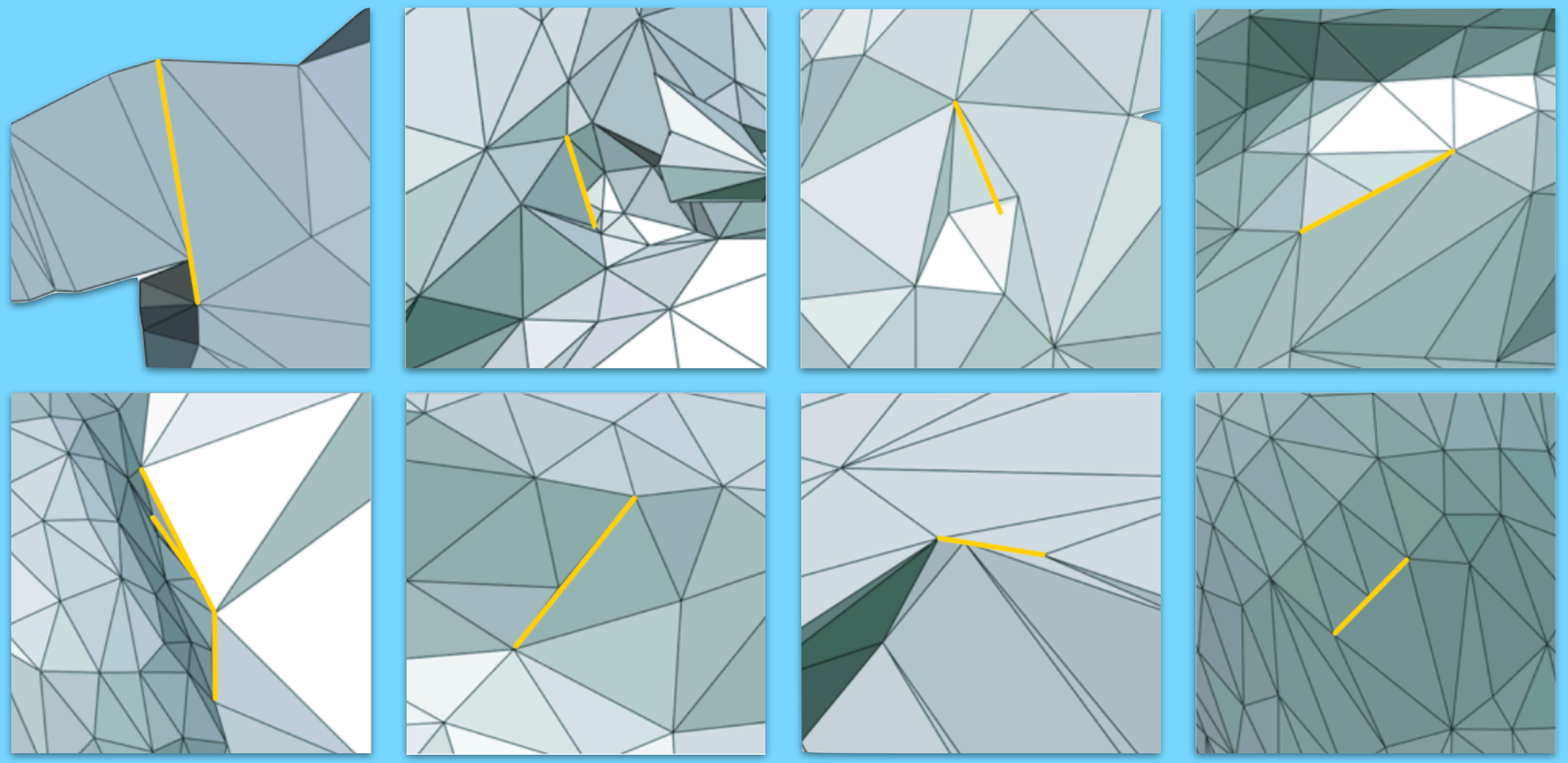




Discrete Uniformization

Conformal Rescaling Can Break Meshes

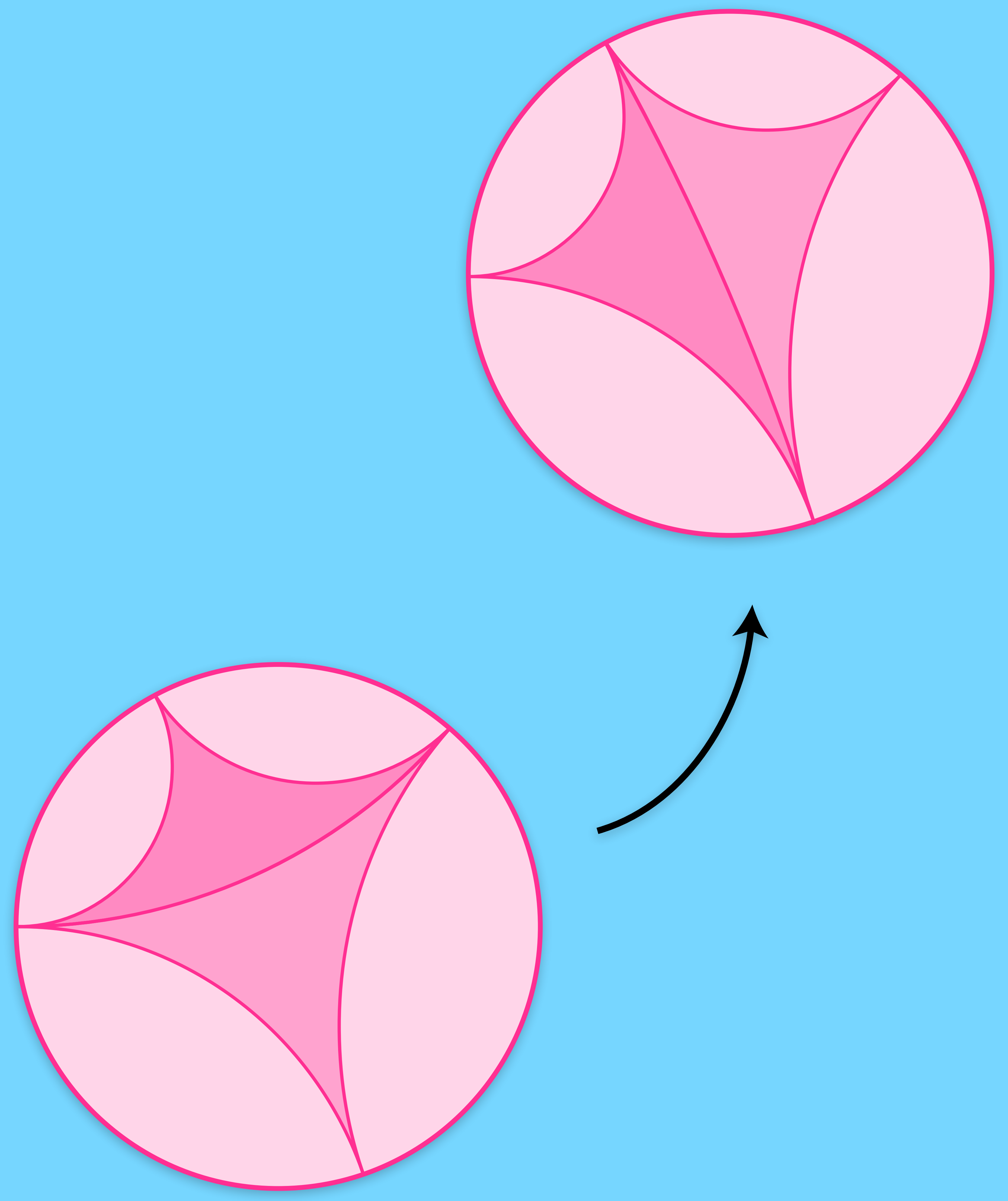




> *"Furthermore, no subset of the [discrete conformal] transformations forms a group".*
>
> M. Roček, R.M. Williams,
> "*The Quantization of Regge Calculus*" (1984)

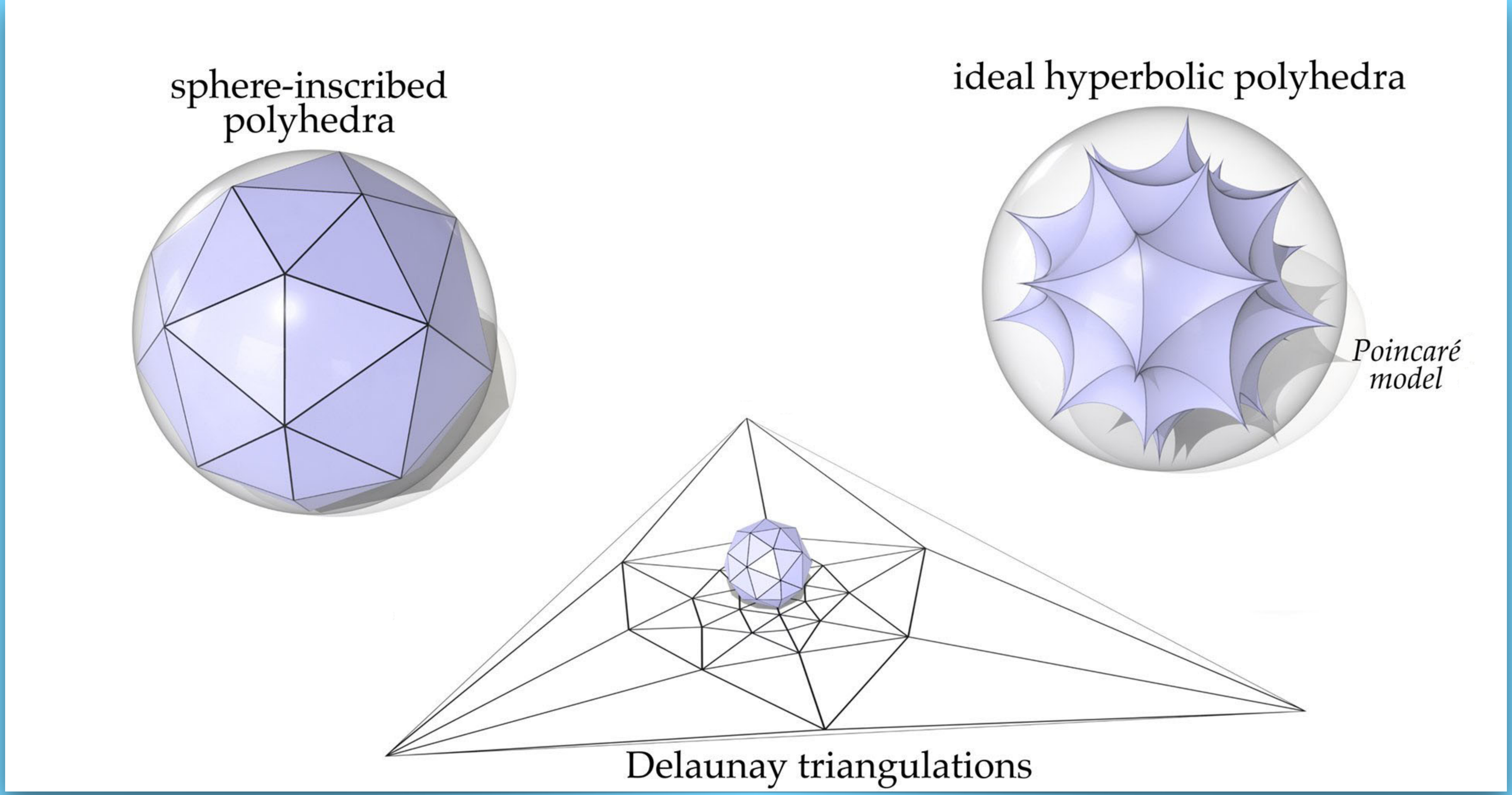
Hyperbolic Edge Flips

- "Degenerate" meshes still define hyperbolic polyhedra
- We can fix degenerate meshes by performing *hyperbolic edge flips*
- Still conformal



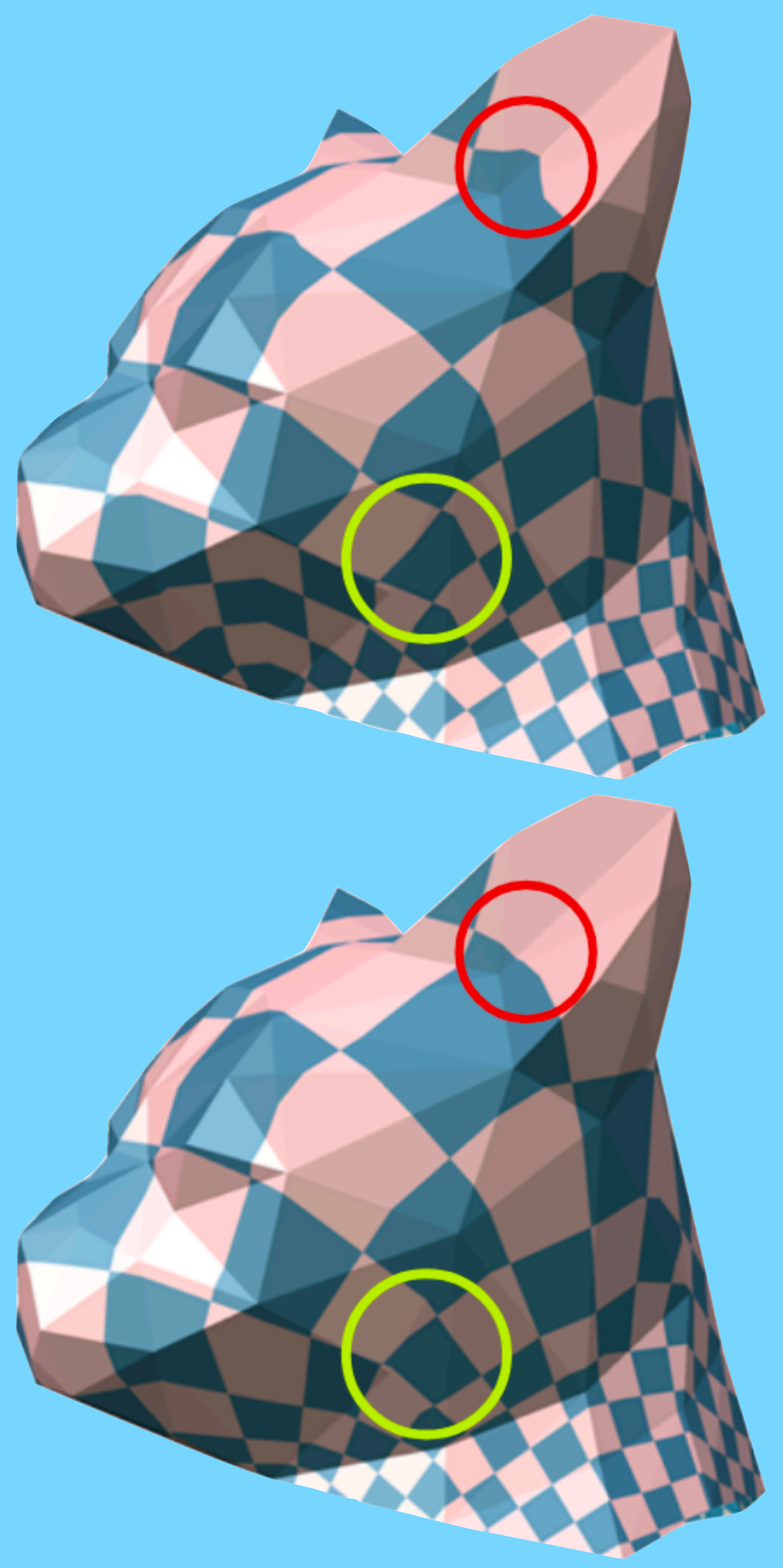
Hyperbolic Edge Flips

- Fact: We can always flip to valid Euclidean edge lengths
  - Hyperbolic Delaunay triangulation



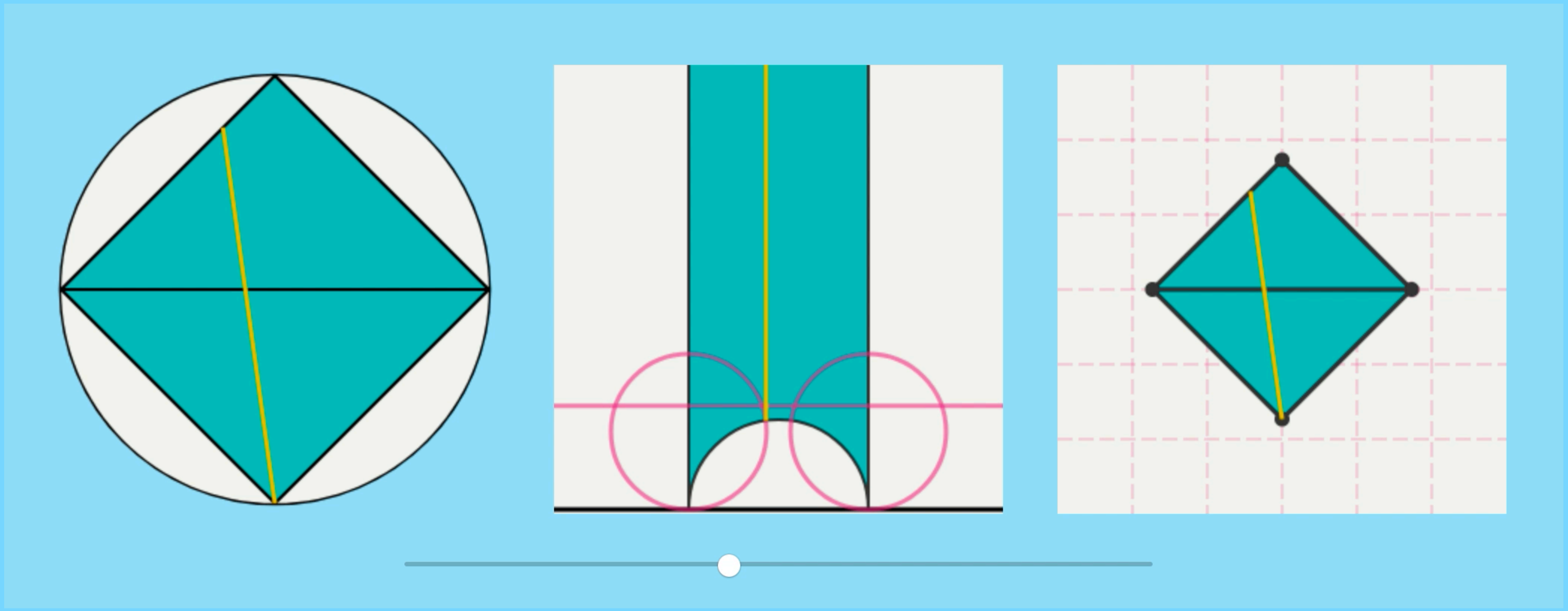
Texture Interpolation with Hyperbolic Maps

- Flattening gives us more than just vertex data
- There's a hyperbolic isometry between the plane and our surface
- Better interpolation

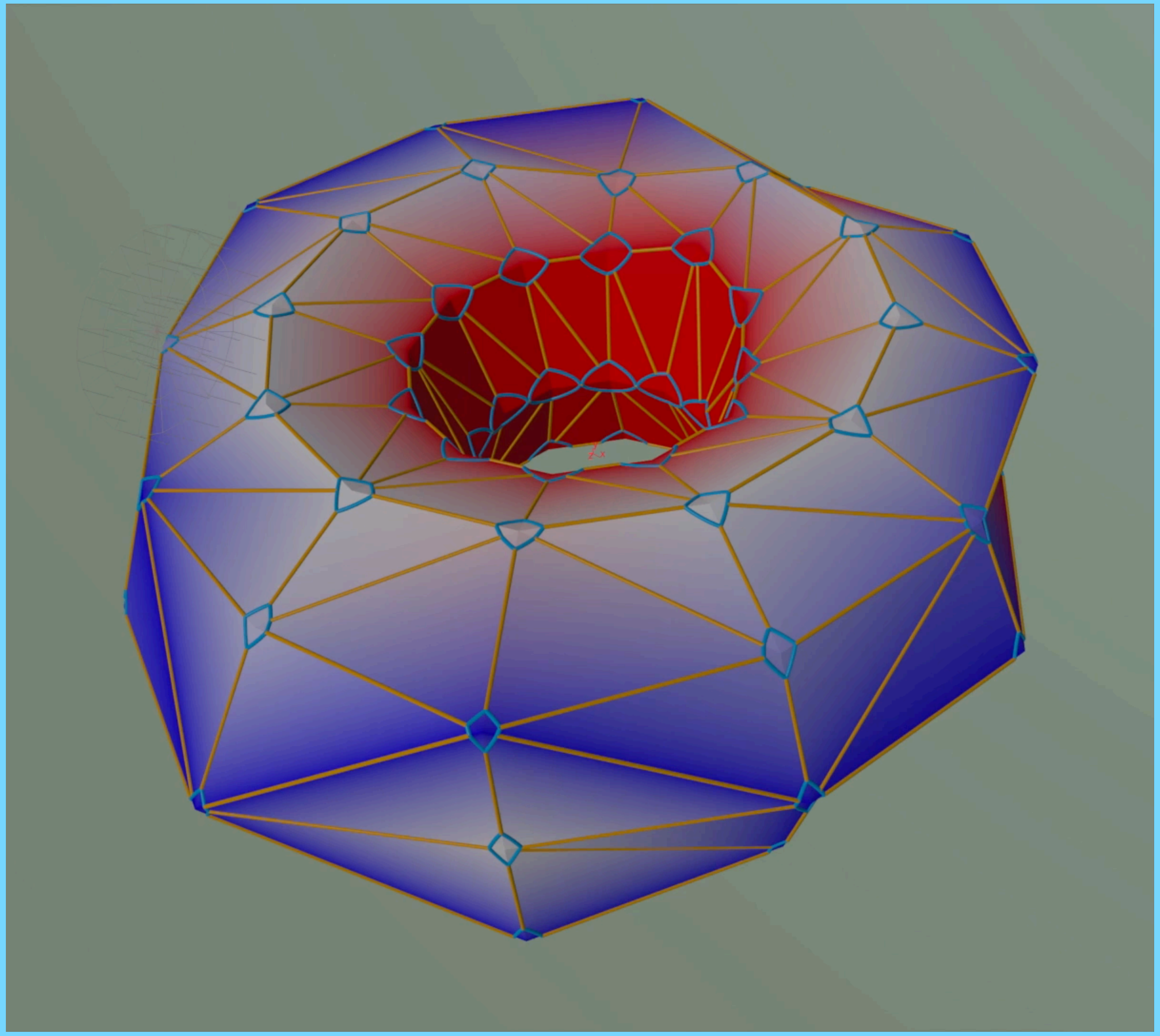


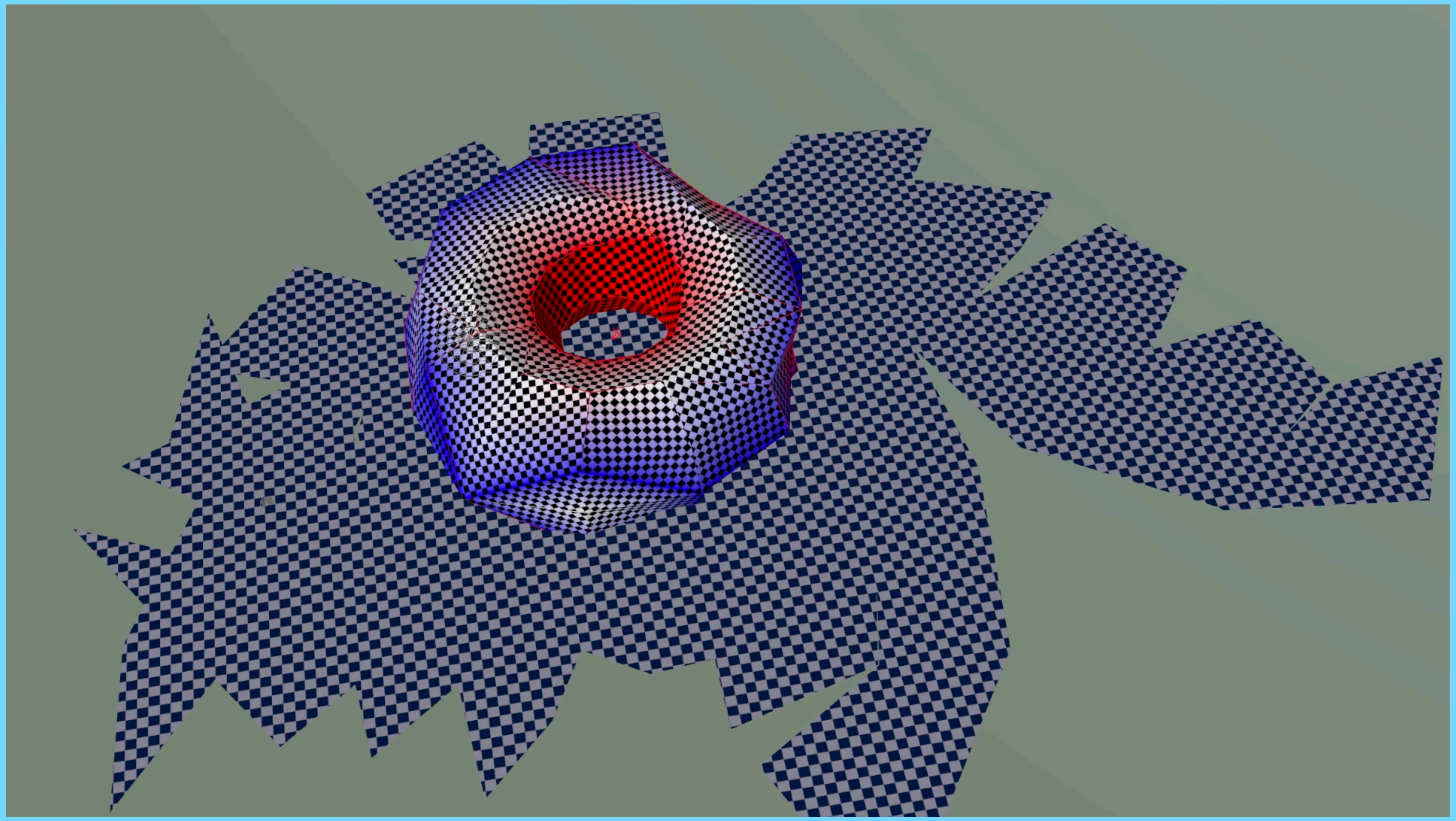
What Do Hyperbolic Edge Flips Look Like?

- An edge is a straight line between vertices
- They can be weird and bendy



Uniformization with Hyperbolic Edge Flips*





Embedding Hyperbolic Polyhedra

- The polyhedra are given intrinsically
    - How do you put them in $\mathbb{H}^3$?
- Conformal flattening!