Perspectives on Winding Numbers

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This short note explores the many different ways one can characterize the *winding number* of a curve Γ around a point p, and why these standard perspectives fail to generalize to curves on surfaces. Ultimately, all perspectives lead back to one of just three analytical descriptions: an integral over the curve Γ, an integral over a circle around the point p, or a particular Laplace equation. On sufaces, however, these formulations have undesirable consequences for curves that do not correspond to region boundaries, helping to motivate the recent *surface winding number* approach of Feng et al. [2023].

Notation and Conventions

We use $|\cdot|$ and $\langle\cdot,\cdot\rangle$ to denote the standard Euclidean norm and inner product for vectors in $\mathbb{R}^2$. We use $J : \mathbb{R}^2 \to \mathbb{R}^2; (x, y) \mapsto (-y, x)$ to denote a quarter turn in the counter-clockwise direction. For any two vectors $u, v \in \mathbb{R}^2$, we define a scalar-valued cross product $u \times v := u_1v_2 - u_2v_1$; note that $\langle Ju, v\rangle = u \times v$. For any function $f(t)$ of a single parameter t, we let $\dot{f}(t) := \frac{d}{dt}f(t)$.

Throughout we consider compact curves Γ on a smooth surface M, possibly with boundary ∂M; an important special case is the Euclidean plane $M = \mathbb{R}^2$. We use S^1 to denote the circle, which serves as the domain for a single closed loop. More generally, we use I to denote the domain of Γ, which may be an open interval, a closed loop, or a larger collection of loops and intervals. We use $w_\Gamma(p)$ to denote the winding number of a curve (or collection of curves) Γ around a point p; when Γ is not closed, this function also describes the *signed solid angle*.

We use Δ to denote the negative-semidefinite *Laplace-Beltrami* operator on M, which locally behaves like the ordinary Laplace operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. A function $u : M \to \mathbb{R}$ is *harmonic* if it is in the kernel of the Laplacian, *i.e.*, if $\Delta u = 0$.

1 TOPOLOGICAL DEGREE

The basic idea of the winding number is that it captures how many times a curve Γ "winds" around a given point p. In particular, for a single closed planar loop $\Gamma : S^1 \to \mathbb{R}^2$, consider the *covering map*

$$\varphi : S^1 \to S^1; t \mapsto \frac{\Gamma(t) - p}{|\Gamma(t) - p|}. \tag{1}$$

The winding number $w_\Gamma(p)$ can then be defined as the *degree* of φ, *i.e.* the number of times φ covers the circle S^1, taking orientation into account. For instance, if φ goes once around the circle in a counter-clockwise direction we get a winding number +1, in the clockwise direction we get −1, and if it goes around the circle multiple times we get a winding number of magnitude greater than one (Figure 1). The winding number is also given by the total *signed length* of the image of S^1 under φ, divided by the circumference of the circle. Since $\varphi(t)$ is always a point on the unit circle, we can express the infinitesimal signed length as $\langle J\varphi(t), \dot{\varphi}(t)\rangle$, *i.e.*, by measuring the length of the tangent vector $\dot{\varphi}(t)$ along the counter-clockwise direction $J\varphi(t)$ tangent to the circle (Figure 2, *left*). The total signed length is then

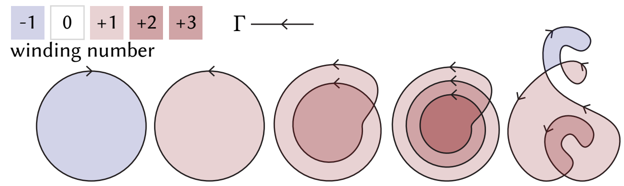


Fig. 1. For curves Γ in the plane, the winding number function $w_\Gamma(p)$ gives the number of times the curve Γ wraps around any given point p.

given by an integral over the circle, namely

$$w_\Gamma(p) := \frac{1}{2\pi}\int_{S^1} \varphi(t) \times \dot{\varphi}(t)\, dt, \tag{2}$$

where we have used the identity $\langle Ju, v\rangle = u \times v$. If the point p is at the origin (which can always be achieved by translating our coordinate system), then substituting Equation 1 into Equation 2 yields a formula for $w_\Gamma(0)$ as an integral over the curve Γ:

$$w_\Gamma(0) = \frac{1}{2\pi}\int_{S^1} \frac{\Gamma(t) \times \dot{\Gamma}(t)}{|\Gamma(t)|^2}\, dt. \tag{3}$$

For curves on surfaces we cannot apply this same idea, since in general the way a curve Γ winds around a point p may not define a continuous map to the circle akin to φ. Consider for instance the curve Γ_2 depicted in Figure 2, *right*, which gets "stuck" if we try to contract it to a small circle around the point q. Indeed, winding numbers are not meaningful for nonbounding loops: by definition they do not bound any region, and do not have a well-defined inside and outside. Hence, any definition of winding numbers for curves on surfaces must carefully account for the topology of the underlying domain.

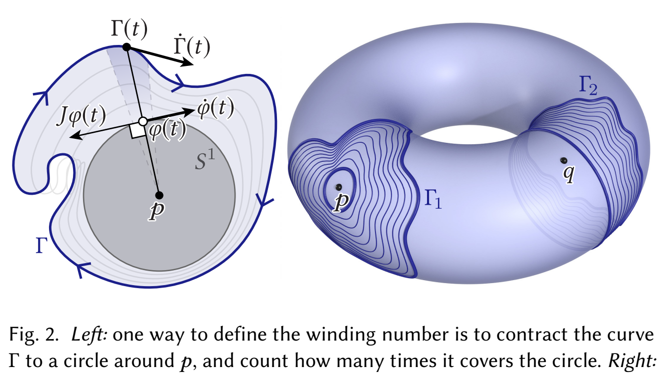


Fig. 2. *Left:* one way to define the winding number is to contract the curve Γ to a circle around p, and count how many times it covers the circle. *Right:* on surfaces, however, not all curves are contractible—consider for instance Γ_2, which cannot be contracted around q.

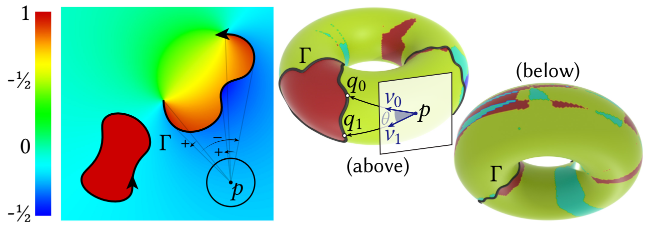


Fig. 3. *Left:* the solid angle function (*a.k.a.* the generalized winding number) is the total *signed* length of the projection of Γ onto a circle around p: positive for counter-clockwise motion, and negative for clockwise motion. *Right:* applying this same idea on surfaces, by using the log map to measure signed angles, yields garbage since the log map jumps discontinuously.

2 SOLID ANGLE

One can interpret the signed length of φ as the total signed angle subtended by the curve Γ over a small circle around p. This idea extends naturally to open curves, in which case the subtended angle is the fraction of the "sky" covered by Γ for an observer standing at p (Figure 3, *left*). Unlike physical solid angle, however, the signed subtended angle is counted with multiplicity, and will be negative whenever Γ and S^1 are oppositely oriented.

On a surface, one might be inclined to compute the subtended angle via the *logarithmic map*. At any point $p \in M$, the logarithmic map[1] $\log_p q$ gives the direction u and *shortest* distance $|u|$ we must walk along a straight path (*i.e.*, geodesic) to reach q. Letting $\theta(t)$ be the angle of the vector $\log_p(\Gamma(t))$, we could then try integrating the quantity $d\theta$ to obtain a notion of winding number. The problem, however, is that there are points at which there is not a *unique* shortest path to points q on the curve Γ. As our point p crosses through this so-called *cut locus*, then, the integral $\int_\Gamma d\theta$ may jump discontinuously, as pictured in Figure 3. Moreover, the exponential map may not be surjective on domains with boundary, hence the log map may not be well-defined for some points on our curve.

3 RAY INTERSECTIONS

Alternatively, one can interpret the degree of the map φ as the number of points along Γ which get mapped to a generic point on the circle (counted again with multiplicity). The number of points which φ maps onto a unit vector v is precisely the number of *signed intersections* $\chi_p(v)$ between Γ and a ray leaving p in the direction v, *i.e.*, +1 if the ray has a positive dot product with the unit normal n of Γ, and −1 otherwise. Hence, one can evaluate the winding number at a point p by shooting a random ray from p and counting the number of intersections with Γ (Figure 4, *left*).

On a surface, the natural analogue of shooting a ray is to evaluate the *exponential map* $\exp_p(tv)$, which traces out a geodesic curve starting at p in the direction v for time t. Unlike the plane, however, a geodesic may intersect a closed curve Γ infinitely many times. Artificially truncating it to a finite length yields an arbitrary answer (Figure 4, *center*); moreover, the number of intersections may also change completely with the ray direction v (Figure 4, *right*).

[1] In terms of the exponential map, discussed in Section 3, the log map gives the vector u of smallest magnitude such that $\exp_p(u) = q$.

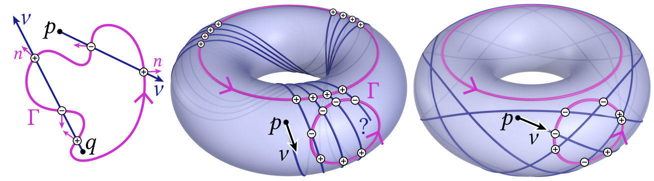


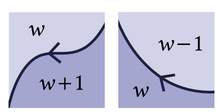
Fig. 4. *Left:* in the plane, the winding number of a closed curve Γ can be found by counting the number of signed intersections with a generic ray—here, $w_p(\Gamma) = -1 + 1 = 0$ and $w_q(\Gamma) = 1 - 1 + 1 = 1$. *Center, right:* on a surface, a geodesic ray may intersect a closed curve infinitely many times, or change completely depending on the initial direction v.

Equivalently, we can take the average number of ray intersections $\chi_p(\theta)$ over all directions $\theta \in [0, 2\pi)$ [Jacobson et al. 2013, Section 4.2]. We can perform a change of measure to integrate this quantity over the curve Γ: when p is at the origin, the angle $d\theta$ subtended by an infinitesimal piece of the curve Γ is inversely proportional to the distance $|\Gamma|$ from the origin, and proportional to the arc length $\dot{\Gamma}\, ds$ crossed with the direction $\widehat{\Gamma} := \Gamma/|\Gamma|$ from the origin to the curve (to obtain signed length). Hence,

$$\frac{1}{2\pi}\int_0^{2\pi} \chi_p(\theta)\, d\theta = \int_I \frac{\Gamma(s) \times \dot{\Gamma}(s)}{|\Gamma(s)|^2}\, ds,$$

i.e., we again recover the winding number integral (Equation 3). Note that this change of measure parallels the one used in rendering to convert between integrals of radiance over the hemisphere and scene surfaces [Veach and Guibas 1995, Section 2.1], *e.g.*, for importance sampling area lights (though here the visibility term is omitted).

4 NUMBERING REGIONS



A more direct way to obtain the winding number for a closed planar curve Γ is via an iterative algorithm: assign the value "0" to the region outside the curve (*i.e.*, the unique unbounded component of $\mathbb{R}^2 \setminus \Gamma$), then increment this value by +1 or −1 whenever we cross Γ from the left or the right, *resp.* (see inset). In the plane, this so-called *Alexander numbering* [Alexander 1928] produces a well-defined function, independent of the order in which one visits the regions bounded by the curve. On general surfaces, however, the procedure may not determine winding numbers in a canonical way. *E.g.*, on a compact surface M no region is clearly "outside" the curve, since all regions bounded by the curve are compact. Moreover, visiting the bounded regions in different orders can yield different numberings—see Figure 5.

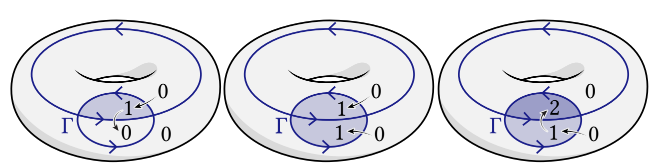


Fig. 5. A naïve approach to assigning winding numbers to regions does not work on surfaces. For instance, even if we always assign "0" to the largest region, the subsequent labeling may depend arbitrarily on the order in which we visit neighboring regions.

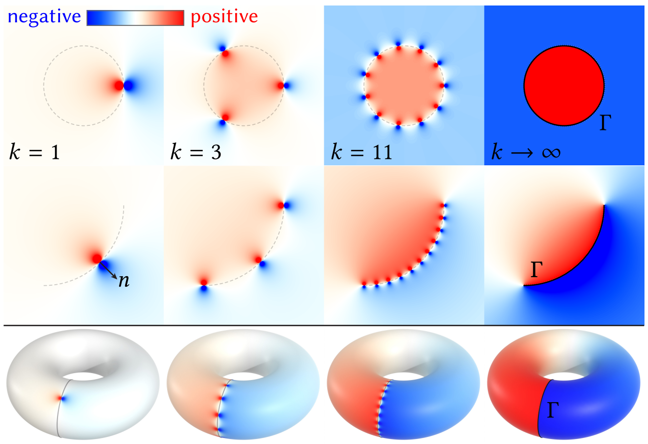
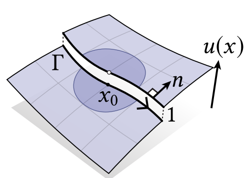


Fig. 6. *Top:* in the plane, the winding number function can be viewed as the electric potential as the number of dipoles k goes to infinity—shown here for an open and closed curve, where normals n determine dipole moments. *Bottom:* on surfaces this same harmonic function is well-defined, but may not provide a meaningful notion of inside/outside even for closed loops.

5 HARMONIC FUNCTIONS

As noted in Feng et al. [2023, Section 1] and in Section 7, the winding number function is also a particular *harmonic function*—for a simple curve Γ in the plane it is a solution to a Laplace equation with jumps, namely,

$$\begin{aligned} \Delta u &= 0, && \text{on } \mathbb{R}^2 \setminus \Gamma, \\ u^+ - u^- &= 1, && \text{on } \Gamma, \\ \partial u^+/\partial n &= \partial u^-/\partial n, && \text{on } \Gamma. \end{aligned} \tag{4}$$



The boundary conditions for this Laplace equation are somewhat unusual: rather than prescribing function values or normal derivatives along Γ (*i.e.*, Dirichlet or Neumann conditions, *resp.*), we have *jump boundary conditions*, which say that the two solution values $u^{\pm}$ on either side of the curve must differ by one, and that the normal derivative must be equal on both sides of the curve (Krutitskii [2001] gives a more formal treatment).

Equation 4 is readily solved for curves on surfaces, and serves as the starting point for the formulation in [Feng et al. 2023]. However, it does not immediately resolve the question of how to define winding numbers on surfaces, since even for closed curves the solution u may not be a piecewise integer function (Figure 6, *bottom right* shows one example).

The PDE perspective can be connected to the standard definition of winding numbers via the idea of a *double layer potential*. Conceptually, we imagine that a collection of equal-magnitude positive and negative electric charges are lined up along the curve Γ. Each positive/negative particle pair has an associated *dipole potential*, and as we pack more and more charges along Γ (as in Figure 6, *top*), the sum of these potentials converges to a harmonic function with a constant jump across Γ (see [Brebbia et al. 1984, pp. 56–58] and [Hsiao and Wendland 2008, Ch. 1] for more formal discussion). More explicitly, in 2D the dipole potential is given by the *Poisson kernel*

$$P(x,y) := -\frac{1}{2\pi}\frac{\langle n, x-y\rangle}{|x-y|^2}. \tag{5}$$

If we assume a constant charge density along an arc-length parameterized curve $\Gamma : I \to \mathbb{R}^2$, then at the origin $x = 0 \in \mathbb{R}^2$ the total potential of the double layer is given by the integral

$$-\frac{1}{2\pi}\int_I \frac{\langle n(t), -\Gamma(t)\rangle}{|-\Gamma(t)|^2}\,dt = \frac{1}{2\pi}\int_I \frac{\langle J\,\dot{\Gamma}(t), \Gamma(t)\rangle}{|\Gamma(t)|^2}\,dt.$$

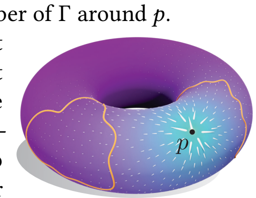
Since $\langle J\,\dot{\Gamma}, \Gamma\rangle = \dot{\Gamma} \times \Gamma$, we recover the usual winding number integral (Equation 3). In other words, the winding number function corresponds to a double-layer potential of constant dipole density, as observed by Maxwell [1881, Article 409]. This connection is central to boundary element methods, as well as the recent method of Barill et al. [2018] for computing generalized winding numbers.

6 ELECTROSTATICS

A more direct connection between winding numbers and electric fields is given by *Gauss' law*, which states that the flux of the electric field $\mathbf{E}$ through a closed surface Γ is given—up to a constant—by the enclosed charge Q:

$$\int_\Gamma \mathbf{E} \cdot \hat{\mathbf{n}}\, dS = \tfrac{1}{\varepsilon_0} Q \tag{6}$$

If we place a single point charge at p, then the resulting electric flux through Γ is precisely the winding number of Γ around p.



The electric field $\mathbf{E}$ induced by a point charge can be written as the gradient of an electric potential ϕ, which can be found as the solution to a Poisson equation. One can extend this procedure to surfaces, solving a Poisson equation for the electric potential of a point charge at p, taking its gradient to obtain the electric field $\mathbf{E}$, and then computing the flux through Γ (inset). This gives the same solution as the jump equation in Section 5, but is less attractive computationally because one must solve a PDE once per evaluation point, rather than once per curve.

7 COMPLEX ANALYSIS

In 2D, every harmonic function is in a sense the "shadow" of a richer complex function. For instance, the winding number integral in Equation 3 is the real part of the complex integral:

$$w_\Gamma^{\mathbb{C}}(p) = \frac{1}{2\pi\imath}\int_\Gamma \frac{1}{z-p}\,dz, \tag{7}$$

where $\imath$ is the imaginary unit, and we view Γ as a complex-valued curve $\Gamma : I \to \mathbb{C}$. By exponentiating, we may obtain a complex function $f(p) = \exp\left(2\pi\imath\, w_\Gamma^{\mathbb{C}}(p)\right)$ whose argument—*i.e.* angle from the origin—is given by the real part of $w_\Gamma^{\mathbb{C}}(p)$ and whose magnitude is determined by the imaginary part of $w_\Gamma^{\mathbb{C}}(p)$. When Γ is an open curve from a to b, we can write $f(p)$ in closed form:

$$f(p) = \frac{z-b}{z-a}, \tag{8}$$

as shown in Figure 7. The function f depends only on Γ's endpoints, oblivious to the location of the curve itself, since angle-valued functions ignore the integer jumps across Γ. In general, f has zeros

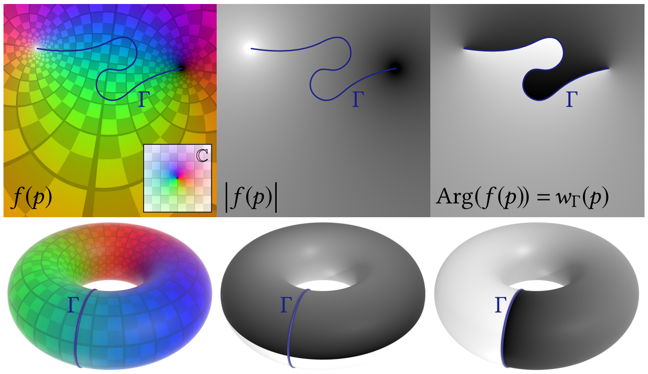


Fig. 7. *Top:* the winding number function $w_\Gamma(p)$ is essentially the "shadow" of a richer complex function $f(p)$ naturally associated with any curve Γ. *Bottom:* though an analogous function can be defined on surfaces, its imaginary part does not yield a useful labeling of inside/outside for all curves Γ.

at positive endpoints of Γ and poles at negative endpoints. Hence, $w_\Gamma^\mathbb{C} = \frac{1}{2\pi\iota}\log f$ has logarithmic singularities at all endpoints of Γ, where it locally looks like the function $\mathrm{Arg}(z)$.

An analogous procedure can be performed on orientable surfaces, constructing a complex function associated with Γ whose argument is given by w_Γ. While the complex perspective does not resolve the fundamental ambiguities of defining winding number on surfaces, the description of the logarithmic singularities around the endpoints of Γ is helpful when discretizing and interpolating jump harmonic functions [Feng et al. 2023, Section 2.3.2].

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