Graphs with $\chi = \Delta$ have big cliques

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Abstract

Brooks' Theorem implies that if a graph has $\Delta \geq 3$ and and $\chi > \Delta$, then $\omega = \Delta + 1$. Borodin and Kostochka conjectured that if $\Delta \geq 9$ and $\chi \geq \Delta$, then $\omega \geq \Delta$. We show that if $\Delta \geq 13$ and $\chi \geq \Delta$, then $\omega \geq \Delta - 3$. For a graph G, let $\mathcal{H}(G)$ denote the subgraph of G induced by vertices of degree Δ. We also show that if $\chi \geq \Delta$, then $\omega \geq \Delta$ or $\omega(\mathcal{H}(G)) \geq \Delta - 5$.

1 Introduction

Our goal in this paper is to prove the following two main results. For a graph G, we write $\Delta(G)$, $\omega(G)$, and $\chi(G)$ to denote the maximum degree, clique number, and chromatic number of G. When the context is clear, we simply write Δ, ω, and χ.

Theorem 1. *If G is a graph with $\chi \geq \Delta \geq 13$, then $\omega \geq \Delta - 3$.*

Theorem 2. *Let G be a graph and let $\mathcal{H}(G)$ denote the subgraph of G induced by vertices of degree Δ. If $\chi \geq \Delta$, then $\omega \geq \Delta$ or $\omega(\mathcal{H}(G)) \geq \Delta - 5$.*

The proofs of Theorems 1 and 2, are both somewhat detailed, so we first prove Theorem 3, which plays a central role in proving our two main theorems. (For a less formal and less notationally dense presentation of these results, see an earlier version of this paper [11].) Brooks' Theorem states that if G is connected and $\chi > \Delta$, then $\omega = \Delta + 1$ or G is an odd cycle; so if $\Delta \geq 3$, then $\chi > \Delta$ implies $\omega = \Delta + 1$. Thus, the interesting case of Theorems 1 and 2 is when $\chi = \Delta$.

Theorem 3. *If G is a critical graph with $\chi \geq \Delta$, then $\omega \geq \Delta - 3$ if $\Delta \equiv 1 \pmod 3)$ and $\omega \geq \Delta - 4$ otherwise.*

When $\Delta = 13$, Theorem 3 implies that either G is 12-colorable or G contains a K_{10}. This result will serve as the base case for a proof of Theorem 1 by induction on Δ. To

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prove Theorem 2, we will further analyze the proof of Theorem 3, and show that we can continue a certain recoloring process unless $\mathcal{H}(G)$ contains a big clique.

Borodin and Kostochka [5] conjectured in 1977 that if G is a graph with $\Delta \geq 9$ and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$. The hypothesis $\Delta \geq 9$ is needed, as witnessed by the following example. Form G from five disjoint copies of K_3, say $D_1, \ldots, D_5$, by adding edges between u and v if $u \in D_i$, $v \in D_j$, and $i - j \equiv 1 \bmod 5$. This graph is 8-regular with $\omega = 6$ and $\chi \geq \lceil 15/2 \rceil = 8$, since each color is used on at most 2 of the 15 vertices; by Brooks' Theorem G is 8-colorable, so $\chi(G) = 8$. Various other examples with $\chi = \Delta$ and $\omega < \Delta$ are known for $\Delta \leq 8$ (see for example [12]). The Borodin-Kostochka Conjecture has been proved for various families of graphs. Reed [30] used probabilistic arguments to prove it for graphs with $\Delta \geq 10^{14}$. The present authors [12] proved it for claw-free graphs (those with no induced $K_{1,3}$).

The contrapositive of the conjecture states that if $\chi \geq \Delta \geq 9$, then $\omega \geq \Delta$. The first result in this direction was due to Borodin and Kostochka [5], who proved that $\omega \geq \lfloor \frac{\Delta+1}{2} \rfloor$ when $\chi \geq \Delta$. Subsequently, Mozhan [25] improved this to $\omega \geq \lfloor \frac{2\Delta+1}{3} \rfloor$ when $\Delta \geq 10$ and Kostochka [20] showed that $\chi \geq \Delta$ implies that $\omega \geq \Delta - 28$. Finally, Mozhan proved that $\omega \geq \Delta - 3$ when $\chi \geq \Delta \geq 31$ (this result was in his Ph.D. thesis, which unfortunately is not readily accessible [30]). Theorem 1 strengthens Mozhan's result, by weakening the condition to $\Delta \geq 13$. Work in the direction of Theorem 2 began in [16], where Kierstead and Kostochka proved that if $\chi \geq \Delta \geq 7$ and $\omega \leq \Delta - 1$, then $\omega(\mathcal{H}(G)) \geq 2$. This was strengthened in [21] to the conclusion $\omega(\mathcal{H}(G)) \geq \lfloor \frac{\Delta-1}{2} \rfloor$. We further strengthen the conclusion to $\omega(\mathcal{H}(G)) \geq \Delta - 5$. We give more background in the introduction to Section 3.

Most of our notation is standard, as in [32]. We write K_t and E_t to denote the complete and empty graphs on t vertices, respectively. We write $[n]$ to denote $\{1, \ldots, n\}$. The *join* of disjoint graphs G and H, denoted $G \vee H$, is formed from the disjoint union of G and H by adding all edges with one endpoint in each of G and H. For a vertex v and a set S (containing v or not) we write $d_S(v)$ to denote $|S \cap N(v)|$. When vertices x and y are adjacent, we write $x \leftrightarrow y$; otherwise $x \not\leftrightarrow y$. If $\mathcal{Z}$ is a set of graphs, we let $V(\mathcal{Z}) = \bigcup_{G \in \mathcal{Z}} V(G)$. A graph G is *k-critical* if $\chi(G) = k$ and $\chi(H) < k$ for every proper subgraph H. A vertex v in a graph G is *critical* if $\chi(G \setminus \{v\}) < \chi(G)$. Note that in a Δ-critical graph, every vertex has degree Δ or $\Delta - 1$. A vertex v is *high* if $d(v) = \Delta$ and *low* otherwise.

2 Mozhan's Partitioned Colorings

In [25], Mozhan used a partition of a graph into groups of color classes to prove bounds on the chromatic number in terms of the degree and clique number. These ideas trace all the way back to the 1966 paper of Lovász [22] where he proves that if G is a graph and $r_1, \ldots, r_k \in \mathbb{N}$ with $\sum_{i \in [k]} r_i \geq \Delta(G) + 1 - k$, then $V(G)$ has a partition $\{V_1, \ldots, V_k\}$ where $\Delta(G[V_i]) \leq r_i$ for all $i \in [k]$. The proof idea is simple; just take a partition minimizing the number of edges within parts (with an appropriate weighting depending on r_i). In [7], Catlin took this idea further by starting with such a minimum partition and then moving vertices around (while preserving minimality) until he achieved a desired property. To get the ability to move vertices around like this, he needed to strengthen the condition on the r_i to $\sum_{i\in[k]} r_i \geq \Delta(G) + 2 - k$.

Mozhan's idea is very similar to Catlin's, but not equivalent. As we will see below, Mozhan considers partitions of $V(G)$ minimizing the number of edges within parts, just like Lovász and Catlin, but he adds the restriction that each part is the disjoint union of color classes in some fixed $\chi(G)$-coloring of G. With this added restriction we get a weaker bound on the degrees within parts, but more information about the coloring. Because of this trade-off Mozhan's method excels when all we care about is coloring the parts, but if we require the parts to have more structure (for example, for them to be degenerate as in Borodin's result [4]), we need to use Catlin's method or some other technique (see [6] for example). There are some cases where either technique will work; Mozhan's method was used in [28] and [21], but the same results were derived in [29] using Catlin's method. The results in this paper require the use of Mozhan's more restrictive partitions, which we define now.

Our proofs only use the partition in the following definition when G is critical. We include non-critical graphs as well because the more general concept is needed to extract an efficient algorithm from our proof. We discuss algorithmic considerations in the final section of the paper.

Definition 1. For $s \in \mathbb{N}_{\geq 2}$ and $r_1, \dots, r_s \in \mathbb{N}_{\geq 3}$, an $(r_1, \dots, r_s)$*-partition* P of a graph G is a partition $(P_1, \dots, P_s)$ of $V(G)$ such that

(1) there is $j \in [s]$ such that $\chi(G[P_i]) = r_i$ for all $i \in [s] \setminus \{j\}$; and

(2) there is $v \in P_j$ so that $\chi(G[P_j] \setminus \{v\}) \leq r_j$.

We refer to j and v by $j(P)$ and $v(P)$ respectively.

For example, if G is 13-critical, then we get a $(3, 3, 3, 3)$-partition of G by removing any $v \in V(G)$, partitioning the color classes of a 12-coloring of $G - v$ into four equal parts and then adding v to one part.

We are interested in $(r_1, \dots, r_s)$-partitions that minimize the total number of edges within parts (without $v(P)$). More precisely, for an $(r_1, \dots, r_s)$-partition P of a graph G, let $\sigma(P) = \left\|G[P_{j(P)}] \setminus \{v(P)\}\right\| + \sum_{i\in[s]\setminus\{j(P)\}} \|G[P_i]\|$; here $\|H\|$ denotes the number of edges in subgraph H. A *minimum* $(r_1, \dots, r_s)$*-partition* of G is an $(r_1, \dots, r_s)$-partition P minimizing $\sigma(P)$.

Lemma 4. *If P is a minimum $(r_1, \dots, r_s)$-partition of a graph G with $\chi(G) = \Delta(G) = 1 + \sum_{i\in[s]} r_i$, then*

(1) $G[P_{j(P)}]$ has a component $\mathcal{A}(P)$, called the active component, that is $K_{r_{j(P)}+1}$ and $\chi(G[P_{j(P)}] \setminus V(\mathcal{A}(P))) \leq r_{j(P)}$; and

(2) for each $u \in V(\mathcal{A}(P))$ and $i \in [s] \setminus \{j(P)\}$ with $d_{P_i}(u) = r_i$, the graph $G[P_i \cup \{u\}]$ has a K_{r_i+1} component (which contains u); and

(3) for each $u \in V(\mathcal{A}(P))$ and $i \in [s] \setminus \{j(P)\}$, if u has at least $d_{P_i}(u) + 1 - r_i$ neighbors in the same component D of $G[P_i]$, then $\chi(G[V(D) \cup \{u\}]) = r_i + 1$; and

(4) if $u \in V(G)$ and $a \in [s]$ so that $d_{P_a}(u) > r_a + 1$, then there is $i \in [s]$ where $d_{P_i}(u) < r_i$. In particular, any r_i-coloring of $G[P_i]$ can be extended to an r_i coloring of $G[P_i \cup \{u\}]$; and

(5) for each $u \in V(\mathcal{A}(P))$ and $i \in [s] \setminus \{j(P)\}$, we have $d_{P_i}(u) \leq r_i + 1$.

Proof. Let P be a minimum $(r_1, \ldots, r_s)$-partition of a graph G with $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$. Let $j = j(P)$ and $v = v(P)$. Let $\mathcal{A}(P)$ be the component of $G[P_j]$ containing v. By construction, $G[P_j \setminus \{v\}]$ has an r_j-coloring. So we may assume that $\chi(\mathcal{A}(P)) = r_j + 1$, since otherwise we get an r_j-coloring of $G[P_j]$, and hence a $(\Delta - 1)$-coloring of G.

To prove (1), it suffices to show that $\mathcal{A}(P)$ is K_{r_j+1}. By Brooks' Theorem, it is enough to show that $\Delta(\mathcal{A}(P)) \leq r_j$. Suppose instead that there exists $u \in V(\mathcal{A}(P))$ with $d_{\mathcal{A}(P)}(u) > r_j$; choose u to minimize the distance in $\mathcal{A}(P)$ from u to v. Uncolor the vertices on a shortest path Q in $\mathcal{A}(P)$ from u to v; move u to some P_k where it has at most r_k neighbors. Color the vertices of Q, starting at v and working along Q; this is possible since each vertex of Q has at most $r_j - 1$ colored neighbors in $\mathcal{A}(P)$ when we color it. The resulting new partition R has fewer edges within color classes, since we lost at least $r_j + 1$ edges incident to u and gained at most r_j incident to v (note that $v(R) = u$). This contradiction implies that $\Delta(\mathcal{A}(P)) \leq r_j$, so $\mathcal{A}(P)$ must be K_{r_j+1} by Brooks' Theorem. Thus (1) holds.

Now we prove (2). Choose such a vertex $u \in V(\mathcal{A}(P))$ and such an $i \in [s] \setminus \{j\}$. Form a new partition R by deleting u from P_j and add it to P_i (now $u = v(R)$); this maintains the total number of edges within parts, so R is another minimum $(r_1, \ldots, r_s)$-partition. By the above proof of (1), u lies in a component of $G[P_i]$ that is K_{r_j+1}. Thus, (2) holds.

If (3) is false, then u has at most $r_i - 1$ neighbors in $G[P_i] \setminus D$, so we may choose an r_i-coloring of $G[P_i] \setminus D$ so that the neighbors of u in $P_i \setminus V(D)$ each get a color in $[r_i - 1]$. Together with an r_i-coloring of $G[V(D) \cup \{u\}]$ where u is colored r_i, this gives an r_i-coloring of $G[V(P_i) \cup \{u\}]$. But then we have a $(\chi(G) - 1)$-coloring of G, a contradiction.

(4) is immediate, since $d_G(u) \leq 1 + \sum_{i \in [s]} r_i$

If (5) is false, then apply (4) and move u to P_i to get a $(\chi(G) - 1)$-coloring of G, a contradiction. □

Definition 2. A *move* is a quadruple (P, v, i, P') where

(1) P is an $(r_1, \ldots, r_s)$-partition of a graph G; and

(2) $v \in V(\mathcal{A}(P))$; and

(3) $i \in [s] \setminus \{j(P)\}$ with $d_{P_i}(v) = r_i$; and

(4) P' is obtained from P by moving v from $P_{j(P)}$ to P_i.

In the proof of part (2) of Lemma 4, we showed that if P is a minimum $(r_1, \ldots, r_s)$-partition and (P, v, i, P') is a move, then P' is a minimum $(r_1, \ldots, r_s)$-partition as well. Moreover, for each $k \in [s]$, the number of components in $G[P_k]$ equals the number of components in $G[P'_k]$.

Definition 3. Let P be an $(r_1, \ldots, r_s)$-partition of a graph G. A *move sequence* starting at P is a sequence of moves $((P^1, v_1, i_1, P^2), \ldots, (P^q, v_q, i_q, P^{q+1}))$ where $P^1 = P$.

Definition 4. Let P be an $(r_1, \ldots, r_s)$-partition of a graph G and

$$\mathcal{S} = ((P^1, v_1, i_1, P^2), \ldots, (P^q, v_q, i_q, P^{q+1}))$$

a move sequence starting at P. For each $i \in [s]$ and component X of $G[P_i]$, let the *club* of X, written $\mathcal{C}_{\mathcal{S}}(X)$, be the sequence $(X_1, X_2, X_3, \ldots, X_{q+1})$ where $X_1 = X$ and for $t \in [q] \setminus \{1\}$

- $X_t = X_{t-1} \setminus \{v_{t-1}\}$ if X_{t-1} is the active component in P^{t-1}; otherwise
- $X_t = X_{t-1} \cup \{v_{t-1}\}$ if $G[V(X_{t-1}) \cup \{v_{t-1}\}]$ is the active component in P^t; otherwise
- $X_t = X_{t-1}$.

Also, if $Y \in \mathcal{C}_{\mathcal{S}}(X)$, then we let $\mathcal{C}_{\mathcal{S}}(Y) = \mathcal{C}_{\mathcal{S}}(X)$. When the move sequence is clear from context, we write $\mathcal{C}(X)$ in place of $\mathcal{C}_{\mathcal{S}}(X)$. We say R *is a club of* $\mathcal{S}$ if $R = \mathcal{C}_{\mathcal{S}}(X)$ for a component X of $G[P_i]$ for some $i \in [s]$. For a club R, we write R_t for the t-th element of R.

Definition 5. Let P be a minimum $(r_1, \ldots, r_s)$-partition of a graph G with $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$. Let

$$\mathcal{S} = ((P^1, v_1, i_1, P^2), \ldots, (P^q, v_q, i_q, P^{q+1}))$$

be a move sequence starting at P. A club R of $\mathcal{S}$ is *full* if R_t is complete and $|R_t| \geq r_{\rho(R)}$ for all $t \in [q+1]$.

We observe a few basic facts about clubs; we omit formal proofs by induction, which are easy exercises.

Observation 1. *Let P be a minimum $(r_1, \ldots, r_s)$-partition of a graph G with $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$. If*

$$\mathcal{S} = ((P^1, v_1, i_1, P^2), \ldots, (P^q, v_q, i_q, P^{q+1}))$$

is a move sequence starting at P, then for a club R of $\mathcal{S}$, we have

(1) if $V(R_1) \subseteq P_i^1$, then $V(R_t) \subseteq P_i^t$ for all $t \in [q+1]$. We call this i the part *of R, written $\rho_{\mathcal{S}}(R)$ (or $\rho(R)$ when context allows).*

(2) if $a, b \in [q+1]$, then R_a is complete if and only if R_b is complete,

(3) if R is full, then $|R_a| = r_{\rho(R)} + 1$ when R_a is active and otherwise $|R_a| = r_{\rho(R)}$.

Lemma 5. *Let H be a graph with induced subgraphs $A_1, \ldots, A_k$ such that $\{V(A_1), \ldots, V(A_k)\}$ partitions $V(H)$ and $\chi(H) = \sum_{i \in [k]} \chi(A_i)$ where $\chi(A_1) \geq 4$ and $\chi(A_i) \geq 3$ for all $i \in [k] \setminus \{1\}$.*

(a) Suppose $u \in V(A_1)$ with $\chi(A_1 - u) < \chi(A_1)$. For each $i \in [k] \setminus \{1\}$, there is a component T_i of A_i such that $d_{V(T_i)}(u) \geq \chi(A_i)$. Let T_1 be u's component in A_1.

(b) Suppose $d_{V(T_k)}(u) = \chi(A_k)$ and $d_{V(A_k)}(u) \leq \chi(A_k)+1$. Put $A^* = V(\{A_1, \ldots, A_{k-1}\})$ and $T^* = V(\{T_1, \ldots, T_{k-1}\})$. Further suppose there is $v \in N(u) \cap V(T_k)$ with $d_{A^*}(v) \leq 1 + \sum_{i \in [k-1]} \chi(A_i)$ and $d_{T^*}(v) \geq 3$. Then there exists $q \in [k-1]$ such that $d_{V(T_q)}(v) \geq \chi(A_q)$.

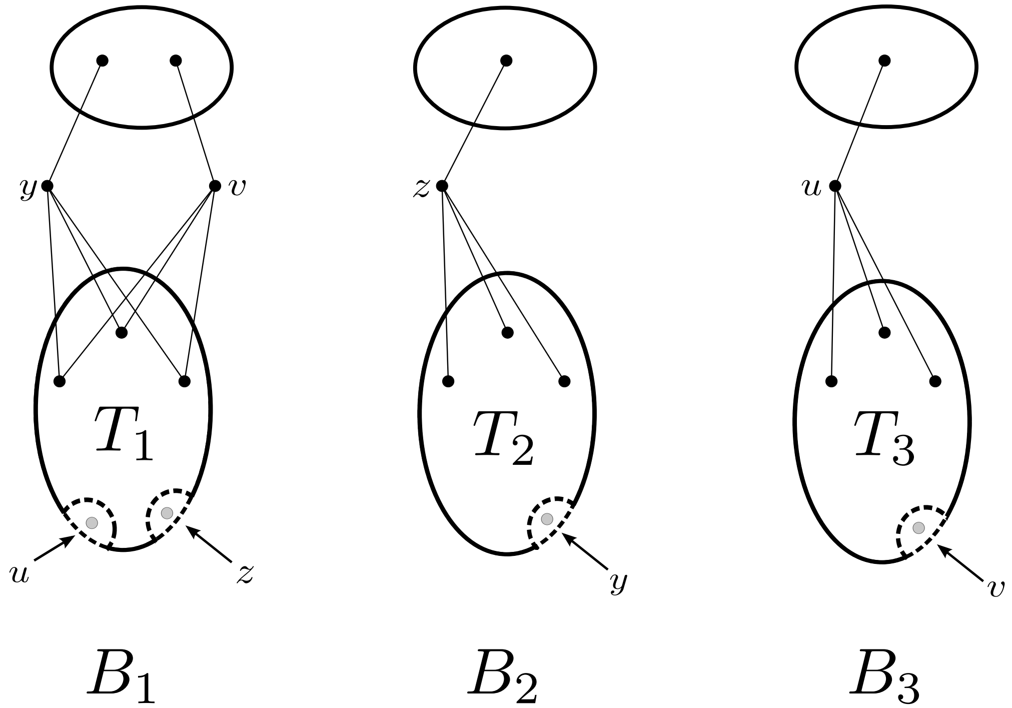

Figure 1: The partition in Claim 2 of Lemma 5. To form B_1, B_2, and B_3 from A_1, A_2, and A_3 (respectively), the vertices circled with dotted lines (and shown in gray) have now been moved to other parts, where they are shown above the T_i's.

(c) *If T^* induces a clique, T_k is complete, and $d_{A^*}(w) \leq |T^*|$ for all $w \in T^*$, then $T^* \cup \{v\}$ induces a clique.*

Proof. First we prove (a). Pick $i \in [k] \setminus \{1\}$. Since $\chi(A_1 \setminus \{u\}) < \chi(A_1)$, we must have $\chi(A'_i) = \chi(A_i) + 1$, where $A'_i = G[V(A_i) \cup \{u\}]$. So, u has at least $\chi(A_i)$ neighbors in some component T_i of A_i, for otherwise we get a $\chi(A_i)$-coloring of A'_i from a $\chi(A_i)$-coloring of A_i by permuting colors in components of A_i. This proves (a).

Now we prove (b). Since $\chi(A_1 \setminus \{u\}) < \chi(A_1)$, we must have $\chi(A'_k) = \chi(A_k) + 1$ and u is critical in A'_k. Then v is also critical in A'_k since $d_{V(T_k)}(u) = \chi(A_k)$ and $d_{V(A_k)}(u) < 2\chi(A_k)$. In particular, $d_{V(A_i)}(v) \geq \chi(A_i)$ for each $i \in [k-1]$.

Put $A'_1 = G\left[V(A_1 \setminus \{u\}) \cup \{v\}\right]$ and $A'_i = G\left[V(A_i) \cup \{v\}\right]$ for each $i \in [k-1] \setminus \{1\}$. Since $\chi(A'_k \setminus \{v\}) < \chi(A'_k)$, we must have $\chi(A'_1) \geq \chi(A_1)$ and $\chi(A'_i) \geq \chi(A_i) + 1$ for $i \in [k-1] \setminus \{1\}$. In particular, v is critical in A'_i for each $i \in [k-1]$. Note that $d_{V(A_i)}(v) \leq \chi(A_i) + 1$ for each $i \in [k-1]$ since $d_{A^*}(v) \leq 1 + \sum_{i \in [k-1]} \chi(A_i)$. Moreover, there is at most one $i \in [k-1]$ for which $d_{V(A_i)}(v) = \chi(A_i) + 1$. Now the remainder of (b) consists of the following claim.

Claim 1. *There exists $q \in [k-1]$ such that $d_{V(T_q)}(v) \geq \chi(A_q)$.*

Pick $w, x \in N(v) \cap T^* \setminus \{u\}$. First, suppose there is $i \in [k-1]$ with $w, x \in V(T_i)$.

Since v is critical in A'_i, it has at least $\chi(A'_i) - 1$ neighbors in some component C of $A'_i \setminus \{v\}$. Since v has two neighbors in T_i, our bounds on $d_{V(A_i)}(v)$ and $\chi(A'_i)$ imply that $C = T_i$. Since $\chi(A'_i) \geq \chi(A_i) + 1$ for $i \in [k-1] \setminus \{1\}$ (and if $i = 1$, v gets u as an extra neighbor), the claim is satisfied.

So, we may assume there are different $i, j \in [k-1]$ with $w \in V(T_i)$ and $x \in V(T_j)$. Since there is at most one $p \in [k-1]$ for which $d_{V(A_p)}(v) = \chi(A_p) + 1$, by symmetry we may assume that $d_{V(A_j)}(v) = \chi(A_j)$. Since v is critical in A'_j, it has at least $\chi(A'_j) - 1$ neighbors in some component C of $A'_j \setminus \{v\}$. Since v has at least one neighbor in T_j, our bounds on $d_{V(A_j)}(v)$ and $\chi(A'_j)$ imply that $C = T_j$. This proves the claim, and completes the proof of (b).

Now we prove (c), which we restate as the following claim.

Claim 2. *If T^* induces a clique, T_k is complete, and $d_{A^*}(w) \leq |T^*|$ for all $w \in T^*$, then $T^* \cup \{v\}$ induces a clique.*

Suppose otherwise that T^* induces a clique, T_k is complete, and $d_{A^*}(w) \leq |T^*|$ for all $w \in T^*$ but $T^* \cup \{v\}$ does not induce a clique. By (b) we have $q \in [k-1]$ such that $d_{V(T_q)}(v) \geq \chi(A_q)$. If $u \notin V(A_q)$, then we could move u into A_q without violating any hypotheses. So, we may assume that $q = 1$. Since $T^* \cup \{v\}$ does not induce a clique, there is some A_p to which v is not joined. By considering only the indices $1, p, k$ we can assume that $k = 3$ and $p = 2$.

By hypothesis $d_{V(T_1)}(v) \geq \chi(A_1)$ and T_1 is complete, so v must be joined to T_1 (otherwise we move v to A_1 and get a good coloring of G). Pick $y \in V(T_2) \setminus N(v)$ and $z \in V(T_1 \setminus \{u\})$. Let $B_1 = G[A_1 \cup \{v, y\} \setminus \{u, z\}]$, $B_2 = G[A_2 \cup \{z\} \setminus \{y\}]$, and $B_3 = G[A_3 \cup \{u\} \setminus \{v\}]$. We derive a contradiction by showing that $\chi(B_1) < \chi(A_1)$ and $\chi(B_2) \leq \chi(A_2)$ and $\chi(B_3) \leq \chi(A_3)$.

We have $d_{V(B_2)}(z) \leq \chi(A_2)$ since $d_{A^*}(z) \leq |T^*|$ and $z \leftrightarrow y$. Since z has exactly $\chi(A_2) - 1$ neighbors in $T_2 \setminus \{y\}$, we see that z has at most $\chi(A_2) - 1$ neighbors in each component of $B_2 \setminus \{z\}$ and hence $\chi(B_2) \leq \chi(A_2)$. Since, by assumption, $d_{V(A_k)}(u) \leq \chi(A_k) + 1$ and T_k is complete, the proof that $\chi(B_3) \leq \chi(A_3)$ is nearly identical.

Suppose $\chi(B_1) \geq \chi(A_1)$. Since $\{u, z\}$ is joined to $\{v, y\}$, we see that $d_{V(B_1)}(v) \leq \chi(A_1) - 1$ and $d_{V(B_1)}(y) \leq \chi(A_1) - 1$. Let $K = G[T_1 \cup \{v, y\} \setminus \{u, z\}]$. Then K is a copy of $K_{\chi(A_1)}$ with the edge vy deleted. First, color $B_1 \setminus V(K)$ with $\chi(A_1) - 1$ colors. Since v and y each have at most one neighbor outside of K in B_1 and $\chi(A_1) \geq 4$, we can finish the coloring on K by choosing the same color for v and y, different from the colors used on their at most 2 (collective) neighbors in $B_1 \setminus V(K)$, and then coloring $K \setminus \{v, y\}$ with the $\chi(A_1) - 2$ other colors (see Figure 1). □

In proving our next few lemmas, we repeatedly use the following helper lemma, which is an easy corollary of Lemma 5.

Lemma 6. *Let P be a minimum $(r_1, \ldots, r_s)$-partition of a graph G with $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$. Let*

$$\mathcal{S} = ((P^1, v_1, i_1, P^2), \ldots, (P^q, v_q, i_q, P^{q+1}))$$

be a move sequence starting at P. Let R and S be full clubs of $\mathcal{S}$ and $t \in [q+1]$. If $R_t = \mathcal{A}(P^t)$, then

(a) *if $u \in V(R_t)$ and u has at least 2 neighbors in S_t, then u is joined to S_t.*

(b) *if $u \in V(R_t)$ and $v \in V(S_t)$ and u has at least 2 neighbors in S_t and v has at least 2 neighbors in $R_t \setminus \{u\}$, then v is joined to R_t.*

Proof. First we prove (a). By symmetry, assume that $V(R_t) \subseteq P_1^t$ and $V(S_t) \subseteq P_2^t$. We apply Lemma 5 (a) with $A_i = G[P_i^t]$ for $i \in [2]$, $H = G[V(A_1) \cup V(A_2)]$ and $T_1 = R_t$. By Lemma 4, $\chi(H) = r_1 + r_2 + 1 = \chi(A_1) + \chi(A_2)$ and $\chi(A_1 - x) < \chi(A_1)$ for all $x \in V(T_1)$. Also by Lemma 4, $d_{A_2}(x) \leq \chi(A_2) + 1$ for all $x \in V(T_1)$. By Lemma 5, u has at least $\chi(A_2)$ neighbors in some component T_2 of A_2. Since $d_{A_2}(u) \leq \chi(A_2) + 1$ and u has at least two neighbors in S_t, we must have $T_2 = S_t$. Since S_t is a $K_{\chi(A_2)}$ this proves (a).

Now we prove (b). If $d_{A_1}(v) > \chi(A_1) + 1$, then there exists some part P_k^t with $d_{P_k^t}(v) < r_k$. By moving v to P_k^t and any vertex in T_1 to P_2^t, we get a $(\chi(G)-1)$-coloring of G, a contradiction. So $d_{A_1}(v) \leq \chi(A_1) + 1$. By (a), $|N(u) \cap V(T_2)| = \chi(A_2)$ and $v \in N(u) \cap V(T_2)$. So, we may apply Lemma 5 (b) to conclude that $|N(v) \cap V(T_1)| \geq \chi(A_1)$. Since T_1 is a $K_{\chi(A_1)}$ this proves (b). □

Lemma 7. *Let P be a minimum $(r_1, \ldots, r_s)$-partition of a graph G with $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$. Let $\mathcal{S}$ be a move sequence starting at P and let R and S be full clubs of $\mathcal{S}$. Then, for any $t_1, t_2 \geq 1$, we have R_{t_1} is joined to S_{t_1} if and only if R_{t_2} is joined to S_{t_2}.*

Proof. Suppose the lemma is false and let

$$\mathcal{S} = ((P^1, v_1, i_1, P^2), \ldots, (P^q, v_q, i_q, P^{q+1}))$$

be the shortest move sequence for which it fails. There must be a $t \in [q]$ such that either R_t is not joined to S_t, but R_{t+1} is joined to S_{t+1} or else R_t is joined to S_t, but R_{t+1} is not joined to S_{t+1}. If $q > 1$, then by starting the move sequence at P^t instead of P^1, we get a shorter counterexample. Hence $\mathcal{S} = ((P^1, v_1, i_1, P^2))$. Since the reverse sequence $(P^2, v_1, j(P^1), P^1)$ is also a counterexample, we may assume that R_1 is not joined to S_1, but R_2 is joined to S_2.

By symmetry between R and S, we may assume that R_1 is the active component. Since R_1 is not joined to S_1, but R_2 is joined to S_2, it must be that $R_2 = R_1 \setminus \{v_1\}$ is joined to $S_2 = S_1$ and there is $u \in V(S_1)$ with $v_1 \not\leftrightarrow u$. Pick $w \in V(R_1 \setminus \{v_1\})$. Now applying Lemma 6(b) to w and u shows that S_1 is joined to R_1, a contradiction. □

Lemma 7 makes it possible for us to talk about full clubs R and S being joined or not joined.

Definition 6. Let P be a minimum $(r_1, \ldots, r_s)$-partition of a graph G. For a club R of a move sequence

$$\mathcal{S} = ((P^1, v_1, i_1, P^2), \ldots, (P^q, v_2, i_2, P^{q+1}))$$

starting at P, we say that R *is active k times* if the number of $t \in [q+1]$ such that R_i is active is k.

Lemma 8. *Let P be a minimum $(r_1, \ldots, r_s)$-partition of a graph G with $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$. Let*

$$\mathcal{S} = ((P^1, v_1, i_1, P^2), \ldots, (P^q, v_q, i_q, P^{q+1}))$$

be a move sequence and S a full club of $\mathcal{S}$ that is active at least once. If R and T are different full clubs of $\mathcal{S}$ such that R is joined to S and S is joined to W, then R is joined to W.

Proof. Pick t such that S_t is active and let $P = P^t$, $T_1 = S_t$, $T_2 = R_t$ and $T_3 = W_t$. By symmetry, we assume that $V(T_1) \subseteq P_1$, $V(T_2) \subseteq P_2$, and $V(T_3) \subseteq P_3$. We will apply Lemma 5 with $A_i = G[P_i]$ for all $i \in [3]$ and $H = G[V(A_1) \cup V(A_2) \cup V(A_3)]$.

Pick $u \in V(T_1)$. By Lemma 4, $\chi(H) = r_1 + r_2 + r_3 + 1 = \chi(A_1) + \chi(A_2) + \chi(A_3)$ and $\chi(A_1 \setminus \{u\}) < \chi(A_1)$. Also by Lemma 4, $d_{V(A_3)}(u) \leq \chi(A_3) + 1$. Since T_3 is a K_{r_3}, we also have $d_{V(T_3)}(u) = \chi(A_3)$. For any $v \in V(T_3)$, we have $d_{A^*}(v) \leq 1 + \chi(A_1) + \chi(A_2)$, for otherwise there exists some part P_q with $d_{P_q}(v) < r_q$. By moving v to P_q and u to P_3, we get a $(\chi(G) - 1)$-coloring of G, a contradiction. Also, $d_{T^*}(v) \geq 3$ since T_1 is joined to T_3. Additionally, T^* induces a clique and T_k is complete. To apply Lemma 5, it remains to check that $d_{A^*}(w) \leq |T^*|$ for all $w \in T^*$. If not, then we could move w to some part P_q with $d_{P_q}(w) < r_q$ and get a $(\chi(G) - 1)$-coloring of G. So, we apply Lemma 5 with each $v \in V(T_3)$ and conclude that T_3 is joined to T_2 as desired. $\square$

Definition 7. Let P be a minimum $(r_1, \ldots, r_s)$-partition of a graph G. For a club R of a move sequence $\mathcal{S}$ starting at P, the *spread* of R is the set of indices of parts to which R sends vertices; more formally,

$$\mathrm{sp}_{\mathcal{S}}(R) = \left\{ i \mid (Q, v, i, Q') \in \mathcal{S} \text{ with } \mathcal{C}(\mathcal{A}(Q)) = R \right\}.$$

The *spread* of $\mathcal{S}$ is $\mathrm{sp}(\mathcal{S}) = \max_R |\mathrm{sp}(R)|$ where the max is over all clubs R of $\mathcal{S}$.

Lemma 9. *Let P be a minimum $(r_1, \ldots, r_s)$-partition of a graph G with $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$. If*

$$\mathcal{S} = ((P^1, v_1, i_1, P^2), \ldots, (P^q, v_q, i_q, P^{q+1}))$$

is a move sequence with $\mathrm{sp}(\mathcal{S}) \leq 2$, then one of the following holds:

(1) $v_i = v_j$ for different $i, j \in [q+1]$ (i.e. some vertex moves more than once); or

(2) there is $t \in [q]$ such that the active component in P^t is joined to the active component in P^{t+1}; or

(3) every club of $\mathcal{S}$ is active at most 3 times.

Proof. Suppose the lemma is false and choose a move sequence

$$\mathcal{S} = ((P^1, v_1, i_1, P^2), \ldots, (P^q, v_q, i_q, P^{q+1}))$$

for which it fails minimizing q. By minimality of q, we have a length three subsequence $((P^1, v_1, i_1, P^2), (P^b, v_b, i_b, P^{b+1}), (P^q, v_q, i_q, P^{q+1}))$ of $\mathcal{S}$ such that

(i) $\mathcal{C}(\mathcal{A}(P^1)) = \mathcal{C}(\mathcal{A}(P^b)) = \mathcal{C}(\mathcal{A}(P^{q+1}))$ and $\mathcal{C}(\mathcal{A}(P^2)) = \mathcal{C}(\mathcal{A}(P^{b+1}))$; and

(ii) there is at most one (P^d, v_d, i_d, P^{d+1}) in $\mathcal{S}$ with $1 < d < b$ such that $\mathcal{C}(\mathcal{A}(P^d)) = \mathcal{C}(\mathcal{A}(P^1))$; and

(iii) $\mathcal{C}(\mathcal{A}(P^2))$ is active at most 3 times.

Let $X = \mathcal{C}(\mathcal{A}(P^1))$ and $Y = \mathcal{C}(\mathcal{A}(P^2))$. We will show that X is joined to Y, which gives a contradiction, since we are assuming (2) does not hold. To simplify notation, let $c = q + 1$. If there does not exist (P^d, v_d, i_d, P^{d+1}) in $\mathcal{S}$ with $1 < d < b$ such that $\mathcal{C}(\mathcal{A}(P^d)) = \mathcal{C}(\mathcal{A}(P^1))$, then let $d = b$.

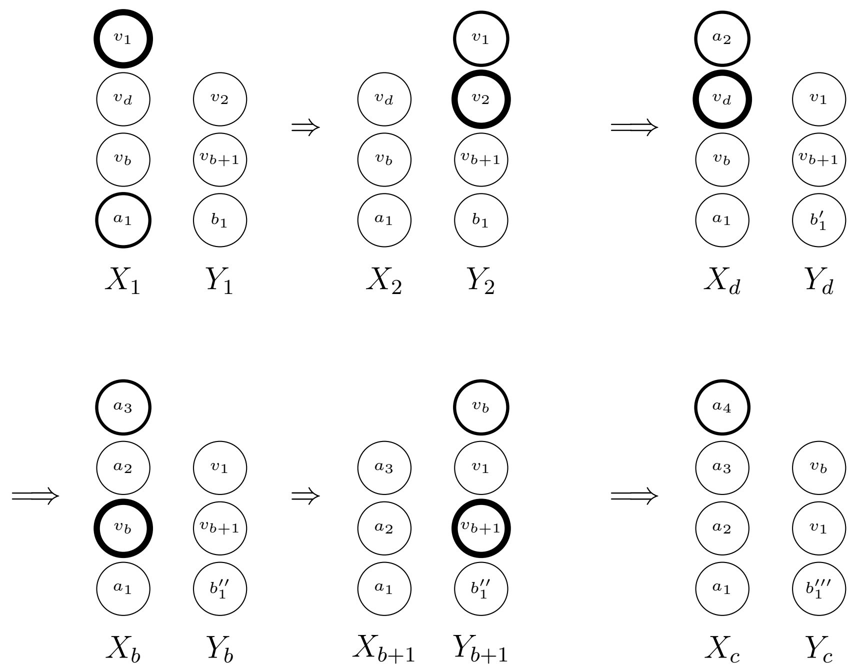

Figure 2: The six key partitions X_i, Y_i in the proof of Lemma 9. In each partition, the next vertex that will move is marked in bold, and the vertex that most recently moved is marked in semi-bold. If a vertex is unnamed in the proof, we denote it as a_i or b_i based on whether it appears in X_j or Y_j.

Claim 1. $\{v_b\}$ *is joined to* $V(Y_d)$.

Since Y becomes active at most once (by (iii)) between move d and move $b+1$, we have $|V(Y_d) \cap V(Y_b)| \geq 2$. One vertex in this intersection is v_1, and another is v_{b+1} (since no vertex is moved twice, by (1)). So v_b is adjacent to v_1 and v_{b+1}, since $v_1, v_b, v_{b+1} \in V(Y_{b+1})$ and Y is full. Applying Lemma 6(a) to X and Y with $t = d$, shows that v_b is joined to $V(Y_d)$, since $v_1, v_{b+1} \in V(Y_d)$.

Claim 2. $\{v_1\}$ *is joined to* $V(X_d)$.

Since $|V(X_d) \cap V(X_1)| \geq 3$, v_1 has at least 3 neighbors in X_d. Now by Claim 1, v_1 is joined to $V(X_d)$ by Lemma 6(b) applied to X and Y with $t = d$.

Claim 3. *$\{v_1\}$ is joined to $V(X_b)$.*

Since Y is full, v_b is joined to $V(Y_b)$. Since $|V(X_d) \cap V(X_b)| \geq 3$ and v_1 is joined to $V(X_d)$, v_1 has at least 3 neighbors in X_b. So v_1 is joined to $V(X_b)$ by Lemma 6(b) applied to X and Y with $t = b$.

Claim 4. *$V(X_{b+1})$ is joined to $V(Y_c)$.*

Since $V(X_{b+1}) \subset V(X_b)$, Claim 3 shows that $\{v_1, v_b\}$ is joined to $V(X_{b+1})$. But, $\{v_1, v_b\} \subset V(Y_c)$, so applying Lemma 6(a) to X and Y with $t = c$ shows that $V(X_{b+1})$ is joined to $V(Y_c)$.

Claim 5. *$V(X_c)$ is joined to $V(Y_c)$. In particular, X is joined to Y.*

Since $|X_{b+1}| \geq 3$, Claim 4 and an application of and Lemma 6(b) to X and Y with $t = c$ shows that $V(X_c)$ is joined to $V(Y_c)$. □

Theorem 3. *If G is a critical graph with $\chi(G) \geq \Delta(G)$, then $\omega(G) \geq \Delta(G) - 3$ if $\Delta(G) \equiv 1 \pmod 3)$ and $\omega(G) \geq \Delta(G) - 4$ otherwise.*

Proof. By Brooks' Theorem, we may assume $\chi(G) = \Delta(G)$. Let $s = \left\lfloor \frac{\Delta(G)-1}{3} \right\rfloor$ and $r_1, \ldots, r_s \in \{3, 4\}$ such that $\Delta(G) = 1 + \sum_{i \in [s]} r_i$. Then G has an $(r_1, \ldots, r_s)$-partition, so we can let P be a minimum $(r_1, \ldots, r_s)$-partition of G. Let

$$\mathcal{S} = ((P^1, v_1, i_1, P^2), \ldots, (P^q, v_q, i_q, P^{q+1}))$$

be a move sequence starting at P with $\operatorname{sp}(\mathcal{S}) \leq 2$ of maximum length such that $v_i \neq v_j$ for different $i, j \in [q+1]$ and for each $t \in [q]$ the active component in P^t is not joined to the active component in P^{t+1}. Let $A = \mathcal{A}(P^{q+1})$. Then by Lemma 9, $\mathcal{C}(A)$ is active at most 3 times in $\mathcal{S}$. Since $r_{i_q} \geq 3$, there is $x \in V(A)$ such that $x \notin \{v_t \mid t \in [q]\}$, i.e., x has never moved during $\mathcal{S}$.

Let $T = \operatorname{sp}(\mathcal{C}(A))$. If there is $i \in T$ with $d_{P_i^{q+1}}(x) = r_i$, then we have a move (P^{q+1}, x, i, Q^i) and by the maximality condition on $\mathcal{S}$, it must be that A is joined to $\mathcal{A}(Q^i)$. But, by assumption, A is not joined to $\mathcal{A}(Q^i)$ for any $i \in T$, so this is impossible.

Since $d_G(x) \leq 1 + \sum_{i \in [s]} r_i$ and x has exactly r_{i_q} neighbors in $P_{i_q}^{q+1}$, there is at most one $i \in [s] \setminus \{i_q\}$ for which $d_{P_i^{q+1}}(x) > r_i$. So, $|T| \leq 1$ and if $|T| = 1$, then T contains the one i with $d_{P_i^{q+1}}(x) > r_i$. By the maximality condition on $\mathcal{S}$, it must be that A is joined to clubs in P_i^{q+1} for all but one $i \in [s] \setminus \{i_q\}$. Since $r_i = 3$ if $\Delta(G) \equiv 1 \pmod 3)$ and $r_i \leq 4$ otherwise, we have the desired large clique by Lemma 8. □

Since any graph G with $\chi(G) \geq \Delta(G) = 13$, contains a critical subgraph H with $\chi(H) \geq \Delta(H) = 13$, as an immediate consequence of Theorem 3, we get the following corollary.

Corollary 9. *If G is a graph with $\chi(G) \geq \Delta(G) = 13$, then G contains K_{10}.*

3 The First Main Theorem

A *hitting set* is an independent set that intersects every maximum clique. If I is a hitting set and also a maximal independent set, then $\Delta(G-I) \le \Delta(G)-1$ and $\chi(G-I) \ge \chi(G)-1$. (In our applications, we can typically assume that $\Delta(G-I) = \Delta(G)-1$, since otherwise we get a good coloring or a big clique from Brooks' Theorem. We give more details in the proof of Theorem 1.) So if $G-I$ has a clique of size $\Delta(G-I)-t$, for some constant t, then also G has a clique of size $\Delta(G)-t$. We repeatedly remove hitting sets to reduce a graph with $\Delta \ge 13$ to one with $\Delta = 13$. Since we proved in Corollary 9 that every graph with $\chi \ge \Delta = 13$ contains K_{10}, this repeated removal of hitting sets allow us to prove that every G with $\chi \ge \Delta \ge 13$ contains $K_{\Delta-3}$.

This idea is not new. Kostochka [20] proved that every graph with $\omega \ge \Delta - \sqrt{\Delta} + \frac{3}{2}$ has a hitting set. Rabern [27] extended this result to the case $\omega \ge \frac{3}{4}(\Delta+1)$, and King [17] strengthened his argument to prove that G has a hitting set if $\omega > \frac{2}{3}(\Delta+1)$. This condition is optimal, as illustrated by the lexicographic product of an odd cycle and a clique. Finally, King's argument was refined by Christofides, Edwards, and King [8] to show that these lexicographic products of odd cycles and cliques are the only sharpness examples; that is, G has a hitting set if $\omega \ge \frac{2}{3}(\Delta+1)$ and G is not the lexicographic product of an odd cycle and a clique. Hitting set reductions have application to other vertex coloring problems. Using this idea (and others), King and Reed [18] gave a short proof that there exists $\epsilon > 0$ such that $\chi \le \lceil (1-\epsilon)(\Delta+1) + \epsilon\omega \rceil$.

To keep this paper largely self-contained, we prove our own hitting set lemma. In the present context, it suffices to find a hitting set when G is a minimal counterexample to Theorem 1 with $\Delta \ge 14$. Such a G is Δ-critical, which facilitates a shorter proof. In [10], we proved a number of results about so called d_1-choosable graphs (defined below), which are certain graphs that cannot appear as induced subgraphs in a Δ-critical graph. We leverage these d_1-choosability results to prove our hitting set lemma, then use the hitting set lemma to reduce to the case $\Delta = 13$, which we proved in Corollary 9. Since the proofs of the d_1-choosability results in [10] are lengthy, we give a short proof of the special case that we need here.

A *list assignment* L is an assignment $L(v)$ of a set of allowable colors to each vertex $v \in V(G)$. An L-coloring is a proper coloring such that each vertex v is colored from $L(v)$. An f-assignment is a list assignment L such that $|L(v)| = f(v)$ for all $v \in V(G)$. In particular, a d_1-assignment is an f-assignment with $f(v) = d(v)-1$ for all v. A graph G is *f-choosable* if G has an L-coloring for every f-assignment L. No Δ-critical graph contains an induced d_1-choosable subgraph H (by criticality, color $G \setminus H$, then extend the coloring to H, since it is d_1-choosable). For a list assignment L, let $Pot(L) = \cup_{v\in V(G)} L(v)$. The following lemma is central in proving each of our d_1-choosability results.

Lemma 10 (Small Pot Lemma, [15, 31])**.** *For a list size function $f : V(G) \to \{0,\dots,|G|-1\}$, a graph G is f-choosable iff G is L-colorable for each list assignment L such that $|L(v)| = f(v)$ for all $v \in V(G)$ and $\left|\cup_{v\in V(G)} L(v)\right| < |G|$.*

Proof. Fix G and f. The "only if" direction is true by definition. Now we prove the "if" direction. Assume that G is L-colorable for each list assignment L such that $|L(v)| = f(v)$ for all v and $\left|\cup_{v\in V(G)}L(v)\right| < |G|$. For any $U \subseteq V(G)$ and any list assignment L, let $L(U)$ denote $\cup_{v\in U}L(v)$. Let L be an f-assignment such that $|L(V)| \geq |G|$ and G is not L-colorable. For each $U \subseteq V(G)$, let $g(U) = |U| - |L(U)|$. Let $\mathcal{B}$ be a bipartite graph, where one part consists of vertices in $V(G)$ and the other part consists of colors in $Pot(L)$, and a vertex v is adjacent to a color c if $c \in L(v)$. Since G is not L-colorable, $\mathcal{B}$ has no matching saturating $V(G)$, so Hall's Theorem implies there exists U with $g(U) > 0$. Choose U to maximize $g(U)$. Let A be an arbitrary set of $|G|-1$ colors containing $L(U)$. Construct L' as follows. For $v \in U$, let $L'(v) = L(v)$. Otherwise, let $L'(v)$ be an arbitrary subset of A of size $f(v)$. Now $|L'(V)| < |G|$, so by hypothesis, G has an L'-coloring. This gives an L-coloring of U. By the maximality of $g(U)$, for all $W \subseteq (V(G)\setminus U)$, we have $|L(W)\setminus L(U)| \geq |W|$. Let $\mathcal{B}' = \mathcal{B}\setminus(\cup_{u\in U}\{u\}\cup N_{\mathcal{B}}(u))$. Thus, by Hall's Theorem $\mathcal{B}'$ has a matching saturating $V(G)\setminus U$; so we can extend the L-coloring of U to all of V. □

Lemma 11 ([10])**.** *For $t \geq 4$, $K_t \vee B$ is not d_1-choosable iff $\omega(B) \geq |B|-1$; or $t = 4$ and B is E_3 or $K_{1,3}$; or $t = 5$ and B is E_3.*

Proof. If $\omega(B) \geq |B|-1$, then assign each $v \in V(K_t \vee B)$ a subset of $\{1,\ldots,t+|B|-2\}$; since $\omega(K_t\vee B) \geq t+|B|-1$, clearly G is not colorable from this list assignment. Now let $G = K_5 \vee E_3$, and note that $K_4 \vee K_{1,3} \cong K_5 \vee E_3$. Consider the following list assignment L for G: each dominating vertex has list $\{1,\ldots,6\}$ and the three other vertices get distinct lists among $\{1,2,3,4\}, \{1,2,5,6\}, \{3,4,5,6\}$. If G has a proper L-coloring, then the dominating vertices use five distinct colors; this leaves only one color for the three remaining vertices, but no color appears in all three lists. Hence, G has no L-coloring. Now form G' from G by deleting one dominating vertex (note that $G' = K_4 \vee E_3$), and let $L' = L\setminus\{6\}$. Since G has no L-coloring, also G' has no L'-coloring. This proves one direction of the lemma; now we consider the other.

Suppose the lemma is false, and let G and L be a minimal counterexample, where $G = K_t \vee B$ and L is a d_1-assignment. If $\omega(B) \leq |B|-2$, then B contains either (i) an independent set $S = \{x_1,x_2,x_3\}$ or (ii) a set $S = \{x_1,x_2,x_3,x_4\}$ with $x_1x_2, x_3x_4 \notin E(B)$. If B contains only (i), then $S = E_3$ and $t \geq 6$ (by moving any dominating vertices from B to K_t). Let $T = V(K_t)$ and denote T by $\{y_1,\ldots,y_t\}$. In Cases (i) and (ii) we assume by minimality that $t = 6$ and $t = 4$, respectively. Also by minimality, we assume that $V(B) = S$ (we can greedily color vertices not in S). By definition $|L(v)| = d(v)-1$; specifically, $|L(x_i)| = d_S(x_i)+t-1$ and $|L(y_j)| = |S|+t-2$ for all $x_i \in S$ and $y_j \in T$. When we have i,j,k with $x_i \not\leftrightarrow x_j$ and $|L(x_i)|+|L(x_j)| > |L(y_k)|$, we often use the following technique, called *saving a color* on y_k via x_i and x_j. If there exists $c \in L(x_i)\cap L(x_j)$, then use c on x_i and x_j. Otherwise, color just one of x_i and x_j with some $c \in (L(x_i)\cup L(x_j))\setminus L(y_k)$. For a set U, let $L(U) = \cup_{v\in U}L(v)$.

Case (i) By the Small Pot Lemma, assume that $|L(G)| \leq 8$. This implies $|L(x_i)\cap L(x_j)| \geq 2$ for all $i,j \in [3]$. If there exist x_i and y_k with $L(x_i) \not\subseteq L(y_k)$, then color x_i to save a color on y_k. Color the remaining x's with a common color; this saves an additional color on each y. Now finish greedily, ending with y_k. Thus, we have $L(x_i) \subset L(y_k)$ for all $i \in [3]$ and $k \in [6]$. This gives $\left|\cup_{i=1}^3 L(x_i)\right| \leq 7$. Since $\sum_{i=1}^3 |L(x_i)| = 15 > 2|\cup_{k=1}^3 L(x_k)|$, we have a color $c \in \cap_{i=1}^3 L(x_i)$. Use c on every x_i and finish greedily.

Case (ii) By the Small Pot Lemma, assume that $|L(G)| \leq 7$. If S induces at least two edges, then $|L(x_1)| + |L(x_2)| \geq 8$. So $L(x_1) \cap L(x_2) \neq \emptyset$. Color x_1 and x_2 with a common color c. If $|L(y_1) \setminus \{c\}| \leq 5$, then save a color on y_1 via x_3 and x_4. Now finish greedily, ending with y_1.

Suppose S induces exactly one edge; by symmetry, say it is x_1x_3. Suppose that $L(x_1) \cap L(x_2) \neq \emptyset$. Similar to the previous argument, use a common color on x_1 and x_2, possibly save on y_1 via x_3 and x_4, then finish greedily. So instead, assume that $L(x_1) \cap L(x_2) = \emptyset$. Since $|L(G)| \leq 7$ and $L(x_1) \cap L(x_2) = \emptyset$, by symmetry (between x_1 and x_3 and also between x_2 and x_4), we may assume that $L(x_1) = L(x_3) = \{a, b, c, d\}$ and $L(x_2) = L(x_4) = \{e, f, g\}$. Also by symmetry, a or e is missing from $L(y_1)$. So color x_1 with a and x_2 and x_4 with e and x_3 arbitrarily; this saves one color on each y_i and a second color on y_1. Now finish greedily, ending with y_1.

So instead $G[S] = E_4$. If a common color appears on 3 vertices of S, use it there, then finish greedily. If not, then by pigeonhole, at least 5 colors appear on pairs of vertices; so, two colors appear on disjoint pairs. Color two such disjoint pairs, each with a common color. Now finish the coloring greedily. $\square$

The following lemma of King enables us to find an independent transversal.

Lemma 12 (Lopsided Transversal Lemma [17])**.** *Let H be a graph and $V_1 \cup \cdots \cup V_r$ a partition of $V(H)$. If there exists $s \geq 1$ such that for each $i \in [r]$ and each $v \in V_i$ we have $d(v) \leq \min\{s, |V_i| - s\}$, then H has an independent transversal I of $V_1, \ldots, V_r$.*

Now we have all the tools to prove our hitting set lemma.

Lemma 13. *Every Δ-critical graph with $\chi \geq \Delta \geq 14$ and $\omega = \Delta - 4$ has a hitting set.*

Proof. Suppose the lemma is false, and let G be a counterexample minimizing $|G|$. Consider distinct intersecting maximum cliques A and B. Since a vertex in their intersection has degree at most Δ, we have $|A \cap B| \geq |A| + |B| - (\Delta + 1) = \Delta - 9 \geq 5$. Since G contains no induced d_1-choosable subgraph, letting $A \cap B = K_t$ in Lemma 11 implies that $\omega(G[A \cup B]) \geq |A \cup B| - 1$. Hence $|A \cap B| = \omega - 1 = \Delta - 5$. Suppose C is another maximum clique intersecting A; let $U = A \cup B \cup C$ and $J = A \cap B \cap C$. We use inclusion-exclusion to bound $|U|$ and $|J|$. First, $|U| = |A \cup B \cup C| = |A \cup B| + |C \setminus (A \cup B)| \leq |A \cup B| + |C \setminus A| = |A \cup B| + |C| - |C \cap A| \leq (\Delta - 5 + 1 + 1) + (\Delta - 4) - (\Delta - 5) = \Delta - 2$. Second, $|J| = |A \cap B| + |C| - |(A \cap B) \cup C| \geq |A \cap B| + |C| - |U| \geq (\Delta - 5) + (\Delta - 4) - (\Delta - 2) = \Delta - 7 \geq 7$.

Since $|J| \geq 7$, by Lemma 11, $\omega(G[U]) \geq |U| - 1$; so $C = A$ or $C = B$, a contradiction. Thus, every maximum clique intersects at most one other maximum clique. Hence we can partition the union of the maximum cliques into sets $D_1, \ldots, D_r$ such that either D_i is a $(\Delta - 4)$-clique C_i or $D_i = C_i \cup \{x_i\}$ for a $(\Delta - 4)$-clique C_i, where x_i is adjacent to all but one vertex of C_i.

For each D_i, if $D_i = C_i$, then let $K_i = C_i$. If $D_i = C_i \cup \{x_i\}$, then let $K_i = C_i \cap N(x_i)$. Consider the subgraph F of G formed by taking the subgraph induced on the union of the K_i and then making each K_i independent. We apply Lemma 12 to F with $s = \frac{\Delta}{2} - 2$. We have two cases to check, when $K_i = C_i$ and when $K_i = C_i \cap N(x_i)$. In the former case, $|K_i| = \Delta - 4$ and for each $v \in K_i$ we have $d_F(v) \leq \Delta(G) + 1 - (\Delta - 4) = 5$. Hence $d_F(v) \leq \frac{\Delta}{2} - 2 = \min\left\{\frac{\Delta}{2} - 2, \Delta - 4 - (\frac{\Delta}{2} - 2)\right\}$ since $\Delta \geq 14$. In the latter case, we have $|K_i| = \Delta - 5$ and since every $v \in K_i$ is adjacent to x_i and to the vertex in $C_i \setminus K_i$, neither of which is in F, we have $d_F(v) \leq \Delta - (\Delta - 4) = 4$. This gives $d_F(v) \leq \frac{\Delta}{2} - 3 = \min\left\{\frac{\Delta}{2} - 2, \Delta - 5 - (\frac{\Delta}{2} - 2)\right\}$ since $\Delta \geq 14$. Now Lemma 12 gives an independent transversal I of the K_i, which is a hitting set. $\square$

Now we can prove the first of our two main results. For convenience, we restate it.

Theorem 1. *Every graph with $\chi \geq \Delta \geq 13$ contains $K_{\Delta-3}$.*

Proof. Let G be a counterexample minimizing $|G|$; note that G is vertex critical. By Corollary 9, we have $\Delta \geq 14$. If $\omega < \Delta - 4$, let I be any maximal independent set; otherwise let I be a hitting set given by Lemma 13 expanded to a maximal independent set. Now $\omega(G - I) < \Delta(G) - 4$, $\Delta(G - I) \leq \Delta(G) - 1$, and $\chi(G - I) \geq \chi(G) - 1$. If $\Delta(G-I) \leq \Delta(G)-3$, then greedy coloring gives $\chi(G-I) \leq \Delta(G-I)+1 \leq \Delta(G)-2$, so $\chi(G) \leq \Delta(G) - 1$. If $\Delta(G - I) = \Delta(G) - 2$, then $\chi(G - I) \leq \Delta(G - I)$ by Brooks' Theorem (since $\omega(G - I) < \Delta(G) - 4$), so $\chi(G) \leq \Delta(G) - 1$. So instead $\Delta(G - I) = \Delta(G) - 1$. Now $\chi(G - I) \geq \Delta(G - I) \geq 13$ and $\omega(G - I) < \Delta(G - I) - 3$ contradicting the minimality of $|G|$. $\square$

We suspect that Theorem 1 holds for all Δ. By Theorem 3 and Theorem 1, the following conjecture is only open when $\Delta \in \{6, 8, 9, 11, 12\}$.

Conjecture 1. Every graph with $\chi \geq \Delta$ contains $K_{\Delta-3}$.

We conclude this section with a nice application of Theorem 1 to the Borodin-Kostochka conjecture for vertex-transitive graphs. Suppose G is a vertex-transitive graph with $\chi(G) \geq \Delta(G) \geq 15$. Then $\omega(G) \geq \Delta(G) - 3$ by Theorem 1. Since G is vertex-transitive, every vertex of G is in a $K_{\Delta(G)-3}$. In [26], it was proved that the Borodin-Kostochka conjecture holds for graphs where every vertex is in a $K_{\frac{2}{3}\Delta(G)+2}$. Now $\Delta(G) - 3 \geq \frac{2}{3}\Delta(G) + 2$ since $\Delta(G) \geq 15$, so we have proved the following.

Corollary 14. *Every vertex-transitive graph with $\chi \geq \Delta \geq 15$ contains K_Δ.*

Corollary 14 should hold for $\Delta \geq 9$ and this may be much easier to prove than the full Borodin-Kostochka conjecture. In a short note [9], we explore these ideas further and prove Corollary 14 for $\Delta \geq 13$. A more general conjecture comes out of these considerations which is worth mentioning because it implies Corollary 14 for $\Delta \geq 9$.

Conjecture 2. Every vertex-transitive graph satisfies $\chi \leq \max\left\{\omega, \lceil\frac{5\Delta+3}{6}\rceil\right\}$.

4 The Second Main Theorem

In this section, we prove our second main theorem. First, we prove a lemma that follows from [10] about list coloring (we use it to forbid a certain subgraph in a Δ-critical graph).

Lemma 15 ([10])**.** *Let $G = K_3 \vee E_2$. If L is a list assignment such that $|L(v)| \geq d(v) - 1$ for all $v \in V(G)$ and for some $w \in V(K_3)$ and some $x \in V(E_2)$ we have $|L(w)| \geq d(w)$ and $|L(x)| \geq d(x)$, then G has an L-coloring.*

Proof. Denote $V(E_2)$ by $\{x, y\}$. By the Small Pot Lemma, we assume $|Pot(L)| \leq 4 < 5 \leq |L(x)|+|L(y)|$. After coloring x and y the same, finish greedily, ending with w. $\square$

In the rest of this section, we extend and refine the ideas in Section 2.

Definition 8. Let P be a minimum $(r_1, \ldots, r_s)$-partition of a graph G with $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$. Let $\mathcal{S}$ be a move sequence starting at P. For a full club S with respect to $\mathcal{S}$, the *clubgroup* $\mathcal{G}_{\mathcal{S}}(S)$ of S is the set consisting of S and the full clubs to which S is joined.

When the move sequence is clear from context, we write $\mathcal{G}(S)$ in place of $\mathcal{G}_{\mathcal{S}}(S)$. Clearly if R and S are full clubs and $R \in \mathcal{G}(S)$, then $S \in \mathcal{G}(R)$. By Lemma 8, we know that if R, S, and T are full clubs, and $R \in \mathcal{G}(S)$ and $S \in \mathcal{G}(T)$, then $R \in \mathcal{G}(T)$. So, the set of full clubs with respect to $\mathcal{S}$ is partitioned into clubgroups. We need a way of differentiating moves that are internal to a clubgroup and moves that go from one clubgroup to another. This motivates the following definition of *internal* and *external* moves.

With the notation we have at this point, referring to objects like "the clubgroup of the club of the active component" is a bit unwieldy. So, we allow ourselves to write $\mathcal{G}_{\mathcal{S}}(A)$ in place of $\mathcal{G}_{\mathcal{S}}(\mathcal{C}_{\mathcal{S}}(A))$.

Definition 9. Let P be a minimum $(r_1, \ldots, r_s)$-partition of a graph G with $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$. Let $\mathcal{S}$ be a move sequence starting at P. Let $M = (P^a, v, i, P^b)$ be a move in $\mathcal{S}$, A^a the active component in P^a and A^b the active component in P^b. Then move M is *internal* if $\mathcal{G}_{\mathcal{S}}(A^a) = \mathcal{G}_{\mathcal{S}}(A^b)$. Otherwise, M is *external.* We write $\mathcal{E}(\mathcal{S})$ for the subsequence of $\mathcal{S}$ consisting of all the external moves of $\mathcal{S}$.

Definition 10. Let P be a minimum $(r_1, \ldots, r_s)$-partition of a graph G with $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$. Let $\mathcal{S} = ((P^1, v_1, i_1, P^2), \ldots, (P^q, v_q, i_q, P^{q+1}))$ be a move sequence starting at P. Let R be a full club of $\mathcal{S}$. We say that the clubgroup $\mathcal{G}_{\mathcal{S}}(R)$ is *activated at least k times* if there is a subsequence $((P^{a_1}, v_{a_1}, i_{a_1}, P^{a_1+1}), \ldots, (P^{a_k}, v_{a_k}, i_{a_k}, P^{a_k+1})$ of $\mathcal{E}(\mathcal{S})$ where the active club in P^{a_i+1} is in $\mathcal{G}_{\mathcal{S}}(R)$ for $i \in [k]$.

Definition 11. Let P be a minimum $(r_1, \ldots, r_s)$-partition of a graph G with $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$. Let $\mathcal{S} = ((P^1, v_1, i_1, P^2), \ldots, (P^q, v_q, i_q, P^{q+1}))$ be a move sequence starting at P. Let R be a full club of $\mathcal{S}$. The *external spread* of R is

$$\operatorname{esp}_{\mathcal{S}}(R) = \left\{ i \mid (Q, v, i, Q') \in \mathcal{E}(\mathcal{S}) \text{ with } \mathcal{C}(\mathcal{A}(Q)) \in \mathcal{G}_{\mathcal{S}}(R) \right\}.$$

The *external spread* of $\mathcal{S}$ is $\operatorname{esp}(\mathcal{S}) = \max_R |\operatorname{esp}(R)|$ where the max is over all full clubs R of $\mathcal{S}$.

In an $(r_1, \ldots, r_s)$-partition of a graph G a clubgroup containing $s-1$ clubs is called a *big* clubgroup. A clubgroup with fewer than $s-1$ clubs is *small.* Our next big lemma will be an analogue of Lemma 9. Intuitively, it says that clubgroups can be thought of much like clubs: in a move sequence with external spread at most 2 (and each vertex moved at most once), each clubgroup is activated at most 3 times. The proof is similar to that of Lemma 9. Not suprisingly, we must first prove an analogue of the helper lemma that played a key role in that proof. This is Lemma 16 which follows quickly from Lemma 5.

Lemma 16. *Let P be a minimum $(r_1, \ldots, r_s)$-partition of a graph G with $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$. Let*

$$\mathcal{S} = ((P^1, v_1, i_1, P^2), \ldots, (P^q, v_q, i_q, P^{q+1}))$$

be a move sequence starting at P. Let R and S be full clubs of $\mathcal{S}$ and $t \in [q+1]$. If $R_t = \mathcal{A}(P^t)$, then

(a) if $u \in V(R_t)$ and u has at least 2 neighbors in S_t, then u is joined to S_t.

(b) if $u \in V(R_t)$ and $v \in V(S_t)$ and u has at least 2 neighbors in S_t and v has at least 2 neighbors in $V(\mathcal{G}(R_t)) \setminus \{u\}$, then v is joined to $V(\mathcal{G}(R_t))$.

Proof. (a) is the same as (a) in Lemma 6; we only restate it here for convenience.

(b): By symmetry, we may assume that $V(\mathcal{G}(R_t))$ intersects each of $P_1^t, \ldots, P_{k-1}^t$ and none of $P_k^t, \ldots, P_s^t$. Moreover, we assume that $V(S_t) \subseteq P_k^t$. Let $A_i = G\left[P_i^t\right]$ for $i \in [k]$. Let $H = G\left[V(\{A_1, \ldots, A_k\})\right]$ and let T_1 be the component of A_1 containing u. Plainly, $\chi(H) = \sum_{i \in [k]} \chi(A_i)$. By Lemma 4, $\chi(A_1 \setminus \{u\}) < \chi(A_1)$ and $d_{A_k}(u) \leq \chi(A_k) + 1$. By Lemma 5 (a), vertex u has at least $\chi(A_k)$ neighbors in some component T_k of A_k. Since $d_{A_k}(u) \leq \chi(A_k) + 1$ and u has at least two neighbors in S_t, we must have $T_k = S_t$.

If $d_{A^*}(v) > 1 + \sum_{i \in [k-1]} \chi(A_i)$, then there exists some part P_q^t with $d_{P_q^t}(v) < r_q$. By moving v to P_q^t and u to P_k^t, we get a $(\chi(G) - 1)$-coloring of G, a contradiction. So $d_{A^*}(v) \leq 1 + \sum_{i \in [k-1]} \chi(A_i) \leq |T^*|$. Similarly, $d_{A^*}(w) \leq |T^*|$ for all $w \in T^*$. To finish the proof of (b), we now apply Lemma 5 (c), with $T^* = V(\mathcal{G}(R_t))$. $\square$

Lemma 17. *Let P be a minimum $(r_1, \ldots, r_s)$-partition of a graph G with $\chi(G) = \Delta(G) = 1 + \sum_{i \in [s]} r_i$. If*

$$\mathcal{S} = ((P^1, v_1, i_1, P^2), \ldots, (P^q, v_q, i_q, P^{q+1}))$$

is a move sequence with $\mathrm{esp}(\mathcal{S}) \leq 2$ *and $v_i \neq v_j$ for different $i, j \in [q+1]$, then:*

(1) every clubgroup of $\mathcal{S}$ is activated at most 3 *times; and*

(2) every big clubgroup of $\mathcal{S}$ is activated at most 2 *times.*

Proof. Suppose the lemma is false and choose a move sequence

$$\mathcal{S} = ((P^1, v_1, i_1, P^2), \ldots, (P^q, v_q, i_q, P^{q+1}))$$

for which it fails minimizing q. By minimality of q (and since $\mathrm{esp}(\mathcal{S}) \leq 2$), we have a length three subsequence $((P^1, v_1, i_1, P^2), (P^b, v_b, i_b, P^{b+1}), (P^q, v_q, i_q, P^{q+1}))$ of $\mathcal{S}$ such that

(i) $\mathcal{G}(\mathcal{A}(P^1)) = \mathcal{G}(\mathcal{A}(P^b)) = \mathcal{G}(\mathcal{A}(P^{q+1}))$ and $\mathcal{C}(\mathcal{A}(P^2)) = \mathcal{C}(\mathcal{A}(P^{b+1}))$; and

(ii) there is at most one (P^d, v_d, i_d, P^{d+1}) in $\mathcal{S}$ with $1 < d < b$ such that $\mathcal{G}(\mathcal{A}(P^d)) = \mathcal{G}(\mathcal{A}(P^1))$; and

(iii) $\mathcal{C}(\mathcal{A}(P^2))$ is active at most 3 times.

Let $X = \mathcal{G}(\mathcal{A}(P^1))$ and $Y = \mathcal{C}(\mathcal{A}(P^2))$. We will show that X is joined to Y; this gives a contradiction, since we are assuming Y is not in the clubgroup of X. To simplify notation, let $c = q + 1$. If there does not exist (P^d, v_d, i_d, P^{d+1}) in $\mathcal{S}$ with $1 < d < b$ such that $\mathcal{C}(\mathcal{A}(P^d)) = \mathcal{C}(\mathcal{A}(P^1))$, then let $d = b$. The proof of (1) is nearly identical to the proof of Lemma 9. The only difference is that each instance of Lemma 6 in that proof is now replaced by Lemma 16; so we omit the proof.

Now for the proof of (2). If a clubgroup is big, then each of its external moves goes to the same part X_i of the partition. Thus, if a big clubgroup becomes active 3 times, then we again have the move subsequence $((P^1, v_1, i_1, P^2), (P^b, v_b, i_b, P^{b+1}), (P^q, v_q, i_q, P^{q+1}))$, with properties (i), (ii), and (iii) above. Hence, the proof of (1) is also valid in this context, and yields a proof of (2). $\square$

Now we can prove our second main theorem (we restate it for convenience), which strengthens Theorem 18 for $\Delta \geq 10$.

Theorem 18 (Kostochka, Rabern, and Stiebitz [21])**.** *If G is a critical graph with $\chi(G) \geq \Delta(G)$ and $\omega(G) < \Delta(G)$, then $\omega(\mathcal{H}(G)) \geq \left\lfloor \frac{\Delta(G)-1}{2} \right\rfloor$.*

Theorem 2. *If G is a critical graph with $\chi(G) \geq \Delta(G)$ and $\omega(G) < \Delta(G)$, then $\omega(\mathcal{H}(G)) \geq \Delta(G) - 4$ if $\Delta(G) \equiv 1 \pmod 3)$ and $\omega(\mathcal{H}(G)) \geq \Delta(G) - 5$ otherwise.*

Proof. Suppose the theorem is false and let G be a critical graph with $\chi(G) \geq \Delta(G)$, $\omega(G) < \Delta(G)$ and $\omega(\mathcal{H}(G)) < \Delta(G) - 4$ if $\Delta(G) \equiv 1 \pmod 3)$ and $\omega(\mathcal{H}(G)) < \Delta(G) - 5$ otherwise. By Brooks' Theorem, we have $\chi(G) = \Delta(G)$. By Theorem 18, $\Delta(G) \geq 10$.

Let $s = \left\lfloor \frac{\Delta(G)-1}{3} \right\rfloor$ and $r_1, \ldots, r_s \in \{3, 4\}$ such that $\Delta(G) = 1 + \sum_{i \in [s]} r_i$. Then G has an $(r_1, \ldots, r_s)$-partition, so we can let P be a minimum $(r_1, \ldots, r_s)$-partition of G. Let $\mathcal{S} = ((P^1, v_1, i_1, P^2), \ldots, (P^q, v_q, i_q, P^{q+1}))$ be a move sequence starting at P with $\text{esp}(\mathcal{S}) \leq 2$ having the maximum number of external moves such that $v_i \neq v_j$ for different $i, j \in [q + 1]$. Let $A = \mathcal{A}(P^{q+1})$.

Suppose $\mathcal{G}(\mathcal{C}(A))$ is small. By Lemma 17, $\mathcal{G}(\mathcal{C}(A))$ is activated at most 3 times in $\mathcal{S}$. Since $r_{i_q} \geq 3$, there is $x \in V(A)$ such that $x \notin \{v_t \mid t \in [q]\}$, i.e., since A has at least 4 vertices, some $x \in V(A)$ has not yet moved. Since $\mathcal{G}(\mathcal{C}(A))$ is small, there is an external move $(P^{q+1}, x, i_{q+1}, P^{q+2})$. If $i_{q+1} \in \text{esp}(\mathcal{C}(A))$, then by maximality of $\mathcal{S}$, we see that $\mathcal{C}(A)$ is joined to a club outside its clubgroup, giving a contradiction by Lemma 8. Since this is true for any such external move, we must have $|\text{esp}(\mathcal{C}(A))| \leq 1$. But then appending the move $(P^{q+1}, x, i_{q+1}, P^{q+2})$ to $\mathcal{S}$ violates the maximality of $\mathcal{S}$, a contradiction.

Hence $\mathcal{G}(\mathcal{C}(A))$ is big. By Lemma 17, $\mathcal{G}(\mathcal{C}(A))$ is activated at most 2 times in $\mathcal{S}$. Consider $K = \bigcup_{Z \in \mathcal{G}(\mathcal{C}(A))} V(Z_{q+1})$. Since $\mathcal{G}(\mathcal{C}(A))$ is big, K is a clique that has vertices in all but one part of P^{q+1}. By renumbering if necessary, we may assume that K has vertices in each of $P_1^{q+1}, \ldots, P_{s-1}^{q+1}$. Then $|K| = 1 + \sum_{i \in [s-1]} r_i$. Hence $|K| = \Delta(G) - 3$ if $\Delta(G) \equiv 1 \pmod 3$ and $|K| \geq \Delta(G) - 4$ otherwise. In either case, K has at least two low vertices by our conditions on $\omega(\mathcal{H}(G))$.

If K contains a low vertex x that has not moved, i.e., $x \in K \setminus \{v_t \mid t \in [q]\}$, then we have an external move $(P^{q+1}, x, i_{q+1}, P^{q+2})$ and hence $\mathcal{C}(A)$ is joined to a club outside its clubgroup, giving a contradiction by Lemma 8. So, since $\mathcal{G}(\mathcal{C}(A))$ is activated at most 2 times in $\mathcal{S}$, K has exactly two low vertices v and w. Moreover, $\mathcal{S}$ contains external moves $(P^{a_1}, v, i_{a_1}, P^{a_1+1})$ and $(P^{a_2}, w, i_{a_2}, P^{a_2+1})$ and in both P^{a_1+1} and P^{a_2+1} the clubgroup $\mathcal{G}(\mathcal{C}(A))$ contains the active club (possibly different each time). By symmetry, assume $a_1 < a_2$ and so $a_2 = q$.

Let B be the active component in P^q. Since $w \in V(B)$ and w is adjacent to at least $\Delta(G) - 5$ vertices in K, we see that $\mathcal{C}(B)$'s clubgroup is $\{\mathcal{C}(B)\}$ (otherwise w would be adjacent to more than 5 vertices coming from $\mathcal{C}(B)$'s clubgroup, which is too many). Suppose that $V(B)$ contains a high vertex that is unmoved, i.e., $z \in V(B) \setminus \{v_t \mid t \in [q-1]\}$. Since $\Delta(G) \geq 10$, we have $s \geq 3$. So there is an external move $M = (P^q, z, i, Q)$ where $i \in [s-1]$. Consider the move sequence formed from $\mathcal{S}$ by removing the last move and appending M. By our considerations in the previous paragraph, this move sequence can be extended (the active club now contains an unmoved low vertex, since the last vertex moved is high), contradicting the maximality condition on $\mathcal{S}$. So, every $z \in V(B) \setminus \{v_t \mid t \in [q-1]\}$ is low.

Since w is low, for every move (Q, z, i, Q') in $\mathcal{S}$ where $\mathcal{C}(B)$ is active in Q, we must have $z \in K$; otherwise w would have at least Δ neighbors. In particular, there are at most two such moves since $\mathcal{G}(\mathcal{C}(A))$ is activated at most twice. So B contains an unmoved vertex, i.e., $|V(B) \setminus \{v_t \mid t \in [q]\}| \geq 1$.

Let $(P^{a_3}, u, i_{a_3}, P^{a_3+1})$ be the first external move in $\mathcal{S}$ after $(P^{a_1}, v, i_{a_1}, P^{a_1+1})$. Let A' be the active component in P^{a_3} and consider $K' = \bigcup_{Z \in \mathcal{G}(\mathcal{C}(A'))} V(Z_{a_3})$. Since $|K'| = |K|$, as we saw before for K, also K' has at least two low vertices v, w'. If u is high, then K would contain low vertices v, w, w', a contradiction. So u is low; in fact, $u = w'$.

We show that $\mathcal{C}(\mathcal{A}(P^{a_3+1})) = \mathcal{C}(B)$. Since v is low, we have the move $M' = (P^{a_3}, v, s, Q')$. Let $B' = \mathcal{A}(Q') \setminus \{v\}$. Since v is adjacent to w (and v is low), we must have $w \in V(B')$. So $\mathcal{C}(B) = \mathcal{C}(B')$. Since $\mathcal{C}(B')$ is active at most twice, v has at least $|B'| - 2 > 0$ neighbors in $\mathcal{C}(B')_{q+1}$. Since v is low, we have the move $M = (P^{q+1}, v, s, Q)$. Now Lemma 4, part (2) shows that $\{v\} \cup V(\mathcal{C}(B')_{q+1})$ induces a K_{r_s+1}. But $u \in P_s^{q+1}$ and v is adjacent to u, so $u \in V(\mathcal{C}(B')_{q+1})$. Therefore, $\mathcal{C}(\mathcal{A}(P^{a_3+1})) = \mathcal{C}(B') = \mathcal{C}(B)$.

Now we have the K_3 on $\{u, v, w\}$ joined to a set of vertices T with $|T| = \Delta(G) - 3$. Namely, $T = (V(K) \setminus \{v, w\}) \cup (V(B) \setminus \{u\})$. Moreover, since $|V(B) \setminus \{v_t \mid t \in [q]\}| \geq 1$, there is a low vertex in $V(B \setminus \{v_q, u\})$ and $V(B \setminus \{v_q, u\}) \subseteq T$. So, by Lemma 15, $\{u, v, w\} \cup T$ induces a $K_{\Delta(G)}$, a contradiction. $\square$

We conjecture that the previous theorem actually holds with $\omega(\mathcal{H}(G)) \geq \Delta - 5$ replaced by $\omega(\mathcal{H}(G)) \geq \Delta - 4$. In [28], the second author proved this result for $\Delta = 6$; later in [21] it was proved for $\Delta = 7$. The condition $\omega(\mathcal{H}(G)) \geq \Delta - 4$ would be tight since the graph O_5 in Figure 3 is a counterexample to $\omega(\mathcal{H}(G)) \geq \Delta - 3$ when $\Delta = 5$. In fact, it was shown in [21] that O_5 is the only counterexample to $\omega(\mathcal{H}(G)) \geq \Delta - 3$ when $\Delta = 5$.

Conjecture 3. Let G be a graph. If $\chi \geq \Delta$, then $\omega \geq \Delta$ or $\omega(\mathcal{H}(G)) \geq \Delta - 4$.

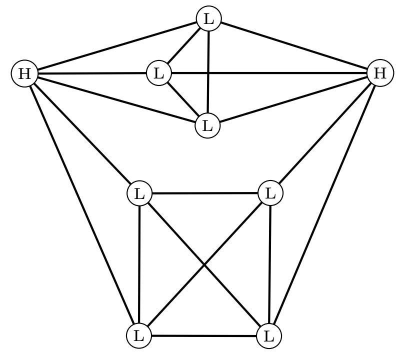

Figure 3: The graph O_5 is a Δ-critical graph with $\Delta = 5$ and $\omega(\mathcal{H}(G)) = 1$.

5 Algorithms

All of our coloring proofs do translate into algorithms to construct the colorings. However these algorithms cannot obviously be made to run in polynomial time. Attempts to do so encounter two main obstacles. The first comes in our proof of Theorem 3, when we consider a critical subgraph H of our given graph G. We do not know an efficient algorithm to find such a critical subgraph; however, we will see how to overcome this difficulty. Our second obstacle comes from King's Lopsided Transversal Lemma. While his proof is constructive, the algorithm it implies may require exponential time. We are not aware of any workaround to efficiently find our hitting set; however, when Δ is sufficiently large, we can use an idea of Alon instead. We implement a modified version of the algorithm from Theorem 3.

Theorem 19. *There is a $\mathcal{O}(V^2E^2)$ time graph algorithm that finds either a $(\Delta - 1)$-coloring or a clique on $\Delta - 4$ vertices ($\Delta - 3$ vertices if $\Delta \equiv 1 \pmod 3$).*

Proof. Let G be an n-vertex graph with $\Delta \geq 10$, and let I be a maximal independent set in G. Let $G_0 = G - I$, and note that $\Delta(G_0) \leq \Delta(G) - 1$. Lovász's proof of Brooks' theorem [23] can be implemented in time $\mathcal{O}(V + E)$ (see [3]). Applying this to G_0 we either get a $\Delta(G)$ clique or a $(\Delta(G) - 1)$-coloring of G_0. In the former case, we are done, so suppose we have a $(\Delta(G) - 1)$-coloring ϕ of G_0.

Let v be an arbitrary vertex in I and put $G_1 = G[V(G_0) \cup \{v\}]$. We give an algorithm that either finds a $(\Delta(G) - 1)$-coloring of G_1 or a clique on $\Delta(G) - 4$ vertices ($\Delta(G) - 3$ vertices if $\Delta(G) \equiv 1 \pmod 3$). Iterating this gives the desired algorithm.

Note that G_1 has an $(r_1, \ldots, r_s)$-partition P, where $s = \left\lfloor \frac{\Delta(G)-1}{3} \right\rfloor$ and $r_1, \ldots, r_s \in \{3, 4\}$; choose an arbitrary such partition which respects the color classes of ϕ. Now we will construct a move sequence as in the proof of Theorem 3, treating the resulting partitions as if they were minimum partitions. For each partition arising from the move sequence, we check whether any property in Lemma 4 is violated; if some property is violated for a partition P, then we can modify P to form a new partition P' such that P' has fewer edge within parts, i.e., $\sigma(P') < \sigma(P)$. When this happens, we begin our move sequence anew, starting from P'. Eventually, we will reach a partition and a move sequence that does not allow us to reduce the number of edges within parts.

Such a move sequence will terminate with either (1) a clique on $\Delta(G)-4$ vertices ($\Delta(G)-3$ vertices if $\Delta(G) \equiv 1 \pmod 3$) or (2) a $(\Delta(G)-1)$-coloring of G_1. In the case of (1), our algorithm halts. In the case of (2), we add a new vertex v' from $I \setminus \{v\}$ and continue.

So, we need only analyze the running time. Each move sequence has length at most n, since each vertex moves at most once. After adding a vertex, we can reduce the number of edges within parts at most $|E(G)|$ times. Hence, after we add a new vertex from I to our partition, we need at most $n\,|E(G)|$ moves until we find either a big clique or a $(\Delta(G)-1)$-coloring. After each move, we can verify that the resulting partition satisfies all the properties of Lemma 4 (or doesn't) and find a vertex to swap with in $\mathcal{O}(V+E)$ time. Since we need to do this at most $n|I|\,|E(G)|$ times, the running time of the algorithm is $\mathcal{O}(V^2E^2)$. □

When $\Delta \not\equiv 1 \pmod 3$, Theorem 19 only finds a $K_{\Delta-4}$; but Theorem 1 guarantees a $K_{\Delta-3}$ when $\Delta \geq 13$. To get an algorithmic version of this result, we need to efficiently find a hitting set when $\chi = \Delta$ and $\omega = \Delta - 4$. We will show how to do this when Δ is sufficiently large. The proof we present here works for $\Delta \geq 37$. We also sketch how to refine this idea to work for $\Delta \geq 33$. Further, using a result of Kolipaka, Szegedy and Xu [19], we show how to get down to $\Delta \geq 26$. The general idea is to find a set of disjoint cliques $\mathcal{A} = \{A_1, A_2, \ldots\}$ such that $|A_i|$ is large for all i and each maximum clique contains some A_i. Following an idea of Alon, we choose one vertex uniformly at random from each A_i and use the Lovasz Local Lemma to prove that with positive probability the chosen vertices form an independent set. Our proof uses one classical lemma each from Hajnal [13] and Kostochka [20].

Lemma 20 (Hajnal [13])**.** *If $\mathcal{S}$ is a collection of maximum cliques in a graph G, then*

$$\left|\bigcup \mathcal{S}\right| + \left|\bigcap \mathcal{S}\right| \geq 2\omega.$$

Proof. We use induction on $|\mathcal{S}|$; the base case $|\mathcal{S}| = 1$ is trivial. Let $S_1 \in \mathcal{S}$ and $\mathcal{S}' = \mathcal{S} - S_1$. Consider the set $(\cap\mathcal{S}' \setminus S_1) \cup (S_1 \cap (\cup\mathcal{S}'))$, which induces a clique. Since S_1 is a maximum clique, $|S_1| \geq |(\cap\mathcal{S}' \setminus S_1) \cup (S_1 \cap (\cup\mathcal{S}'))|$, which yields $|S_1 \setminus (\cup\mathcal{S}')| \geq |(\cap\mathcal{S}') \setminus S_1|$. By hypothesis, $|\cup\mathcal{S}'| + |\cap\mathcal{S}'| \geq 2\omega$. Adding this to the previous inequality gives the desired result. □

Now we need the following definition. Given a collection $\mathcal{S}$ of sets, the *intersection graph* $X_{\mathcal{S}}$ has one vertex for each set of $\mathcal{S}$ and two vertices are adjacent if their sets intersect.

Lemma 21 (Kostochka [20])**.** *Let G be a graph with $\omega(G) > \frac{2}{3}(\Delta(G)+1)$. If $\mathcal{S}$ is a collection of maximum cliques in G and the intersection graph $X_{\mathcal{S}}$ is connected, then $|\bigcap \mathcal{S}| \geq 2\omega(G) - (\Delta(G)+1)$.*

Proof. We use induction on $|\mathcal{S}|$; the base case $|S| = 1$ is trivial. The key is to show that $|\bigcap \mathcal{S}| > 0$, for then $|\bigcup \mathcal{S}| \leq \Delta(G)+1$, so the lemma follows directly from Lemma 20. Let $S_1 \in \mathcal{S}$ be a noncutvertex of $X_{\mathcal{S}}$, and choose $S_2 \in \mathcal{S}$ that intersects S_1. Lemma 20 for the set $\{S_1, S_2\}$ implies $|S_1 \setminus S_2| = |S_1| - |S_1 \cap S_2| \leq \omega(G) - (2\omega(G) - (\Delta(G)+1)) =$

$\Delta(G)+1-\omega(G)$. Let $\mathcal{S}'=\mathcal{S}-S_1$. Now $X_{\mathcal{S}'}$ is connected, so by hypothesis, the lemma holds for $\mathcal{S}'$. Choose $v\in\bigcap\mathcal{S}'$. Now $|\bigcup\mathcal{S}'|\le d_G(v)+1\le\Delta(G)+1$. Thus, $|\bigcup\mathcal{S}|\le|\bigcup\mathcal{S}'|+|S_1\setminus S_2|\le(\Delta(G)+1)+(\Delta(G)+1-\omega(G))<2\omega(G)$. By Lemma 20, $|\bigcap S|>0$, so the lemma follows. $\square$

In [20], Kostochka used Lemma 20 and Lemma 21 to prove that a hitting set always exists when $\omega\ge\Delta+\frac{3}{2}-\sqrt{\Delta}$. Using an independent transversal result of Haxell [14], this was improved to $\omega\ge\frac{3}{4}(\Delta+1)$ in [27] and finally to the best possible $\omega>\frac{2}{3}(\Delta+1)$ in [17]. Using an independent transversal result of Alon [1] (see also [2], p. 70), we get $\omega\ge\frac{2e+1}{2e+2}(\Delta+1)$. Since Alon's proof is based on the Local Lemma, we can use the efficient algorithms developed by Moser and Tardos [24].

Lemma 22. *If G is a graph with $\omega\ge\frac{2e+1}{2e+2}(\Delta+1)$, then G contains an independent set I such that I intersects every maximum clique in G.*

Proof. Let $\mathcal{S}$ be the set of maximum cliques in G and let $\mathcal{S}_i$ be the set of vertices in one component C_i of $X_{\mathcal{S}}$. For each i, Lemma 21 gives $|\bigcap\mathcal{S}_i|\ge 2\omega-(\Delta+1)\ge\frac{e}{e+1}(\Delta+1)$.

Let $k=\lceil\frac{e}{e+1}(\Delta+1)\rceil$. For each component C_i, let A_i be a set of k vertices that lie in every clique of C_i. Use the Local Lemma (see [2], p. 64–65) to choose the desired independent set. From each A_i, choose a vertex uniformly at random. For each edge uv with $u\in A_i$ and $v\in A_j$ (and $i\ne j$), let E_{uv} be the bad event that both u and v are chosen for I; event E_{uv} occurs with probability $p=1/(|A_i|\,|A_j|)=k^{-2}$. Each E_{uv} is independent of all other bad events except for those corresponding to edges with an endpoint in A_i or A_j. Since each u has at least $\omega-1$ neighbors in $\mathcal{S}_i$ and v has at least $\omega-1$ neighbors in $\mathcal{S}_j$, the degree d of E_{uv} in the dependency graph is at most $(\Delta+1-\omega)(|A_i|+|A_j|)-1\le\frac{2k}{2e+2}(\Delta+1)-1=\frac{k}{e+1}(\Delta+1)-1$. This gives $ep(d+1)\le 1$, so the desired independent set I exists. $\square$

Corollary 23. *If G is a graph with $\Delta\ge 37$ and $\omega=\Delta-4$, then G contains an independent set I such that I intersects every maximum clique in G. Furthermore, I can be found in polynomial time.*

Proof. If $\Delta\ge 37$, then we have $\omega=\Delta-4\ge\frac{2e+1}{2e+2}(\Delta+1)$, so we can apply Lemma 22. All that remains is to show that we can implement its proof in polynomial time. We can find the set of all maximum cliques by considering each $(\Delta-4)$-element subset of the closed neighborhood of each vertex. We use a union-find algorithm to find the components of the intersection graph of this set of maximum cliques. Now consider a set $\mathcal{S}$ of maximum cliques such that the intersection graph $X_{\mathcal{S}}$ is connected. We can slightly modify the union-find algorithm so that it also returns $\cap\mathcal{S}$. To now find our hitting set, we apply the algorithm for the Local Lemma from Moser and Tardos [24]. $\square$

With a more complicated algorithm we can do better. Specifically, instead of using Lemma 20 and Lemma 21, we use Lemma 11 as in the proof of Lemma 13. Basically, we just need to do a preprocessing step where we find and remove all d_1-choosable induced subgraphs on at most 9 vertices (we can color them after coloring the rest). Once we have a graph with none of these d_1-choosable induced subgraphs, we know, as in the proof of Lemma 13, that the components of $X_{\mathcal{S}}$ have at most two vertices. So, we can replace our estimate $|\bigcap \mathcal{S}_i| \geq 2\omega - (\Delta + 1)$ with $|\bigcap \mathcal{S}_i| \geq \omega - 1$. This improves the needed condition in Lemma 22 to $\omega \geq \frac{2e}{2e+1}\Delta + 1$ and thus allows Corollary 23 to work for $\Delta \geq 33$.

Using a recent result of Kolipaka, Szegedy and Xu [19] we can do a bit better. The idea is that the local lemma can be strengthened when the dependency graph has nice structure. In our case, the dependency graph is the line graph of a multigraph (the multigraph formed by contracting all the A_i in $G\left[\bigcup_i A_i\right]$). Because of this structure, we may apply the Clique Lovász Local Lemma from [19] to prove Lemma 22 with $\omega \geq \frac{4}{5}\Delta + 1$. Since there is an efficient algorithm for the Clique Lovász Local Lemma as well, we get Corollary 23 for $\Delta \geq 26$. So, we can prove the following conjecture for $\Delta \geq 25$.

Conjecture 4. For $\Delta \geq 13$, there is a polynomial time graph algorithm that finds either a $(\Delta - 1)$-coloring or a clique on $\Delta - 3$ vertices.

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