Improved lower bounds on the number of edges in list critical and online list critical graphs

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Abstract

We prove that every k-list-critical graph ($k \geq 7$) on $n \geq k+2$ vertices has at least $\frac{1}{2}\left(k-1+\frac{k-3}{(k-c)(k-1)+k-3}\right)n$ edges where $c=(k-3)\left(\frac{1}{2}-\frac{1}{(k-1)(k-2)}\right)$. This improves the bound established by Kostochka and Stiebitz [13]. The same bound holds for online k-list-critical graphs, improving the bound established by Riasat and Schauz [16]. Both bounds follow from a more general result stating that either a graph has many edges or it has an Alon-Tarsi orientable induced subgraph satisfying a certain degree condition.

1 Introduction

A *k-coloring* of a graph G is a function $\pi\colon V(G) \to [k]$ such that $\pi(x) \neq \pi(y)$ for each $xy \in E(G)$. The least k for which G has a k-coloring is the *chromatic number* $\chi(G)$ of G. We say that G is *k-chromatic* when $\chi(G) = k$. A graph G is *k-critical* if G is not $(k-1)$-colorable, but every proper subgraph of G is $(k-1)$-colorable. A k-critical graph G is k-chromatic since for any vertex v, a $(k-1)$-coloring of $G-v$ extends to a k-coloring of G by giving v a new color. If G is k-chromatic, then any minimal k-chromatic subgraph of G is k-critical. In this way, many questions about k-chromatic graphs can be reduced to questions about k-critical graphs which have more structure. The study of critical graphs was initiated by Dirac [4] in 1951. It is easy to see that a k-critical graph G must have minimum degree at least $k-1$ and hence $2\left\|G\right\| \geq (k-1)\left|G\right|$. The problem of determining the minimum number of edges in a k-critical graph has a long history. First, in 1957, Dirac [5] generalized Brooks' theorem [3] by showing that any k-critical graph G with $k \geq 4$ and $|G| \geq k+2$ must satisfy

$$2\left\|G\right\| \geq (k-1)\left|G\right| + k - 3.$$

In 1963, this bound was improved for large $|G|$ by Gallai [7]. Put

$$g_k(n,c) := \left(k-1+\frac{k-3}{(k-c)(k-1)+k-3}\right)n.$$

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Gallai showed that every k-critical graph G with $k \geq 4$ and $|G| \geq k+2$ satisfies $2\|G\| \geq g_k(|G|, 0)$. In 1997, Krivelevich [14] improved Gallai's bound by replacing $g_k(|G|, 0)$ with $g_k(|G|, 2)$. Then, in 2003, Kostochka and Stiebitz [13] improved this by showing that a k-critical graph with $k \geq 6$ and $|G| \geq k+2$ must satisfy $2\|G\| \geq g_k(|G|, (k-5)\alpha_k)$ where

$$\alpha_k := \frac{1}{2} - \frac{1}{(k-1)(k-2)}.$$

Table 1 gives the values of these bounds for small k. In 2012, Kostochka and Yancey [11] achieved a drastic improvement by showing that every k-critical graph G with $k \geq 4$ must satisfy

$$\|G\| \geq \left\lceil \frac{(k+1)(k-2)|G| - k(k-3)}{2(k-1)} \right\rceil.$$

Moreover, they show that their bound is tight for $k = 4$ and $n \geq 6$ as well as for infinitely many values of $|G|$ for any $k \geq 5$. This bound has many interesting coloring applications such as a very short proof of Grötsch's theorem on the 3-colorability of triangle-free planar graphs [10] and short proofs of the results on coloring with respect to Ore degree in [9, 15, 12].

Given the applications to coloring theory, it makes sense to investigate the same problem for more general types of coloring. In this article, we obtain improved lower bounds on the number of edges for both the list coloring and online list coloring problems. To state our results we need some definitions.

List coloring was introduced by Vizing [19] and independently Erdős, Rubin and Taylor [6]. Let G be a graph. A list assignment on G is a function L from $V(G)$ to the subsets of $\mathbb{N}$. A graph G is *L-colorable* if there is $\pi\colon V(G) \to \mathbb{N}$ such that $\pi(v) \in L(v)$ for each $v \in V(G)$ and $\pi(x) \neq \pi(y)$ for each $xy \in E(G)$. A graph G is *L-critical* if G is not L-colorable, but every proper subgraph H of G is $L\big|_{V(H)}$-colorable. For $f\colon V(G) \to \mathbb{N}$, a list assignment L is an *f-assignment* if $|L(v)| = f(v)$ for each $v \in V(G)$. If $f(v) = k$ for all $v \in V(G)$, then we also call an f-assignment a k-assignment. We say that G is *f-choosable* if G is L-colorable for every f-assignment L. We say that G is k-list-critical if G is L-critical for some k-list assignment L. The best, known-lower bound on the number of edges in a k-list-critical graph, was given by Kostochka and Stiebitz [13] in 2003. It states that for $k \geq 9$ and every graph $G \neq K_k$ if G is a k-list-critical graph, then $2\|G\| \geq g_k(|G|, \frac{1}{3}(k-4)\alpha_k)$. We improve their bound to $2\|G\| \geq g_k(|G|, (k-3)\alpha_k)$ for $k \geq 7$ (see Table 1).

Online list coloring was independently introduced by Zhu [20] and Schauz [17] (Schauz called it *paintability*). Let G be a graph and $f\colon V(G) \to \mathbb{N}$. We say that G is *online f-choosable* if $f(v) \geq 1$ for all $v \in V(G)$ and for every $S \subseteq V(G)$ there is an independent set $I \subseteq S$ such that $G - I$ is online f'-choosable where $f'(v) := f(v)$ for $v \in V(G) - S$ and $f'(v) := f(v) - 1$ for $v \in S - I$. Observe that if a graph is online f-choosable then it is f-choosable. When $f(v) := k-1$ for all $v \in V(G)$, we say that G is *online k-list-critical* if G is not online f-choosable, but every proper subgraph H of G is online $f\big|_{V(H)}$-choosable. In 2012, Riasat and Schauz [16] showed that Gallai's bound $2\|G\| \geq g_k(|G|, 0)$ holds for online k-list-critical graphs. We improve this for $k \geq 7$ by proving the same bound as we have for list coloring: $2\|G\| \geq g_k(|G|, (k-3)\alpha_k)$.

Our main theorem shows that a graph either has many edges or an induced subgraph which has a certain kind of good orientation. To describe these good orientations we need a few definitions. A subgraph H of a directed multigraph D is called *Eulerian* if $d_H^-(v) = d_H^+(v)$ for every $v \in V(H)$. We call H *even* if $\|H\|$ is even and *odd* otherwise. Let $EE(D)$ be the number of even, spanning, Eulerian subgraphs of D and $EO(D)$ the number of odd, spanning, Eulerian subgraphs of D. Note that the edgeless subgraph of D is even and hence we always have $EE(D) > 0$.

Let G be a graph and $f\colon V(G) \to \mathbb{N}$. We say that G is *f-Alon-Tarsi* (for brevity, *f-AT*) if G has an orientation D where $f(v) \geq d_D^+(v) + 1$ for all $v \in V(D)$ and $EE(D) \neq EO(D)$. One simple way to achieve $EE(D) \neq EO(D)$ is to have D be acyclic since then we have $EE(D) = 1$ and $EO(D) = 0$. In this case, ordering the vertices so that all edges point the same direction and coloring greedily shows that G is f-choosable. If we require f to be constant, we get the familiar *coloring number* $\operatorname{col}(G)$; that is, $\operatorname{col}(G)$ is the smallest k for which G has an acyclic orientation D with $k \geq d_D^+(v) + 1$ for all $v \in V(D)$. Alon and Tarsi [1] generalized from the acyclic case to arbitrary f-AT orientations.

Lemma 1.1. *If a graph G is f-AT for $f\colon V(G) \to \mathbb{N}$, then G is f-choosable.*

Schauz [18] extended this result to online f-choosability.

Lemma 1.2. *If a graph G is f-AT for $f\colon V(G) \to \mathbb{N}$, then G is online f-choosable.*

For a graph G, we define $d_0\colon V(G) \to \mathbb{N}$ by $d_0(v) := d_G(v)$. The d_0-choosable graphs were first characterized by Borodin [2] and independently by Erdős, Rubin and Taylor [6]. The connected graphs which are not d_0-choosable are precisely the Gallai trees (connected graphs in which every block is complete or an odd cycle). The generalization to a characterization of d_0-AT graphs was first given in [8] by Hladkỳ, Král and Schauz.

We prove the following general theorem saying that either a graph has many edges or has an induced f_H-AT subgraph H where f_H basically gives the number of colors we would expect the vertices to have left in their lists after $\delta(G)$-coloring $G - H$.

Definition 1. A graph G is *AT-reducible* to H if H is a nonempty induced subgraph of G which is f_H-AT where $f_H(v) := \delta(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$. If G is not AT-reducible to any nonempty induced subgraph, then it is *AT-irreducible.*

Theorem 4.4. *If G is an AT-irreducible graph with $\delta(G) \geq 4$ and $\omega(G) \leq \delta(G)$, then $2\|G\| \geq g_{\delta(G)+1}(|G|, c)$ where $c := (\delta(G) - 2)\alpha_{\delta(G)+1}$ when $\delta(G) \geq 6$ and $c := (\delta(G) - 3)\alpha_{\delta(G)+1}$ when $\delta(G) \in \{4, 5\}$.*

The *Alon-Tarsi number* of a graph $AT(G)$ is the least k such that G is f-AT where $f(v) := k$ for all $v \in V(G)$. We have $\chi(G) \leq \operatorname{ch}(G) \leq \operatorname{ch}_{OL}(G) \leq AT(G) \leq \operatorname{col}(G)$. We say that G is k-AT-critical if $\mathrm{AT}(G) \geq k$ and $AT(H) < k$ for all proper induced subgraphs H of G. From Theorem 4.4 we can conclude the following.

Corollary 5.3. *For $k \geq 5$ and $G \neq K_k$ a k-AT-critical graph, we have $2\|G\| \geq g_k(|G|, c)$ where $c := (k-3)\alpha_k$ when $k \geq 7$ and $c := (k-4)\alpha_k$ when $k \in \{5, 6\}$.*

Similarly, applying Lemma 1.1 gives the following.

	k-Critical G				k-ListCritical G	
k	Gallai [7] $d(G) \geq$	Kriv [14] $d(G) \geq$	KS [13] $d(G) \geq$	KY [11] $d(G) \geq$	KS [13] $d(G) \geq$	Here $d(G) \geq$
4	3.0769	3.1429	—	3.3333	—	—
5	4.0909	4.1429	—	4.5000	—	**4.0984**
6	5.0909	5.1304	5.0976	5.6000	—	**5.1053**
7	6.0870	6.1176	6.0990	6.6667	—	**6.1149**
8	7.0820	7.1064	7.0980	7.7143	—	**7.1128**
9	8.0769	8.0968	8.0959	8.7500	8.0838	**8.1094**
10	9.0722	9.0886	9.0932	9.7778	9.0793	**9.1055**
15	14.0541	14.0618	14.0785	14.8571	14.0610	**14.0864**
20	19.0428	19.0474	19.0666	19.8947	19.0490	**19.0719**

Table 1: History of lower bounds on the average degree $d(G)$ of k-critical and k-list-critical graphs G.

Corollary 5.1. *For $k \geq 5$ and $G \neq K_k$ a k-list-critical graph, we have $2\|G\| \geq g_k(|G|, c)$ where $c := (k-3)\alpha_k$ when $k \geq 7$ and $c := (k-4)\alpha_k$ when $k \in \{5, 6\}$.*

This improves the bound given by Kostochka and Stiebitz in [13]; for k-list-critical graphs, they have $2\|G\| \geq g_k(|G|, \frac{1}{3}(k-4)\alpha_k)$ for $k \geq 9$. Now, applying Lemma 1.2 gives the following.

Corollary 5.2. *For $k \geq 5$ and $G \neq K_k$ an online k-list-critical graph, we have $2\|G\| \geq g_k(|G|, c)$ where $c := (k-3)\alpha_k$ when $k \geq 7$ and $c := (k-4)\alpha_k$ when $k \in \{5, 6\}$.*

2 Critical graphs are AT-irreducible

Instead of proving lower bounds on the number of edges in critical graphs directly, we prove our bound for AT-irreducible graphs and show that graphs that are critical with respect to choice number, online choice number and Alon-Tarsi number are all AT-irreducible. In this section, we take on the easier task of proving that the various critical graphs are AT-irreducible.

Lemma 2.1. *If G is a k-list-critical graph, then G is AT-irreducible.*

Proof. Suppose G is AT-reducible to H. Let L be a $(k-1)$-assignment on G such that G is L-critical. Let π be a coloring of $G - H$ from L and let L' be the list assignment on H defined by $L'(v) := L(v) - \pi(N(v) \cap V(G-H))$ for $v \in V(H)$. Then $|L'(v)| \geq |L(v)| - (d_G(v) - d_H(v)) = k - 1 + d_H(v) - d_G(v)$. By Lemma 1.1, H is f_H-choosable and hence H is L'-colorable. Therefore G is L-colorable, a contradiction. □

For online list coloring, we use the following lemma from [17] allowing us to patch together online list colorability of parts into online list colorability of the whole.

Lemma 2.2. *Let G be a graph and $f\colon V(G) \to \mathbb{N}$. If H is an induced subgraph of G such that $G - H$ is online $f\big|_{V(G-H)}$-choosable and H is online f_H-choosable where $f_H(v) := f(v) + d_H(v) - d_G(v)$, then G is online f-choosable.*

Lemma 2.3. *If G is an online k-list-critical graph, then G is AT-irreducible.*

Proof. Immediate from Lemma 2.2 and Lemma 1.2. □

To prove that k-AT-critical graphs are AT-irreducible, we need a lemma that serves the same purpose as Lemma 2.2 for orientations.

Lemma 2.4. *Let G be a graph and $f\colon V(G) \to \mathbb{N}$. If H is an induced subgraph of G such that $G - H$ is $f\big|_{V(G-H)}$-AT and H is f_H-AT where $f_H(v) := f(v) + d_H(v) - d_G(v)$, then G is f-AT.*

Proof. Take an orientation of $G-H$ demonstrating that it is $f\big|_{V(G-H)}$-AT and an orientation of H demonstrating that it is f_H-AT. Now orient all the edges between H and $G-H$ into $G-H$. Call the resulting oriented graph D. Then D satisfies the out degree requirements of being f-AT since the out degree of the vertices in $G-H$ haven't changed and the out degree of each $v \in V(H)$ has increased by $d_G(v) - d_H(v)$. Since no directed cycle in D has vertices in both H and $D-H$, the Eulerian subgraphs of D are just all pairings of Eulerian subgraphs of H and $D-H$. Therefore $EE(D) - EO(D) = EE(H)EE(D-H) + EO(H)EO(D-H) - (EE(H)EO(D-H) + EO(H)EE(D-H)) = (EE(H) - EO(H))(EE(D-H) - EO(D-H)) \neq 0$. Hence G is f-AT. □

Lemma 2.5. *If G is a k-AT-critical graph, then G is AT-irreducible.*

Proof. Immediate from Lemma 2.4. □

3 Extending Alon-Tarsi orientations

In [13] Kostochka and Stiebitz gave a method for extending list colorings into Gallai trees. We generalize these ideas in terms of extensions of orientations. Let $\mathcal{T}_k$ be the Gallai trees with maximum degree at most $k-1$, excepting K_k. For a graph G, let $W^k(G)$ be the set of vertices of G that are contained in some K_{k-1} in G.

Lemma 3.1. *Let G be a multigraph without loops and $f\colon V(G) \to \mathbb{N}$. If there are $F \subseteq G$ and $Y \subseteq V(G)$ such that:*

1. *any multiple edges in G are contained in $G[Y]$; and*
2. *$f(v) \geq d_G(v)$ for all $v \in V(G) - Y$; and*
3. *$f(v) \geq d_{G[Y]}(v) + d_F(v) + 1$ for all $v \in Y$; and*
4. *For each component T of $G-Y$ there are different $x_1, x_2 \in V(T)$ where $N_T[x_1] = N_T[x_2]$ and $T - \{x_1, x_2\}$ is connected such that either:*
    - *(a) there are $x_1y_1, x_2y_2 \in E(F)$ where $y_1 \neq y_2$ and $N(x_i) \cap Y = \{y_i\}$ for $i \in [2]$; or*

(b) $|N(x_2) \cap Y| = 0$ *and there is* $x_1y_1 \in E(F)$ *where* $N(x_1) \cap Y = \{y_1\}$,

then G *is* f*-AT.*

Proof. Suppose not and pick a counterexample (G, f, F, Y) minimizing $|G - Y|$. If $|G - Y| = 0$, then $Y = V(G)$ and thus $f(v) \geq d_G(v) + 1$ for all $v \in V(G)$ by (3). Pick an acyclic orientation D of G. Then $EE(D) = 1$, $EO(D) = 0$ and $d_D^+(v) \leq d_G(v) \leq f(v) - 1$ for all $v \in V(D)$. Hence G is f-AT. So, we must have $|G - Y| > 0$.

Pick a component T of $G - Y$ and pick $x_1, x_2 \in V(T)$ as guaranteed by (4). First, suppose (4a) holds. Put $G' := (G - T) + y_1y_2$, $F' := F - T$, $Y' := Y$ and let f' be f restricted to $V(G')$. Then G' has an orientation D' where $f'(v) \geq d_{D'}^+(v) + 1$ for all $v \in V(D')$ and $EE(D') \neq EO(D')$, for otherwise (G', f', F', Y') would contradict minimality. By symmetry we may assume that the new edge y_1y_2 is directed toward y_2. Now we use the orientation of D' to construct the desired orientation of D. First, we use the orientation on $D' - y_1y_2$ on $G - T$. Now, order the vertices of T as $x_1, x_2, z_1, z_2, \ldots$ so that every vertex has at least one neighbor to the right. Orient the edges of T left-to-right in this ordering. Finally, we use y_1x_1 and x_2y_2 and orient all other edges between T and $G - T$ away from T. Plainly, $f(v) \geq d_D^+(v) + 1$ for all $v \in V(D)$. Since y_1x_1 is the only edge of D going into T, any Eulerian subgraph of D that contains a vertex of T must contain y_1x_1. So, any Eulerian subgraph of D either contains (i) neither y_1x_1 nor x_2y_2, (ii) both y_1x_1 and x_2y_2, or (iii) y_1x_1 but not x_2y_2. We first handle (i) and (ii) together. Consider the function h that maps an Eulerian subgraph Q of D' to an Eulerian subgraph $h(Q)$ of D as follows. If Q does not contain y_1y_2, let $h(Q) = \iota(Q)$ where $\iota(Q)$ is the natural embedding of $D' - y_1y_2$ in D. Otherwise, let $h(Q) = \iota(Q - y_1y_2) + \{y_1x_1, x_1x_2, x_2y_2\}$. Then h is a parity-preserving injection with image precisely the union of those Eulerian subgraphs of D in (i) and (ii). Hence if we can show that exactly half of the Eulerian subgraphs of D in (iii) are even, we will conclude $EE(D) \neq EO(D)$, a contradiction. To do so, consider an Eulerian subgraph A of D containing y_1x_1 and not x_2y_2. Since x_1 must have in-degree 1 in A, it must also have out-degree 1 in A. We show that A has a mate A' of opposite parity. Suppose $x_2 \notin A$ and $x_1z_1 \in A$; then we make A' by removing x_1z_1 from A and adding $x_1x_2z_1$. If $x_2 \in A$ and $x_1x_2z_1 \in A$, we make A' by removing $x_1x_2z_1$ and adding x_1z_1. Hence exactly half of the Eulerian subgraphs of D in (iii) are even and we conclude $EE(D) \neq EO(D)$, a contradiction.

Now suppose (4b) holds. Put $G' := G - T$, $F' := F - T$, $Y' := Y$ and define f' by $f'(v) = f(v)$ for all $v \in V(G' - y_1)$ and $f'(y_1) = f(y_1) - 1$. Then G' has an orientation D' where $f'(v) \geq d_{D'}^+(v) + 1$ for all $v \in V(D')$ and $EE(D') \neq EO(D')$, for otherwise (G', f', F', Y') would contradict minimality. We orient $G - T$ according to D, orient T as in the previous case, again use y_1x_1 and orient all other edges between T and $G - T$ away from T. Since we decreased $f'(y_1)$ by 1, the extra out edge of y_1 is accounted for and we have $f(v) \geq d_D^+(v) + 1$ for all $v \in V(D)$. Again any additional Eulerian subgraph must contain y_1x_1 and since x_2 has no neighbor in $G - T$ we can use x_2 as before to build a mate of opposite parity for any additional Eulerian subgraph. Hence $EE(D) \neq EO(D)$ giving our final contradiction. □

Lemma 3.2. *Let* $r \geq 0$, $k \geq r + 4$ *and* $G \neq K_k$ *be a graph with* $x \in V(G)$ *such that:*

1. $G - x \in \mathcal{T}_k$; *and*

2. $d_G(x) \geq r + 2$*; and*

3. $\left|N(x) \cap W^k(G - x)\right| \geq 1$*; and*

4. $d_G(v) \leq k - 1$ *for all* $v \in V(G - x)$.

Then G *is* f*-AT where* $f(x) = d_G(x) - r$ *and* $f(v) = d_G(v)$ *for all* $v \in V(G - x)$.

Proof. Suppose not and choose a counterexample minimizing $|G|$. Let Q be the set of non-separating vertices in $G - x$. Suppose we have $y \in Q$ such that $G - y$ satisfies all the hypotheses of the theorem. Then minimality of $|G|$ shows that $G - y$ is f'-AT where $f'(v) := f(v) + d_{G-y}(v) - d_G(v)$ for $v \in V(G)$. Create an orientation D of G from the orientation of $G - y$ by directing all edges incident to y into y. These new edges are on no cycle and thus the Eulerian subgraph counts did not change. Also, we have increased the out degree of any vertex v by at most $d_G(v) - d_{G-y}(v)$. Hence G is f-AT, a contradiction. Therefore $G - y$ must fail some hypothesis for each $y \in Q$; note that it is only possible for $G - y$ to fail (2) or (3).

We show that $Q \subseteq N(x)$. Suppose otherwise that we have $y \in Q - N(x)$. Since (2) is satisfied for $G - y$, (3) must fail and hence y is contained in a K_{k-1}, call it B, in $G - x$ such that $N(x) \cap B \neq \emptyset$. Pick $z \in N(x) \cap B$. Since $d_G(z) \leq k - 1$ we must have $N_{G-x}(z) \subseteq B$ and hence $z \in Q$. Since $y \in Q$ and $G - x \in \mathcal{T}_k$, we must have $N_{G-x}(y) \subseteq B$. But then the conditions of Lemma 3.1 are satisfied with $F := G[x, z]$ and $Y := \{x\}$ since $f(x) \geq d_G(x) - r \geq 2 = d_{G[Y]}(x) + d_F(x) + 1$. This is a contradiction and hence we must have $Q \subseteq N(x)$.

Now, by (3), $G - x$ has at least one K_{k-1}, call it B, such that $N(x) \cap V(B) \neq \emptyset$. If $V(G-x) = B$, then $B = Q \subseteq N(x)$ and $G = K_k$, impossible. Hence we may pick $y \in Q - B$. Then $G - y$ satisfies (3) and hence must not satisfy (2). We conclude that $d_G(x) = r + 2$ and hence $|Q| \leq r + 2$. But $|Q| \geq \Delta(G - x) = k - 1$ and hence $k \leq r + 3$, a contradiction. □

We will need to know what happens when we patch two d_0-choosable graphs together at a vertex. To determine this we first need to understand the structure of d_0-choosable graphs. The d_0-choosable graphs were first characterized by Borodin [2] and independently by Erdős, Rubin and Taylor [6]. The generalization to a characterization of d_0-AT graphs was first given in [8] by Hladkỳ, Král and Schauz. This generalization is easily derived from the following lemma from [6] that is often referred to as "Rubin's Block Theorem".

Lemma 3.3 (Rubin [6])**.** *A 2-connected graph is either complete, an odd cycle or contains an induced even cycle with at most one chord.*

Lemma 3.4. *For a connected graph* G*, the following are equivalent:*

1. G *is not a Gallai tree,*

2. G *contains an even cycle with at most one chord,*

3. G *is* d_0*-choosable,*

4. G *is* d_0*-AT,*

5. *G has an orientation D where $d_G(v) \geq d_D^+(v) + 1$ for all $v \in V(D)$, $EE(D) \in \{2, 3\}$ and $EO(D) \in \{0, 1\}$.*

Proof. That (1), (2) and (3) are equivalent is the characterization of d_0-choosable graphs in [2] and [6]. Since (5) implies (4) and (4) implies (3) it will suffice to show that (2) implies (5). The proof we give of (5) is the same as in [8]. Suppose (2) holds and let H be an induced even cycle with at most one chord in G. Orient the even cycle in H clockwise and the (possible) other edge arbitrarily. Contract H to a single vertex x_H to form H' and take a spanning tree T of H' with root x_H. Orient the remaining edges in G away from the root in this tree to get D. Then every vertex has in degree at least 1 in D and hence $d_G(v) \geq d_D^+(v) + 1$ for all $v \in V(D)$. Also, since the orientation of $D - H$ is acyclic, the only spanning Eulerian subgraphs of D are the edgeless graph, the graph with just the edges from the even cycle in H and possibly one other using the chord in H. Hence $EE(D) \in \{2, 3\}$ and $EO(D) \in \{0, 1\}$, thus (5) holds. □

Lemma 3.5. *If $\{A, B\}$ is a separation of G such that $G[A]$ and $G[B]$ are connected d_0-AT graphs and $A \cap B = \{x\}$, then G is f-AT where $f(v) = d_G(v)$ for all $v \in V(G) - x$ and $f(x) = d_G(x) - 1$.*

Proof. By Lemma 3.4 we may choose an orientation D_A of A with $d^+(v) < d(v)$ for all $v \in V(D_A)$ and $EE(D_A) \neq EO(D_A)$ and an orientation D_B of B with $d^+(v) < d(v)$ for all $v \in V(D_B)$ and $EE(D_B) \neq EO(D_B)$. Together these give the desired orientation D of G since no cycle has vertices in both $A - x$ and $B - x$ and thus $EE(D) - EO(D) = EE(D_A)EE(D_B) + EO(D_A)EO(D_B) - (EE(D_A)EO(D_B) + EO(D_A)EE(D_B)) = (EE(D_A) - EO(D_A))(EE(D_B) - EO(D_B)) \neq 0$. □

Lemma 3.2 restricts the interaction of a high vertex and a single low component. Similarly to [13] we'll use the following lemma to restrict a high vertex's interaction with two low components.

Lemma 3.6. *Let $k \geq 4$ and let G be a graph with $x \in V(G)$ such that:*

1. *$G - x$ has two components $H_1, H_2 \in \mathcal{T}_k$; and*
2. *$|N(x) \cap V(H_i)| = 2$ for $i \in [2]$; and*
3. *$\left|N(x) \cap W^k(H_i)\right| \geq 1$ for $i \in [2]$.*

Then G is f-AT where $f(x) = d_G(x) - 1$ and $f(v) = d_G(v)$ for all $v \in V(G - x)$.

Proof. Using Lemma 3.5, we just need to show that $Q_i := G[\{x\} \cup V(H_i)]$ is d_0-AT for $i \in [2]$; that is show that Q_i is not a Gallai tree. If Q_i is a Gallai tree, then x's two neighbors in H_i must be in the same block in H_i and this block must be a K_{k-1}, but this creates a diamond since $k \geq 4$, impossible. □

Combining Lemma 3.2 and Lemma 3.6 gives the following.

Lemma 3.7. *Let $k \geq 5$ and let G be a graph with $x \in V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*

2. $G - x$ has t components $H_1, H_2, \dots, H_t$, and all are in $\mathcal{T}_k$; and

3. $d_G(v) \le k - 1$ for all $v \in V(G - x)$; and

4. $\left|N(x) \cap W^k(H_i)\right| \ge 1$ for $i \in [t]$; and

5. $d_G(x) \ge t + 2$.

Then G is f-AT where $f(x) = d_G(x) - 1$ and $f(v) = d_G(v)$ for all $v \in V(G - x)$.

Proof. Since $d_G(x) \ge t + 2$, either x has 3 neighbors in some H_i or x has two neighbors in each of H_i, H_j. In either case, let $C_1, \dots, C_q$ be the other components of $G - x$. For each $i \in [q]$, pick $z_i \in N(x) \cap V(C_i)$. Then order the vertices of C_i with z_i first and orient all the edges in C_i to the right with respect to this ordering. Now orient all edges between C_i and $G - C_i$ into C_i. Note that each vertex in C_i has in-degree at least one and no cycle passes through C_i. Hence we can complete the orientation using one of Lemma 3.2 or Lemma 3.6 to get our desired orientation D of G. □

To deal with more than one high vertex we need to define the following auxiliary bipartite graph. For a graph G, $\{X, Y\}$ a partition of $V(G)$ and $k \ge 4$, let $\mathcal{B}_k(X, Y)$ be the bipartite graph with one part Y and the other part the components of $G[X]$. Put an edge between $y \in Y$ and a component T of $G[X]$ iff $N(y) \cap W^k(T) \ne \emptyset$. Lemma 3.9 gives the substantive improvement over [13] on the lower bound on the number of edges in a list critical graph. Before proceeding we need a lemma about orientations.

Let $G = (V, E)$ be a multigraph. A function $A : V \to \wp(E)$ is called an *incidence preference.* Set $d(v, A) = d_G(v, A) = |E(v) \cap A(v)|$. Call an edge uv A-good (or just good) if $uv \in A(u) \cap A(v)$, and let $A(G)$ be the set of good edges of G. If D is an orientation of G, set $d^-(v, A) = |\{(u, v) \in E(D) : \{u, v\} \in A(v)\}|$.

Lemma 3.8. *Let G be a graph with incidence preference A, $S \subseteq V(G)$ and $g \colon S \to \mathbb{N}$. Then G has an orientation such that $d^-(v, A) \ge g(v)$ for all $v \in S$ iff for every $H \trianglelefteq G[S]$*

$$\sum_{v \in V(H)} d(v, A) - |A(H)| \ge \sum_{v \in V(H)} g(v).$$

Proof. First, suppose G has such an orientation D with $d^-(v, A) \ge g(v)$ for all $v \in S$. Consider any $H \trianglelefteq G[S]$. Then the second sum in (1) equals $|\{uv \in E(D) : v \in V(H) \text{ and } uv \in A(v)\}|$, and the third sum equals $|\{uv \in E(G) : v \in V(H) \text{ and } uv \in A(v)\}|$. So

$$\sum_{v \in V(H)} g(v) \le \sum_{v \in V(H)} d_H^-(v, A) \le \sum_{v \in V(H)} d(v, A) - |A(H)|. \tag{1}$$

For the other direction, pick an orientation D of G minimizing

$$\Theta := \sum_{v \in S} \max\left\{0, g(v) - d^-(v, A)\right\}.$$

It suffices to show $\Theta = 0$. If not then there is $x_0 \in S$ with $d^-(x_0) < g(x_0)$. Put

$$X := \{v \in V(G) : (\exists P_v := x_0 x_1 \dots x_t \text{ with } v = x_t)(\forall i \in [t])[v_{i-1} v_i \in E(D) \cap A(v_{i-1})]\}.$$

Every $v \in X$ satisfies $d^-(v, A) \leq g(v)$ for otherwise reversing all the edges on P_v violates the minimality of Θ. By definition, all edges $vw \in E(G) \cap A(v)$ with $v \in X$ and $w \in G - X$ are directed into X, so with $H := G[X]$ we have the contradiction

$$\sum_{v \in X} d(v, A) - |A(H)| = \sum_{v \in V(H)} d^-(v, A) < g(x_0) + \sum_{v \in V(H) - x_0} d^-(v, A) \leq \sum_{v \in V(H)} g(v). \qquad \square$$

For a graph G, let $\bar{S}(G)$ be the subset of non-separating vertices of G.

Lemma 3.9. *Let $k \geq 7$ and let G be a graph with $Y \subseteq V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*
2. *the components of $G - Y$ are in $\mathcal{T}_k$; and*
3. *$d_G(v) \leq k - 1$ for all $v \in V(G - Y)$; and*
4. *with $\mathcal{B} := \mathcal{B}_k(V(G - Y), Y)$ we have $\delta(\mathcal{B}) \geq 3$.*

Then G has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G' - Y)$.

Proof. Suppose not and pick a counterexample G minimizing $|G|$. Note that $\|w, Y\|_G \leq 1$ for every $w \in W^k(T)$, and if $\|w, Y\|_G = 1$ then $w \in \bar{S}(T)$; so if $y \in Y$ and T is a component of $G - Y$ then $N(y) \cap W^k(T) \subseteq \bar{S}(T)$. By Lemma 3.7, $\|y, T\|_G \leq 2$ for each edge yT of $\mathcal{B}$ since otherwise $G' = G[N_{\mathcal{B}}[y]]$ satisfies the conclusion of the lemma. Call an edge yT of $\mathcal{B}$ *heavy* if $\|y, T\|_G = 2$. Let $\mathcal{H}$ be the set of heavy edges, and $H = \bigcup_{yT \in \mathcal{H}} \{yx \in E(G) : x \in V(T)\}$. For $v \in S \subseteq V(\mathcal{B})$, set $h(v) = |E_{\mathcal{B}}(v) \cap \mathcal{H}|$ and $h(S) = \sum_{v \in S} h(v)$. By Lemma 3.7, $h(y) \leq 1$ for all $y \in Y$ since otherwise $G' = G[N_{\mathcal{B}}[y]]$ satisfies the conclusion of the lemma.

Suppose a component T of $G - Y$ has an endblock B with $B \neq K_{k-1}$ or $E(\bar{S}(B), Y) = \emptyset$. Then $G' := G - \bar{S}(B)$ still satisfies the hypotheses of the theorem since the degrees in $\mathcal{B}$ are not affected. Hence, by minimality of $|G|$, there is an induced subgraph $G'' \subseteq G'$ that is f-AT where $f(y) = d_{G''}(y) - 1$ for $y \in Y$ and $f(v) = d_{G''}(v)$ for all $v \in V(G'' - Y)$. But G'' is also an induced subgraph of G, a contradiction. Hence every endblock B of every component T of $G - Y$ is a K_{k-1} and $E(\bar{S}(B), Y) = \emptyset$. Let $x_B y_B \in E(\bar{S}(B), Y)$.

To each component T of $G - Y$ we associate a set of edges $u(T) \subseteq E(W^k(T), Y)$ as well as a *type*, where $\text{type}(T) \in \{1, 2a, 2b, 2c, 3\}$. Call a block B of T *saturated* if $\|v, Y\| \neq 0$ for all $v \in \bar{S}(B)$. For each component T of $G - Y$, order the endblocks of T as $B_1, \ldots, B_t$ so that the saturated blocks come first. Define $u(T)$ and $\text{type}(T)$ as follows:

1. B_1 is saturated.
   (a) $t = 1$
      - put $u(T) = E(T, Y)$ and $\text{type}(T) = 2a$.

   (b) $t \geq 2$
      i. B_2 is saturated
         - put $u(T) = E(\bar{S}(B_1 \cup B_2), Y)$ and $\text{type}(T) = 3$.

ii. B_2 is unsaturated
   - put $u(T) = E(\bar{S}(B_1), Y) \cup \{x_{B_2}y_{B_2}\}$ and $\text{type}(T) = 2b$.

2. Every endblock is unsaturated.

   (a) $t = 1$
   - since $\delta(\mathcal{B}) \geq 3$, there are three edges $e_1, e_2, e_3 \in E(T, Y)$ with distinct ends in Y, put $u(T) = \{e_1, e_2, e_3\}$ and $\text{type}(T) = 1$.

   (b) $t = 2$

   i. for some $i \in [2]$, there are two edges $e_1, e_2 \in E(\bar{S}(B_i), Y)$ with distinct ends in Y
      - put $u(T) = \{e_1, e_2, x_{B_{3-i}}y_{B_{3-i}}\}$ and $\text{type}(T) = 1$.

   ii. otherwise, since $\delta(\mathcal{B}) \geq 3$, there is an internal block $B_0 = K_{k-1}$ with an edge $x_{B_0}y_{B_0} \in E(\bar{S}(B), Y - y_{B_1} - y_{B_2})$

   A. B_0 is saturated
      - put $u(T) = \{x_{B_1}y_{B_1}, x_{B_2}y_{B_2}\} \cup E(\bar{S}(B_0), Y)$ and $\text{type}(T) = 2c$.

   B. B_0 is unsaturated
      - put $u(T) = \{x_{B_1}y_{B_1}, x_{B_2}y_{B_2}, x_{B_0}y_{B_0}\}$ and $\text{type}(T) = 1$.

   (c) $t \geq 3$
   - put $u(T) = \{x_{B_1}y_{B_1}, x_{B_2}y_{B_2}, x_{B_3}y_{B_3}\}$ and $\text{type}(T) = 1$.

Every type other than type 1 results from a unique case of this definition. If $\text{type}(T) \in \{2a, 2b, 2c\}$ we also say $\text{type}(T) = 2$ (but type 2 vertices arise in three cases). If $\text{type}(T) = i$ then any i-set of independent edges of $u(T)$ either contains an edge ending in an unsaturated block or two edges ending in the same block.

Let $\mathcal{H}(T) = \{e = yT \in \mathcal{H} : E_G(y, T) \cap u(T) \neq \emptyset\}$ and $h'(T) = |\mathcal{H}(T)|$. For $S \subseteq \mathcal{B} - Y$, let $h'(S) = \sum_{T \in S} h'(T)$. A component T of $G - Y$ is *heavy* if $\text{type}(T) \leq h'(T)$; else T is light. Define a function

$$\begin{aligned} g : V(\mathcal{B}) &\to \mathbb{N} \\ v &\mapsto \begin{cases} 2 - h(v) & \text{if } v \in Y \\ i - h'(T) & \text{if } v = T,\ T \text{ is light and } \text{type}(T) = i \\ 0 & \text{if } v = T \text{ and } T \text{ is heavy.} \end{cases} \end{aligned}$$

Let A be an incidence preference for $\mathcal{B}$ with $A(T) = \{yT \in E(\mathcal{B}) : E_G(y, T) \cap u(T) \setminus H \neq \emptyset\}$ if T is light, $A(T) = \emptyset$ if T is heavy, and $A(y) = \{yT \in E(\mathcal{B}) : E_G(y, T) \setminus H \neq \emptyset\}$ if $y \in Y$. We claim:

$$\text{There is an orientation } \mathcal{D} \text{ of } \mathcal{B} \text{ with } d^-_{\mathcal{D}}(v, A) \geq g(v) \text{ for all } v \in V(\mathcal{B}). \tag{2}$$

By Lemma 3.8, it suffices to show every induced subgraph $\mathcal{B}' \subseteq \mathcal{B}$ satisfies

$$\eta := \sum_{v \in V(\mathcal{B}')} d_{\mathcal{B}}(v, A) - |A(\mathcal{B}')| - \sum_{v \in V(\mathcal{B}')} g(v) \geq 0.$$

Fix such a $\mathcal{B}'$. Let $Y' = Y \cap V(\mathcal{B}')$, Q be the light vertices of type 1 in $\mathcal{B}'$, P be the light vertices of type 2 in $\mathcal{B}'$ and R be the light vertices of type 3 in $\mathcal{B}'$. Recall $\delta(\mathcal{B}) \geq 3$. For a light component T of $G - Y$,

$$d_{\mathcal{B}}(T, A) = \begin{cases} |\bar{S}(B_1(T))| - 2h'(T) = k - 1 - 2h'(T), & \text{if type}(T) = 2a \\ |\bar{S}(B_1(T))| - 2h'(T) + 1 = k - 1 - 2h'(T), & \text{if type}(T) = 2b \\ |\bar{S}(B_0(T))| - 2h'(T) + 2 = k - 1 - 2h'(T), & \text{if type}(T) = 2c \\ |\bar{S}(B_1(T)) \cup \bar{S}(B_2(T))| - 2h'(T) = 2k - 4 - 2h'(T), & \text{if type}(T) = 3. \end{cases}$$

So, if $T \in P$ then $d_{\mathcal{B}}(T, A) = k - 1 - 2h'(T)$ in $\mathcal{B}$. Thus

$$\begin{aligned} \sum_{v \in V(\mathcal{B}')} d(v, A) &= \sum_{v \in Y'} d(v, A) + \sum_{v \in Q \cup P \cup R} d(v, A) && (3) \\ \sum_{v \in Y'} d(v, A) &\geq 3|Y'| - h(Y'); && (4) \\ \sum_{v \in Q \cup P \cup R} d(v, A) &\geq 3|Q| + (k-1)|P| + (2k-4)|R| - 2h'(P \cup R); && (5) \\ |A(\mathcal{B}')| &\leq \min\left\{ \sum_{v \in Y'} d(v, A), \sum_{v \in Q \cup P \cup R} d(v, A) \right\}; \text{ and} && (6) \\ \sum_{v \in V(\mathcal{B}')} g(v) &= 2|Y'| + |Q| + 2|P| + 3|R| - h(Y') - h'(P) - h'(R). && (7) \end{aligned}$$

Using (11, 14, 12, 15) yields

$$\begin{aligned} \eta &= \sum_{v \in V(\mathcal{B}')} d(v, A) - |A(\mathcal{B}')| - \sum_{v \in V(\mathcal{B}')} g(v) \\ &\geq |Y'| - |Q| - 2|P| - 3|R| + h'(P \cup R). && (8) \end{aligned}$$

Replacing (12) with (13) yields

$$\eta \geq -2|Y'| + 2|Q| + (k-3)|P| + (2k-7)|R| + h(Y') - h'(P \cup R). \quad (9)$$

Adding twice (16) to (17) yields

$$3\eta \geq (k-7)|P| + 2(k-6.5)|R| + h(Y') + h'(P \cup R).$$

Since $k \geq 7$, this implies $\eta \geq 0$. So there exists an orientation $\mathcal{D}$ satisfying 10.

Finally we use $\mathcal{D}$ to construct the subgraph $F \subseteq G$ needed in Lemma 3.1. For an edge $e = yT \in A(T) \cup \mathcal{H}(T)$, there is an edge $e' \in E_G(y, T)$ such that $e' \in u(T)$ if e is light. If e is heavy then there is another edge $e'' \in E_G(y, T)$. Let

$$F = \{e' : e = yT \in A(T) \text{ and } yT \in E(\mathcal{D})\} \cup \{e' : e = yT \in \mathcal{H}\}.$$

We claim F satisfies (4) of Lemma 3.1. Consider any component $T \in G - Y$; say type$(T) = i$. Then there are at least i edges $e'_1 = x_1y_1, \ldots, e'_i = x_iy_i \in F$ with $y_i \in T$. Moreover, these edges are independent.

Suppose $i = 1$. Then $x_1 \in \bar{S}(B)$ for an unsaturated block $B \subseteq T$. As B is unsaturated, there is a vertex $x \in \bar{S}(B) - x_1$ with no neighbor in Y. So $N[x_1] = N[x]$, and e'_1 and x witness (4b).

Suppose $i = 2$. If type$(T) = 2a$, then T has only one block B_1. So $\bar{S}(B_1) = T$, and e'_1 and e'_2 witness (4a). If type$(T) \in \{2b, 2c\}$, then (4a) is satisfied if x_1 and x_2 are in the same block of T; else one of them ends in an unsaturated block, and (4b) is satisfied.

Finally, suppose $i = 3$. Then (4a) is satisfied since two of x_1, x_2, x_3 are in the same block.

Also, as each $y \in Y$ satisfies $d^-(y, A) \geq 2 - h(y)$, we have —$E(y, G - Y) \setminus F| \geq 2$. Thus $f(y) = d_G(y) - 1 \geq d_{G[Y]}(y) + d_F(y) + E(y, G - Y) \setminus E(F) - 1$. So (3) holds. ☐

With a slightly simpler argument we get the following version with asymmetric degree condition on $\mathcal{B}$. The point here is that this works for $k \geq 5$. As we'll see in the next section, the consequence is that we trade a bit in our size bound for the proof to go through with $k \in \{5, 6\}$.

Lemma 3.10. *Let $k \geq 5$ and let G be a graph with $Y \subseteq V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*
2. *the components of $G - Y$ are in $\mathcal{T}_k$; and*
3. *$d_G(v) \leq k - 1$ for all $v \in V(G - Y)$; and*
4. *with $\mathcal{B} := \mathcal{B}_k(V(G - Y), Y)$ we have $d_{\mathcal{B}}(y) \geq 4$ for all $y \in Y$ and $d_{\mathcal{B}}(T) \geq 2$ for all components T of $G - Y$.*

Then G has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G' - Y)$.

Proof. Suppose not and pick a counterexample G minimizing $|G|$. Note that $\|w, Y\|_G \leq 1$ for every $w \in W^k(T)$, and if $\|w, Y\|_G = 1$ then $w \in \bar{S}(T)$; so if $y \in Y$ and T is a component of $G - Y$ then $N(y) \cap W^k(T) \subseteq \bar{S}(T)$. By Lemma 3.7, $\|y, T\|_G \leq 2$ for each edge yT of $\mathcal{B}$ since otherwise $G' = G[N_{\mathcal{B}}[y]]$ satisfies the conclusion of the lemma. Call an edge yT of $\mathcal{B}$ *heavy* if $\|y, T\|_G = 2$. Let $\mathcal{H}$ be the set of heavy edges, and $H = \bigcup_{yT \in \mathcal{H}} \{yx \in E(G) : x \in V(T)\}$. For $v \in S \subseteq V(\mathcal{B})$, set $h(v) = |E_{\mathcal{B}}(v) \cap \mathcal{H}|$ and $h(S) = \sum_{v \in S} h(v)$. By Lemma 3.7, $h(y) \leq 1$ for all $y \in Y$ since otherwise $G' = G[N_{\mathcal{B}}[y]]$ satisfies the conclusion of the lemma.

Suppose a component T of $G - Y$ has an endblock B with $B \neq K_{k-1}$ or $E(\bar{S}(B), Y) = \emptyset$. Then $G' := G - \bar{S}(B)$ still satisfies the hypotheses of the theorem since the degrees in $\mathcal{B}$ are not affected. Hence, by minimality of $|G|$, there is an induced subgraph $G'' \subseteq G'$ that is f-AT where $f(y) = d_{G''}(y) - 1$ for $y \in Y$ and $f(v) = d_{G''}(v)$ for all $v \in V(G'' - Y)$. But G'' is also an induced subgraph of G, a contradiction. Hence every endblock B of every component T of $G - Y$ is a K_{k-1} and $E(\bar{S}(B), Y) = \emptyset$. Let $x_B y_B \in E(\bar{S}(B), Y)$.

To each component T of $G - Y$ we associate a set of edges $u(T) \subseteq E(W^k(T), Y)$ as well as a *type*, where type$(T) \in \{1, 2a, 2b, 3\}$. Call a block B of T *saturated* if $\|v, Y\| \neq 0$ for all $v \in \bar{S}(B)$. For each component T of $G - Y$, order the endblocks of T as $B_1, \ldots, B_t$ so that the saturated blocks come first. Define $u(T)$ and type(T) as follows:

1. B_1 is saturated.

(a) $t = 1$

- put $u(T) = E(T, Y)$ and $\text{type}(T) = 2a$.

(b) $t \geq 2$

i. B_2 is saturated

- put $u(T) = E(\bar{S}(B_1 \cup B_2), Y)$ and $\text{type}(T) = 3$.

ii. B_2 is unsaturated

- put $u(T) = E(\bar{S}(B_1), Y) \cup \{x_{B_2}y_{B_2}\}$ and $\text{type}(T) = 2b$.

2. B_1 is unsaturated.

(a) $t = 1$

- since $\delta(\mathcal{B}) \geq 2$, there are two edges $e_1, e_2 \in E(T, Y)$ with distinct ends in Y, put $u(T) = \{e_1, e_2\}$ and $\text{type}(T) = 1$.

(b) $t \geq 2$

- put $u(T) = \{x_{B_1}y_{B_1}, x_{B_2}y_{B_2}\}$ and $\text{type}(T) = 1$.

Every type other than type 1 results from a unique case of this definition. If $\text{type}(T) \in \{2a, 2b\}$ we also say $\text{type}(T) = 2$ (but type 2 vertices arise in three cases). If $\text{type}(T) = i$ then any i-set of independent edges of $u(T)$ either contains an edge ending in an unsaturated block or two edges ending in the same block.

Let $\mathcal{H}(T) = \{e = yT \in \mathcal{H} : E_G(y, T) \cap u(T) \neq \emptyset\}$ and $h'(T) = |\mathcal{H}(T)|$. For $S \subseteq \mathcal{B} - Y$, let $h'(S) = \sum_{T \in S} h'(T)$. A component T of $G - Y$ is *heavy* if $\text{type}(T) \leq h'(T)$; else T is light. Define a function

$$\begin{aligned} g : V(\mathcal{B}) &\to \mathbb{N} \\ v &\mapsto \begin{cases} 2 - h(v) & \text{if } v \in Y \\ i - h'(T) & \text{if } v = T,\ T \text{ is light and } \text{type}(T) = i \\ 0 & \text{if } v = T \text{ and } T \text{ is heavy.} \end{cases} \end{aligned}$$

Let A be an incidence preference for $\mathcal{B}$ with $A(T) = \{yT \in E(\mathcal{B}) : E_G(y, T) \cap u(T) \setminus H \neq \emptyset\}$ if T is light, $A(T) = \emptyset$ if T is heavy, and $A(y) = \{yT \in E(\mathcal{B}) : E_G(y, T) \setminus H \neq \emptyset\}$ if $y \in Y$. We claim:

$$\text{There is an orientation } \mathcal{D} \text{ of } \mathcal{B} \text{ with } d^-_{\mathcal{D}}(v, A) \geq g(v) \text{ for all } v \in V(\mathcal{B}). \tag{10}$$

By Lemma 3.8, it suffices to show every induced subgraph $\mathcal{B}' \subseteq \mathcal{B}$ satisfies

$$\eta := \sum_{v \in V(\mathcal{B}')} d_{\mathcal{B}}(v, A) - |A(\mathcal{B}')| - \sum_{v \in V(\mathcal{B}')} g(v) \geq 0.$$

Fix such a $\mathcal{B}'$. Let $Y' = Y \cap V(\mathcal{B}')$, Q be the light vertices of type 1 in $\mathcal{B}'$, P be the light vertices of type 2 in $\mathcal{B}'$ and R be the light vertices of type 3 in $\mathcal{B}'$. Recall $d_{\mathcal{B}}(y) \geq 4$ for all $y \in Y$ and $d_{\mathcal{B}}(T) \geq 2$ for all components T of $G - Y$. For a light component T of $G - Y$,

$$d_{\mathcal{B}}(T, A) = \begin{cases} |\bar{S}(B_1(T))| - 2h'(T) = k - 1 - 2h'(T), & \text{if } \text{type}(T) = 2a \\ |\bar{S}(B_1(T))| - 2h'(T) + 1 = k - 1 - 2h'(T), & \text{if } \text{type}(T) = 2b \\ |\bar{S}(B_1(T)) \cup \bar{S}(B_2(T))| - 2h'(T) = 2k - 4 - 2h'(T), & \text{if } \text{type}(T) = 3. \end{cases}$$

So, if $T \in P$ then $d_{\mathcal{B}}(T, A) = k - 1 - 2h'(T)$ in $\mathcal{B}$. Thus

$$\begin{aligned}
\sum_{v \in V(\mathcal{B}')} d(v,A) &= \sum_{v \in Y'} d(v,A) + \sum_{v \in Q \cup P \cup R} d(v,A) && (11)\\
\sum_{v \in Y'} d(v,A) &\geq 4|Y'| - h(Y'); && (12)\\
\sum_{v \in Q \cup P \cup R} d(v,A) &\geq 2|Q| + (k-1)|P| + (2k-4)|R| - 2h'(P \cup R); && (13)\\
|A(\mathcal{B}')| &\leq \min\left\{\sum_{v \in Y'} d(v,A), \sum_{v \in Q \cup P \cup R} d(v,A)\right\}; \text{ and} && (14)\\
\sum_{v \in V(\mathcal{B}')} g(v) &= 2|Y'| + |Q| + 2|P| + 3|R| - h(Y') - h'(P) - h'(R). && (15)
\end{aligned}$$

Using (11, 14, 12, 15) yields

$$\begin{aligned}
\eta &= \sum_{v \in V(\mathcal{B}')} d(v,A) - |A(\mathcal{B}')| - \sum_{v \in V(\mathcal{B}')} g(v)\\
&\geq 2|Y'| - |Q| - 2|P| - 3|R| + h'(P \cup R). && (16)
\end{aligned}$$

Replacing (12) with (13) yields

$$\eta \geq -2|Y'| + |Q| + (k-3)|P| + (2k-7)|R| + h(Y') - h'(P \cup R). \tag{17}$$

Adding (16) to (17) yields

$$2\eta \geq (k-5)|P| + 2(k-5)|R| + h(Y').$$

Since $k \geq 5$, this implies $\eta \geq 0$. So there exists an orientation $\mathcal{D}$ satisfying 10.

Finally we use $\mathcal{D}$ to construct the subgraph $F \subseteq G$ needed in Lemma 3.1. For an edge $e = yT \in A(T) \cup \mathcal{H}(T)$, there is an edge $e' \in E_G(y,T)$ such that $e' \in u(T)$ if e is light. If e is heavy then there is another edge $e'' \in E_G(y,T)$. Let

$$F = \{e' : e = yT \in A(T) \text{ and } yT \in E(\mathcal{D})\} \cup \{e' : e = yT \in \mathcal{H}\}.$$

We claim F satisfies (4) of Lemma 3.1. Consider any component $T \in G - Y$; say $\text{type}(T) = i$. Then there are at least i edges $e'_1 = x_1y_1, \ldots, e'_i = x_iy_i \in F$ with $y_i \in T$. Moreover, these edges are independent.

Suppose $i = 1$. Then $x_1 \in \bar{S}(B)$ for an unsaturated block $B \subseteq T$. As B is unsaturated, there is a vertex $x \in \bar{S}(B) - x_1$ with no neighbor in Y. So $N[x_1] = N[x]$, and e'_1 and x witness (4b).

Suppose $i = 2$. If $\text{type}(T) = 2a$, then T has only one block B_1. So $\bar{S}(B_1) = T$, and e'_1 and e'_2 witness (4a). If $\text{type}(T) = 2b$, then (4a) is satisfied if x_1 and x_2 are in the same block of T; else one of them ends in an unsaturated block, and (4b) is satisfied.

Finally, suppose $i = 3$. Then (4a) is satisfied since two of x_1, x_2, x_3 are in the same block.

Also, as each $y \in Y$ satisfies $d^-(y,A) \geq 2 - h(y)$, we have —$E(y, G-Y) \smallsetminus F| \geq 2$. Thus $f(y) = d_G(y) - 1 \geq d_{G[Y]}(y) + d_F(y) + E(y, G-Y) \smallsetminus E(F) - 1$. So (3) holds. □

4 Main theorem: AT-irreducible graphs have many edges

The rest of the proof is basically taken verbatim from [13]. We need the following definitions:

$$
\begin{aligned}
\mathcal{L}_k(G) &:= G\left[x \in V(G) \mid d_G(x) < k\right], \\
\mathcal{H}_k(G) &:= G\left[x \in V(G) \mid d_G(x) \geq k\right], \\
\sigma_k(G) &:= \left(k - 2 + \frac{2}{k-1}\right) |\mathcal{L}_k(G)| - 2\left\|\mathcal{L}_k(G)\right\|, \\
\tau_{k,c}(G) &:= 2\left\|\mathcal{H}_k(G)\right\| + \left(k - c - \frac{2}{k-1}\right) \sum_{y \in V(\mathcal{H}_k(G))} \left(d_G(y) - k\right), \\
\alpha_k &:= \frac{1}{2} - \frac{1}{(k-1)(k-2)}, \\
q_k(G) &:= \alpha_k \sum_{v \in V(G) \setminus W^k(G)} \left(k - 1 - d_G(v)\right), \\
g_k(n, c) &:= \left(k - 1 + \frac{k-3}{(k-c)(k-1) + k - 3}\right) n.
\end{aligned}
$$

As proved in [13], a computation gives the following.

Lemma 4.1. *Let G be a graph with $\delta := \delta(G) \geq 3$ and $0 \leq c \leq \delta + 1 - \frac{2}{\delta}$. If $\sigma_{\delta+1}(G) + \tau_{\delta+1,c}(G) \geq c\left|\mathcal{H}_{\delta+1}(G)\right|$, then $2\left\|G\right\| \geq g_{\delta+1}(|G|, c)$.*

We need the following degeneracy lemma.

Lemma 4.2. *Let G be a graph and $f\colon V(G) \to \mathbb{N}$. If $\|G\| > \sum_{v \in V(G)} f(v)$, then G has an induced subgraph H such that $d_H(v) > f(v)$ for each $v \in V(H)$.*

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Then $|G| \geq 3$ and we have $x \in V(G)$ with $d_G(x) \leq f(x)$. But now $\|G - x\| > \sum_{v \in V(G-x)} f(v)$, contradicting minimality of $|G|$. $\square$

We'll also need the following consequence of Lemma 2.3 in [13] giving a lower bound on $\sigma_k(T)$ for $T \in \mathcal{T}_k$. Lemma 2.3 in [13] is only proved for $k \geq 6$, but we need our lemma to work for $k = 5$ as well, so we prove that here. Notice that when $T \in \mathcal{T}_k$, we have $\mathcal{L}_k(T) = T$. We also use the following simple fact (Lemma 2.1(b) in [13]): if B is an endblock of $T \in \mathcal{T}_k$ and x is the unique cutvertex of T in $V(B)$, then $\sigma_k(T) = \sigma_k(T - (B - x)) + \sigma_k(B) - (k - 2 + \frac{2}{k-1})$.

Lemma 4.3. *Let $k \geq 5$ and $T \in \mathcal{T}_k$. If $K_{k-1} \subseteq T$, then $\sigma_k(T) \geq 2 + q_k(T)$; otherwise $\sigma_k(T) \geq 2 - \alpha_k + q_k(T)$.*

Proof. Suppose the lemma is false and choose a counterexample $T \in \mathcal{T}_k$ minimizing $|T|$. By Lemma 2.3 in [13], we have $k = 5$. Then $\alpha_5 = \frac{5}{12}$ and $2 - \alpha_5 = \frac{19}{12}$. Also, $\sigma_5(T) = \frac{7}{2}|T| - 2\left\|T\right\|$ and $q_5(T) = \frac{5}{12} \sum_{v \in V(G) \setminus W^5(G)} (4 - d_T(v))$. Suppose T has only one block. First, suppose $T = K_t$ for $t \in \{2, 3\}$. Then $\sigma_5(T) - q_5(T) = \frac{7}{2}t - t(t-1) - \frac{5}{12}t(5-t) \geq \frac{19}{12}$, a contradiction.

Also, $T \neq K_4$ since then $\sigma_5(T) = 2$ and $q_5(T) = 0$. If T is an odd cycle of length ℓ, then $\sigma_5(T) - q_5(T) = \frac{3}{2}\ell - \frac{5}{12}(2\ell) = \frac{2}{3}\ell \geq \frac{19}{12}$, a contradiction.

So, T must have at least two blocks. For an endblock B of T, let x_B be the unique cut vertex of T in $V(B)$. Consider $T_B := T - (B - x)$. Clearly (and by Lemma 2.1(b) in [13]), we have $\sigma_5(T) = \sigma_5(T_B) + \sigma_5(B) - \frac{7}{2}$. Suppose B has an endblock $B \neq K_4$. Since $B \neq K_4$, any K_4 in T is in T_B, so minimality of $|T|$ yields $\sigma_5(T_B) \geq 2 + q_5(T_B)$ if $K_4 \subseteq T$ and $\sigma_5(T_B) \geq 2 - \alpha_5 + q_5(T_B)$ otherwise. Since T is a counterexample, we must have $\sigma_5(B) - \frac{7}{2} + q_5(T_B) < q_5(T)$. Since B is regular, this gives

$$\begin{aligned}
\frac{7}{2}|B| - 2\,\|B\| - \frac{7}{2} &= \sigma_5(B) - \frac{7}{2} \\
&< \alpha_5\left(-d_B(x_B) + \sum_{v \in V(B - x_B)} 4 - d_B(v)\right) \\
&= \frac{5}{12}\left(-\Delta(B) + (|B| - 1)(4 - \Delta(B))\right).
\end{aligned}$$

Therefore,

$$\frac{7}{2}(|B| - 1) - |B|\Delta(B) < \frac{5}{12}\left(-\Delta(B) + (|B| - 1)(4 - \Delta(B))\right).$$

This simplifies to the following which is a contradiction since $\Delta(B) \in \{1, 2\}$:

$$\Delta(B) > \frac{22}{7}\left(1 - \frac{1}{|B|}\right).$$

Therefore, every endblock of T is K_4. Choose an endblock $B = K_4$ of T. Then the other block containing x_B is a K_2, let y be the other vertex in this K_2. Consider $T' = T - B$. Then $K_4 \subseteq T'$ since T had another endblock which must be K_4. By minimality of $|T|$, we conclude $\sigma_5(T') \geq 2 + q_5(T')$. Since T has 4 more vertices and 7 more edges than T', we have $\sigma_5(T) = \sigma_5(T') + 4\frac{7}{2} - (2)(7) = \sigma_5(T')$. Also, $q_5(T') = q_5(T)$ if y is in a K_{k-1} and $q_5(T') = q_5(T) + \alpha_5$ otherwise (since all the vertices in B are in a K_{k-1}, they do not contribute to $q_5(T)$). Hence $\sigma_5(T) = \sigma_5(T') \geq 2 + q_5(T') \geq 2 + q_5(T)$, a contradiction. $\square$

We are now ready to prove the main theorem.

Theorem 4.4. *If G is an AT-irreducible graph with $\delta(G) \geq 4$ and $\omega(G) \leq \delta(G)$, then $2\,\|G\| \geq g_{\delta(G)+1}(|G|, c)$ where $c := (\delta(G) - 2)\alpha_{\delta(G)+1}$ when $\delta(G) \geq 6$ and $c := (\delta(G) - 3)\alpha_{\delta(G)+1}$ when $\delta(G) \in \{4, 5\}$.*

Proof. Put $k := \delta(G) + 1$, $\mathcal{L} := \mathcal{L}_k(G)$ and $\mathcal{H} := \mathcal{H}_k(G)$. Plainly, $c \leq \delta(G) + 1 - \frac{2}{\delta(G)}$. So, using Lemma 4.1, we just need to show that $\sigma_k(G) + \tau_{k,c}(G) \geq c\,|\mathcal{H}|$. Put $W := W^k(\mathcal{L})$, $L' := V(\mathcal{L}) \setminus W$ and $H' := \{v \in V(\mathcal{H}) : d_G(v) = k\}$. For $y \in V(\mathcal{H})$, put

$$\tau_{k,c}(y) := d_{\mathcal{H}}(y) + \left(k - c + \frac{2}{k-1}\right)(d_G(y) - k).$$

We have

$$\begin{aligned}
\tau_{k,c}(G) &= \sum_{y \in V(\mathcal{H})} \tau_{k,c}(y) \\
&\geq \sum_{y \in H'} d_{\mathcal{H}}(y) + \sum_{y \in V(\mathcal{H}) \setminus H'} \left(d_{\mathcal{H}}(y) + k - c + \frac{2}{k-1} \right) \\
&\geq \sum_{y \in H'} d_{\mathcal{H}}(y) + \left(k - c + \frac{2}{k-1} \right) |\mathcal{H} - H'| \\
&\geq \sum_{y \in H'} d_{\mathcal{H}}(y) + c\,|\mathcal{H} - H'|,
\end{aligned}$$

where the last inequality follows since $c \leq (k-3)\alpha_k = (k-3)\left(\frac{1}{2} - \frac{1}{(k-1)(k-2)}\right) \leq \frac{k}{2}$. Therefore, it will be sufficient to prove that $S := \sigma_k(G) + \sum_{y \in H'} d_{\mathcal{H}}(y) \geq c\,|H'|$.

Let $\mathcal{D}$ be the components of $\mathcal{L}$ containing K_{k-1} and $\mathcal{C}$ the components of $\mathcal{L}$ not containing K_{k-1}. Then $\mathcal{D} \cup \mathcal{C} \subseteq \mathcal{T}_k$ for otherwise some $T \in \mathcal{D} \cup \mathcal{C}$ is d_0-AT and hence f_T-AT and G is AT-reducible. By Lemma 4.3, we have $\sigma_k(T) \geq 2 + q_k(T)$ for if $T \in \mathcal{D}$ and $\sigma_k(T) \geq 2 - \alpha_k + q_k(T)$ if $T \in \mathcal{C}$. Hence, we have $\sigma_k(G) = \sum_{T \in \mathcal{D}} \sigma_k(T) + \sum_{T \in \mathcal{C}} \sigma_k(T) \geq 2\,|\mathcal{D}| + (2 - \alpha_k)\,|\mathcal{C}| + \alpha_k \sum_{v \in L'} (k - 1 - d_{\mathcal{L}}(v))$.

Now we define an auxiliary bipartite graph F with parts A and B where:

1. $B = H'$ and A is the disjoint union of the following sets A_1, A_2 and A_3,

2. $A_1 = \mathcal{D}$ and each $T \in \mathcal{D}$ is adjacent to all $y \in H'$ where $N(y) \cap W^k(T) \neq \emptyset$,

3. For each $v \in L'$, let $A_2(v)$ be a set of $|N(v) \cap H'|$ vertices connected to $N(v) \cap H'$ by a matching in F. Let A_2 be the disjoint union of the $A_2(v)$ for $v \in L'$,

4. For each $y \in H'$, let $A_3(y)$ be a set of $d_{\mathcal{H}}(y)$ vertices which are all joined to y in F. Let A_3 be the disjoint union of the $A_3(y)$ for $y \in H'$.

Case 1. $\delta \geq 6$.

Define $f\colon V(F) \to \mathbb{N}$ by $f(v) = 1$ for all $v \in A_2 \cup A_3$ and $f(v) = 2$ for all $v \in B \cup A_1$. First, suppose $\|F\| > \sum_{v \in V(F)} f(v)$. Then by Lemma 4.2, F has an induced subgraph Q such that $d_Q(v) > f(v)$ for each $v \in V(Q)$. In particular, $V(Q) \subseteq B \cup A_1$ and $\delta(Q) \geq 3$. Put $Y := B \cap V(Q)$ and let X be $\bigcup_{T \in V(Q) \cap A_1} V(T)$. Now $H := G[X \cup Y]$ satisfies the hypotheses of Lemma 3.9, so H has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for $v \in X$. Since $Y \subseteq H'$ and $X \subseteq \mathcal{L}$, we have $f(v) = \delta(G) + d_{G'}(v) - d_G(v)$ for all $v \in V(G')$. Hence, G is AT-reducible to G', a contradiction.

Therefore $\|F\| \leq \sum_{v \in V(F)} f(v) = 2(|H'| + |\mathcal{D}|) + |A_2| + |A_3|$. By Lemma 3.7, for each $y \in B$ we have $d_F(y) \geq k-1$. Hence $\|F\| \geq (k-1)\,|H'|$. This gives $(k-3)\,|H'| \leq 2\,|\mathcal{D}| + |A_2| + |A_3|$. By our above estimate we have $S \geq 2\,|\mathcal{D}| + \alpha_k \sum_{v \in L'} (k - 1 - d_{\mathcal{L}}(v)) + \sum_{y \in H'} d_{\mathcal{H}}(y) = 2\,|\mathcal{D}| + \alpha_k\,|A_2| + |A_3| \geq \alpha_k(2\,|\mathcal{D}| + |A_2| + |A_3|)$. Hence $S \geq \alpha_k(k-3)\,|H'|$. Thus our desired bound holds by Lemma 4.1.

Case 2. $\delta \in \{4, 5\}$.

Define $f\colon V(F) \to \mathbb{N}$ by $f(v) = 1$ for all $v \in A_1 \cup A_2 \cup A_3$ and $f(v) = 3$ for all $v \in B$. First, suppose $\|F\| > \sum_{v \in V(F)} f(v)$. Then by Lemma 4.2, F has an induced subgraph Q such that $d_Q(v) > f(v)$ for each $v \in V(Q)$. In particular, $V(Q) \subseteq B \cup A_1$ and $d_Q(v) \geq 4$ for $v \in B \cap V(Q)$ and $d_Q(v) \geq 2$ for $v \in A_1 \cap V(Q)$. Put $Y := B \cap V(Q)$ and let X be $\bigcup_{T \in V(Q) \cap A_1} V(T)$. Now $H := G[X \cup Y]$ satisfies the hypotheses of Lemma 3.10, so H has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for $v \in X$. Since $Y \subseteq H'$ and $X \subseteq \mathcal{L}$, we have $f(v) = \delta(G) + d_{G'}(v) - d_G(v)$ for all $v \in V(G')$. Hence, G is AT-reducible to G', a contradiction.

Therefore $\|F\| \leq \sum_{v \in V(F)} f(v) = 3\,|H'| + |\mathcal{D}| + |A_2| + |A_3|$. By Lemma 3.7, for each $y \in B$ we have $d_F(y) \geq k-1$. Hence $\|F\| \geq (k-1)\,|H'|$. This gives $(k-4)\,|H'| \leq |\mathcal{D}| + |A_2| + |A_3|$. By our above estimate we have $S \geq 2\,|\mathcal{D}| + \alpha_k \sum_{v \in L'} (k - 1 - d_{\mathcal{L}}(v)) + \sum_{y \in H'} d_{\mathcal{H}}(y) = 2\,|\mathcal{D}| + \alpha_k\,|A_2| + |A_3| \geq \alpha_k(|\mathcal{D}| + |A_2| + |A_3|)$. Hence $S \geq \alpha_k(k-4)\,|H'|$. Thus our desired bound holds by Lemma 4.1. □

We note a corollary of the above proof that will be useful in a later paper. When $\mathcal{H}_k(G)$ is edgeless, A_3 is empty and $S = \sigma_k(G)$. Also from the proof, we have $\sigma_k(G) \geq 2\,|\mathcal{D}| + (2-\alpha_k)\,|\mathcal{C}| + \alpha_k \sum_{v \in L'} (k-1-d_{\mathcal{L}}(v)) \geq \alpha_k(2\,|\mathcal{D}| + |A_2|) + 2(1-\alpha_k)\,|\mathcal{D}| + (2-\alpha_k)\,|\mathcal{C}|$. We write $c(G)$ for the number of components of G. Since $(2-\alpha_k) \geq 2(1-\alpha_k)$, we have $\sigma_k(G) \geq (k-3)\alpha_k\,|\mathcal{H}_k(G)| + 2(1-\alpha_k)c(\mathcal{L}(G))$.

Corollary 4.5. *If G is an AT-irreducible graph with $\delta := \delta(G) \geq 6$ and $\omega(G) \leq \delta$ such that $\mathcal{H}_{\delta+1}(G)$ is edgeless, then $\sigma_{\delta+1}(G) \geq (\delta-2)\alpha_{\delta+1}\,|\mathcal{H}_{\delta+1}(G)| + 2(1-\alpha_{\delta+1})c(\mathcal{L}(G))$.*

5 Corollaries: Critical graphs have many edges

Corollary 5.1. *For $k \geq 5$ and $G \neq K_k$ a k-list-critical graph, we have $2\,\|G\| \geq g_k(|G|, c)$ where $c := (k-3)\alpha_k$ when $k \geq 7$ and $c := (k-4)\alpha_k$ when $k \in \{5, 6\}$.*

Proof. Let L be a $(k-1)$-assignment such that G is L-critical. Since G is L-critical, we have $\delta(G) \geq k-1 \geq 5$. If $\delta(G) \geq k$, then $2\,\|G\| \geq k\,|G| \geq g_k(|G|, k)$ and we are done. Hence we may assume that $\delta(G) = k-1$. Since $G \neq K_k$ and G is L-critical, we have $K_{\delta(G)+1} \not\subseteq G$. By Lemma 2.1, G is AT-irreducible, so Lemma 4.4 proves the corollary. □

Note that applying Lemma 2.2 where H has a single vertex shows that $\delta(G) \geq k-1$ for an online k-list-critical graph.

Corollary 5.2. *For $k \geq 5$ and $G \neq K_k$ an online k-list-critical graph, we have $2\,\|G\| \geq g_k(|G|, c)$ where $c := (k-3)\alpha_k$ when $k \geq 7$ and $c := (k-4)\alpha_k$ when $k \in \{5, 6\}$.*

Proof. Since G is online k-list-critical, we have $\delta(G) \geq k-1 \geq 5$. If $\delta(G) \geq k$, then $2\,\|G\| \geq k\,|G| \geq g_k(|G|, k)$ and we are done. Hence we may assume that $\delta(G) = k-1$. Since $G \neq K_k$ and G is online k-list-critical, we have $K_{\delta(G)+1} \not\subseteq G$. By Lemma 2.3, G is AT-irreducible, so Lemma 4.4 proves the corollary. □

Note that applying Lemma 2.4 where H has a single vertex shows that $\delta(G) \geq k-1$ for a k-AT-critical graph G.

Corollary 5.3. *For $k \geq 5$ and $G \neq K_k$ a k-AT-critical graph, we have $2\|G\| \geq g_k(|G|, c)$ where $c := (k-3)\alpha_k$ when $k \geq 7$ and $c := (k-4)\alpha_k$ when $k \in \{5, 6\}$.*

Proof. Since G is k-AT-critical, we have $\delta(G) \geq k-1 \geq 5$. If $\delta(G) \geq k$, then $2\|G\| \geq k|G| \geq g_k(|G|, k)$ and we are done. Hence we may assume that $\delta(G) = k-1$. Since $G \neq K_k$ and G is k-AT-critical, we have $K_{\delta(G)+1} \not\subseteq G$. By Lemma 2.5, G is AT-irreducible, so Lemma 4.4 proves the corollary. □

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