Volumetric Basis Reduction for Global Seamless Parameterization of Meshes

Introduction

In this supplement, we describe how to construct positional and translational functions for satisfying the seam constraints. We assume that we are given a mesh M and a set of (oriented) seams $\sigma \in \Sigma$ with associated rotations $r_\sigma \in \{1, \mathbf{i}, -1, -\mathbf{i}\}$ satisfying $r_{\sigma^{-1}} = r_\sigma^{-1}$. For a given choice of consistent translations shifts across seams, $\{t_\sigma\} \subset \mathbb{C}$ (with $t_{\sigma^{-1}} = -r_\sigma^{-1} t_\sigma$), our goal is to define a function space making it easy to express functions $F : M \to \mathbb{C}$ that satisfy the following condition:

Seam Constraint (Mesh): For any point p on any seam σ, the limits of the function values as we approach the point p from below and above the seam satisfy:

$$\lim_{p^+ \to p} F(p^+) = r_\sigma \left(\lim_{p^- \to p} F(p^-) \right) + t_\sigma,$$

where t_σ is the prescribed translation across σ.

Local Domain Graph

We consider a single basis function $B : M \to \mathbb{C}$, suppressing its index for simplicity as in the paper. This function is supported on the open, connected region D, which we cut into components with the seams in Σ:

$$\bigcup_i D_i = D - \bigcup_{\sigma \in \Sigma} \sigma.$$

Using region D_j's indicator function $\chi_j(p)$, we can decompose $B(p)$ into a sum of individual basis functions $B_j(p) = \chi_j(p) B(p)$ as in the paper.

We define D's local domain graph, $\mathcal{G}_D$, with nodes $v \in V(\mathcal{G}_D)$ corresponding to the components D_v and with edges $\{v, w\}$ for any two regions sharing a seam (i.e. $\overline{D_v} \cap \overline{D_w} \cap \sigma \neq \emptyset$). $\mathcal{G}_D$ is a directed graph since we care about the direction in which we cross a seam, but because its edges always come in pairs $e = \{v, w\}, e^{-1} = \{w, v\}$, we pretend that it is undirected when discussing spanning trees and cycles.

For convenience, we consider functions on $\mathcal{G}_D$'s vertices that are equivalent to the mesh functions spanned by B_j due to the mapping:

$$\begin{array}{ccc} \{V(\mathcal{G}_D) \to \mathbb{C}\} & \longrightarrow & \{M \to \mathbb{C}\} \\ F(v) & \mapsto & F(p) = \sum_{v \in V(\mathcal{G}_D)} F(v) B_v(p). \end{array} \qquad (1)$$

Now we can replace the seam constraints for a function on the mesh with the constraint that $F : V(\mathcal{G}_D) \to \mathbb{C}$ satisfies:

Seam Constraint (Graph): For any edge $\{v, w\} = e \in E(\mathcal{G}_D)$ we must have:

$$F(w) = r_{\ell(e)} F(v) + t_{\ell(e)}$$

where $\ell(e) \in \Sigma$ is the seam corresponding to edge e.[1]

Notation: Given a path $\pi \subset E(\mathcal{G}_D)$ and given an edge along the path, $e \in \pi$, we denote by π_e^- and π_e^+ the two halves of the path on either side of e:

$$\pi = \pi_e^+ \circ e \circ \pi_e^-.$$

[1] Note that this formulation implicitly assumes that the basis functions form a partition of unity.

Definition: We define the product of rotations along the path $\pi \subset E(\mathcal{G}_D)$ as:

$$R(\pi) = \prod_{e \in \pi} r_{\ell(e)}.$$

For consistency, we set $R(\emptyset) = 1$.

Definition: Similarly, for a given seam $\sigma \in \Sigma$, we define:

$$T^\sigma(\pi) = \sum_{e \in \pi} R(\pi_e^+) \delta_{\sigma,e} \text{ with } \delta_{\sigma,e} = \begin{cases} 1 & \text{if } \sigma = \ell(e) \\ -r_\sigma^{-1} & \text{if } \sigma = \ell(e^{-1}) \\ 0 & \text{otherwise} \end{cases}.$$

For consistency, we set $T^\sigma(\emptyset) = 0$.

Proof of Proposition 1: Letting $\pi_2 \circ \pi_1$ denote the concatenation of two paths, and letting π^{-1} denote the reverse of a path, we have:

$$\begin{aligned} R(\pi_2 \circ \pi_1) = R(\pi_1) \cdot R(\pi_2) & \quad T^\sigma(\pi_2 \circ \pi_1) = R(\pi_2) T^\sigma(\pi_1) + T^\sigma(\pi_2) \\ R(\pi^{-1}) = R^{-1}(\pi) & \quad T^\sigma(\pi^{-1}) = -R^{-1}(\pi) T^\sigma(\pi) \\ R(\pi^{-1} \circ \pi) = 1 & \quad T^\sigma(\pi^{-1} \circ \pi) = 0. \end{aligned} \qquad (2)$$

Thus, the definitions of $R(\pi)$ and $T^\sigma(\pi)$ are homotopy-invariant.

Proof of Proposition 2

Once we know graph function F's value on some vertex $u \in \mathcal{G}_D$, the seam constraints for the edges of a spanning tree, $\tilde{\mathcal{G}}_D$, determine the values at all $v \in \mathcal{G}_D$ because $\mathcal{G}_D$ is strongly connected. We show that these values are given by

$$\tilde{F}(v) = c_u \tilde{F}_u(v) + \sum_{\sigma \in \Sigma} t_\sigma \tilde{F}_u^\sigma(v), \qquad (3)$$

with positional and translational functions $\tilde{F}_u$ and $\tilde{F}_u^\sigma$ defined below. Of course, when $\tilde{\mathcal{G}}_D \neq \mathcal{G}_D$ (because of cycles), there will be an additional constraint added by each "undirected" edge not in this spanning tree, but this is not considered for Proposition 2.

Traversing a path π_{vu} in $\tilde{\mathcal{G}}_D$ from u to v, we accumulate a rotation of $R(\pi_{vu})$ and a contribution of $T^\sigma(\pi_{vu})$ to t_σ's coefficient. This motivates the following definitions:

$$\tilde{F}_u(v) = R(\pi_{vu}), \qquad \tilde{F}_u^\sigma(v) = \frac{T^\sigma(\pi_{vu})}{2}, \qquad (4)$$

which indeed satisfy the constraints as we prove in Lemma 1. Since all paths on tree $\tilde{\mathcal{G}}_D$ are homotopy equivalent and $R(\pi_{vu})$ and $T^\sigma(\pi_{vu})$ are homotopy-invariant, these functions are uniquely defined. The division by two is needed because of our redundant use of both t_σ and $t_{\sigma^{-1}}$: for every edge $e \in \pi_{vu}$, we accumulate not only translation $t_{\ell(e)}$ but also the identical translation $-r_{\ell(e)} t_{\ell(e^{-1})}$.

Lemma 1. *Function $\tilde{F}(v)$ in (3) with translational and positional functions from (4) satisfies every seam constraint in $\tilde{\mathcal{G}}_D$.*

Proof. Taking any $\{v,w\} = e \in E(\tilde{\mathcal{G}}_D)$,

$$\begin{aligned}\tilde{F}(w) &= c_u R(e\circ\pi_{vu}) + \sum_{\sigma\in\Sigma} t_\sigma \frac{T^\sigma(e\circ\pi_{vu})}{2}\\ &= r_{\ell(e)} c_u R(\pi_{vu}) + \sum_{\sigma\in\Sigma} t_\sigma \frac{r_{\ell(e)} T^\sigma(\pi_{vu}) + \delta_{\sigma,e}}{2}\\ &= r_{\ell(e)}\left[c_u R(\pi_{vu}) + \sum_{\sigma\in\Sigma} t_\sigma \frac{T^\sigma(\pi_{vu})}{2}\right] + \frac{t_{\ell(e)} - r_{\ell(e)} t_{\ell(e^{-1})}}{2}\\ &= r_{\ell(e)}\tilde{F}(v) + t_{\ell(e)},\end{aligned}$$

so we see that the edge's constraint is satisfied. □

Completing the Proof: Picking $u = 0$, corresponding to the reference domain D_0, and applying mapping (1), the constrained mesh function can be written as:

$$\begin{aligned}F(p) &= \sum_j \left[c_0 R(\pi_{j0}) + \sum_{\sigma\in\Sigma} t_\sigma \frac{T^\sigma(\pi_{j0})}{2}\right] B_j(p)\\ &= c_0\left[\sum_j R(\pi_{j0}) B_j(p)\right] + \sum_{\sigma\in\Sigma} t_\sigma \left[\sum_j \frac{T^\sigma(\pi_{j0})}{2} B_j(p)\right],\end{aligned}$$

revealing $\tilde{B}$ and $\hat{B}^\sigma_{loc}$ as presented in Proposition 2.

Accounting for Cycles in $\mathcal{G}_D$

The constraints on edges $E^{\mathrm{cyc}} := E(\tilde{\mathcal{G}}_D)\setminus E(\mathcal{G}_D)$ (if any exist) generally remain unsatisfied by (3): for $\{v,w\} = e \in E^{\mathrm{cyc}}$,

$$\begin{aligned}r_{\ell(e)}\tilde{F}(v) + t_{\ell(e)} &= r_{\ell(e)}\left[c_u R(\pi_{vu}) + \sum_{\sigma\in\Sigma} t_\sigma \frac{T^\sigma(\pi_{vu})}{2}\right] + t_{\ell(e)}\\ &\neq c_u R(\pi_{wu}) + \sum_{\sigma\in\Sigma} t_\sigma \frac{T^\sigma(\pi_{wu})}{2} = \tilde{F}(w).\end{aligned}$$

However, we can construct a new graph function F satisfying this constraint. The clearest way is to first traverse back to the source vertex u on both sides of the constraint by following π_{wu}^{-1}:

$$\begin{aligned}&c_u R(\pi_{wu}^{-1}\circ e\circ\pi_{vu}) + \sum_{\sigma\in\Sigma} t_\sigma \frac{T^\sigma(\pi_{wu}^{-1}\circ e\circ\pi_{vu})}{2}\\ &\qquad \stackrel{!}{=} c_u R(\pi_{wu}^{-1}\circ\pi_{wu}) + \sum_{\sigma\in\Sigma} t_\sigma \frac{T^\sigma(\pi_{wu}^{-1}\circ\pi_{wu})}{2} = c_u.\end{aligned}$$

Rearranging, we arrive at the linear equality constraint on the positional and translational coefficients:

$$c_u(R(\zeta_e) - 1) + \sum_{\sigma\in\Sigma} t_\sigma \frac{T^\sigma(\zeta_e)}{2} \stackrel{!}{=} 0,$$

where $\zeta_e = \pi_{wu}^{-1}\circ e\circ\pi_{vu}$ is the cycle formed by adding edge e to $\tilde{\mathcal{G}}_D$. Note that if $R(\zeta_e) = 1$, positional coefficient c_u disappears from the constraint, and we're left with a constraint on the translation variables, t_σ, only. However, when $R(\zeta_e) \neq 1$, we choose to eliminate c_u using the constraint:

$$c_u = \frac{1}{1 - R(\zeta_e)} \sum_{\sigma\in\Sigma} t_\sigma \frac{T^\sigma(\zeta_e)}{2}.$$

In doing this, we remove $\tilde{F}_u^{\sigma_m}$ (mesh function $\tilde{B}$) as a distinct basis function, but we fold its contribution into the translational basis functions that c_u depends on.

We repeat this procedure for every $e \in E^{\mathrm{cyc}}$, collecting all translational coefficient constraints into a linear system. We then solve for a set of independent translations, t_n for $n \in I^t_{ind}$, determining dependent translations t_d for $d \in I^t_{dep}$ by a linear combination: $t_d = \sum_{n\in I^t_{ind}} w_{dn} t_n$. For convenience, we extend this weight matrix W to compute all translations (t_m for $m \in I^t_{all} = I^t_{ind}\cup I^t_{dep}$) by adding rows $w_{mn} = \delta_{mn}$ (Kronecker delta) for $m,n \in I^t_{ind}$.

If all cycles have $R(\zeta_e) = 1$, our final graph function F is given by expressing $\tilde{F}$ in terms of the independent translations t_n for $n \in I^t_{ind}$:

$$\begin{aligned}F(v) &= c_u\tilde{F}_u(v) + \sum_{m\in I^t_{all}}\left[\sum_{n\in I^t_{ind}} w_{mn} t_n\right]\tilde{F}_u^{\sigma_m}(v)\\ &= c_u\tilde{F}_u(v) + \sum_{n\in I^t_{ind}} t_n \left[\sum_{m\in I^t_{all}} w_{mn}\tilde{F}_u^{\sigma_m}(v)\right].\end{aligned}$$

Here we used the fact that each translational coefficient t_m corresponds to some seam σ_m. If any of the cycles has $R(\zeta_e) \neq 1$, eliminating the positional coefficient c_u gives:

$$\begin{aligned}F(v) &= \sum_{m\in I^t_{all}}\left[\sum_{n\in I^t_{ind}} t_n w_{mn}\right]\left[\tilde{F}_u^{\sigma_m}(v) + \frac{1}{1-R(\zeta_e)}\frac{T^{\sigma_m}(\zeta_e)}{2}\tilde{F}_u(v)\right]\\ &= \sum_{n\in I^t_{ind}} t_n \left(\sum_{m\in I^t_{all}} w_{mn}\left[\tilde{F}_u^{\sigma_m}(v) + \frac{1}{1-R(\zeta_e)}\frac{T^{\sigma_m}(\zeta_e)}{2}\tilde{F}_u(v)\right]\right).\end{aligned}$$

By mapping these into mesh functions and repeating for each general basis function B, we arrive at Proposition 3.