Planar Elastica

Julian Panetta

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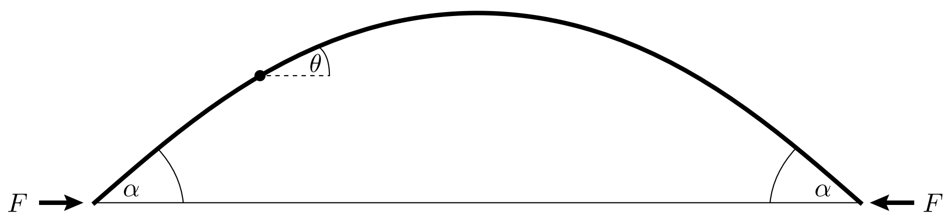

This document derives an expression for the curve traced out by a buckled elastic rod of length L whose endpoints have been pinned at a distance $a < L$ apart (the endpoints cannot move, but the tangents at these points are free to rotate). The pin constraints apply an unknown force F along the line connecting the endpoints.

This curve is the unique minimizer of the elastic bending energy:

$$E_b = \frac{1}{2}\int_0^L \kappa^2 \, \mathrm{d}s,$$

where $\kappa := \|\frac{\mathrm{d}n}{\mathrm{d}s}\|$ is the curvature, or the rate at which the normal/tangent vector change as we traverse the curve at unit speed. For simplicity, we have set the rod's bending stiffness, YI to 1. See Section 3 for a discussion on how the results change for different stiffnesses.

1 Deriving an ODE

First, we derive an ODE describing the curve's shape using the calculus of variations. We simplify the problem by switching to a formulation where a curve is represented by function $\theta(s)$ giving the angle between the tangent vector and the horizontal direction at arc length s along the curve. Once we find such a function, we can recover a parametric expression for the curve by integrating:

$$\begin{bmatrix} x(\bar{s}) \\ y(\bar{s}) \end{bmatrix} = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} + \int_0^{\bar{s}} \begin{bmatrix} \cos\big(\theta(s)\big) \\ \sin\big(\theta(s)\big) \end{bmatrix} \, \mathrm{d}s. \tag{1}$$

This representation is nice since it yields a simple curvature expression $\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s}$, making the bending energy:

$$E_b[\theta] = \frac{1}{2}\int_0^L \left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 \, \mathrm{d}s.$$

We now seek the function $\theta(s)$ that minimizes $E_b[\theta]$. Notice that the bending energy is invariant to rigid motion of the curve. The representation $\theta(s)$ already factors out the global translation of the curve (which the integration in (1) reintroduces by choosing the initial endpoint coordinates $x(0)$ and $y(0)$), but it still allows global rotations. We pin down the global rotation by requiring the second endpoint to also touch the x axis. By symmetry, this means the curve tangents will make an angle of α and $-\alpha$ with the x axis at the beginning and end, respectively.

We enforce the pin constraints on the endpoints using Lagrange multipliers:

$$\mathcal{L}(\theta, \lambda) = \frac{1}{2}\int_0^L \left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 \, \mathrm{d}s + \lambda(x(L) - a),$$

where we can neglect the Lagrange multiplier for the constraint $y(L) = 0$ since it will be zero—this constraint simply specifies the curve's global rotation, meaning it acts in the nullspace of the bending energy.

The minimizer must satisfy:

$$\left\langle \frac{\partial \mathcal{L}}{\partial \theta}(\theta, \lambda), \psi \right\rangle := \frac{\partial}{\partial t}\bigg|_{t=0} \mathcal{L}(\theta + t\psi, \lambda) = \int_0^L \frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}\psi}{\mathrm{d}s} \, \mathrm{d}s - \lambda \int_0^L \sin\big(\theta(s)\big)\psi \, \mathrm{d}s = 0 \quad \forall \psi.$$

Integrating the first term by parts:

$$0 = -\int_0^L \frac{\mathrm{d}^2\theta}{\mathrm{d}s^2}\psi \, \mathrm{d}s + \left[\frac{\mathrm{d}\theta}{\mathrm{d}s}\psi\right]_0^L - \lambda \int_0^L \sin\big(\theta(s)\big)\psi \, \mathrm{d}s = 0 \quad \forall \psi.$$

Varying ψ in the interior $(0, L)$ and at the endpoints, we find:

$$\begin{aligned} -\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} &= \lambda \sin\big(\theta(s)\big) \quad \text{for } s \in (0, L) \\ \frac{\mathrm{d}\theta}{\mathrm{d}s}(0) &= \frac{\mathrm{d}\theta}{\mathrm{d}s}(L) = 0. \end{aligned}$$

The boundary conditions at the two endpoints mean that the ends have no curvature (this is expected: the tangent is free to rotate at the endpoints to relieve any curvature). Finally, we note the physical significance of the Lagrange multiplier λ: it is actually the horizontal force F applied to pin the endpoints (it is multiplied by $x(L)$ in the Lagrangian to compute the work done by pin constraint). So we write our final ODE as:

$$\begin{aligned} -\theta'' &= F\sin(\theta) \quad \text{for } s \in (0, L) \\ \theta'(0) &= \theta'(L) = 0, \end{aligned} \tag{2}$$

where F is the *unknown* force applied at the endpoints and we've used $'$ to denote differentiation with respect to s. For a given F, this ODE uniquely defines the planar elastica curve—we simply need to find the F that holds the endpoints at distance a apart.

Remarkably, (2) is also describes the motion of a nonlinear (finite amplitude) pendulum: the equilibrium curve's tangent vector "swings" from angle α to $-\alpha$ as we move at unit speed along the curve as exactly like a pendulum would oscillate.

2 Integrating the ODE

2.1 Determining F and α

We now integrate (2) to obtain an expression for the minimizing curve in terms of the (unknown) applied force F. We use an integrating factor, multiplying both sides of the ODE by θ':

$$\begin{aligned} \theta''\theta' + F\sin(\theta)\theta' &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{2}(\theta')^2 - F\cos(\theta)\right) = 0 \\ \implies \quad \left[\frac{1}{2}(\theta')^2 - F\cos(\theta)\right]_0^s &= \frac{1}{2}(\theta')^2 - F\cos(\theta) - F\cos(\alpha) = 0 \\ \implies \quad \theta' &= -\sqrt{2F\big(\cos(\theta) - \cos(\alpha)\big)}, \end{aligned} \tag{3}$$

where we used the fact that $\theta(0) = \alpha$ by our choice for the curve's global rotation (α currently unknown), and chose the sign of θ' by realizing that the angle decreases monotonically as we traverse the curve.

Next, we integrate from $(\theta = \alpha, s = 0)$, to distance $\overline{s} \in [0, L]$ along the curve (where $\theta = \overline{\theta}$):

$$\int_{\alpha}^{\overline{\theta}} -\frac{\mathrm{d}\theta}{\sqrt{2F\big(\cos(\theta) - \cos(\alpha)\big)}} = \int_0^{\overline{s}} \mathrm{d}s = \overline{s}.$$

The integral on the left is an elliptical integral, which we express in standard form by applying some transformations. First, we use the identity $\cos(x) = 1 - 2\sin^2\left(\frac{x}{2}\right)$:

$$\int_{\overline{\theta}}^{\alpha} \frac{\mathrm{d}\theta}{\sqrt{\sin^2\left(\frac{\alpha}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}} = 2\overline{s}\sqrt{F}. \tag{4}$$

Next, we change variables, introducing φ such that:

$$\underbrace{\sin\left(\frac{\alpha}{2}\right)}_{:=k} \sin(\varphi) = \sin\left(\frac{\theta}{2}\right),$$

where we defined the *elliptic modulus* κ. Differentiating this equation to relate $\mathrm{d}\varphi$ and $\mathrm{d}\theta$:

$$\begin{aligned} k\cos(\varphi)\mathrm{d}\varphi = \frac{1}{2}\cos\left(\frac{\theta}{2}\right)\mathrm{d}\theta = \frac{1}{2}\sqrt{1 - \sin^2\left(\frac{\theta}{2}\right)}\mathrm{d}\theta = \frac{1}{2}\sqrt{1 - k^2\sin^2(\varphi)}\mathrm{d}\theta \\ \implies \quad \mathrm{d}\theta = \frac{2k\cos(\varphi)}{\sqrt{1 - k^2\sin^2(\varphi)}}\mathrm{d}\varphi, \end{aligned} \tag{5}$$

allowing us to rewrite the integral in (4):

$$\begin{aligned} \int_{\overline{\varphi}}^{\frac{\pi}{2}} \frac{\cancel{2}\cancel{k\cos(\varphi)}}{\sqrt{1 - k^2\sin^2(\varphi)}\cancel{\sqrt{k^2 - k^2\sin^2(\varphi)}}}\mathrm{d}\varphi = \cancel{2}\overline{s}\sqrt{F} \\ \implies \quad \int_{\overline{\varphi}}^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2\sin^2(\varphi)}}\mathrm{d}\varphi = \overline{s}\sqrt{F}, \end{aligned} \tag{6}$$

where $\overline{\varphi} = \arcsin\left(\frac{\sin(\overline{\theta}/2)}{\sin(\alpha/2)}\right)$. Equation (6) now gives us a semi-explicit relationship between the tangent's angle with the horizontal and the arc length along the curve. But currently there are two unknown quantities: the force F and the initial/final angle α. We can use (6) to determine F in terms of α: we know that $\overline{\varphi} = \arcsin\left(\frac{\sin(-\alpha/2)}{\sin(\alpha/2)}\right) = -\frac{\pi}{2}$ when $\overline{s} = L$. This means:

$$2\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2\sin^2(\varphi)}}\mathrm{d}\varphi = 2K\left(\sin\left(\frac{\alpha}{2}\right)\right) = L\sqrt{F} \quad \implies \quad \boxed{F = \frac{4K\left(\sin\left(\frac{\alpha}{2}\right)\right)^2}{L^2},}$$

where K is the complete elliptic integral of the first kind.

Finally, we use the constraint $x(L) - x(0) = \int_0^L \cos(\theta)\,\mathrm{d}s = a$ to find α. From (3) and (5), we have:

$$\mathrm{d}s = \frac{\cancel{2}\cancel{k\cos(\varphi)}}{\cancel{2}\sqrt{F}\sqrt{1 - k^2\sin^2(\varphi)}\cancel{\sqrt{k^2 - k^2\sin^2(\varphi)}}}\mathrm{d}\varphi = \frac{\mathrm{d}\varphi}{\sqrt{F}\sqrt{1 - k^2\sin^2(\varphi)}}. \tag{7}$$

Using the identity $\cos(\theta) = 1 - 2\sin^2\left(\frac{\theta}{2}\right) = 1 - 2k^2\sin^2(\varphi)$:

$$\Big(x(L) - x(0)\Big)\sqrt{F} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 - 2\kappa^2\sin^2(\varphi)}{\sqrt{1 - k^2\sin^2(\varphi)}}\,\mathrm{d}\varphi = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 - \kappa^2\sin^2(\varphi)}{\sqrt{1 - k^2\sin^2(\varphi)}}\,\mathrm{d}\varphi - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2\sin^2(\varphi)}}\,\mathrm{d}\varphi \quad (8)$$

$$\implies \quad a\sqrt{F} = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - k^2\sin^2(\varphi)}\,\mathrm{d}\varphi - 2K(k) = 4E(k) - 2K(k),$$

where $E(k)$ is the complete elliptic integral of the second kind. Plugging in our expression for F, we obtain:

$$\boxed{\frac{a}{L} = 2\frac{E(k)}{K(k)} - 1.} \quad (9)$$

For a given relative span $\frac{a}{L}$, this nonlinear equation can be solved efficiently with Newton's method to determine the elliptic modulus k as well as the initial orientation of the curve tangent, $\boxed{\alpha = 2\arcsin(k).}$

2.2 Euler's Critical Load

We can determine the force at which the rod initiates buckling by evaluating F at $\frac{a}{L} = 1$. In this case, solving (9) gives $k = 0$, so unsurprisingly $\alpha = 0$, and the force is $F_{\mathrm{cr}} = \frac{\pi^2}{L^2}$.

2.3 Parametric Curve Solution

With the elliptic modulus k finally in hand, we can express the bending energy minimizer as a parametric curve $(x(\overline{\varphi}), y(\overline{\varphi}))$. Note that this is not an arc length parametrization—that would require inverting the $\varphi \mapsto s$ relationship (6). To simplify the formula, we place the initial endpoint at $\left(-\frac{a}{2}, 0\right)$ so that the curve is symmetric around the y axis. Then we can compute the x coordinate function by integrating from $\varphi = \theta = 0$ (which happens at $x = 0$) to the desired value of $\overline{\varphi}$:

$$x(\overline{\varphi}) = \frac{-1}{\sqrt{F}}\Bigg(\underbrace{2\int_0^{\overline{\varphi}} \sqrt{1 - k^2\sin^2\phi}\,\mathrm{d}\varphi - \int_0^{\overline{\varphi}} \frac{1}{\sqrt{1 - k^2\sin^2(\varphi)}}\,\mathrm{d}\varphi}_{\text{Incomplete integral version of (8)}}\Bigg) = \frac{L}{2K(k)}\Big(F(\overline{\varphi}; k) - 2E(\overline{\varphi}; k)\Big),$$

where $F(x; k)$ and $E(x; k)$ are the incomplete elliptic integrals of the first and second kind, respectively. The sign is due to the fact that increasing φ actually moves *leftward*, not rightward.

We can obtain the $y(\overline{\varphi})$ coordinate function similarly. Starting from some unknown height $y(0) = h$, the curve descends to $y\left(\frac{\pi}{2}\right) = y\left(-\frac{\pi}{2}\right) = 0$ as we traverse either left or right:

$$\begin{aligned}
y(\overline{\varphi}) &= h - \int_0^{\overline{s}(\overline{\varphi})} |\sin(\theta(s))|\,\mathrm{d}s = h - \int_0^{\overline{s}(\overline{\varphi})} \sqrt{1 - \cos^2(\theta(s))}\,\mathrm{d}s = h - \int_0^{\overline{s}(\overline{\varphi})} \sqrt{1 - \left(1 - 2\sin^2\left(\frac{\theta}{2}\right)\right)^2}\,\mathrm{d}s \\
&= h - \int_0^{\overline{s}(\overline{\varphi})} \sqrt{4\sin^2\left(\frac{\theta}{2}\right) - 4\sin^4\left(\frac{\theta}{2}\right)}\,\mathrm{d}s = h - \int_0^{\overline{s}(\overline{\varphi})} 2\sin\left(\frac{\theta}{2}\right)\sqrt{1 - \sin^2\left(\frac{\theta}{2}\right)}\,\mathrm{d}s \\
&= h - \int_0^{\overline{s}(\overline{\varphi})} 2k\sin(\varphi)\sqrt{1 - k^2\sin^2(\varphi)}\,\mathrm{d}s = h - \int_0^{\overline{\varphi}} 2k\sin(\varphi)\cancel{\sqrt{1 - k^2\sin^2(\varphi)}}\frac{\mathrm{d}\varphi}{\sqrt{F}\cancel{\sqrt{1 - k^2\sin^2(\varphi)}}} \\
&= h - \frac{2k}{\sqrt{F}}(1 - \cos(\overline{\varphi})).
\end{aligned}$$

We can find h by evaluating at an endpoint:

$$0 = y\left(\frac{\pi}{2}\right) = h - \frac{2k}{\sqrt{F}}\left(1 - \cos\left(\frac{\pi}{2}\right)\right) = h - \frac{kL}{K(k)} \quad \Longrightarrow \quad \boxed{h = \frac{kL}{K(k)},}$$

and the full parametric curve representation is:

$$\boxed{\begin{bmatrix} x(\overline{\varphi}) \\ y(\overline{\varphi}) \end{bmatrix} = \begin{bmatrix} \frac{L}{2K(k)}\Big(F(\overline{\varphi}; k) - 2E(\overline{\varphi}; k)\Big) \\ \frac{kL}{K(k)}\cos(\overline{\varphi}) \end{bmatrix}.}$$

3 Scaling With Stiffness

If we introduce the bending stiffness YI, where Y is the Young's modulus and I is the cross section's moment of inertia, the bending energy scales to:

$$E_b = \frac{1}{2}\int_0^L YI\ \kappa^2\ \mathrm{d}s,$$

and some of the quantities we've derived above change. The force applied to the endpoints scales linearly with YI to become:

$$F = \frac{4YIK\left(\sin\left(\frac{\alpha}{2}\right)\right)^2}{L^2}.$$

The critical buckling load also scales to $F_{\mathrm{cr}} = \frac{\pi^2 YI}{L^2}$ However, the buckled curve itself and all measurements of it (like h, α, and k) are invariant to bending stiffness; they depend only on a and L.