Material Derivative of Summed Curvature

Julian Panetta

February 6, 2018

This document computes the material derivative of summed curvature as a surface is advected by some velocity field. It later compares against the derivation of (5.42) in [1] to point out some errors. We use the notation of [1], denoting the surface normal as $\boldsymbol{\nu}$ and using the formula $\kappa := \nabla_\Gamma \cdot \boldsymbol{\nu} = \kappa_1 + \kappa_2$ as our definition of summed curvature (other references use the opposite sign convention).

We note also that very similar derivations to the ones given here can also easily obtain the material derivative of the Gaussian curvature $\kappa_1\kappa_2$ due to the relationship $2\kappa_1\kappa_2 = (\kappa_1+\kappa_2)^2 - \kappa_1^2 - \kappa_2^2 = \kappa^2 - \|\nabla_\Gamma \boldsymbol{\nu}\|_F^2$.

1 Material Derivative Formula

We consider the first order change in the normal and summed curvature of surface Γ as it is advected by the map $\phi_\epsilon \mathbf{x} \mapsto \mathbf{x} + \epsilon \mathbf{V}$ to surface Γ_ϵ. We represent these surface quantities as fields over all $\mathbb{R}^n$ (meant to be evaluated at points on the moving surface), and denote quantities for Γ_ϵ with subscript ϵ. For convenience, we also define: $\tilde{\boldsymbol{\nu}}_\epsilon := \boldsymbol{\nu}_\epsilon \circ \phi_\epsilon$, which evaluates the perturbed surface's normal at a fixed material point $\mathbf{x} \in \Gamma$.

The material derivative of the unit normal vector is given by:

$$\dot{\boldsymbol{\nu}} := \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \tilde{\boldsymbol{\nu}}_\epsilon = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \frac{\nabla\phi_\epsilon{}^{-T}\boldsymbol{\nu}}{\|\nabla\phi_\epsilon{}^{-T}\boldsymbol{\nu}\|} = -(\nabla \mathbf{V})^T\boldsymbol{\nu} + \boldsymbol{\nu}\cdot\left((\nabla\mathbf{V})^T\boldsymbol{\nu}\right) = -(\nabla_\Gamma \mathbf{V})^T \boldsymbol{\nu}.$$

We compute summed curvature on Γ_ϵ as:

$$\kappa_\epsilon \circ \phi_\epsilon = \operatorname{tr}\left(\nabla_{\Gamma_\epsilon}\boldsymbol{\nu}_\epsilon\right)\circ\phi_\epsilon = \operatorname{tr}\left(\nabla\tilde{\boldsymbol{\nu}}_\epsilon(\nabla\phi_\epsilon)^{-1}\left[I - \tilde{\boldsymbol{\nu}}_\epsilon \otimes \tilde{\boldsymbol{\nu}}_\epsilon\right]\right),$$

and its material derivative as:

$$\begin{aligned}
\dot{\kappa} := \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}(\kappa_\epsilon\circ\phi_\epsilon) &= \operatorname{tr}\left(\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\left(\nabla\tilde{\boldsymbol{\nu}}_\epsilon(\nabla\phi_\epsilon)^{-1}\left[I - \tilde{\boldsymbol{\nu}}_\epsilon\otimes\tilde{\boldsymbol{\nu}}_\epsilon\right]\right)\right) \\
&= \operatorname{tr}\left(\nabla\dot{\boldsymbol{\nu}}[I - \boldsymbol{\nu}\otimes\boldsymbol{\nu}] + \nabla\boldsymbol{\nu}\left.\frac{\mathrm{d}(\nabla\phi_\epsilon)^{-1}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}[I - \boldsymbol{\nu}\otimes\boldsymbol{\nu}] - \nabla\boldsymbol{\nu}\left[\dot{\boldsymbol{\nu}}\otimes\boldsymbol{\nu} + \boldsymbol{\nu}\otimes\dot{\boldsymbol{\nu}}\right]\right) \\
&= \operatorname{tr}\left(\nabla_\Gamma\dot{\boldsymbol{\nu}} - \nabla\boldsymbol{\nu}\nabla_\Gamma\mathbf{V} - \nabla\boldsymbol{\nu}\left[\dot{\boldsymbol{\nu}}\otimes\boldsymbol{\nu} + \boldsymbol{\nu}\otimes\dot{\boldsymbol{\nu}}\right]\right) \\
&= \operatorname{tr}\left(-\nabla_\Gamma[(\nabla_\Gamma\mathbf{V})^T\boldsymbol{\nu}] - \nabla\boldsymbol{\nu}\nabla_\Gamma\mathbf{V} - \nabla\boldsymbol{\nu}\left(\dot{\boldsymbol{\nu}}\otimes\boldsymbol{\nu} + \underbrace{\boldsymbol{\nu}\otimes[(\nabla_\Gamma\mathbf{V})^T\boldsymbol{\nu}]}_{(\boldsymbol{\nu}\otimes\boldsymbol{\nu})\nabla_\Gamma\mathbf{V}}\right)\right) \\
&= \operatorname{tr}\left(-\nabla_\Gamma[(\nabla_\Gamma\mathbf{V})^T\boldsymbol{\nu}] - \nabla\boldsymbol{\nu}(I - \boldsymbol{\nu}\otimes\boldsymbol{\nu})\nabla_\Gamma\mathbf{V} - [(\nabla\boldsymbol{\nu})\dot{\boldsymbol{\nu}}]\otimes\boldsymbol{\nu}\right) \\
&= \operatorname{tr}\left(-\nabla_\Gamma[(\nabla_\Gamma\mathbf{V})^T\boldsymbol{\nu}] - \nabla_\Gamma\boldsymbol{\nu}\nabla_\Gamma\mathbf{V} - [(\nabla_\Gamma\boldsymbol{\nu})\dot{\boldsymbol{\nu}}]\otimes\boldsymbol{\nu}\right).
\end{aligned}$$

In the last line, we used the fact that $\boldsymbol{\nu}\cdot\dot{\boldsymbol{\nu}} = 0$ to replace the final $\nabla\boldsymbol{\nu}$ with $\nabla_\Gamma\boldsymbol{\nu}$. Now, evaluating the trace:

$$\begin{aligned}
\dot{\kappa} &= -\nabla_\Gamma\cdot[(\nabla_\Gamma\mathbf{V})^T\boldsymbol{\nu}] - \nabla_\Gamma\boldsymbol{\nu} : \nabla_\Gamma\mathbf{V} - \cancelto{0}{\boldsymbol{\nu}\cdot(\nabla_\Gamma\boldsymbol{\nu})}\dot{\boldsymbol{\nu}} \qquad (1) \\
&= -\nabla_\Gamma\cdot[\nabla_\Gamma(\mathbf{V}\cdot\boldsymbol{\nu}) - (\nabla_\Gamma\boldsymbol{\nu})^T\mathbf{V}] - \nabla_\Gamma\boldsymbol{\nu} : \nabla_\Gamma\mathbf{V} \\
&= -\triangle_\Gamma(\mathbf{V}\cdot\boldsymbol{\nu}) + \nabla_\Gamma\cdot[(\nabla_\Gamma\boldsymbol{\nu})^T\mathbf{V}] - \nabla_\Gamma\boldsymbol{\nu} : \nabla_\Gamma\mathbf{V} \\
&= -\triangle_\Gamma(\mathbf{V}\cdot\boldsymbol{\nu}) + [\nabla_\Gamma\cdot(\nabla_\Gamma\boldsymbol{\nu})^T]\cdot\mathbf{V}.
\end{aligned}$$

Applying identity (6) derived below, we find:

$$\boxed{\dot{\kappa} = -\,\triangle_\Gamma\,(\mathbf{V}\cdot\boldsymbol{\nu}) - (\mathbf{V}\cdot\boldsymbol{\nu})\|\nabla_\Gamma\boldsymbol{\nu}\|_F^2 + (\mathbf{V}\cdot\nabla_\Gamma)\kappa.} \tag{2}$$

This formula neatly separates the effects of tangential and normal velocity; decomposing the perturbation velocity into normal and tangential components, $\mathbf{V} = v_n\boldsymbol{\nu} + \mathbf{v}_t$, we find:

$$\dot{\kappa} = -\,\triangle_\Gamma\,v_n - v_n\|\nabla_\Gamma\boldsymbol{\nu}\|_F^2 + (\mathbf{v}_t\cdot\nabla_\Gamma)\kappa.$$

We could instead obtain the formula in [1] by applying the divergence product rule to the first term of (1):

$$\boxed{\dot{\kappa} = -(\triangle_\Gamma\mathbf{V})\cdot\boldsymbol{\nu} - 2\nabla_\Gamma\boldsymbol{\nu} : \nabla_\Gamma\mathbf{V}.} \tag{3}$$

2 Errata for [1]

The derivation of $\dot{\kappa}$ in [1] has two errors that cancel each other out. The first error is in the second term of

$$\dot{\kappa}_{\text{book}} = \kappa' + (\mathbf{V}\cdot\nabla_\Gamma)\kappa, \tag{4}$$

which neglects κ's normal variation ($\kappa = \nabla\cdot\boldsymbol{\nu}$ is generally *not* a constant normal extension). In fact, under a constant normal extension of $\boldsymbol{\nu}$, we will find $\frac{\partial\kappa}{\partial\nu} = -\|\nabla_\Gamma\boldsymbol{\nu}\|_F^2$ (see (7)). The correct formula should read:

$$\dot{\kappa}_{\text{correct}} = \kappa' + (\mathbf{V}\cdot\nabla)\kappa. \tag{5}$$

The second error occurs when manipulating κ', where $[\nabla_\Gamma\cdot(\nabla_\Gamma\boldsymbol{\nu})^T]\cdot\mathbf{V}$ is replaced by $(\mathbf{V}\cdot\nabla_\Gamma)(\nabla_\Gamma\cdot\boldsymbol{\nu})$. We now derive the correct version of this identity, denoting components of $\boldsymbol{\nu}$ by n_i and components of $\mathbf{V}$ by v_i.

$$\begin{aligned}
&[\nabla_\Gamma\cdot(\nabla_\Gamma\boldsymbol{\nu})^T]\cdot\mathbf{V}\\
&\quad= \mathbf{V}\cdot\operatorname{tr}\left(\left(\nabla\big[(I-\boldsymbol{\nu}\otimes\boldsymbol{\nu})(\nabla\boldsymbol{\nu})^T\big]\right)(I-\boldsymbol{\nu}\otimes\boldsymbol{\nu})\right)\\
&\quad= v_i\left(-\frac{\partial n_i n_l}{\partial x_k}\frac{\partial n_j}{\partial x_l} + (\delta_{il}-n_in_l)\frac{\partial}{\partial x_k}\frac{\partial n_j}{\partial x_l}\right)(\delta_{jk}-n_jn_k)\\
&\quad= v_i\left(-\frac{\partial n_i n_l}{\partial x_k}(\delta_{jk}-n_jn_k)\frac{\partial n_j}{\partial x_l} + (\delta_{il}-n_in_l)(\delta_{jk}-n_jn_k)\frac{\partial}{\partial x_l}\frac{\partial n_j}{\partial x_k}\right)\\
&\quad= -\underbrace{v_i\frac{\partial n_i n_l}{\partial x_k}(\delta_{jk}-n_jn_k)\frac{\partial n_j}{\partial x_l}}_{A} + \underbrace{v_i(\delta_{il}-n_in_l)\frac{\partial}{\partial x_l}\left((\delta_{jk}-n_jn_k)\frac{\partial n_j}{\partial x_k}\right)}_{(\mathbf{V}\cdot\nabla_\Gamma)(\nabla_\Gamma\cdot\boldsymbol{\nu})} + \underbrace{v_i(\delta_{il}-n_in_l)\frac{\partial n_jn_k}{\partial x_l}\frac{\partial n_j}{\partial x_k}}_{B}
\end{aligned}$$

$$\begin{aligned}
A &= -v_i\frac{\partial n_i}{\partial x_k}n_l(\delta_{jk}-n_jn_k)\frac{\partial n_j}{\partial x_l} - v_in_i\frac{\partial n_l}{\partial x_k}(\delta_{jk}-n_jn_k)\frac{\partial n_j}{\partial x_l}\\
&= -\mathbf{V}^T(\nabla_\Gamma\boldsymbol{\nu})(\nabla\boldsymbol{\nu})\boldsymbol{\nu} - (\mathbf{V}\cdot\boldsymbol{\nu})(\nabla_\Gamma\boldsymbol{\nu})^T : \nabla\boldsymbol{\nu}\\
&= -\mathbf{V}^T(\nabla_\Gamma\boldsymbol{\nu})(\nabla\boldsymbol{\nu})\boldsymbol{\nu} - (\mathbf{V}\cdot\boldsymbol{\nu})\nabla_\Gamma\boldsymbol{\nu} : \nabla_\Gamma\boldsymbol{\nu},\\
B &= v_i(\delta_{il}-n_in_l)\frac{\partial n_j}{\partial x_l}n_k\frac{\partial n_j}{\partial x_k} + v_i(\delta_{il}-n_in_l)n_j\frac{\partial n_k}{\partial x_l}\frac{\partial n_j}{\partial x_k}\\
&= [(\nabla_\Gamma\boldsymbol{\nu})\mathbf{V}]\cdot[(\nabla\boldsymbol{\nu})\boldsymbol{\nu}] + \boldsymbol{\nu}\cdot(\nabla\boldsymbol{\nu})\underbrace{(\nabla_\Gamma\boldsymbol{\nu})\mathbf{V}}_{\text{tangential}}\\
&= \mathbf{V}^T(\nabla_\Gamma\boldsymbol{\nu})^T(\nabla\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{\nu}\cdot\underbrace{(\nabla_\Gamma\boldsymbol{\nu})(\nabla_\Gamma\boldsymbol{\nu})\mathbf{V}}_{\text{tangential}}\\
&= \mathbf{V}^T(\nabla_\Gamma\boldsymbol{\nu})(\nabla\boldsymbol{\nu})\boldsymbol{\nu}.
\end{aligned}$$

Adding everything together:

$$[\nabla_\Gamma \cdot (\nabla_\Gamma \boldsymbol{\nu})^T] \cdot \mathbf{V} = (\mathbf{V} \cdot \nabla_\Gamma)\kappa - (\mathbf{V} \cdot \boldsymbol{\nu})\|\nabla_\Gamma \boldsymbol{\nu}\|_F^2. \tag{6}$$

Finally, if $\boldsymbol{\nu}$ is a constant normal extension,

$$\begin{aligned} 0 = \frac{\partial}{\partial x_i}\left(\frac{\partial n_i}{\partial x_k} n_k\right) = \frac{\partial^2 n_i}{\partial x_i \partial x_k} n_k + \frac{\partial n_i}{\partial x_k}\frac{\partial n_k}{\partial x_i} = n_k \frac{\partial}{\partial x_k}\frac{\partial n_i}{\partial x_i} + \frac{\partial n_i}{\partial x_k}\frac{\partial n_i}{\partial x_k} = (\boldsymbol{\nu} \cdot \nabla_\Gamma)\kappa + \|\nabla_\Gamma \boldsymbol{\nu}\|_F^2 \\ \Rightarrow \quad \frac{\partial \kappa}{\partial \nu} = -\|\nabla_\Gamma \boldsymbol{\nu}\|_F^2, \end{aligned} \tag{7}$$

and (6) takes an even simpler form:

$$[\nabla_\Gamma \cdot (\nabla_\Gamma \boldsymbol{\nu})^T] \cdot \mathbf{V} = (\mathbf{V} \cdot \nabla_\Gamma)\kappa + (\mathbf{V} \cdot \boldsymbol{\nu})\frac{\partial \kappa}{\partial \nu} = (\mathbf{V} \cdot \nabla_\Gamma)\kappa. \tag{8}$$

Notice that, if substituted into the book's derivation, corrected identity (8) would properly cancel with the corrected advection term in (5), just as the book's expression cancelled the advection term in (4).

References

[1] Shawn Walker. *The Shapes of Things*, chapter 5, pages 87–104.