Analytic Eigensystems for Isotropic Membrane Energies

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This document follows the approach of [2] to derive the Hessian eigenvalues and eigenmatrices for isotropic membrane energy densities $\psi(F)$, where F is a 3×2 deformation gradient. We assume that the energy is expressed in terms of the following generalizations for 3×2 matrices of the 2×2 tensor invariants[1]:

$$\begin{aligned} I_1^{3\times 2} &:= \sigma_1 + \sigma_2 \\ I_2^{3\times 2} &:= F : F = \sigma_1^2 + \sigma_2^2 \\ I_3^{3\times 2} &:= \sigma_1 \sigma_2. \end{aligned}$$

In these definitions, σ_1 and σ_2 are the singular values of F obtained from the singular value decomposition:

$$F = U \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}}_{\Sigma} V^T \qquad U \in O(3), V \in O(2).$$

We note that the third column of U is the deformed surface normal $\hat{\boldsymbol{n}}$.

1 Differentiating the SVD

We will need formulas for how U, Σ, and V change as F is perturbed with "velocity" $\dot{F}$, which we find by differentiating both sides of the SVD:

$$\dot{F} = \dot{U}\Sigma V^T + U\dot{\Sigma}V^T + U\Sigma\dot{V}^T \quad \Longrightarrow \quad U^T \dot{F} V = U^T \dot{U} \Sigma + \dot{\Sigma} + \Sigma \dot{V}^T V. \tag{1}$$

Differentiating the relationships $U^T U = \text{Id}_{3\times 3}$ and $V^T V = \text{Id}_{2\times 2}$ reveals that $U^T \dot{U}$ and $\dot{V}^T V$ are skew symmetric and can be written as the infinitesimal rotations:

$$U^T \dot{U} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}, \qquad \dot{V}^T V = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}.$$

Plugging these into (1), we obtain a formula for the infinitesimal rotations and singular value perturbations induced by $\dot{F}$:

$$U^T \dot{F} V = \begin{bmatrix} \dot{\sigma_1} & -(\sigma_2\omega_z + \sigma_1\alpha) \\ \sigma_1\omega_z + \sigma_2\alpha & \dot{\sigma_2} \\ -\sigma_1\omega_y & \sigma_2\omega_x \end{bmatrix}. \tag{2}$$

Geometrically, ω_z indicates a rotation of the surface element about the current normal $\hat{\boldsymbol{n}}$, while ω_x and ω_y are rotations around the principal stretch axes. When $\omega_x = \omega_y = 0$, the deformed surface element simply rotates in-plane around $\hat{\boldsymbol{n}}$ (and $\hat{\boldsymbol{n}}$ does not change). However, nonzero ω_x and ω_y indicate that $\dot{F}$ induces a rotation of $\hat{\boldsymbol{n}}$.

[1] The I_2 invariant used here is from [2]; the other standard definition of principal invariant $I_2 = \frac{1}{2}\left(\text{tr}(A)^2 - \|A\|_F^2\right)$ actually coincides with I_3 in the 2D case

1.1 Example Perturbations

According to (2), a perturbation of the form

$$\dot{F} = U \begin{bmatrix} a & b \\ c & d \\ 0 & 0 \end{bmatrix} V^T$$

leaves $\hat{\boldsymbol{n}}$ unchanged as it stretches/rotates the surface element in-plane. Specifically, we have $\dot{\sigma_1} = a$, $\dot{\sigma_2} = d$ and the following system for ω_z and α:

$$\begin{aligned} \sigma_2\omega_z + \sigma_1\alpha &= -b \\ \sigma_1\omega_z + \sigma_2\alpha &= c \end{aligned} \tag{3}$$

On the other hand, perturbation

$$\dot{F} = U \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ e & f \end{bmatrix} V^T$$

rotates the surface element's normal by angular velocities $\omega_x = f/\sigma_2, \omega_y = -e/\sigma_1$ without any in-plane stretch/rotation.

2 Gradients of the Invariants

We can now use the formulas for $\dot{\sigma_1}$ and $\dot{\sigma_2}$ to differentiate the invariants:

$$\frac{\partial I_1^{3\times2}}{\partial F} : \dot{F} = \dot{\Sigma} : \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \left(U^T \dot{F} V\right) : \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \dot{F} : \left(U \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} V^T\right) \implies \frac{\partial I_1^{3\times2}}{\partial F} = U \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} V^T,$$

$$\frac{\partial I_3^{3\times2}}{\partial F} : \dot{F} = \dot{\Sigma} : \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_1 \\ 0 & 0 \end{bmatrix} = \left(U^T \dot{F} V\right) : \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_1 \\ 0 & 0 \end{bmatrix} = \dot{F} : \left(U \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_1 \\ 0 & 0 \end{bmatrix} V^T\right) \implies \frac{\partial I_3^{3\times2}}{\partial F} = U \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_1 \\ 0 & 0 \end{bmatrix} V^T,$$

$$\frac{\partial I_2^{3\times2}}{\partial F} : \dot{F} = 2F : \dot{F} \implies \frac{\partial I_2^{3\times2}}{\partial F} = 2F.$$

3 Hessians of the Invariants

We evaluate the Hessian applied to an arbitrary perturbation $\dot{F}$. First, the easy invariant:

$$\frac{\partial^2 I_2^{3\times2}}{\partial F^2} : \dot{F} = 2\dot{F},$$

which means $\frac{\partial^2 I_2^{3\times2}}{\partial F^2}$ is a multiple of the fourth order identity tensor. Any orthogonal basis can be chosen as a set of eigenmatrices, and their corresponding eigenvalues are all 2.

Next, we consider $I_1^{3\times2}$:

$$U^T \left(\frac{\partial^2 I_1^{3\times2}}{\partial F^2} : \dot{F}\right) V = U^T \dot{U} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \dot{V}^T V = \begin{bmatrix} 0 & -(\omega_z + \alpha) \\ \omega_z + \alpha & 0 \\ -\omega_y & \omega_x \end{bmatrix}.$$

We plug in $\dot{F} = U \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} V^T$ and note that summing the equations in (3) yields $\omega_z + \alpha = \frac{c-b}{\sigma_1+\sigma_2}$. Thus:

$$\frac{\partial^2 I_1^{3\times2}}{\partial F^2} : \dot{F} = U \begin{bmatrix} 0 & \frac{b-c}{\sigma_1+\sigma_2} \\ \frac{c-b}{\sigma_1+\sigma_2} & 0 \\ \frac{e}{\sigma_1} & \frac{f}{\sigma_2} \end{bmatrix} V^T.$$

From this expression, we see there is a three dimensional null space with $e = f = 0$ and $b = c$. We can pick the following orthonormal basis for this subspace:

$$\frac{1}{\sqrt{2}}U\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}V^T,\quad \frac{1}{\sqrt{2}}U\begin{bmatrix}1 & 0\\0 & -1\\0 & 0\end{bmatrix}V^T,\quad \frac{1}{\sqrt{2}}U\begin{bmatrix}0 & 1\\1 & 0\\0 & 0\end{bmatrix}V^T \qquad (\lambda = 0).$$

We further deduce the three eigenmatrices with nonzero eigenvalues:

$$\underbrace{\frac{1}{\sqrt{2}}U\begin{bmatrix}0 & -1\\1 & 0\\0 & 0\end{bmatrix}V^T}_{\lambda=\frac{2}{\sigma_1+\sigma_2}},\quad \underbrace{U\begin{bmatrix}0 & 0\\0 & 0\\1 & 0\end{bmatrix}V^T}_{\lambda=\frac{1}{\sigma_1}},\quad \underbrace{U\begin{bmatrix}0 & 0\\0 & 0\\0 & 1\end{bmatrix}V^T}_{\lambda=\frac{1}{\sigma_2}}.$$

Finally, we consider $I_3^{3\times 2}$:

$$U^T\left(\frac{\partial^2 I_3^{3\times 2}}{\partial F^2} : \dot{F}\right)V = U^T\dot{U}\begin{bmatrix}\sigma_2 & 0\\0 & \sigma_1\\0 & 0\end{bmatrix} + \begin{bmatrix}\dot{\sigma}_2 & 0\\0 & \dot{\sigma}_1\\0 & 0\end{bmatrix} + \begin{bmatrix}\sigma_2 & 0\\0 & \sigma_1\\0 & 0\end{bmatrix}\dot{V}^TV = \begin{bmatrix}\dot{\sigma}_2 & -(\sigma_1\omega_z + \sigma_2\alpha)\\ \sigma_2\omega_z + \sigma_1\alpha & \dot{\sigma}_1\\ -\sigma_2\omega_y & \sigma_1\omega_x\end{bmatrix}.$$

Again plugging in $\dot{F} = U\begin{bmatrix}a & b\\c & d\\e & f\end{bmatrix}V^T$ and using the formulas from Section 1.1, we find:

$$\frac{\partial^2 I_3^{3\times 2}}{\partial F^2} : \dot{F} = U\begin{bmatrix}d & -c\\-b & a\\ \frac{\sigma_2}{\sigma_1}e & \frac{\sigma_1}{\sigma_2}f\end{bmatrix}V^T$$

We deduce the following eigenmatrices and eigenvalues:

$$\underbrace{\frac{1}{\sqrt{2}}U\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}V^T,\quad \frac{1}{\sqrt{2}}U\begin{bmatrix}0 & -1\\1 & 0\\0 & 0\end{bmatrix}V^T}_{\lambda=1},\quad \underbrace{\frac{1}{\sqrt{2}}U\begin{bmatrix}1 & 0\\0 & -1\\0 & 0\end{bmatrix}V^T,\quad \frac{1}{\sqrt{2}}U\begin{bmatrix}0 & 1\\1 & 0\\0 & 0\end{bmatrix}V^T}_{\lambda=-1},$$

$$\underbrace{U\begin{bmatrix}0 & 0\\0 & 0\\1 & 0\end{bmatrix}V^T}_{\lambda=\frac{\sigma_2}{\sigma_1}},\quad \underbrace{U\begin{bmatrix}0 & 0\\0 & 0\\0 & 1\end{bmatrix}V^T}_{\lambda=\frac{\sigma_1}{\sigma_2}}.$$

We note that for all invariants, four of the six Hessian eigenmatrices are simply padded versions of the 2D eigenmatrices from [2], while the last two are new and concern the rotation of the surface element's normal.

4 Example: Incompressible neo-Hookean Sheet

We consider the membrane energy of a thin sheet of incompressible neo-Hookean material [1]:

$$\psi_{\text{IncNeo}}(F_{3\text{D}}) = \frac{\mu}{2}\left(\text{tr}(F_{3\text{D}}^TF_{3\text{D}}) - 3\right) = \frac{\mu}{2}\left(I_2^{3\text{D}} - 3\right)$$

When the sheet experiences an in-plane deformation gradient $F \in \mathbb{R}^{3\times 2}$, it stretches or compresses in the normal direction to maintain $J = 1$. We can solve for the normal stretch as $\frac{1}{I_3^{3\times 2}}$ and express ψ_{IncNeo} directly in terms of F's invariants:

$$\psi_{\text{sheet}}(F) = \frac{\mu}{2}\left(I_2^{3\times 2} + \left(\frac{1}{I_3^{3\times 2}}\right)^2 - 3\right).$$

The Hessian of this energy density is:

$$\begin{aligned}\frac{\partial^2 \psi_{\text{sheet}}}{\partial F^2} &= \frac{\mu}{2}\left[\frac{\partial^2 I_2^{3\times 2}}{\partial F^2} + 6\left(\frac{1}{I_3^{3\times 2}}\right)^4 \frac{\partial I_2^{3\times 2}}{\partial F} \otimes \frac{\partial I_2^{3\times 2}}{\partial F} - 2\left(\frac{1}{I_3^{3\times 2}}\right)^3 \frac{\partial^2 I_3^{3\times 2}}{\partial F^2}\right] \\ &= \mu\left[\text{Id}_4 + 3\left(\frac{1}{I_3^{3\times 2}}\right)^4 \left(U \begin{bmatrix}\sigma_2 & 0\\ 0 & \sigma_1\\ 0 & 0\end{bmatrix} V^T\right) \otimes \left(U \begin{bmatrix}\sigma_2 & 0\\ 0 & \sigma_1\\ 0 & 0\end{bmatrix} V^T\right) - \left(\frac{1}{I_3^{3\times 2}}\right)^3 \frac{\partial^2 I_3^{3\times 2}}{\partial F^2}\right].\end{aligned}$$

We note that $\frac{\partial I_2^{3\times 2}}{\partial F}$ is orthogonal to all but two of the eigenmatrices of $\frac{\partial^2 I_3^{3\times 2}}{\partial F^2}$ (and eigenmatrices for the fourth order identity tensor Id_4 can be chosen arbitrarily), so we immediately get the following four eigenpairs:

$$\underbrace{\frac{1}{\sqrt{2}} U \begin{bmatrix}0 & -1\\ 1 & 0\\ 0 & 0\end{bmatrix} V^T}_{\lambda = \mu - \mu\left(\frac{1}{I_3^{3\times 2}}\right)^3}, \quad \underbrace{\frac{1}{\sqrt{2}} U \begin{bmatrix}0 & 1\\ 1 & 0\\ 0 & 0\end{bmatrix} V^T}_{\lambda = \mu + \mu\left(\frac{1}{I_3^{3\times 2}}\right)^3}, \quad \underbrace{U \begin{bmatrix}0 & 0\\ 0 & 0\\ 1 & 0\end{bmatrix} V^T}_{\lambda = \mu - \mu\left(\frac{1}{I_3^{3\times 2}}\right)^3 \frac{\sigma_2}{\sigma_1}}, \quad \underbrace{U \begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix} V^T}_{\lambda = \mu - \mu\left(\frac{1}{I_3^{3\times 2}}\right)^3 \frac{\sigma_1}{\sigma_2}}.$$

Because $\frac{\partial I_2^{3\times 2}}{\partial F}$ is generally not orthogonal to either of the remaining two eigenmatrices of $\frac{\partial^2 I_3^{3\times 2}}{\partial F^2}$ (whose eigenvalues are distinct) we must diagonalize the projection of $\frac{\partial^2 \psi_{\text{sheet}}}{\partial F^2}$ onto their span to obtain the final two eigenpairs. We obtain simpler expressions using the basis $D_1 := U \begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix} V^T$ and $D_2 := U \begin{bmatrix}0 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix} V^T$ for this subspace, which results in the reduced Hessian:

$$\begin{bmatrix} D_1 : \frac{\partial^2 \psi_{\text{sheet}}}{\partial F^2} : D_1 & D_1 : \frac{\partial^2 \psi_{\text{sheet}}}{\partial F^2} : D_2 \\ D_2 : \frac{\partial^2 \psi_{\text{sheet}}}{\partial F^2} : D_1 & D_2 : \frac{\partial^2 \psi_{\text{sheet}}}{\partial F^2} : D_2 \end{bmatrix} = \mu \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{\mu}{\left(I_3^{3\times 2}\right)^4} \begin{bmatrix}3\sigma_2^2 & 2I_3^{3\times 2}\\ 2I_3^{3\times 2} & 3\sigma_1^2\end{bmatrix}.$$

The eigendecomposition of this 2×2 matrix can be expressed by introducing quantities $\beta := 3(\sigma_2^2 - \sigma_1^2)$ and $\gamma := \sqrt{16\left(I_3^{3\times 2}\right)^2 + \beta^2}$:

$$\mathbf{v}_1 = \begin{bmatrix}\beta - \gamma\\ 4I_3^{3\times 2}\end{bmatrix}, \quad \lambda_1 = \mu + \mu\frac{3I_2^{3\times 2} + \gamma}{2\left(I_3^{3\times 2}\right)^4}, \qquad \mathbf{v}_2 = \begin{bmatrix}\beta + \gamma\\ 4I_3^{3\times 2}\end{bmatrix}, \quad \lambda_2 = \mu + \mu\frac{3I_2^{3\times 2} + \gamma}{2\left(I_3^{3\times 2}\right)^4},$$

making the final two eigenpairs of $\frac{\partial^2 \psi_{\text{sheet}}}{\partial F^2}$:

$$\underbrace{U \begin{bmatrix}\beta - \gamma & 0\\ 0 & 4I_3^{3\times 2}\\ 0 & 0\end{bmatrix} V^T}_{\lambda = \mu + \mu\frac{3I_2^{3\times 2} + \gamma}{2\left(I_3^{3\times 2}\right)^4}}, \quad \underbrace{U \begin{bmatrix}\beta + \gamma & 0\\ 0 & 4I_3^{3\times 2}\\ 0 & 0\end{bmatrix} V^T}_{\lambda = \mu + \mu\frac{3I_2^{3\times 2} - \gamma}{2\left(I_3^{3\times 2}\right)^4}}.$$

Note that these eigenmatrices do not have unit norm and should be normalized.

References

[1] Javier Bonet and Richard D Wood. *Nonlinear continuum mechanics for finite element analysis.* Cambridge university press, 1997.

[2] Breannan Smith, Fernando De Goes, and Theodore Kim. Analytic eigensystems for isotropic distortion energies. *ACM Trans. Graph.*, 38(1):3:1–3:15, February 2019.