Electromagnetic Theory

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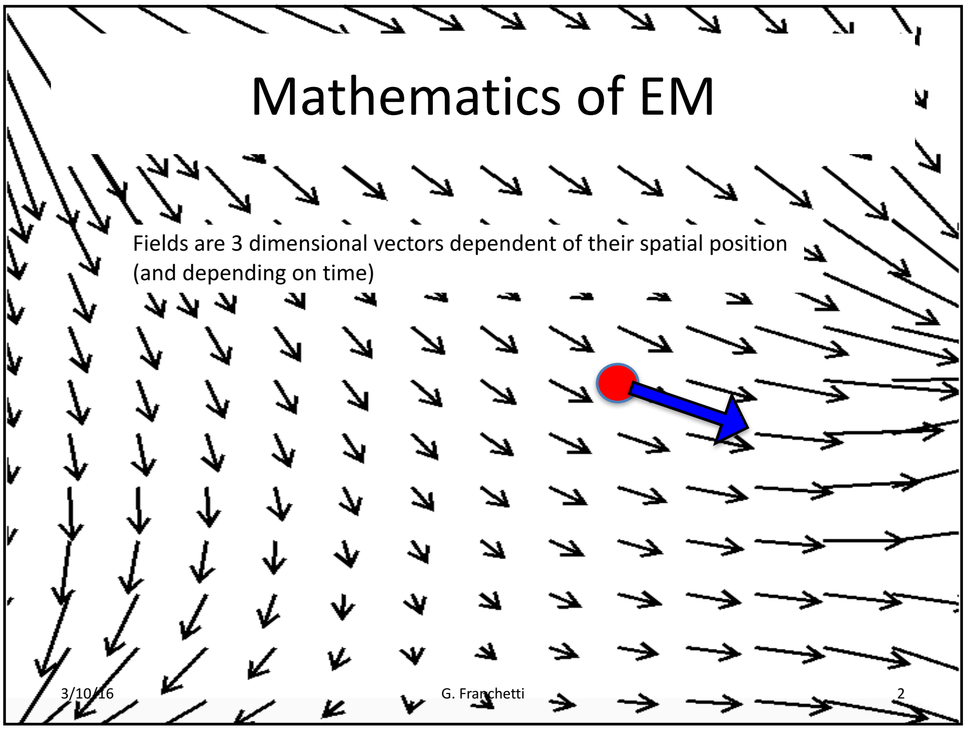
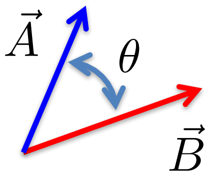
Mathematics of EM



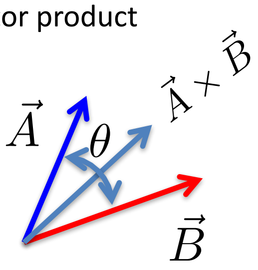
Fields are 3 dimensional vectors dependent of their spatial position (and depending on time)

Products

Scalar product



Vector product



$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

$$\vec{A}\times\vec{B} = AB\sin\theta\hat{v}$$

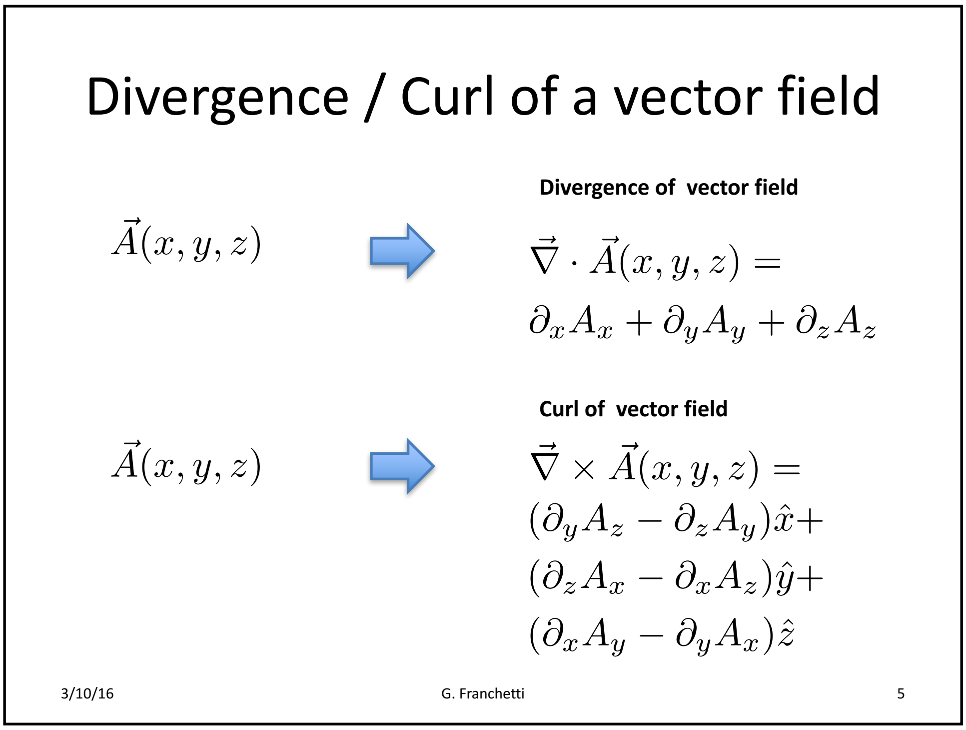
The gradient operator

$$\vec{\nabla} = (\partial_x, \partial_y, \partial_z)$$

Is an operator that transform space dependent scalar in vector

Example: given $f(x,y,z)$

$$\vec{\nabla}f(x,y,z) = (\partial_x f, \partial_y f, \partial_z f)$$



Divergence / Curl of a vector field

$\vec{A}(x,y,z)$ ⇨

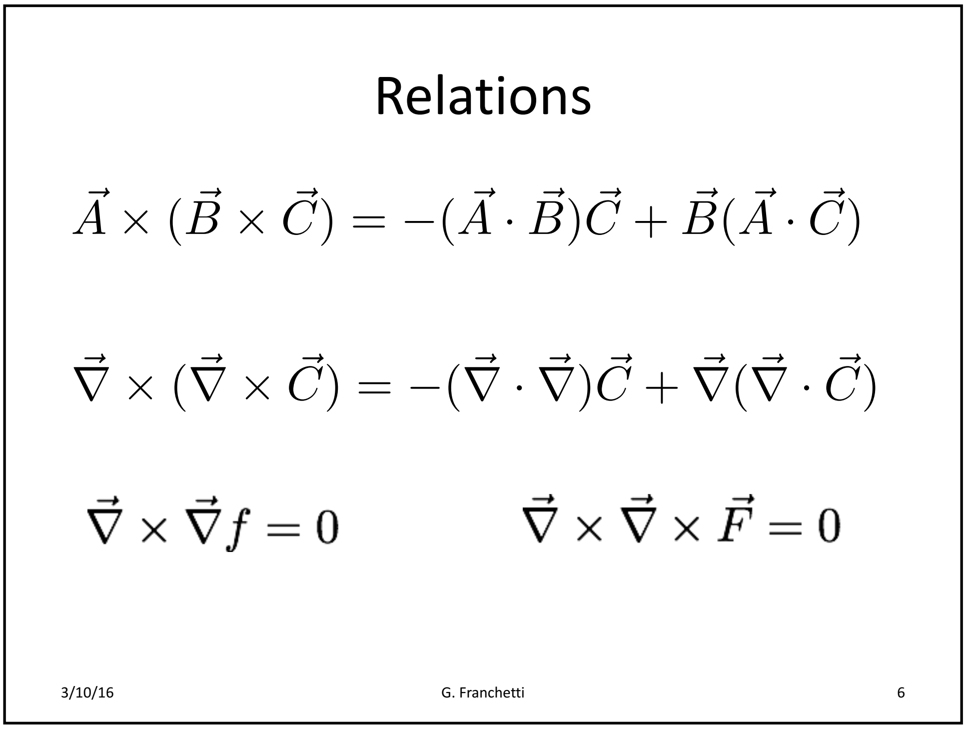
Divergence of vector field

$$\vec{\nabla}\cdot\vec{A}(x,y,z) =$$
$$\partial_x A_x + \partial_y A_y + \partial_z A_z$$

$\vec{A}(x,y,z)$ ⇨

Curl of vector field

$$\vec{\nabla}\times\vec{A}(x,y,z) =$$
$$(\partial_y A_z - \partial_z A_y)\hat{x}+$$
$$(\partial_z A_x - \partial_x A_z)\hat{y}+$$
$$(\partial_x A_y - \partial_y A_x)\hat{z}$$



Relations

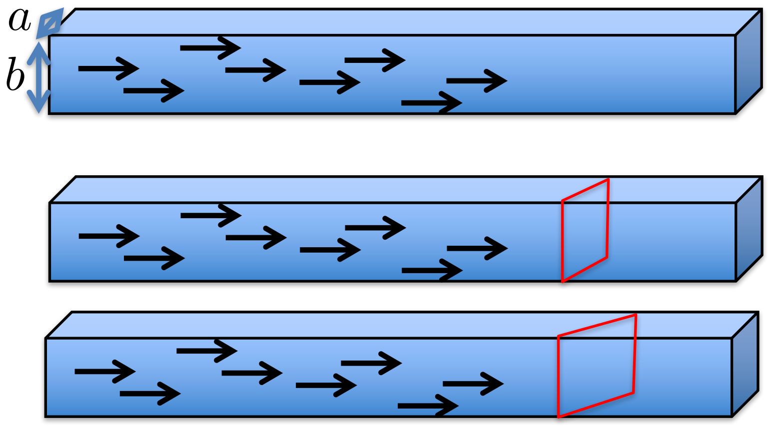
$$\vec{A}\times(\vec{B}\times\vec{C}) = -(\vec{A}\cdot\vec{B})\vec{C} + \vec{B}(\vec{A}\cdot\vec{C})$$

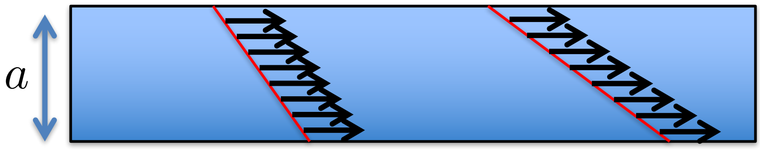
$$\vec{\nabla}\times(\vec{\nabla}\times\vec{C}) = -(\vec{\nabla}\cdot\vec{\nabla})\vec{C} + \vec{\nabla}(\vec{\nabla}\cdot\vec{C})$$

$$\vec{\nabla}\times\vec{\nabla}f = 0 \qquad \vec{\nabla}\times\vec{\nabla}\times\vec{F} = 0$$

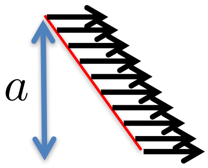
Flux Concept

Example with water



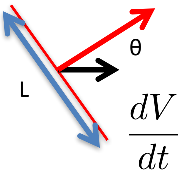


Volume per second

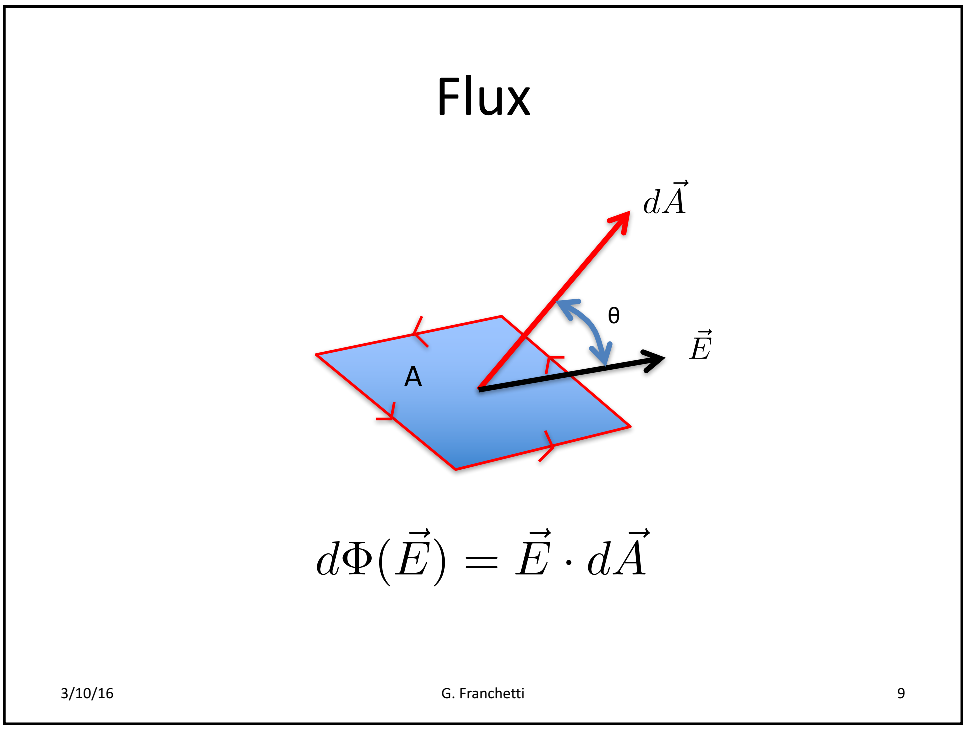


$$\frac{dV}{dt} = av$$

or

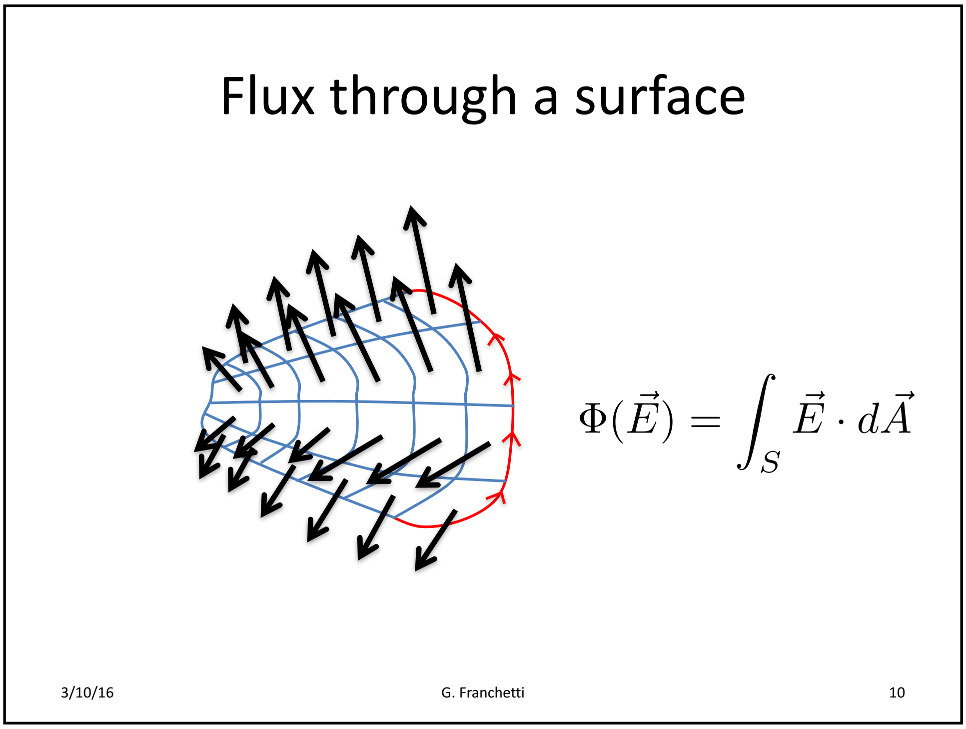


$$\frac{dV}{dt} = Lbvcos\theta$$



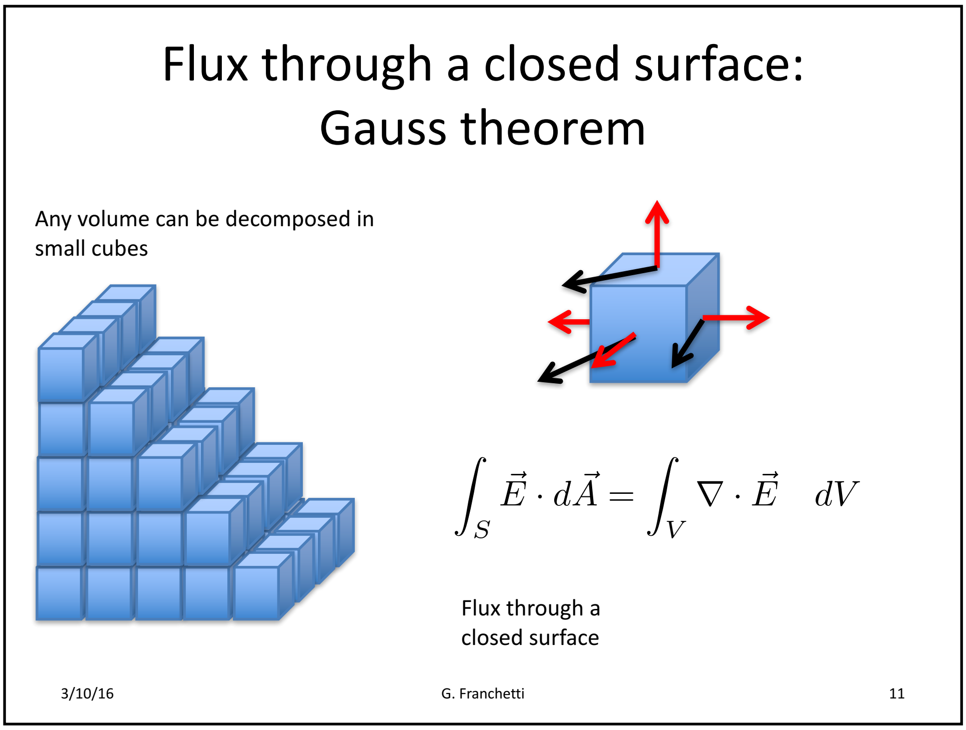
Flux

$$d\Phi(\vec{E}) = \vec{E} \cdot d\vec{A}$$



Flux through a surface

$$\Phi(\vec{E}) = \int_S \vec{E} \cdot d\vec{A}$$

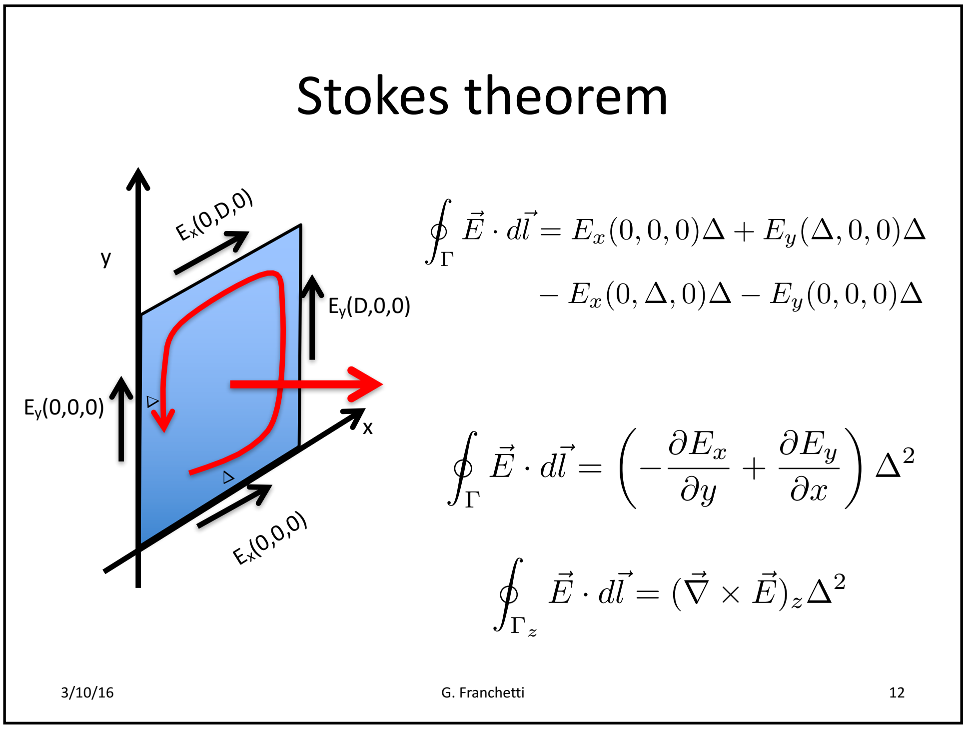


Flux through a closed surface: Gauss theorem

Any volume can be decomposed in small cubes

$$\int_S \vec{E}\cdot d\vec{A} = \int_V \nabla\cdot\vec{E} \quad dV$$

Flux through a closed surface

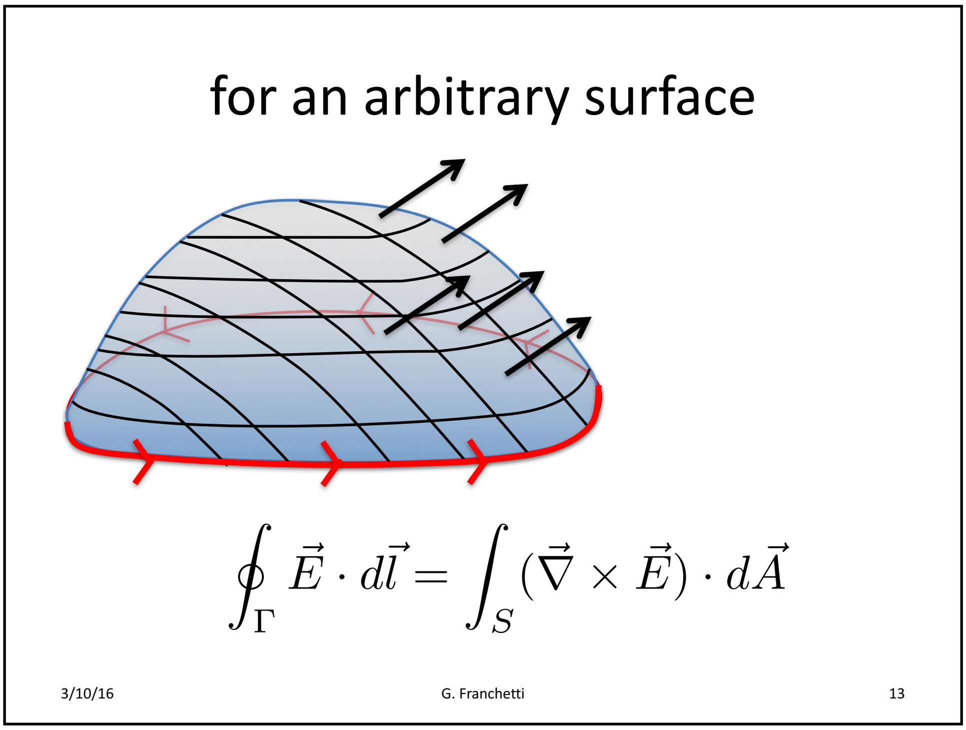


Stokes theorem

$$\oint_\Gamma \vec{E}\cdot d\vec{l} = E_x(0,0,0)\Delta + E_y(\Delta,0,0)\Delta - E_x(0,\Delta,0)\Delta - E_y(0,0,0)\Delta$$

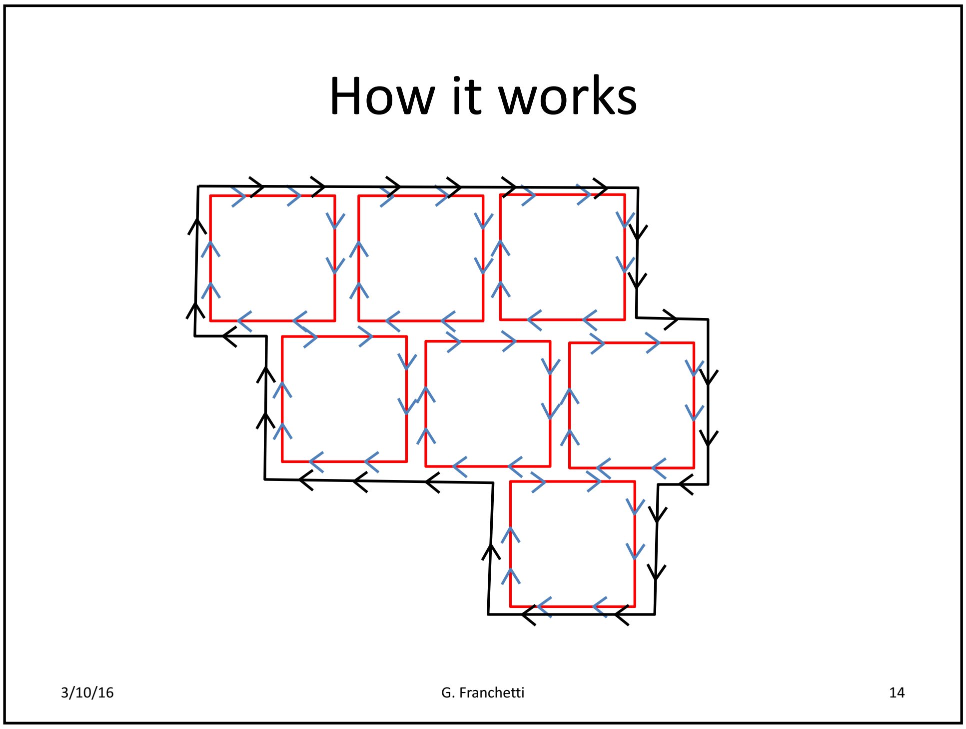
$$\oint_\Gamma \vec{E}\cdot d\vec{l} = \left(-\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x}\right)\Delta^2$$

$$\oint_{\Gamma_z} \vec{E}\cdot d\vec{l} = (\vec{\nabla}\times\vec{E})_z\Delta^2$$

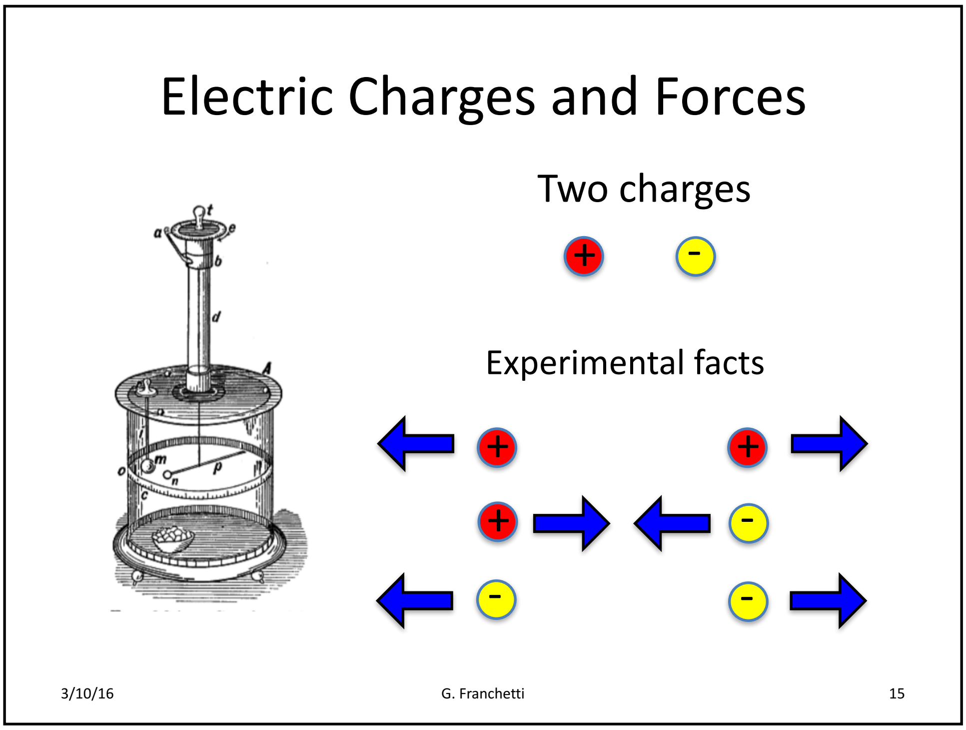


for an arbitrary surface

$$\oint_{\Gamma} \vec{E} \cdot d\vec{l} = \int_{S} (\vec{\nabla} \times \vec{E}) \cdot d\vec{A}$$



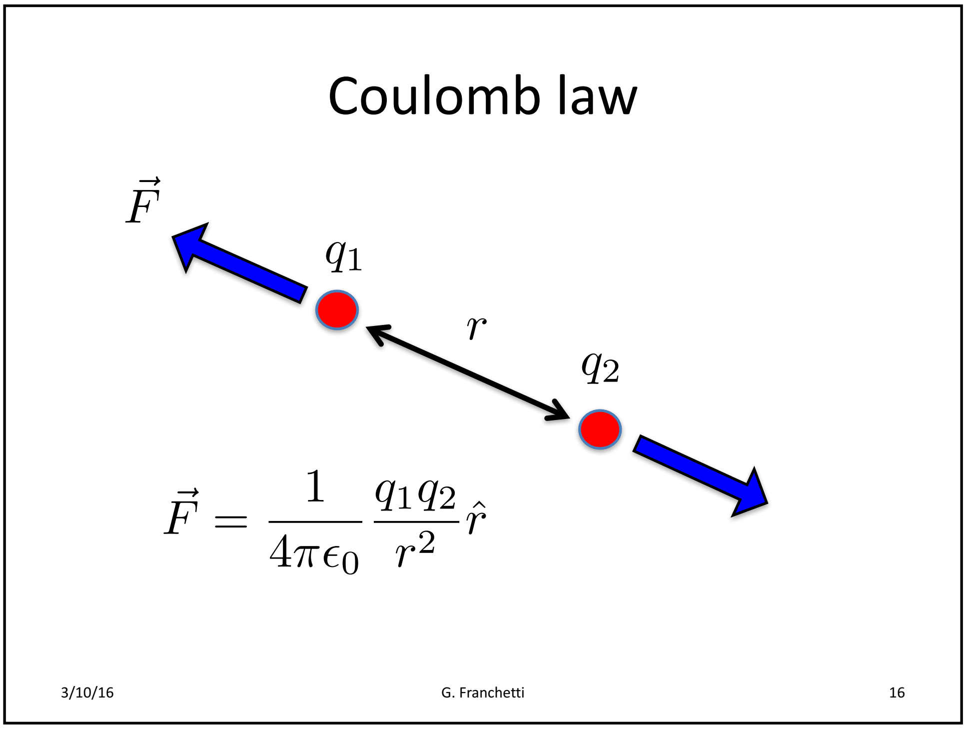
How it works



Electric Charges and Forces

Two charges

Experimental facts



Coulomb law

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$$

Units

System SI

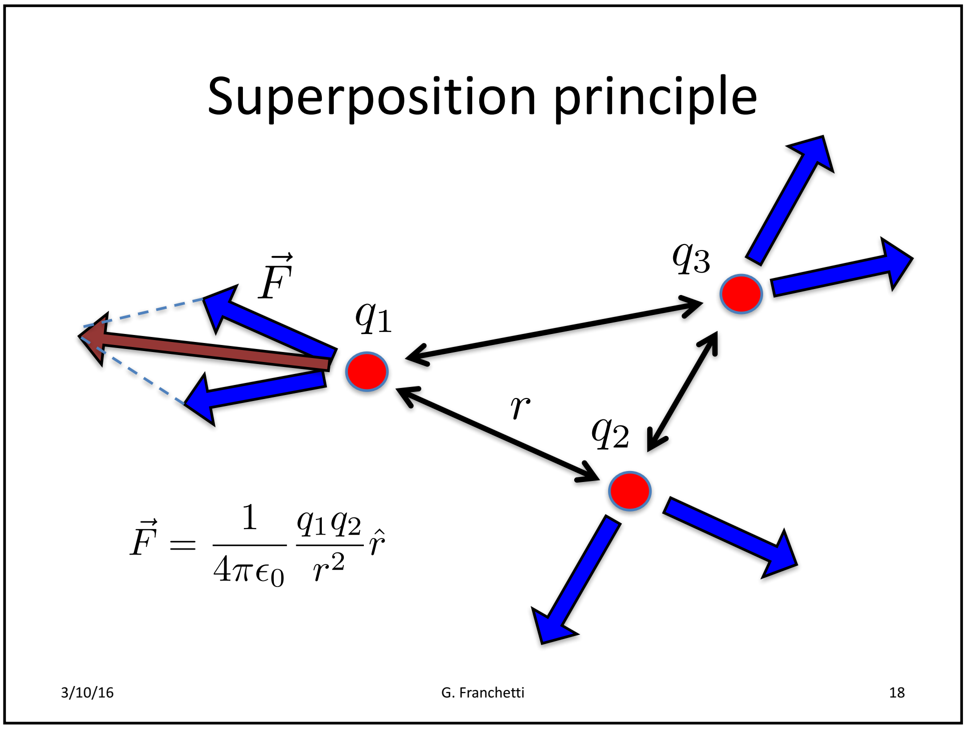
$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$$

$\vec{F}$	Newton
q_1	Coulomb
r	Meters

$$\epsilon_0 = 8.8541 \times 10^{-12}$$

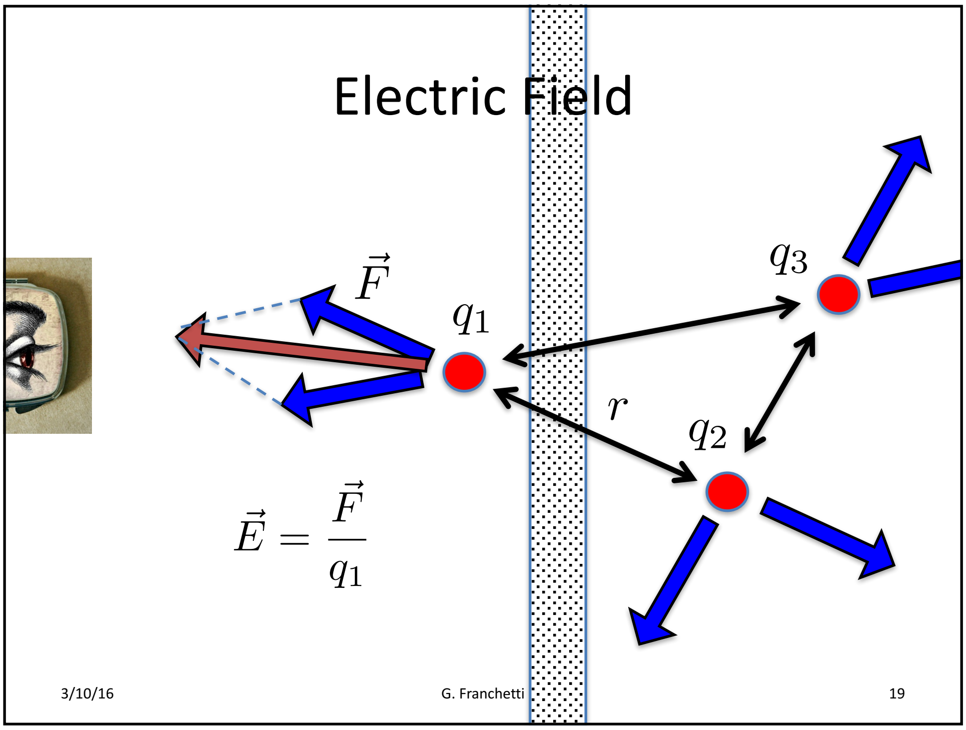
$C^2\ N^{-1}\ m^{-2}$

permettivity of free space



Superposition principle

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$$



Electric Field

$$\vec{E} = \frac{\vec{F}}{q_1}$$

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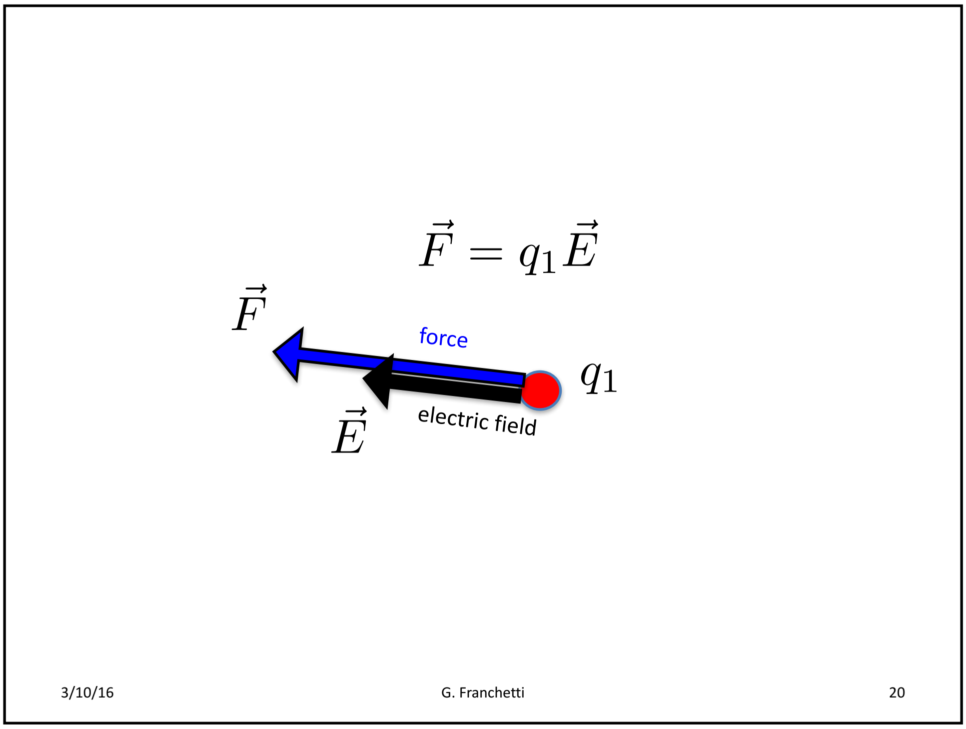

$$\vec{F} = q_1 \vec{E}$$

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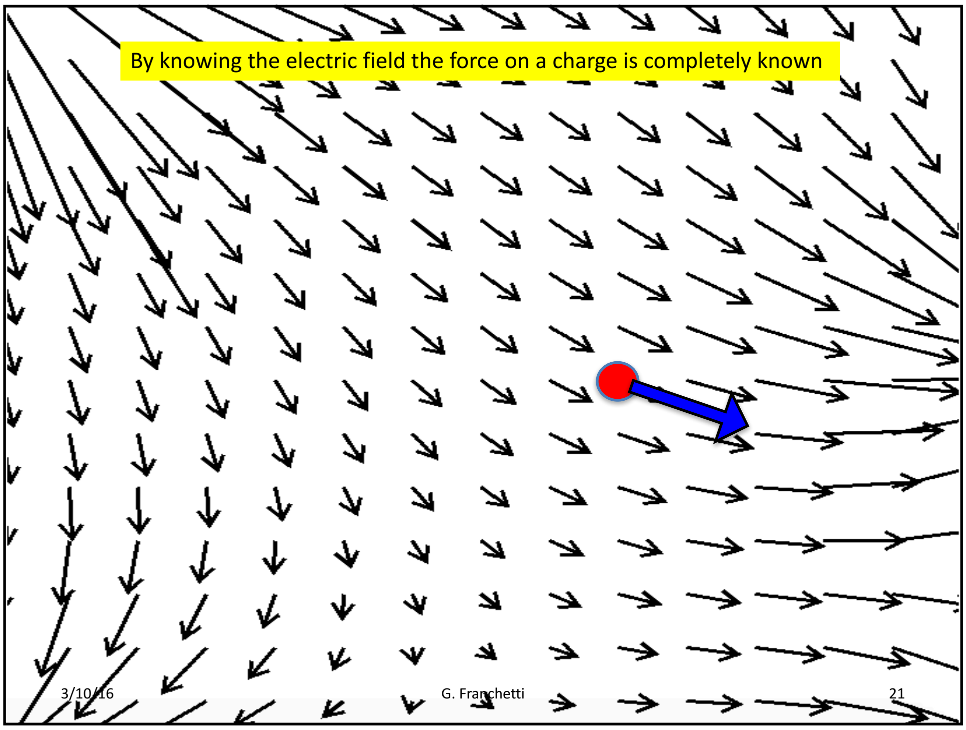
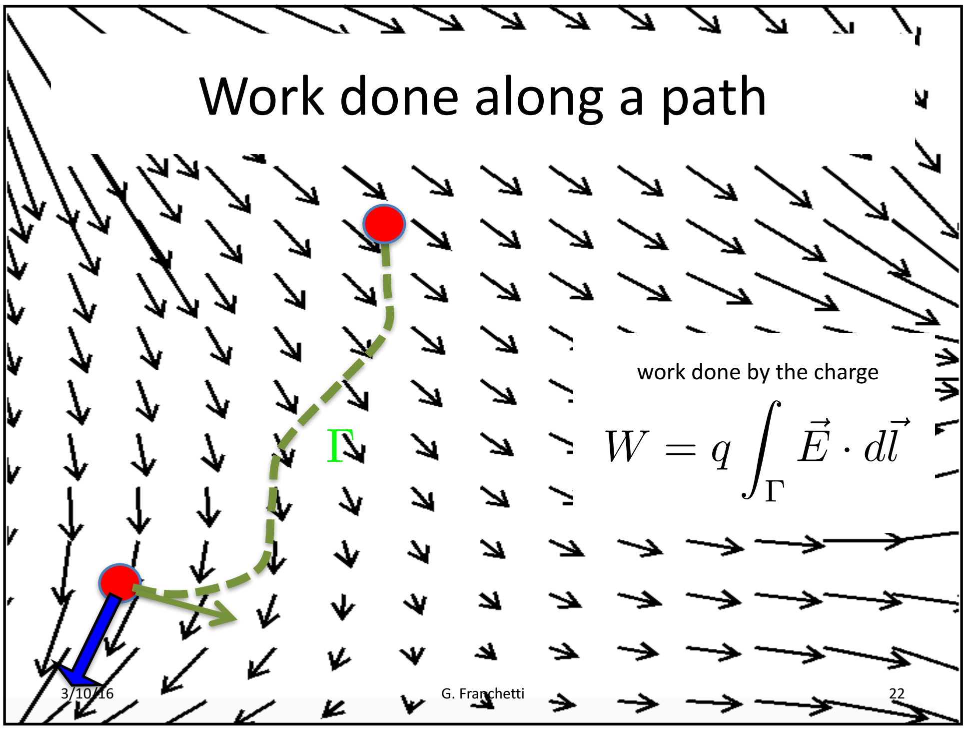


By knowing the electric field the force on a charge is completely known



Work done along a path

work done by the charge

$$W = q \int_{\Gamma} \vec{E} \cdot d\vec{l}$$

Electric potential

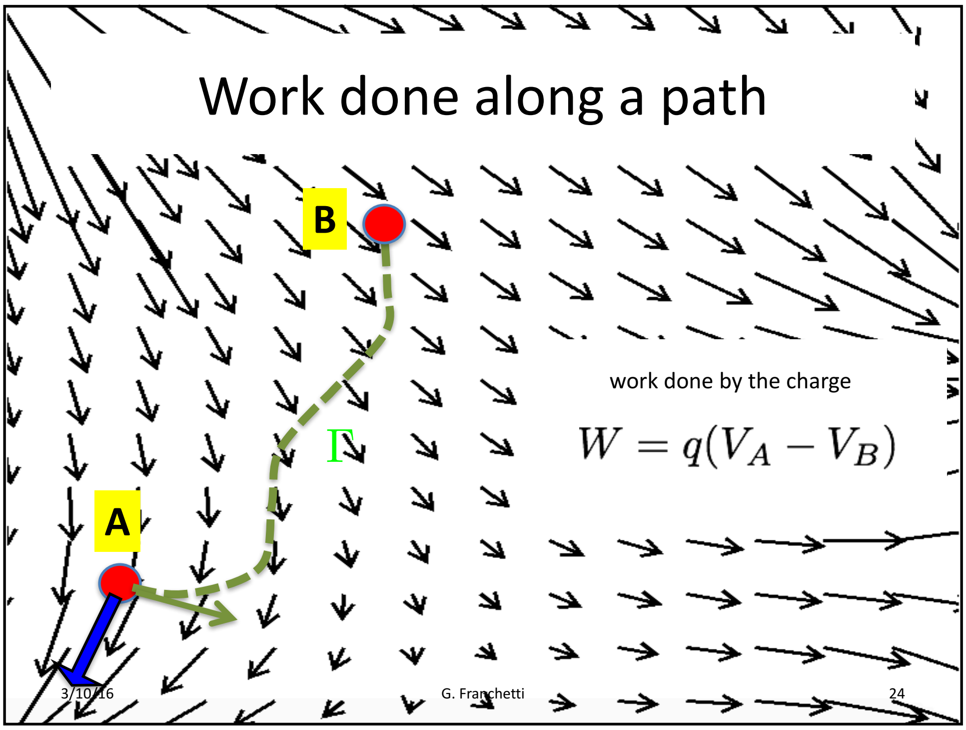
$$V(P) = -\int_{\infty}^{P} \vec{E} \cdot d\vec{l}$$

For conservative field V(P) does
not depend on the path !

Central forces are conservative

UNITS: Joule / Coulomb = Volt

Work done along a path



work done by the charge

$$W = q(V_A - V_B)$$

Electric Field ←→ Electric Potential

$$E_x = -\frac{\partial}{\partial x} V(x,y,z)$$

$$E_y = -\frac{\partial}{\partial y} V(x,y,z)$$

$$E_z = -\frac{\partial}{\partial z} V(x,y,z)$$

In vectorial notation

$$\vec{E} = -\vec{\nabla} V$$

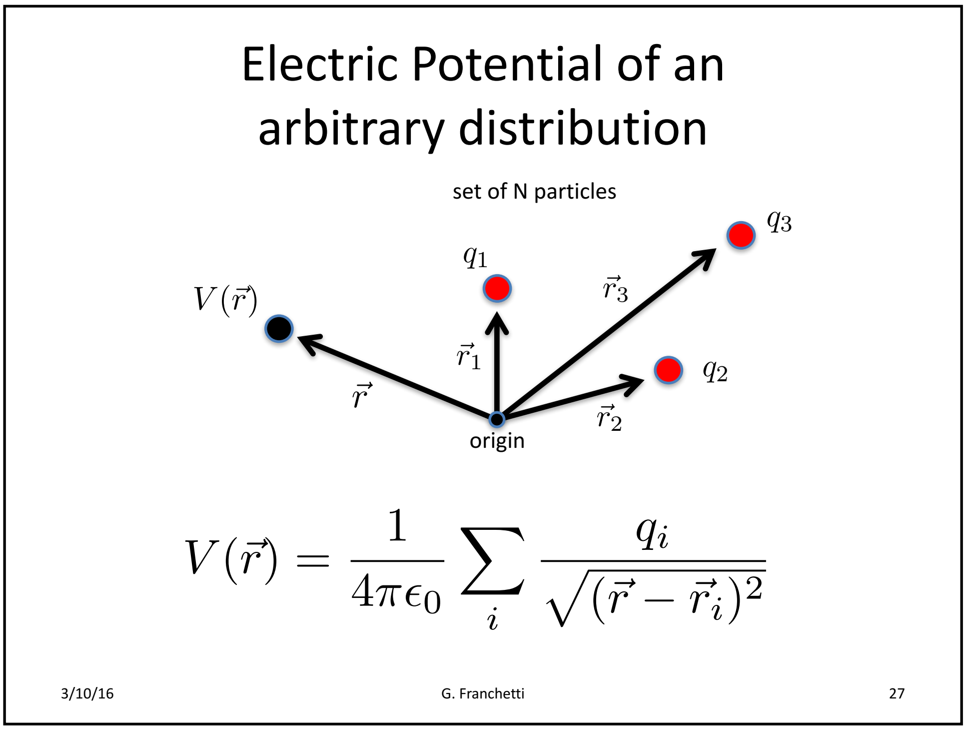
Electric potential by one charge

Take one particle located at the origin, then

$$V(\vec{r}) = -\int_{\infty}^{\vec{r}} \frac{1}{4\pi\epsilon_0} \frac{q}{(r_1 - r_0)^3} (\vec{r}_1 - \vec{r}_0) \cdot d\vec{l}$$

$$\Downarrow$$

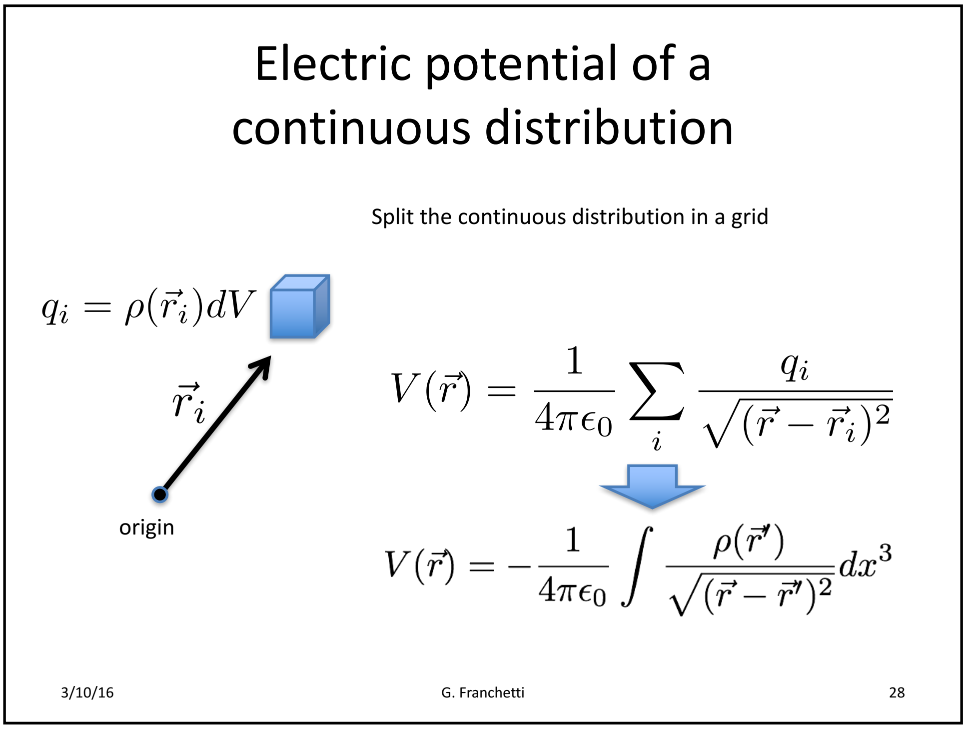
$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$



Electric Potential of an arbitrary distribution

set of N particles

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{\sqrt{(\vec{r}-\vec{r}_i)^2}}$$



Electric potential of a continuous distribution

Split the continuous distribution in a grid

$q_i = \rho(\vec{r}_i)dV$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{\sqrt{(\vec{r}-\vec{r}_i)^2}}$$

$$V(\vec{r}) = -\frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')}{\sqrt{(\vec{r}-\vec{r}')^2}}dx^3$$

Energy of a charge distribution

it is the work necessary to bring the charge distribution from infinity

$$U = \sum_j q_j \sum_{i=1,j} V_i(\vec{r}_j)$$

More simply $$U = \sum_j q_j \sum_{i=1,j} \frac{1}{4\pi\epsilon_0} \frac{q_i}{|\vec{r}_j - \vec{r}_i|}$$

More simply $$U = \frac{1}{2} \sum_{i \neq j} \frac{1}{4\pi\epsilon_0} \frac{q_i q_j}{|\vec{r}_j - \vec{r}_i|}$$

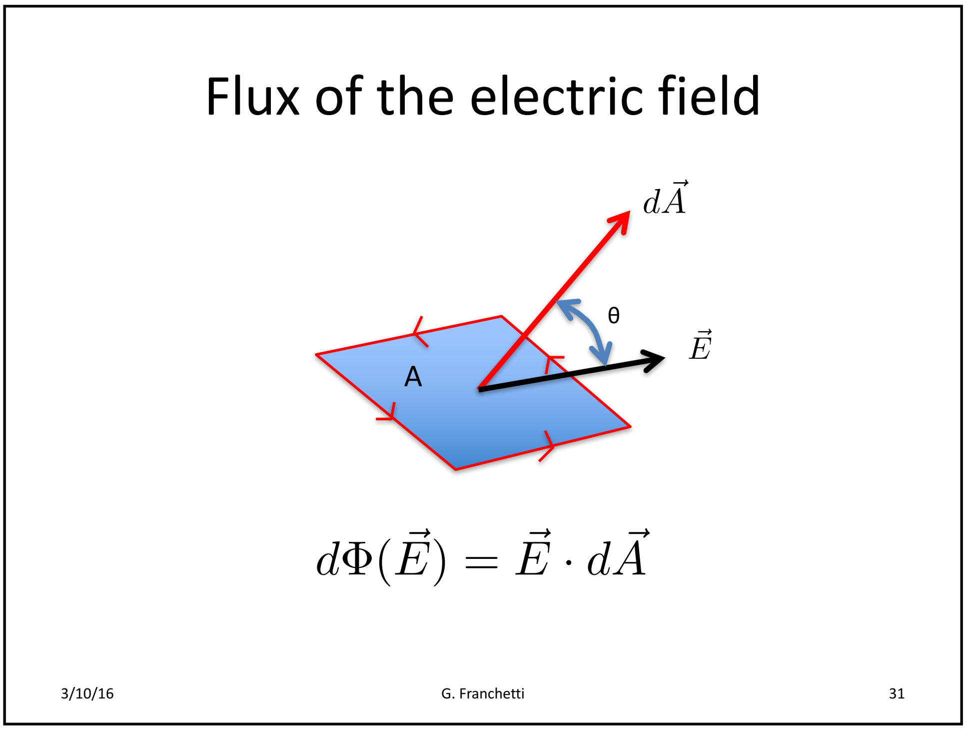
In integral form

$$U = \frac{1}{2} \sum_i q_i V(\vec{r}_i) = \int \rho V dx^3$$

Using $\vec{E} = -\vec{\nabla} V$ and the divergence theorem it can be proved that

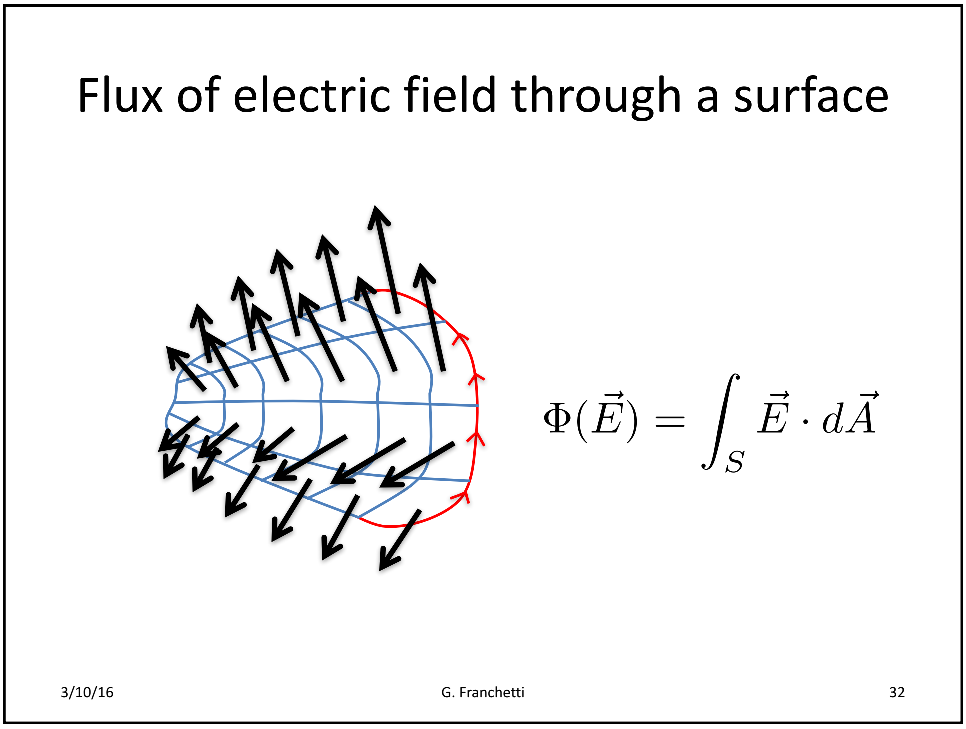
$$U = \int \epsilon_0 \frac{E^2}{2} dx^3$$ ➡

$\epsilon_0 \dfrac{E^2}{2}$ is the density of energy of the electric field



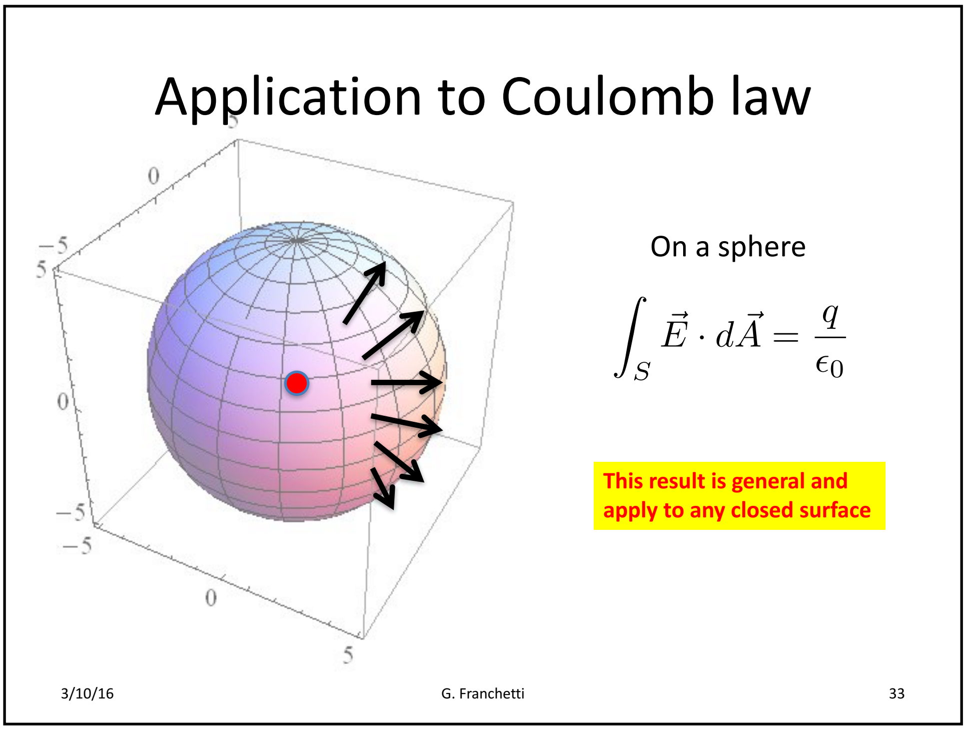
Flux of the electric field

$$d\Phi(\vec{E}) = \vec{E} \cdot d\vec{A}$$



Flux of electric field through a surface

$$\Phi(\vec{E}) = \int_S \vec{E} \cdot d\vec{A}$$

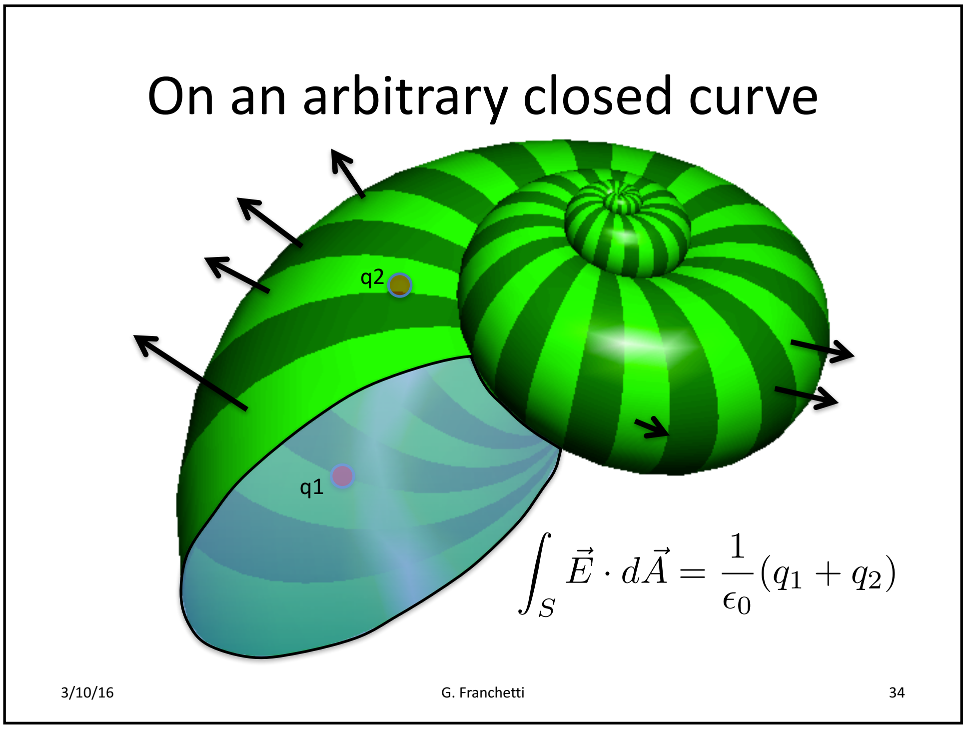


Application to Coulomb law

On a sphere

$$\int_S \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

This result is general and apply to any closed surface



On an arbitrary closed curve

$$\int_S \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0}(q_1 + q_2)$$

First Maxwell Law

integral form

$$\int_S \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

for a infinitesimal
small volume

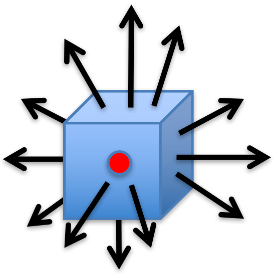
$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

differential form

(try to derive it. Hint: used Gauss theorem)

Physical meaning

If there is a charge in one place, the electric flux is different than zero



One charge create an electric flux.

$$\Phi(\vec{E}) = \frac{q}{\epsilon_0}$$

Poisson and Laplace Equations

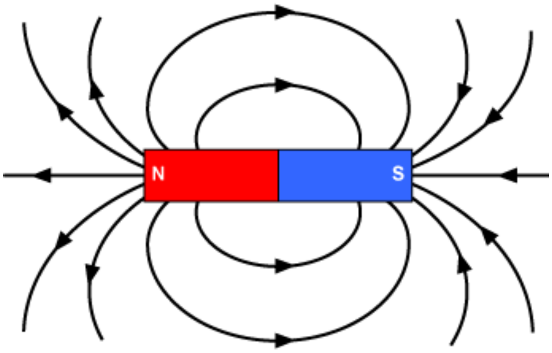
As $\vec{E} = -\vec{\nabla} V$ and $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

combining both we find $\vec{\nabla} \cdot \vec{\nabla} V = -\frac{\rho}{\epsilon_0}$

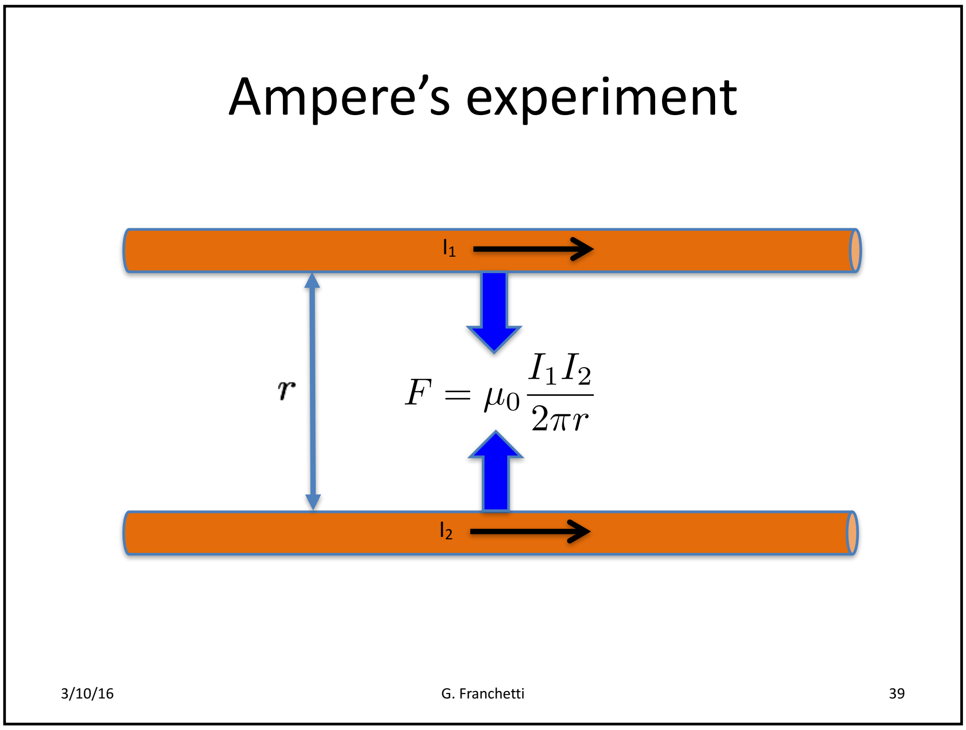
$\nabla^2 V = -\frac{\rho}{\epsilon_0}$ Poisson

In vacuum: $\nabla^2 V = 0$ Laplace

Magnetic Field

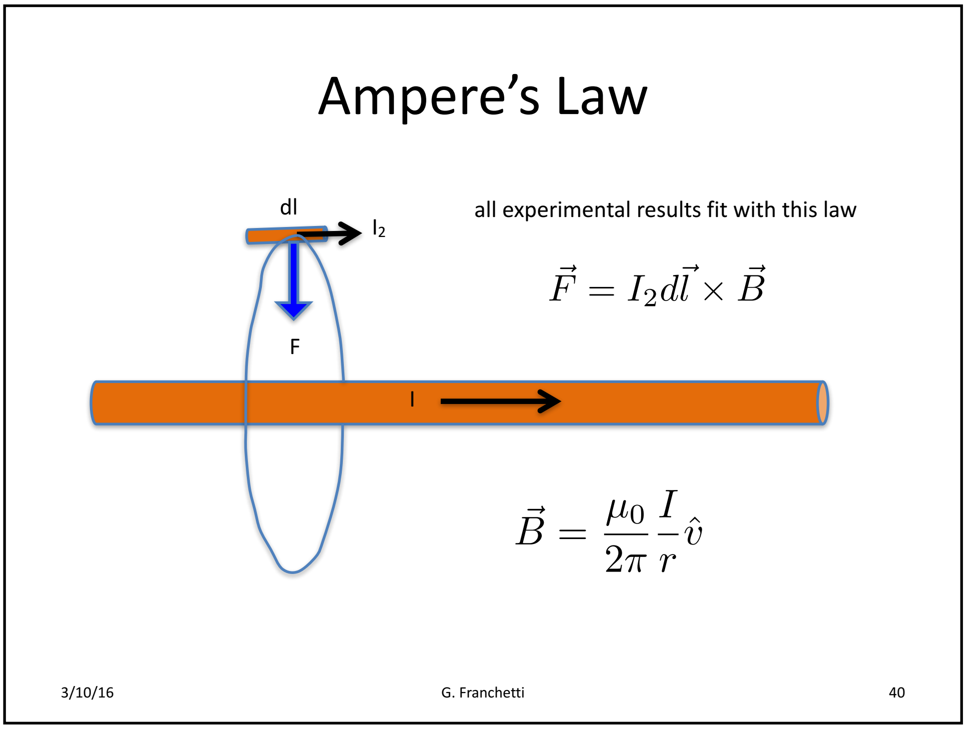


There exist not a magnetic charge!
(Find a magnetic monopole and you get the Nobel Prize)



Ampere's experiment

$$F = \mu_0 \frac{I_1 I_2}{2\pi r}$$



Ampere's Law

all experimental results fit with this law

$$\vec{F} = I_2 d\vec{l} \times \vec{B}$$

$$\vec{B} = \frac{\mu_0}{2\pi} \frac{I}{r} \hat{v}$$

Units

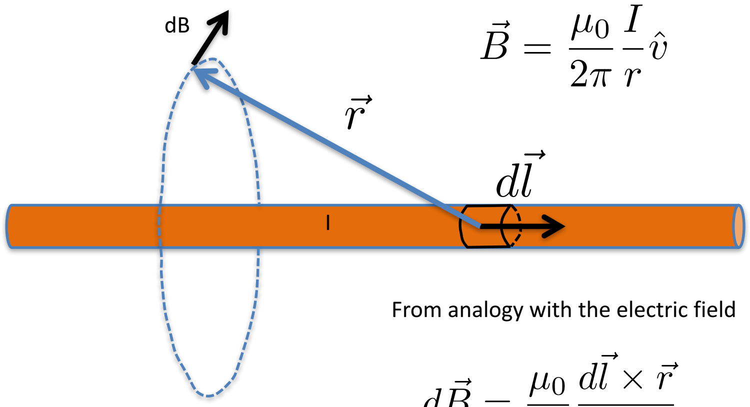
From $\vec{F} = dl\vec{I}_2 \times \vec{B}$ $\qquad \frac{N}{Am} = T$ [Tesla]

From $\vec{B} = \frac{\mu_0}{2\pi}\frac{I}{r}\hat{v}$ follows

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N\ A}^{-2}$$

To have 1T at 10 cm
with one cable ⇒ I = 5 x 10^5 Amperes !!

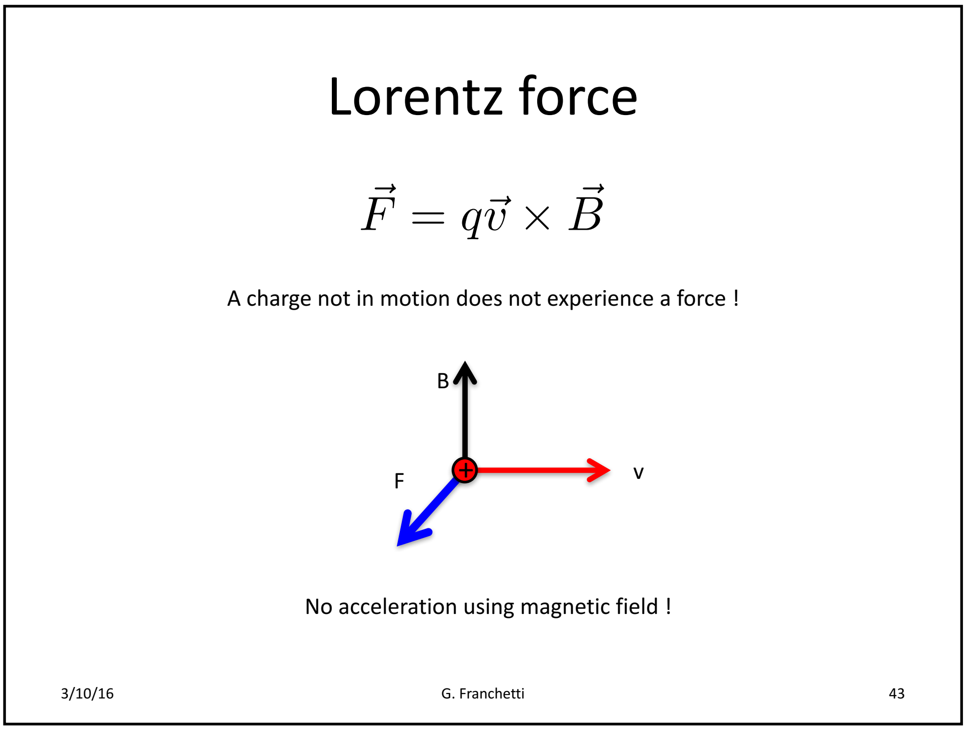
Biot-Savart Law



$$\vec{B} = \frac{\mu_0}{2\pi}\frac{I}{r}\hat{v}$$

From analogy with the electric field

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{d\vec{l} \times \vec{r}}{r^3}$$

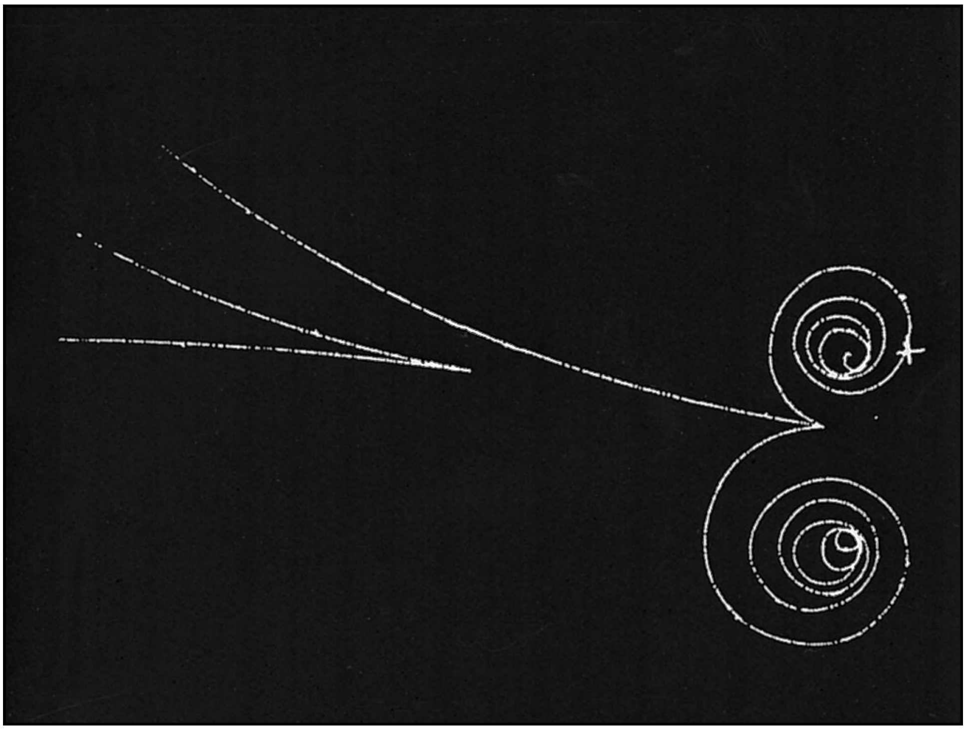


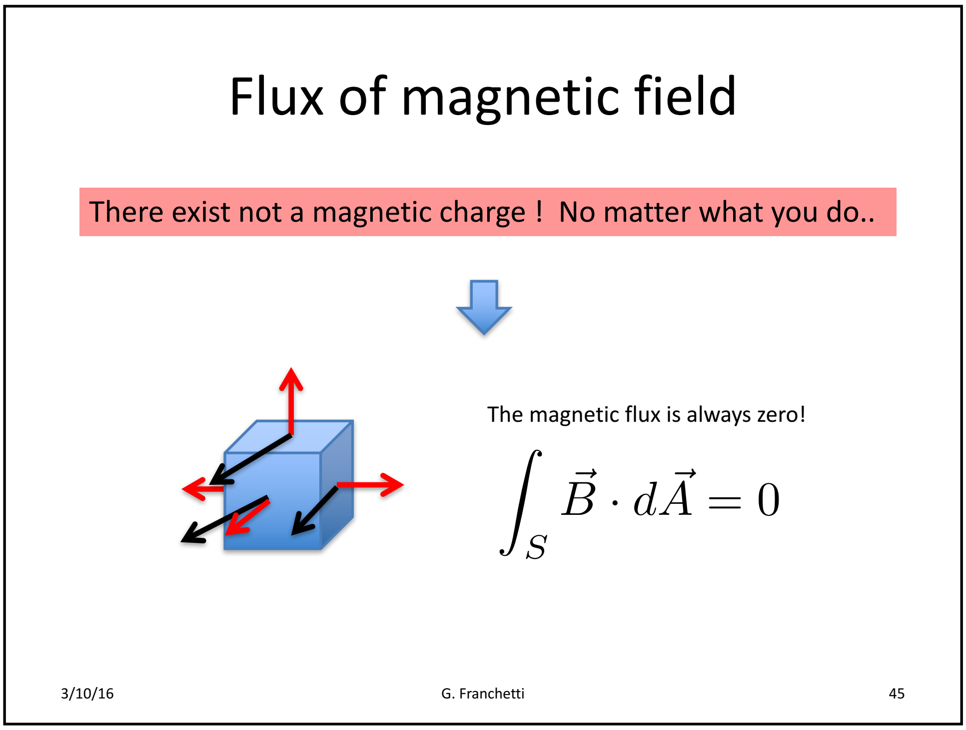
Lorentz force

$$\vec{F} = q\vec{v} \times \vec{B}$$

A charge not in motion does not experience a force !

No acceleration using magnetic field !





Flux of magnetic field

There exist not a magnetic charge ! No matter what you do..

The magnetic flux is always zero!

$$\int_S \vec{B} \cdot d\vec{A} = 0$$

Second Maxwell Law

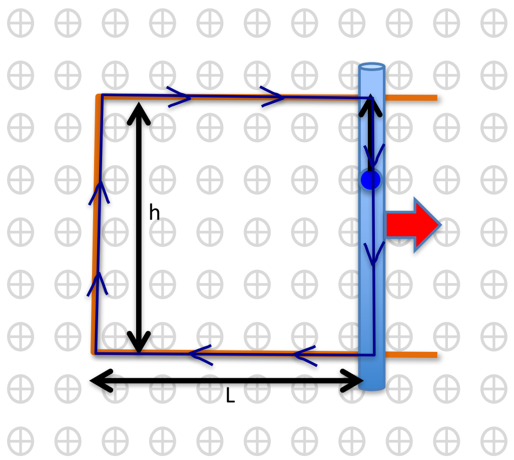
Integral form

$$\int_S \vec{B} \cdot d\vec{A} = 0$$

Differential from

$$\nabla \cdot \vec{B} = 0$$

Changing the magnetic Flux...



$$\vec{E} = \vec{v} \times \vec{B}$$

Magnetic flux

$$\Phi(\vec{B}) = hLB$$

$$E = \frac{1}{h}\frac{d\Phi(\vec{B})}{dt}$$

Following the path

$$\int_\Gamma \vec{E} \cdot d\vec{l} = -\frac{d\Phi(\vec{B})}{dt}$$

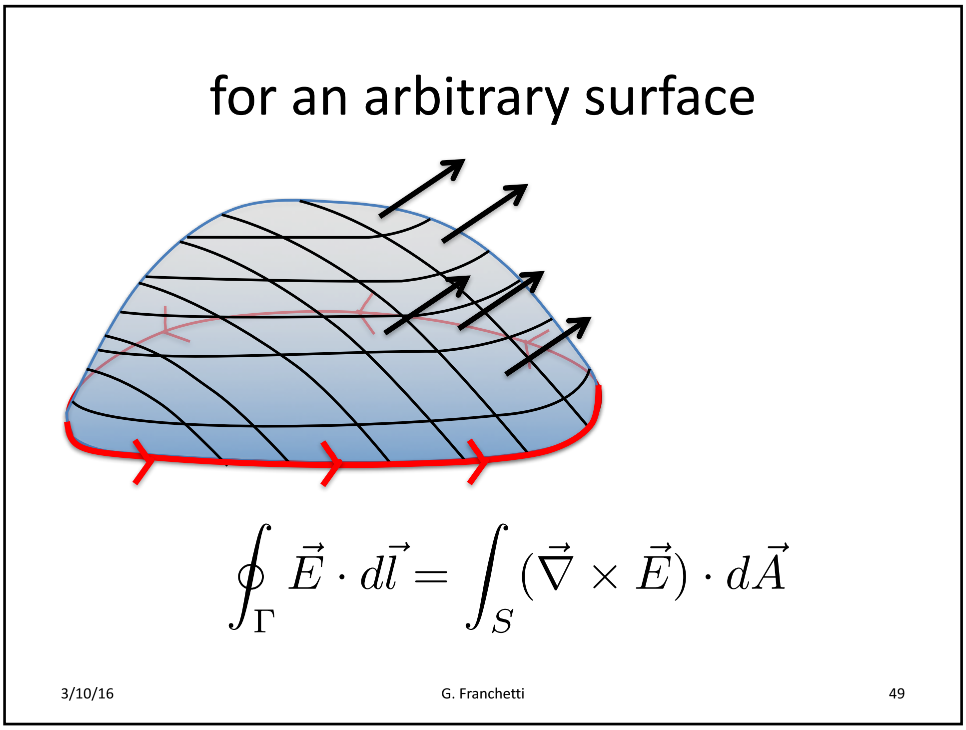
Faraday's Law

integral form

$$\int_\Gamma \vec{E} \cdot d\vec{l} = -\frac{d\Phi(\vec{B})}{dt}$$

valid in any way
the magnetic flux
is changed !!!

(Really not obvious !!)



for an arbitrary surface

$$\oint_\Gamma \vec{E}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{A}$$

Faraday’s Law in differential form

$$\int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{A} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{A}$$

⇩

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Summary Faraday's Law

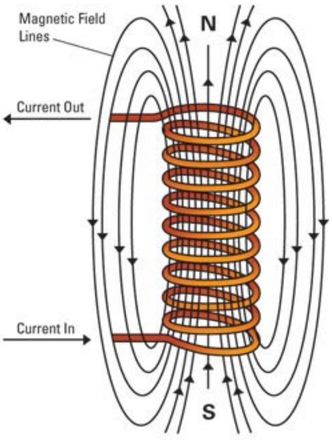
Integral form

$$\oint_\Gamma \vec{E} \cdot d\vec{l} = -\frac{d\Phi(\vec{B})}{dt}$$

differential form

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Important consequence



Current creates magnetic field

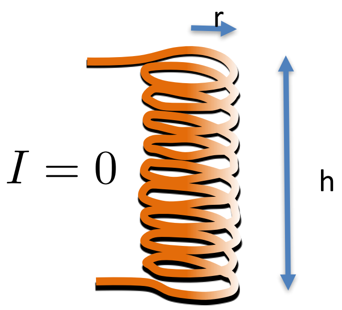
magnetic field create magnetic flux

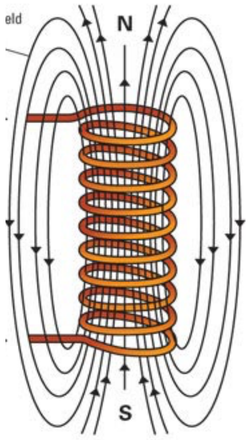
$$\Phi(B) = LI$$

L = inductance [Henry]

Changing the magnetic flux creates an induced emf

$$\oint_\Gamma \vec{E} \cdot d\vec{l} = -\frac{d\Phi(\vec{B})}{dt}$$





$$\epsilon_{emf} = -\frac{d\Phi(B)}{dt} \qquad dU = \epsilon_{emf} I dt = -\frac{d\Phi(B)}{dt} I dt$$

energy necessary to create the magnetic field $U = \frac{1}{2} L I^2$

Field inside the solenoid $B = \mu_0 N I$

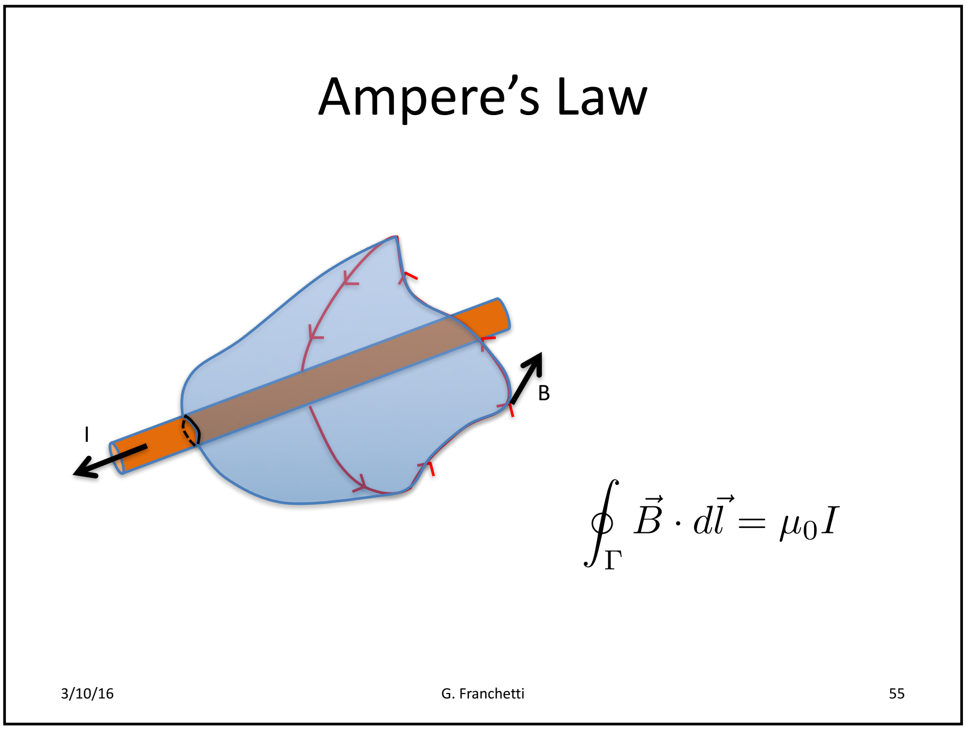
Magnetic flux $\Phi(B) = \pi r^2 B N h$

$$\Phi(B) = Volume \frac{B^2}{\mu_0 I} \Rightarrow Volume \frac{B^2}{\mu_0 I} = LI$$

Therefore

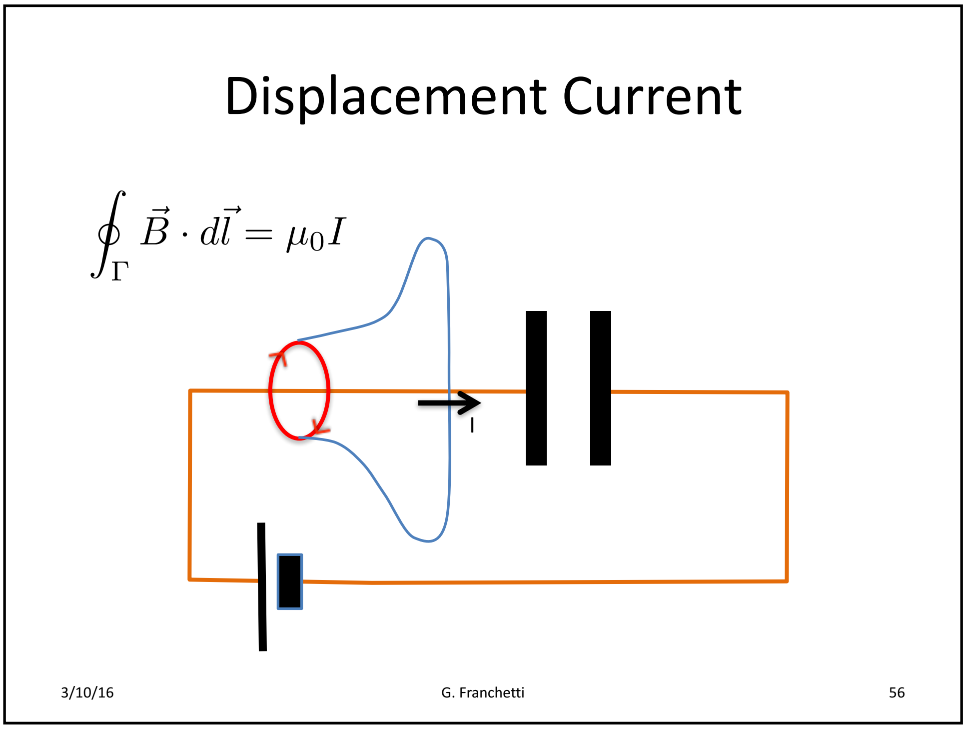
$$U = \frac{1}{2} L I^2 = Volume \frac{B^2}{2\mu_0}$$

Energy density of the magnetic field $\frac{B^2}{2\mu_0}$



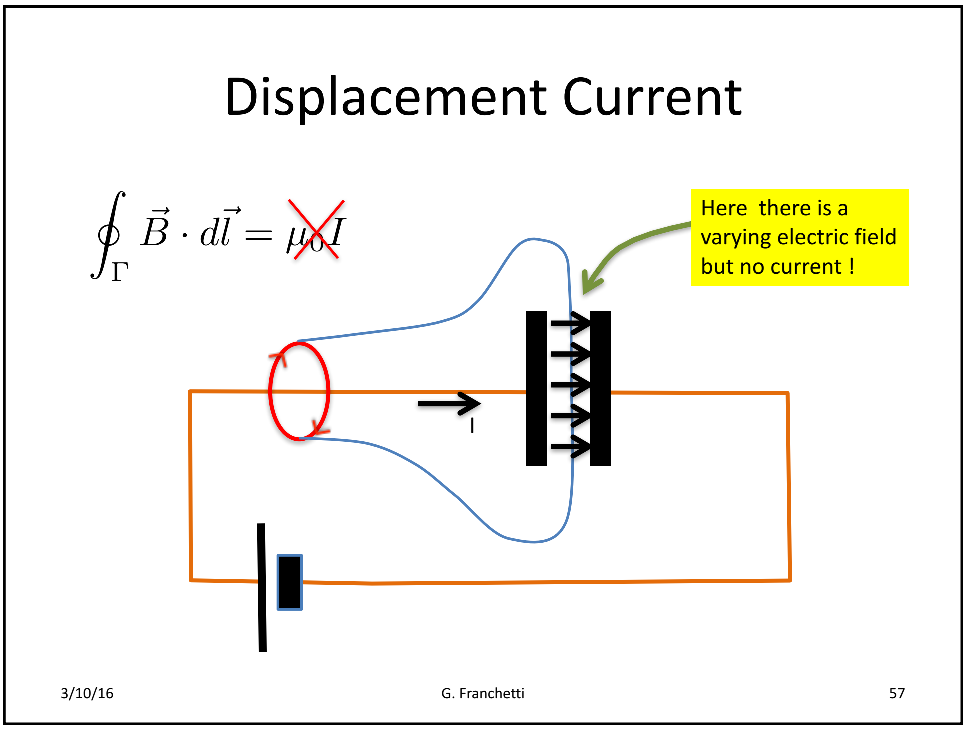
Ampere's Law

$$\oint_\Gamma \vec{B}\cdot d\vec{l} = \mu_0 I$$



Displacement Current

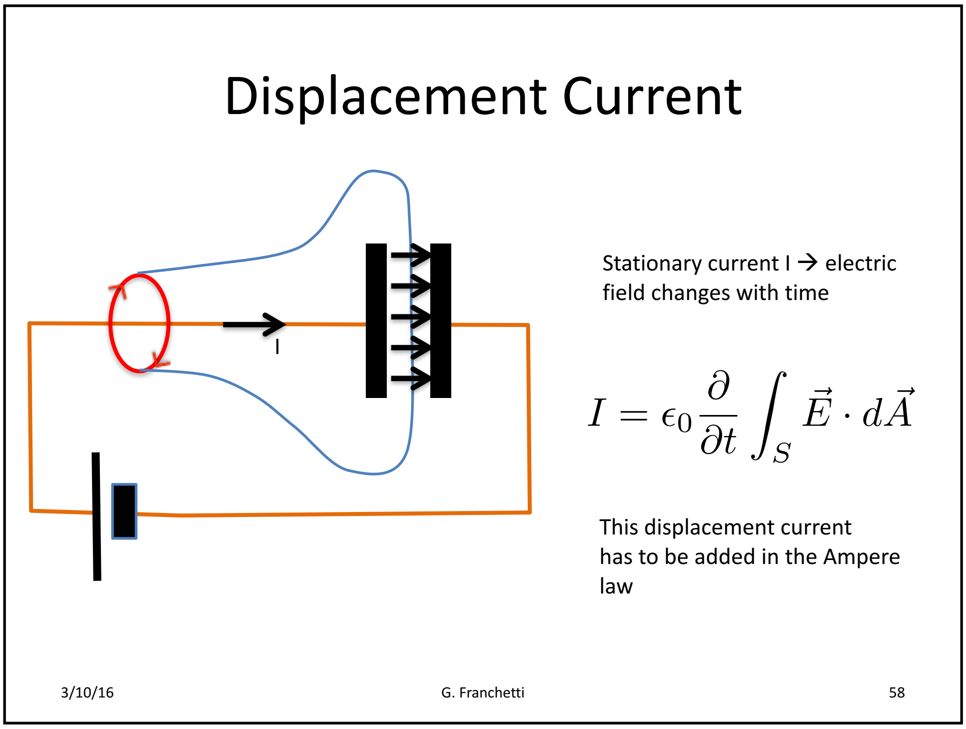
$$\oint_\Gamma \vec{B}\cdot d\vec{l} = \mu_0 I$$



Displacement Current

$$\oint_\Gamma \vec{B}\cdot d\vec{l} = \cancel{\mu_0 I}$$

Here there is a varying electric field but no current !



Displacement Current

Stationary current I → electric field changes with time

$$I = \epsilon_0 \frac{\partial}{\partial t}\int_S \vec{E}\cdot d\vec{A}$$

This displacement current has to be added in the Ampere law

Final form of the Ampere law

integral form

$$\oint_\Gamma \vec{B} \cdot d\vec{l} = \mu_0 \left(I + \epsilon_0 \frac{\partial}{\partial t} \int_S \vec{E} \cdot d\vec{A} \right)$$

differential form

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{j} + \epsilon_0 \frac{\partial}{\partial t} \vec{E} \right)$$

Maxwell Equations in vacuum

Integral form	Differential form
$\int_S \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$	$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
$\int_S \vec{B} \cdot d\vec{A} = 0$	$\nabla \cdot \vec{B} = 0$
$\oint_\Gamma \vec{E} \cdot d\vec{l} = -\frac{d\Phi(\vec{B})}{dt}$	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
$\oint_\Gamma \vec{B} \cdot d\vec{l} = \mu_0 \left(I + \epsilon_0 \frac{\partial}{\partial t} \int_S \vec{E} \cdot d\vec{A} \right)$	$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{j} + \epsilon_0 \frac{\partial}{\partial t} \vec{E} \right)$

Magnetic potential ?

Can we find a “potential” such that $\vec{B} = -\vec{\nabla} V$?

$$\vec{\nabla} \cdot \vec{B} = -\nabla^2 V$$

Maxwell equation

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\nabla^2 V = 0$$

But $\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{\nabla} V = 0$

it means that we cannot include currents !!

Example: 2D multipoles

For 2D static magnetic field in vacuum (only B_x, B_y)

$$\vec{B} = -\vec{\nabla} V$$
$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{B} = (-\partial_x V, -\partial_y V, 0)$$
$$\vec{B} = (\partial_y A_z, -\partial_x A_z, 0)$$

$$-\partial_x V = \partial_y A_z$$
$$\partial_y V = \partial_x A_z$$

These are the Cauchy-Reimann
That makes the function

$$A + iV$$

analytic

$$B_y + iB_x = -\partial_x (A + iV)$$

$$\boxed{B_y + iB_x = B \sum_n (b_n + ia_n) z^n}$$

Vector Potential

In general we require

$$\boxed{\begin{aligned}\vec{B} &= \vec{\nabla}\times\vec{A}\\ \vec{\nabla}\cdot\vec{A} &= 0\end{aligned}}$$

(this choice is always possible)

$$\Downarrow$$

Automatically

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \quad\Rightarrow\quad \nabla^2\vec{A} = -\mu_0\vec{J}$$

Solution

Electric potential

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

$$\Downarrow$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}_i)}{|\vec{r}-\vec{r}_i|}dV$$

Magnetic potential

$$\nabla^2\vec{A} = -\mu_0\vec{J}$$

$$\Downarrow$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r'})}{|\vec{r'}-\vec{r}|}dV$$

Effect of matter

Electric field

Conductors
Dielectric

Magnetic field

Diamagnetism
Paramagnetism
Ferrimagnetism

Maxwell equation in vacuum are always valid, even when we consider the effect of matter

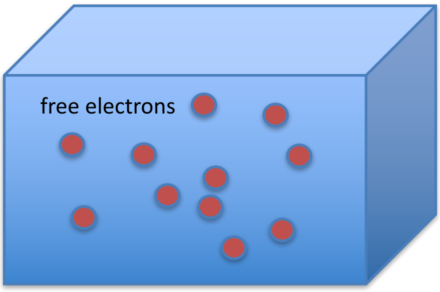
Microscopic field

That is the field is "local"
between atoms and moving charges

Averaged field

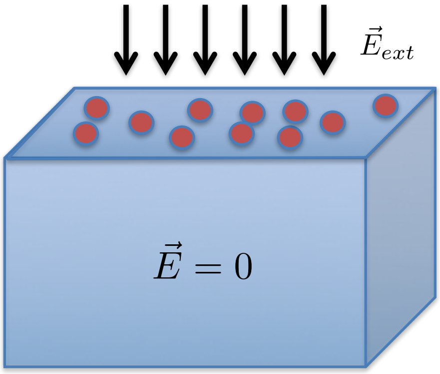
this is a field averaged
over a volume that contain
many atoms or molecules

Conductors



bounded to
be inside the
conductor

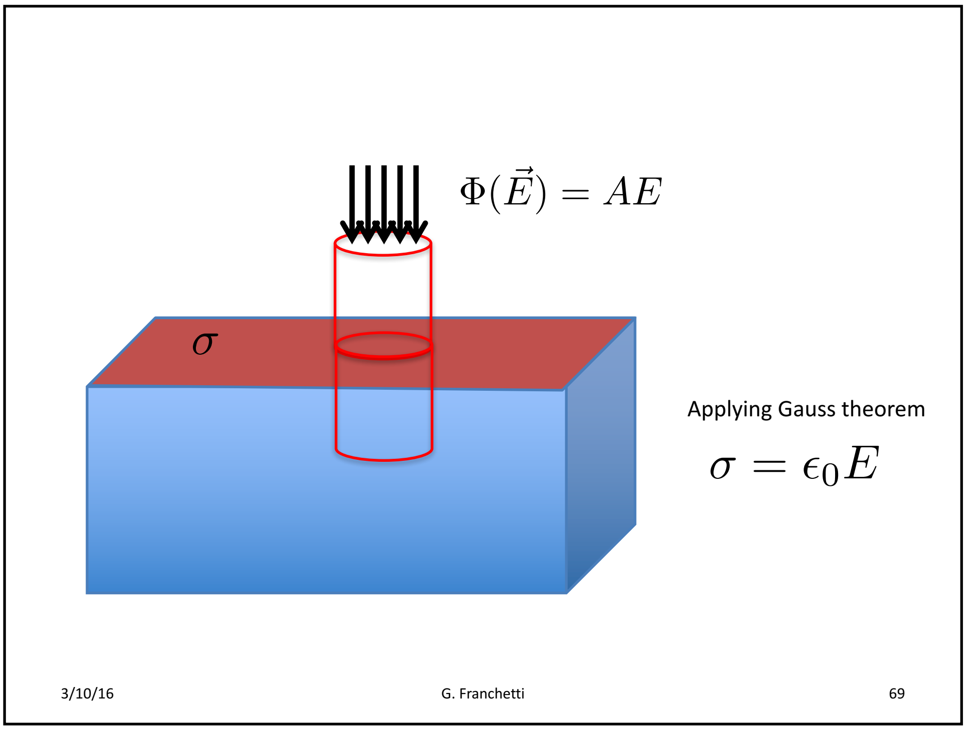
Conductors and electric field



bounded to
be inside the
conductor

on the surface
the electric field is
always perpendicular

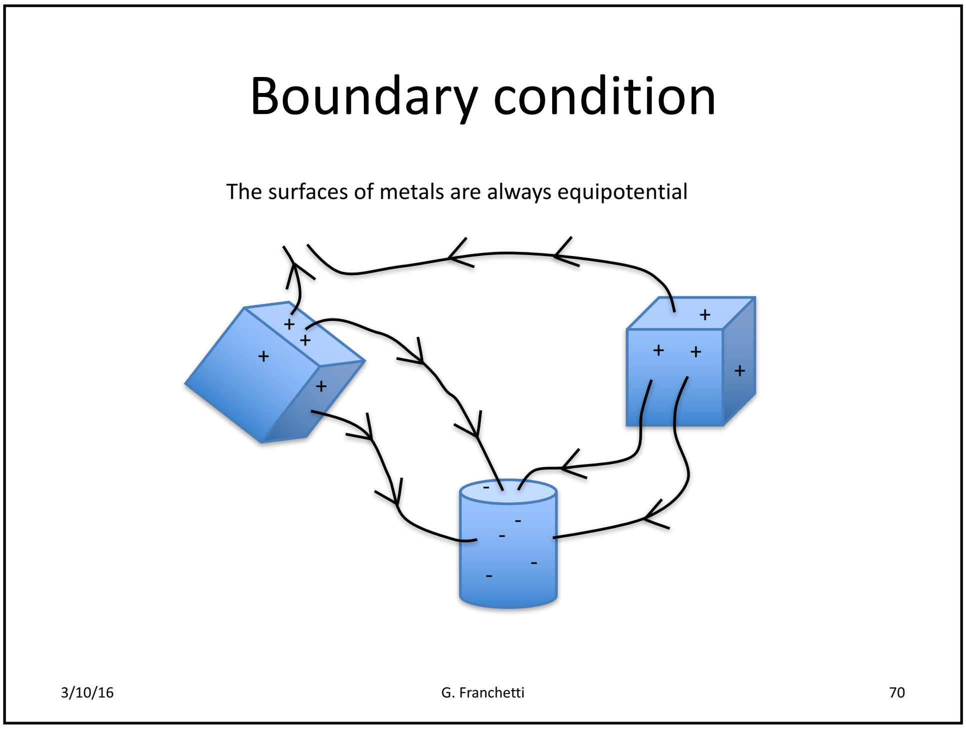
surface distribution
of electrons



$$\Phi(\vec{E}) = AE$$

Applying Gauss theorem

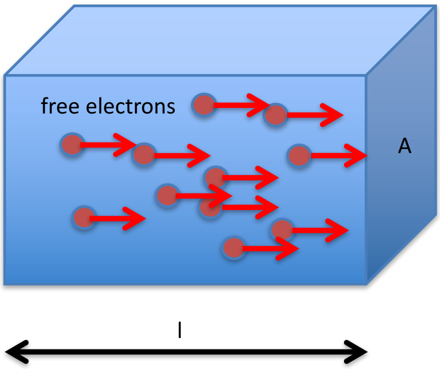
$$\sigma = \epsilon_0 E$$



Boundary condition

The surfaces of metals are always equipotential

Ohm's Law



$$R = \frac{l}{A}\rho \qquad [\Omega]$$

$$\rho \quad \text{resistivity} \qquad [\Omega\text{m}]$$

$$\sigma = \frac{1}{\rho} \quad \text{conductivity}$$

$$\vec{E} = \rho\vec{J}$$

or

$$\vec{J} = \sigma\vec{E}$$

Who is who ?