Bootstrapping gauge theories (QCD)

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based on 2309.12402 and WIP with Martin Kruczenski

Low energy physics of asymptotically free gauge theory

asymptotically free gauge theory $SU(N_c)$

chiral symmetry breaking and confinement

N_f massive quarks $m_q \ll \Lambda_{\mathrm{QCD}}$ fundamental representation of gauge group

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$$\mathcal{L} = i\sum_j^{N_f} \bar{q}_j \not{D} q_j - \sum_j^{N_f} m_q \bar{q}_j q_j - \frac{1}{4} G_a^{\mu\nu} G^a_{\mu\nu} + \text{gauge fixing} + \text{ghost}$$

gauge theory parameters: N_c N_f m_q $\Lambda_{\rm QCD}$

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What is the low energy physics?

Pion physics

chiral symmetry breaking

explicitly

quark mass $m_q \bar{q}_j q_j$

$$SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$$

dynamically

condensate $\langle \bar{q}_j q_j \rangle$

(approximate) Goldstone boson dominate the low energy physics

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dominate the low energy physics

e.g. $N_f = 2$ pions $\pi_0 = \pi^3 \quad \pi_\pm = \frac{1}{\sqrt{2}}(\pi^1 \pm i\pi^2)$ $\longrightarrow$ $U = e^{i\frac{\vec{\tau}\cdot\vec{\pi}}{f_\pi}}$

pion decay constant

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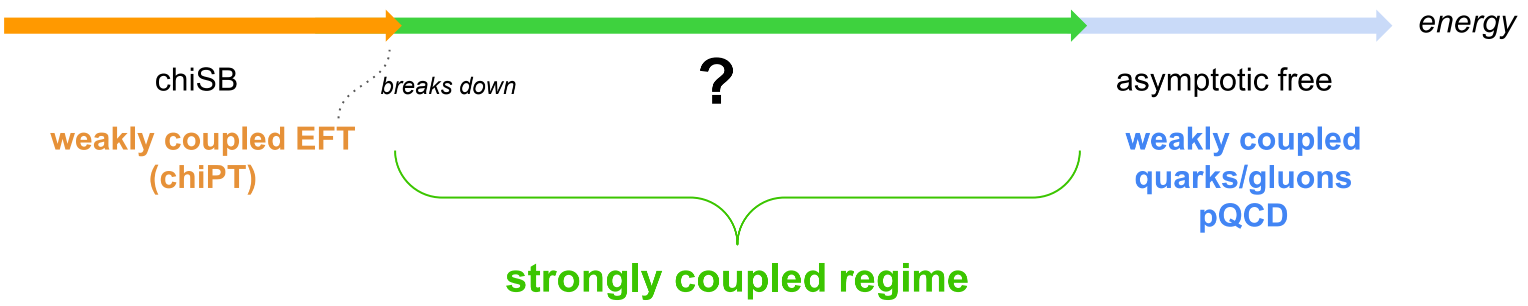
very low energy – effective Lagrangian (lowest order): $\mathcal{L} = \frac{f_\pi^2}{4}\left\{\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + m_\pi^2 \mathrm{Tr}\left(U + U^\dagger\right)\right\}$

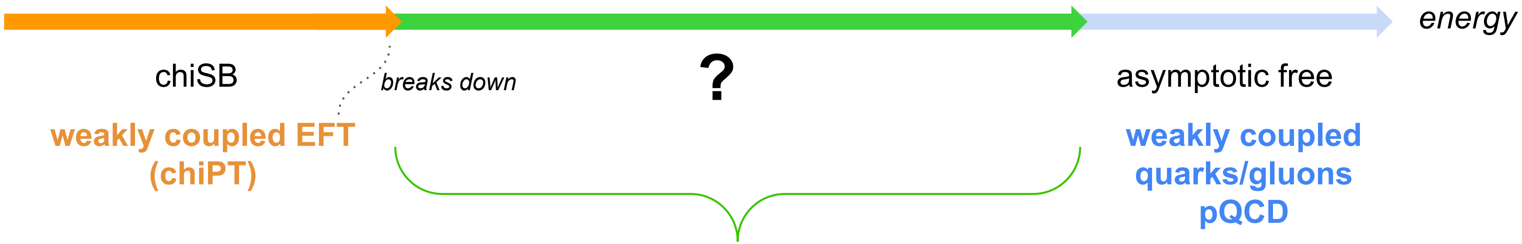
$$\mathcal{L}_2^{2\pi} = \frac{1}{2}\partial_\mu \vec{\pi}\cdot\partial^\mu\vec{\pi} - \frac{1}{2}m_\pi^2 \vec{\pi}^2 \qquad \mathcal{L}_2^{4\pi} = \frac{1}{6f_\pi^2}\Big((\vec{\pi}\cdot\partial_\mu\vec{\pi})^2 - \vec{\pi}^2(\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi})\Big) + \frac{m_\pi^2}{24 f_\pi^2}(\vec{\pi}^2)^2 \quad \ldots$$

The problem of strongly coupled QCD

The problem of strongly coupled QCD



The problem of strongly coupled QCD

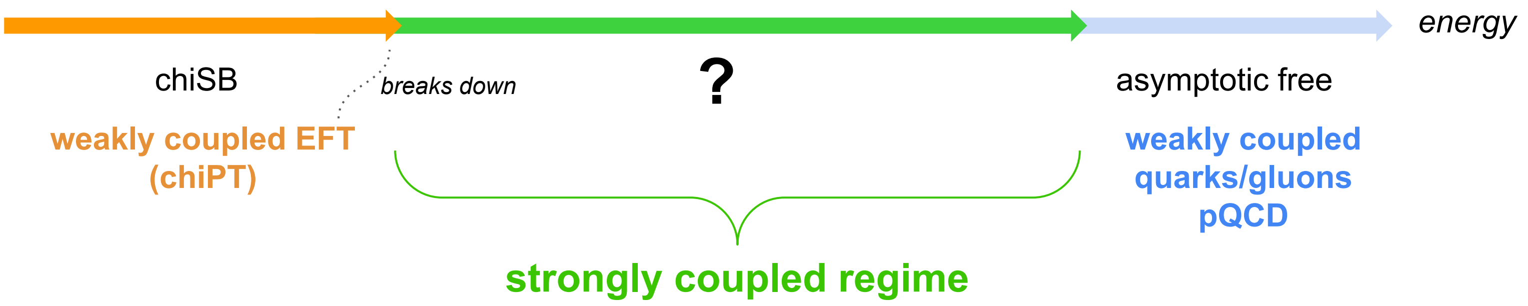


strongly coupled regime

bootstrap

compute the S-matrix of pions

The problem of strongly coupled QCD



bootstrap

compute the S-matrix of pions

rules of the game: chiral symmetry breaking, confinement, gauge theory parameters

as few as possible low energy parameters

Bootstrap approach

- **Pure S-matrix bootstrap:**

symmetry, analyticity, crossing, unitarity

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very low energy behavior (weakly coupled EFT)

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gauge theory information

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Numerical test of the method: $N_f = 2 \quad N_c = 3$

can be compared with QCD experimental data

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formalism is general — can be compared with lattice data

partial waves

$$f_\ell^I(s)$$

form factors

$$F_\ell^I(s)$$

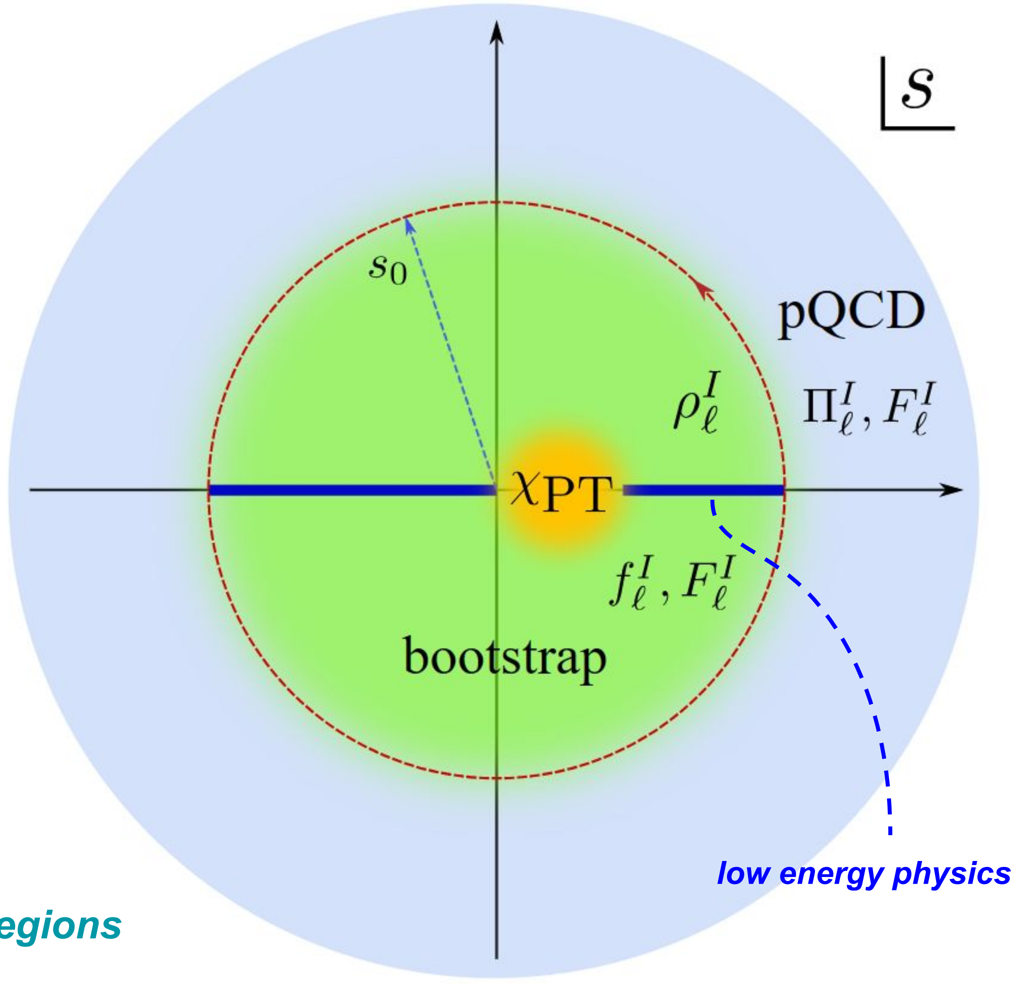
2-point functions

$$\Pi_\ell^I(s)$$

spectral density

$$\rho_\ell^I(s)$$

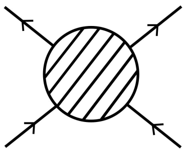
(partial waves, form factors and 2-point functions:) *analytic function in s*



control in different regions

S-matrix bootstrap

modern S-matrix bootstrap: [Paulos, Penedones, Toledo, van Rees, Vieira, 2016 & 2017]



in the context of pion scattering: $\pi_a(p_1) + \pi_b(p_2) \to \pi_c(p_3) + \pi_d(p_4)$

$$\langle p_1, a; p_2, b|\mathbf{T}|p_3, c; p_4, d\rangle = A(s,t,u)\delta_{ab}\delta_{cd} + A(t,s,u)\delta_{ac}\delta_{bd} + A(u,t,s)\delta_{ad}\delta_{bc}$$

($\times$ momentum conservation)

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_3)^2$$
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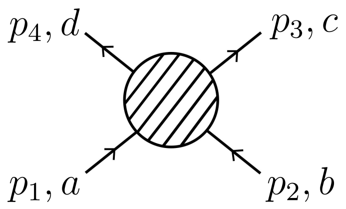

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analyticity cuts $s, t, u > 4$

$$m_\pi = 1$$

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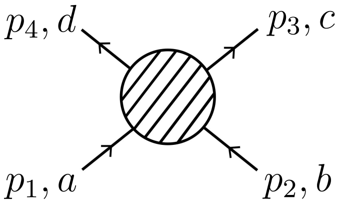

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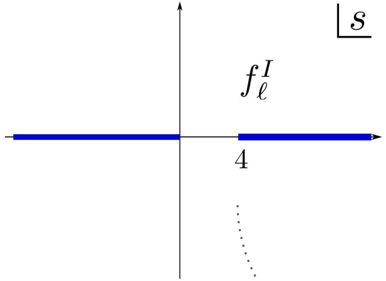
$$A(s,t,u) = T_0 + \frac{1}{\pi}\int_4^\infty dx\, \frac{\sigma_1(x)}{x-s} + \frac{1}{\pi}\int_4^\infty dx\, \sigma_2(x)\left[\frac{1}{x-t} + \frac{1}{x-u}\right]$$
$$+ \frac{1}{\pi^2}\int_4^\infty dx \int_4^\infty dy\, \frac{\rho_1(x,y)}{x-s}\left[\frac{1}{y-t} + \frac{1}{y-u}\right] + \frac{1}{\pi^2}\int_4^\infty dx \int_4^\infty dy\, \frac{\rho_2(x,y)}{(x-t)(y-u)}$$

parameters: $T_0, \quad \sigma_{\alpha=1,2}(x), \quad \rho_{\alpha=1,2}(x,y)$

S-matrix bootstrap



$$
\begin{aligned}
T^{I=0}(s,t,u) &= 3A(s,t,u) + A(t,s,u) + A(u,t,s) \\
T^{I=1}(s,t,u) &= A(t,s,u) - A(u,t,s) \\
T^{I=2}(s,t,u) &= A(t,s,u) + A(u,t,s)
\end{aligned}
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$SU(2)_V$ isospin

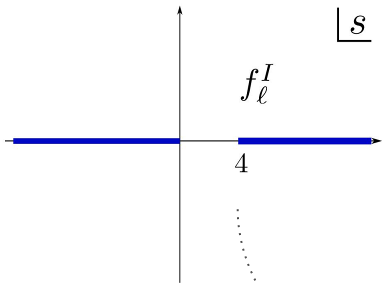
symmetry

partial waves:

$$f_\ell^I(s) = \frac{1}{4}\int_{-1}^{+1} d\mu\, P_\ell(\mu)\, T^I(s,t)$$

analytic function of s

S-matrix bootstrap



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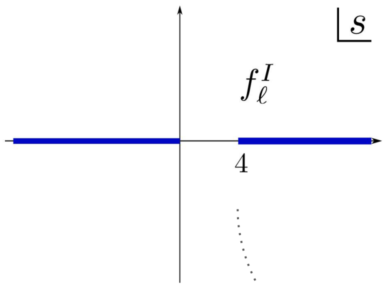
analytic function of s

phase shift

$$|S_\ell^I(s^+)| \leq 1, \;\; s > 4 \quad \forall \ell, I$$

unitarity

S-matrix bootstrap



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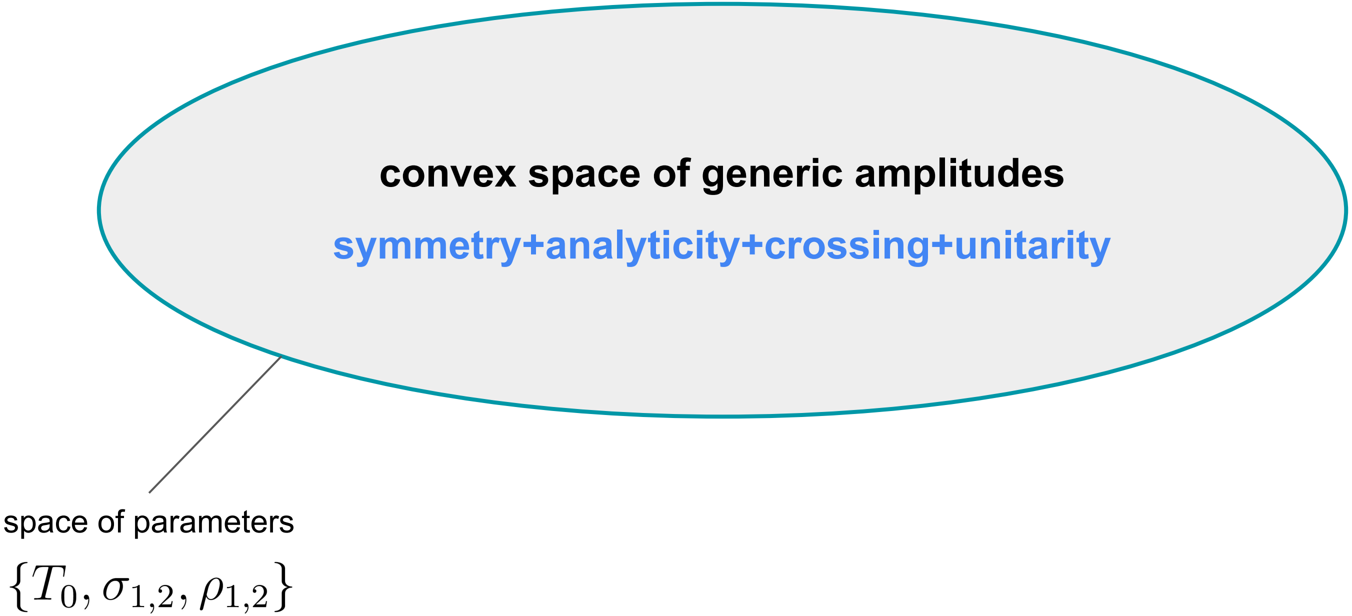
unitarity

$$\begin{pmatrix} 1 & S_\ell^I(s) \\ S_\ell^{I*}(s) & 1 \end{pmatrix} \succeq 0$$

positive semidefinite → convex space of amplitudes

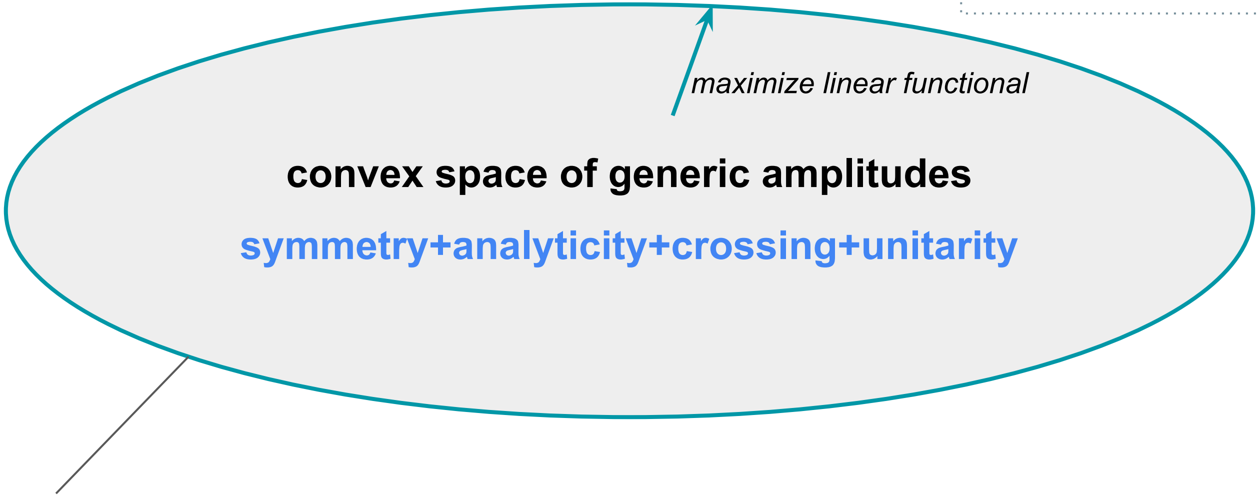
convex optimization

S-matrix bootstrap



S-matrix bootstrap

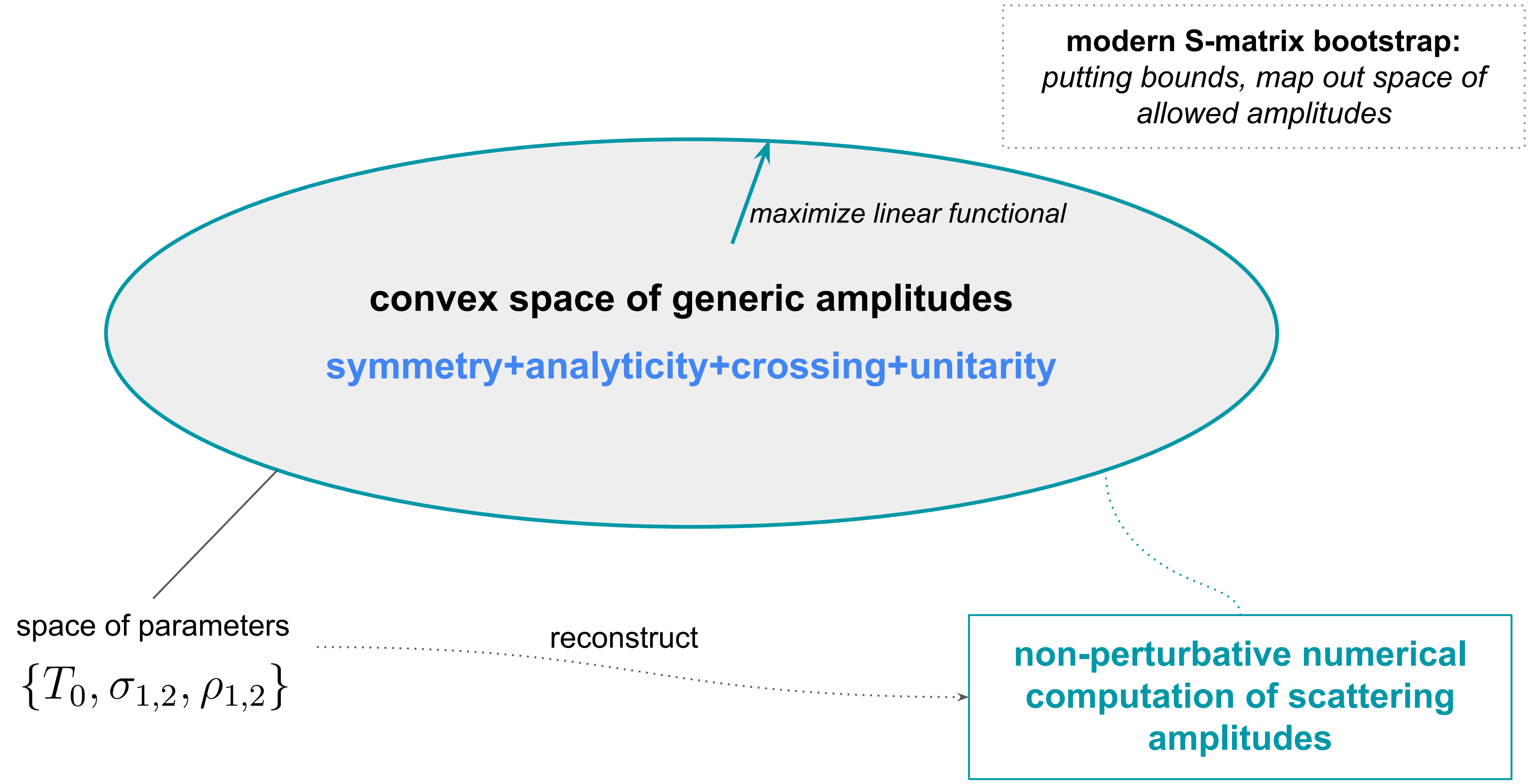
modern S-matrix bootstrap:
putting bounds, map out space of allowed amplitudes



space of parameters

$$\{T_0, \sigma_{1,2}, \rho_{1,2}\}$$

S-matrix bootstrap

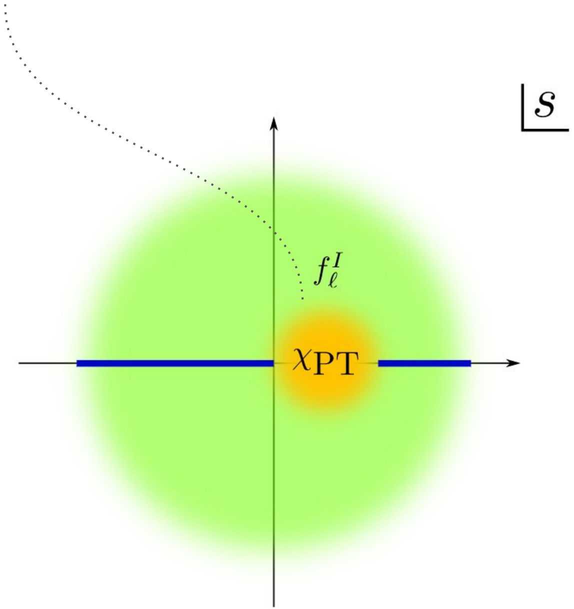


Chiral symmetry breaking

EFT gives very good control in the very low energy subthreshold region

interaction: $$\mathcal{L}_2^{4\pi} = \frac{1}{6f_\pi^2}\Big((\vec{\pi}\cdot\partial_\mu\vec{\pi})^2 - \vec{\pi}^2(\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi})\Big) + \frac{m_\pi^2}{24f_\pi^2}(\vec{\pi}^2)^2$$

tree-level amplitude: $$A(s,t,u) = \frac{4}{\pi}\frac{s-m_\pi^2}{32\pi f_\pi^2}$$ [Weinberg, 1966]



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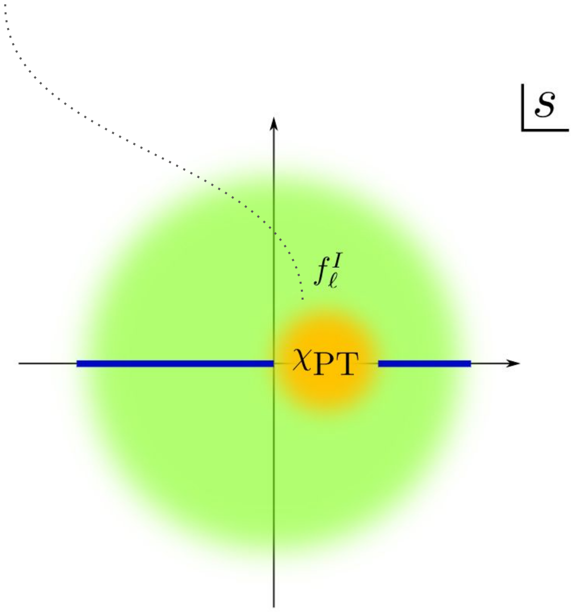
$$f_0^0(s) = \frac{2}{\pi}\frac{2s-m_\pi^2}{32\pi f_\pi^2}, \quad f_1^1(s) = \frac{2}{\pi}\frac{s-4m_\pi^2}{96\pi f_\pi^2}, \quad f_0^2(s) = \frac{2}{\pi}\frac{2m_\pi^2-s}{32\pi f_\pi^2}$$

$S0$ $P1$ $S2$

approximate linear subthreshold behavior: input in bootstrap



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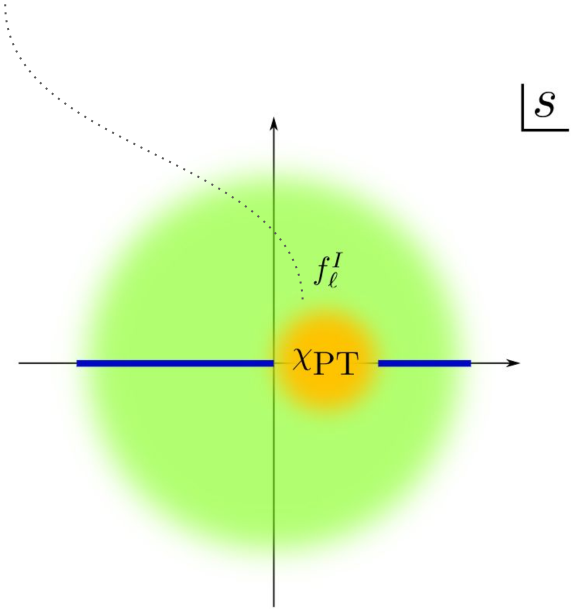

approximate linear subthreshold behavior: input in bootstrap

can consider various values of the pion decay constant f_π

approximate linearity to be valid: $\lambda_{\text{eff}} \sim \frac{s}{f_\pi^2}$ small in the subthreshold region $0 < s < 4m_\pi^2$ ➡ f_π/m_π bounded from below

Form factor bootstrap

[Karateev, Kuhn, Penedones, 2019]

an important development in modern S-matrix bootstrap:

$$|\psi_1\rangle = |p_1, p_2\rangle_{in}\,, \qquad |\psi_2\rangle = |p_1, p_2\rangle_{out}\,, \qquad |\psi_3\rangle = \int dx e^{-i(p_1+p_2)\cdot x}\mathcal{O}(x)|0\rangle$$

asymptotic states – IR

state created by local operator – UV

positive semidefinite matrix

$$\langle\psi_a|\psi_b\rangle = \begin{pmatrix} 1 & S & \mathcal{F} \\ S^* & 1 & \mathcal{F}^* \\ \mathcal{F}^* & \mathcal{F} & \rho \end{pmatrix} \succeq 0$$

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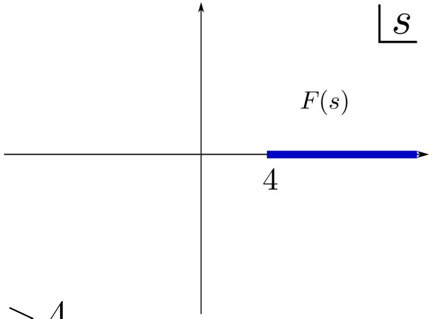

2-particle form factor: $${}_{\text{out}}\langle p_1, p_2|\mathcal{O}(0)|0\rangle = F(s)$$ *analytic function of s*

spectral density: $$\int \frac{d^4x}{(2\pi)^4} e^{iPx}\langle 0|\mathcal{O}^\dagger(x)\mathcal{O}(0)|0\rangle = \rho(s)$$

support at $s > 4$

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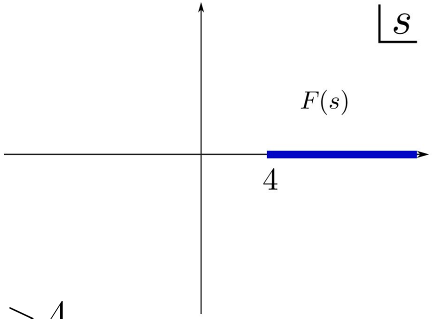

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2d applications: bound UV central charge

allow connection with UV theory

Current correlators from the UV theory

will use form factor bootstrap to connect with UV gauge theory

$$
\begin{array}{c} \langle \text{in}|_{P',I,\ell} \\ \langle \text{out}|_{P',I,\ell} \\ \langle 0|\mathcal{O}^\dagger_{P',I,\ell} \end{array}
\overset{\begin{array}{ccc} |\text{in}\rangle_{P,I,\ell} & |\text{out}\rangle_{P,I,\ell} & \mathcal{O}_{P,I,\ell}|0\rangle \end{array}}{\begin{pmatrix} 1 & S_\ell^I(s) & \mathcal{F}_\ell^I \\ S_\ell^{I*}(s) & 1 & \mathcal{F}_\ell^{I*} \\ \mathcal{F}_\ell^{I*} & \mathcal{F}_\ell^I & \rho_\ell^I(s) \end{pmatrix}} \succeq 0 \qquad s > 4 \quad \forall \ell, I
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construct operators from gauge theory with desired quantum numbers and lowest scaling dimension

e.g.

$$S0 \quad : \quad j_S(x) = m_q(\bar{u}u + \bar{d}d)$$

$$P1 \quad : \quad j_V^\mu(x) = \frac{1}{2}(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d)$$

Current correlators from the UV theory

will use form factor bootstrap to connect with UV gauge theory

$$\begin{array}{c} \\ \langle \text{in}|_{P',I,\ell} \\ \langle \text{out}|_{P',I,\ell} \\ \langle 0|\mathcal{O}^{\dagger}_{P',I,\ell} \end{array} \begin{array}{c} \begin{array}{ccc} |\text{in}\rangle_{P,I,\ell} & |\text{out}\rangle_{P,I,\ell} & \mathcal{O}_{P,I,\ell}|0\rangle \end{array} \\ \begin{pmatrix} 1 & S^I_\ell(s) & \mathcal{F}^I_\ell \\ S^{I*}_\ell(s) & 1 & \mathcal{F}^{I*}_\ell \\ \mathcal{F}^{I*}_\ell & \mathcal{F}^I_\ell & \rho^I_\ell(s) \end{pmatrix} \end{array} \succeq 0 \qquad s > 4 \quad \forall \ell, I$$

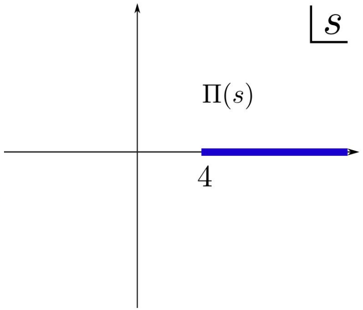
construct operators from gauge theory with desired quantum numbers and lowest scaling dimension

$$\rho^I_\ell(s) = 2\,\text{Im}\Pi^I_\ell(x + i\epsilon)$$

e.g.

$$S0 \quad : \quad j_S(x) = m_q(\bar{u}u + \bar{d}d) \qquad \Pi^0_0(s) = i\int \frac{d^4x}{(2\pi)^4} e^{iPx} \langle 0|\hat{T}\left\{j_S(x) j_S(0)\right\}|0\rangle$$

$$P1 \quad : \quad j^\mu_V(x) = \frac{1}{2}(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d) \qquad \Pi^1_1(s) = i\int \frac{d^4x}{(2\pi)^4} e^{iPx} \langle 0|\hat{T}\left\{j^\dagger_\sigma(x) j_\sigma(0)\right\}|0\rangle$$



Current correlators from the UV theory

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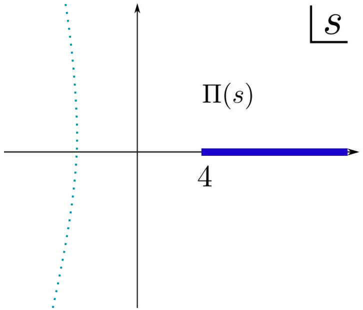

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large spacelike momenta — asymptotic free region with pQCD computation

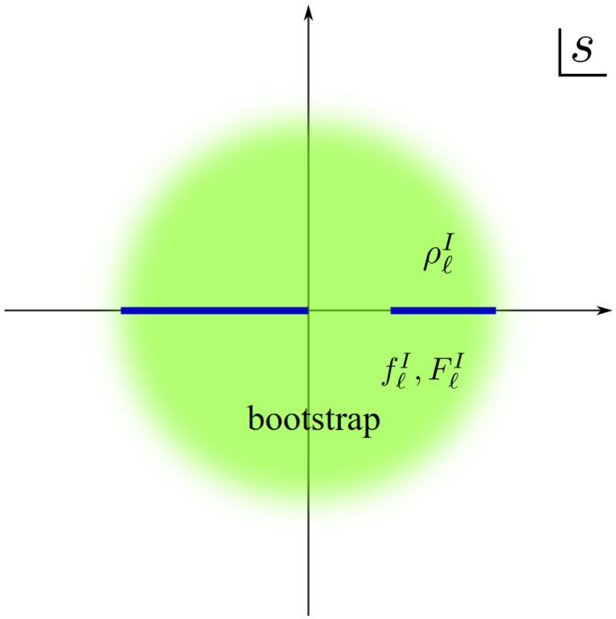
Form factor bootstrap – saturation

positive semidefinite
$$\begin{pmatrix} 1 & S & \mathcal{F} \\ S^* & 1 & \mathcal{F}^* \\ \mathcal{F}^* & \mathcal{F} & \rho \end{pmatrix} \succeq 0 \qquad \forall I,\ \ell,\ s$$

iff all its principal minors are non-negative

$$\rho + S^*\mathcal{F}^2 + S(\mathcal{F}^*)^2 - 2|\mathcal{F}|^2 - \rho|S|^2 \geq 0$$

$$\rho \geq 0 \qquad |\mathcal{F}|^2 \leq \rho \qquad |S|^2 \leq 1$$



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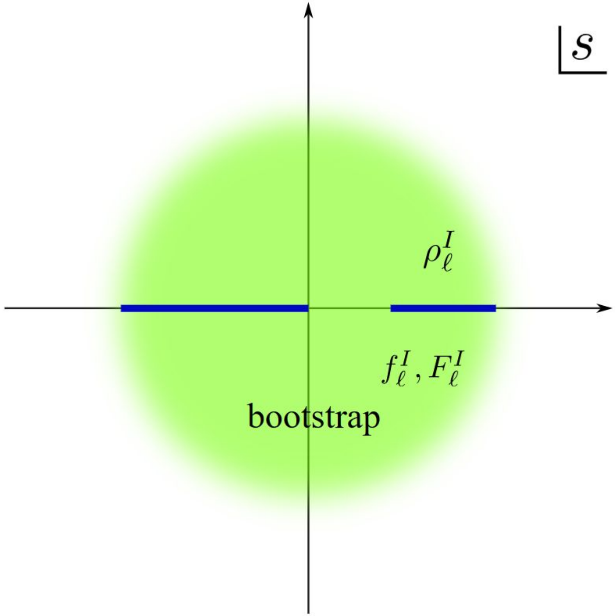
$$|S| = 1 \quad S = \frac{\mathcal{F}}{\mathcal{F}^*}$$

Watson / Muskhelishvili-Omnes



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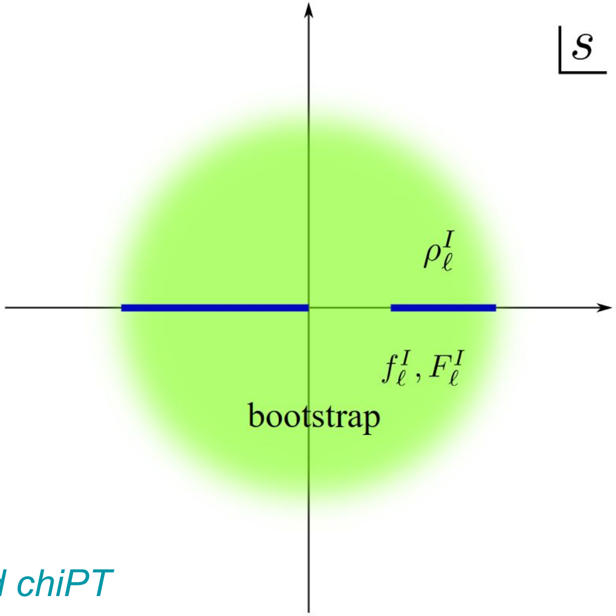
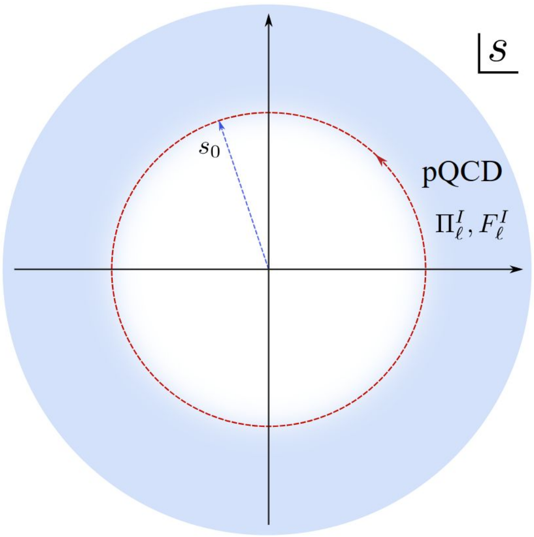


saturation in bootstrap connects quantities controlled by pQCD and chiPT

SVZ expansion

[Shifman, Vainshtein, Zakharov, 1979]

$s \to -\infty$ *perturbative current correlator, e.g.* $\Pi_0^0(s) \simeq \frac{N_c N_f m_q^2}{(2\pi)^4} \frac{(-s)}{8\pi^2} \ln(-\frac{s}{\mu^2})$ $\Pi_1^1(s) \simeq \frac{N_c}{(2\pi)^4} \frac{(-s)}{24\pi^2} \ln(-\frac{s}{\mu^2})$



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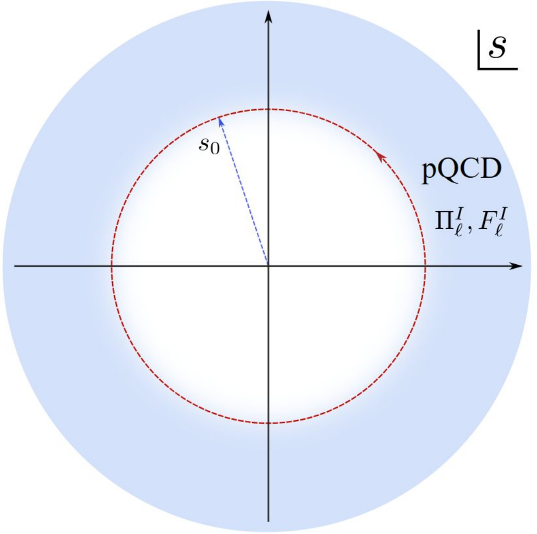
LO in PT

OPE: $$T\{j(x)j(0)\} = C_{\mathbb{1}}(x)\, \mathbb{1} + \sum_{\mathcal{O}} C_{\mathcal{O}}(x)\, \mathcal{O}(0)$$



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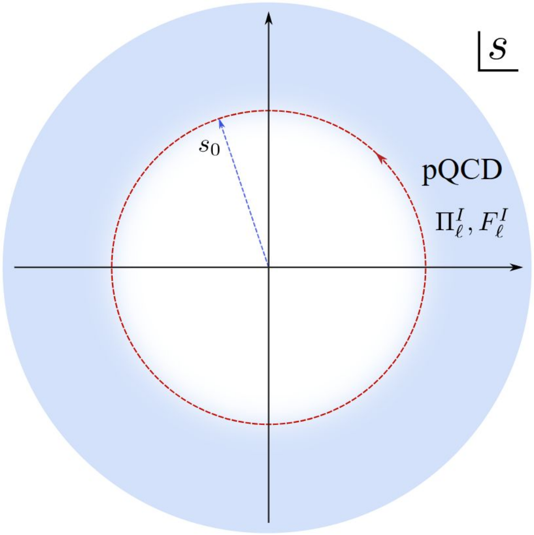

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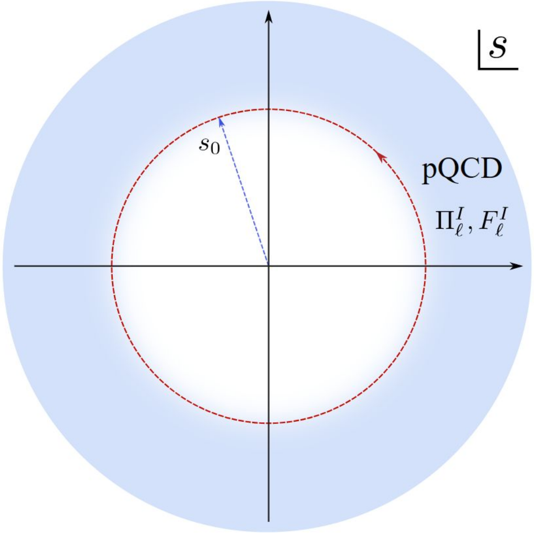
quark condensate

gluon condensate

SB vacuum

pQCD



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quark condensate

gluon condensate

SB vacuum

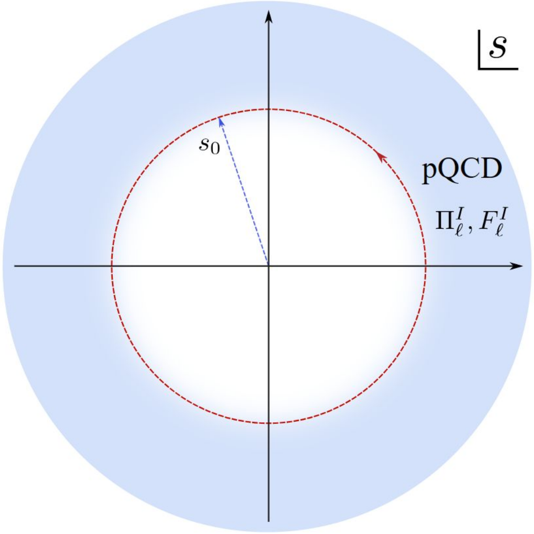
Fourier transform

pQCD

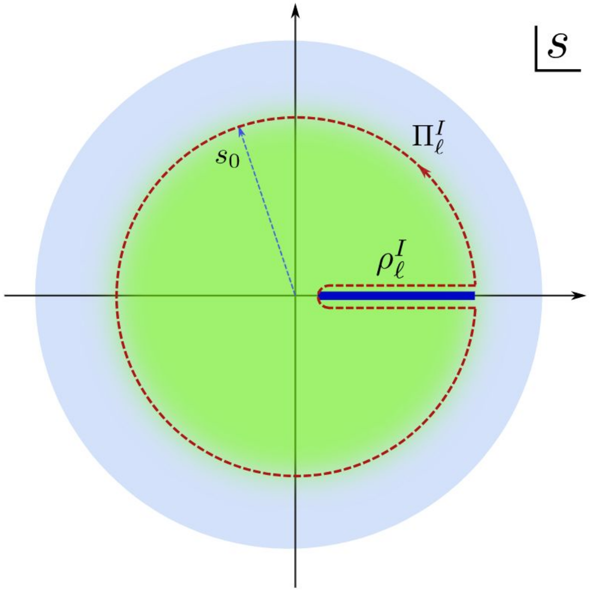
$$N_c = 3$$

$$\Pi_0^0(s) \simeq \frac{N_f m_q^2}{(2\pi)^4}\left\{-\frac{3}{8\pi^2}\left(1+\frac{13}{3}\frac{\alpha_s}{\pi}\right)s\ln(-\frac{s}{\mu^2}) - \frac{1}{8s}\langle\frac{\alpha_s}{\pi}G^2\rangle - \frac{3}{2s}\langle j_S\rangle\right\}$$

$$\Pi_1^1(s) \simeq \frac{1}{2}\frac{1}{(2\pi)^4}\left\{-\frac{1}{4\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)s\ln(-\frac{s}{\mu^2}) + \frac{1}{12s}\langle\frac{\alpha_s}{\pi}G^2\rangle + \frac{1}{s}\langle j_S\rangle\right\}$$

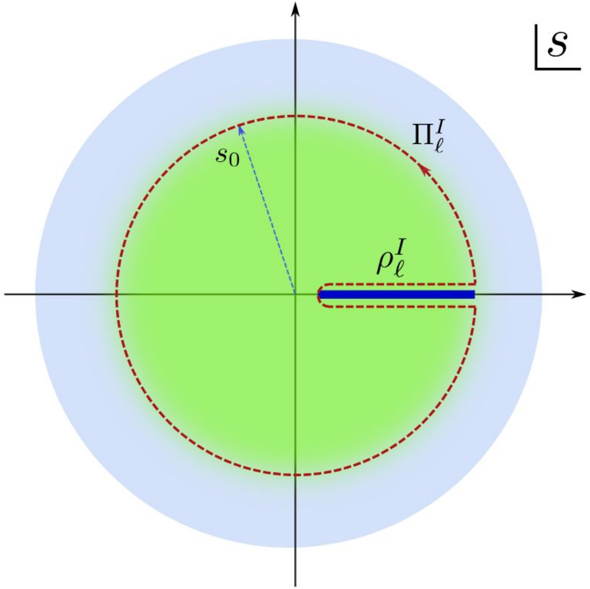


Finite energy sum rule



connect pQCD with bootstrap at s_0

Finite energy sum rule



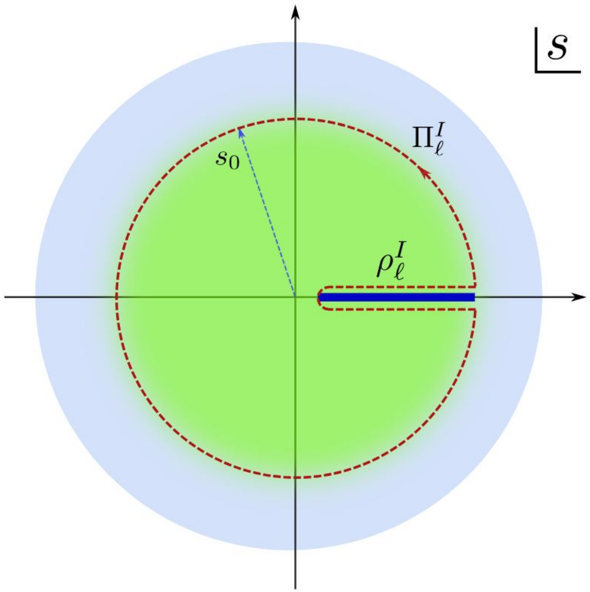
connect pQCD with bootstrap at s_0

integrate $s^n \Pi(s)$ around contour

$$\int_4^{s_0} \rho(x) x^n dx = -s_0^{n+1} \int_0^{2\pi} e^{i(n+1)\varphi} \Pi(s_0 e^{i\varphi}) d\varphi$$

linear constraints on the bootstrap parameter

Finite energy sum rule



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$$S0 \quad : \quad \int_4^{s_0} \rho_0^0(x) x^n dx \;=\; \frac{s_0^{n+1} N_f m_q^2}{(2\pi)^4} \left\{ \frac{3 s_0}{4\pi(n+2)} \left(1 + \frac{13}{3}\frac{\alpha_s}{\pi}\right) + \delta_n \frac{\pi}{4 s_0} \langle \frac{\alpha_s}{\pi} G^2 \rangle + \delta_n \frac{3\pi}{s_0} \langle j_S \rangle \right\}, \quad n \geq 0$$

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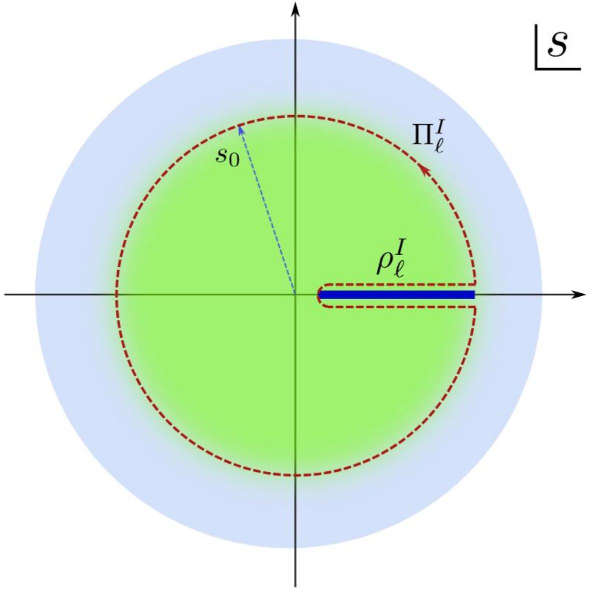

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linear constraints on the bootstrap parameter

m_q N_c $\Lambda_{\rm QCD}$ additional IR parameters

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QCD parameters in our numerical example

explicit QCD parameters used in our test example:

gauge theory info:

$$N_f = 2 \qquad N_c = 3$$

for comparison with experiments

$$s_0 = (1.2\,\mathrm{GeV})^2, \quad \alpha_s = 0.4, \quad m_u = 4\,\mathrm{MeV} \quad m_d = 7.3\,\mathrm{MeV}$$

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can be extracted from lattice computation
numerically not significant in our working precision

possible bootstrap target?

Numerical pure S-matrix bootstrap

analytic & crossing symmetric amplitude $A(s,t,u)$ parametrized by $T_0, \quad \sigma_{\alpha=1,2}(x), \quad \rho_{\alpha=1,2}(x,y)$

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discretize ↓

$$f_\ell^I(s) = \frac{1}{4}\int_{-1}^{+1} d\mu\, P_\ell(\mu)\, T^I(s,t)$$

← compute p.w.

$\{T_0, \sigma_{\alpha,i}, \rho_{\alpha,ij}\}, \quad \alpha = 1,2$

bootstrap variables

Numerical pure S-matrix bootstrap

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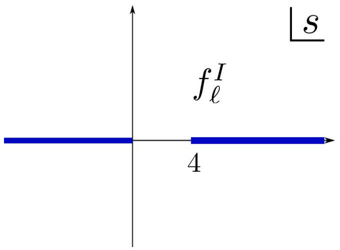
evaluate in unphysical region

$f_\ell^I(0 < s < 4)$

linear functionals and chiSB input (next step)



Numerical pure S-matrix bootstrap

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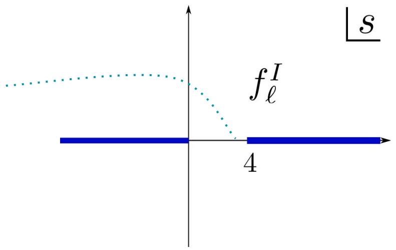
$f_\ell^I(0 < s < 4)$

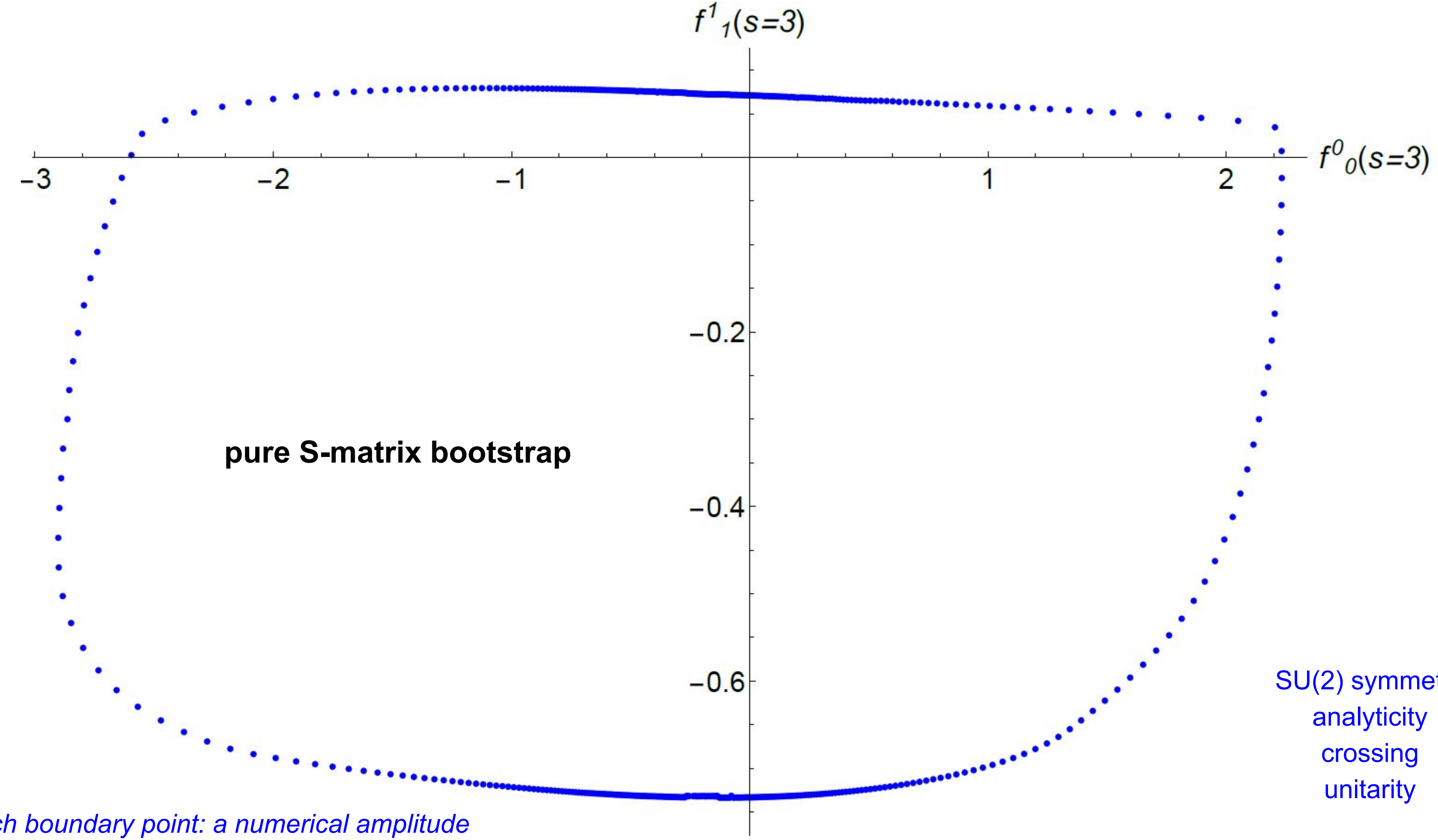
$$\left(f_0^0(s = 3), \; f_1^1(s = 3)\right)$$



linear functionals and chiSB input (next step)

project out space of amplitudes
symmetry, analyticity, crossing, unitarity

boundary: non-perturbative computation of amplitudes

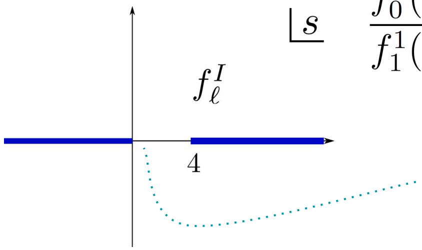


SU(2) symmetry
analyticity
crossing
unitarity

each boundary point: a numerical amplitude

Chiral symmetry breaking input

requires p.w. in subthreshold region to match weakly coupled EFT

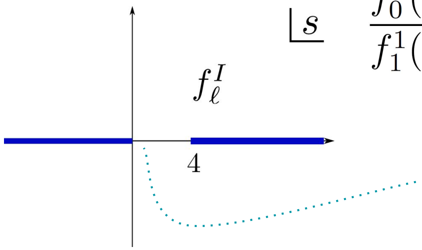


$$\frac{f_0^2(s)}{f_1^1(s)} \simeq \frac{3(2-s)}{s-4} \qquad \frac{f_0^0(s)}{f_1^1(s)} \simeq \frac{3(2s-1)}{s-4}$$

impose ratios at a few points in unphysical very low energy region

Chiral symmetry breaking input

requires p.w. in subthreshold region to match weakly coupled EFT



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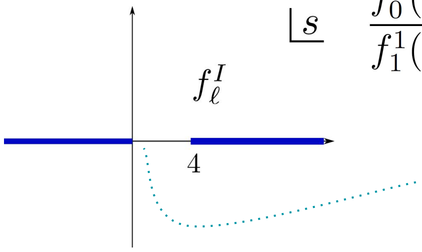
impose ratios at a few points in unphysical very low energy region

selecting pion scattering with various f_π *not too small*

Chiral symmetry breaking input

requires p.w. in subthreshold region to match weakly coupled EFT



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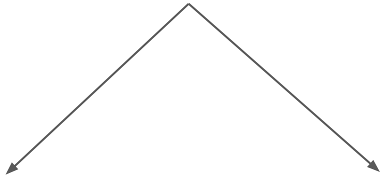
impose ratios at a few points in unphysical very low energy region

selecting pion scattering with various f_π not too small

numerics 1: linear constraints with some tolerance

ϵ^χ



too loose:
large deviation from approximate linear form

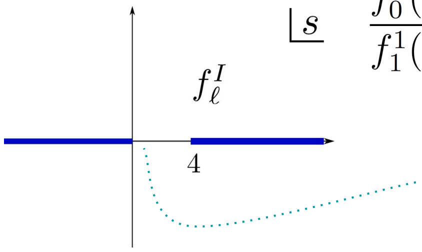
too tight:
exclude the desired theory

a series of tolerance

use $f_\pi = 92\mathrm{MeV}$ to select appropriate tolerance

Chiral symmetry breaking input

requires p.w. in subthreshold region to match weakly coupled EFT



$$\frac{f_0^2(s)}{f_1^1(s)} \simeq \frac{3(2-s)}{s-4} \qquad \frac{f_0^0(s)}{f_1^1(s)} \simeq \frac{3(2s-1)}{s-4}$$

impose ratios at a few points in unphysical very low energy region

selecting pion scattering with various f_π not too small

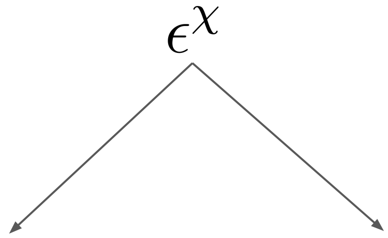
numerics 2: in practice, only 4 points $s \leq 2$

$$s_j = 1/2, 1, 3/2, 2$$

$$\left(f_0^0(s=3),\ f_1^1(s=3)\right)$$

allowed space near EFT prediction

numerics 1: linear constraints with some tolerance

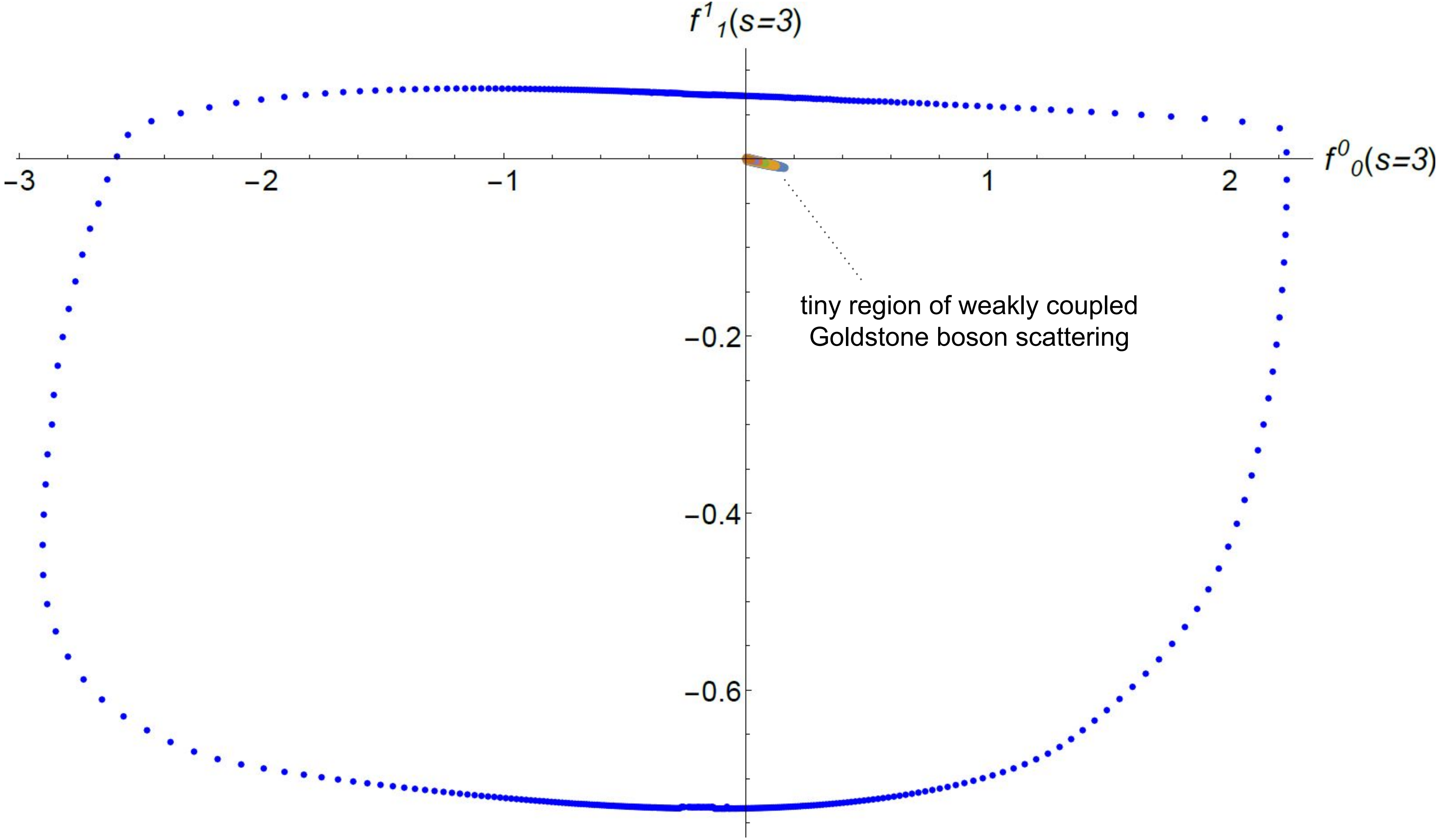


too loose: large deviation from approximate linear form

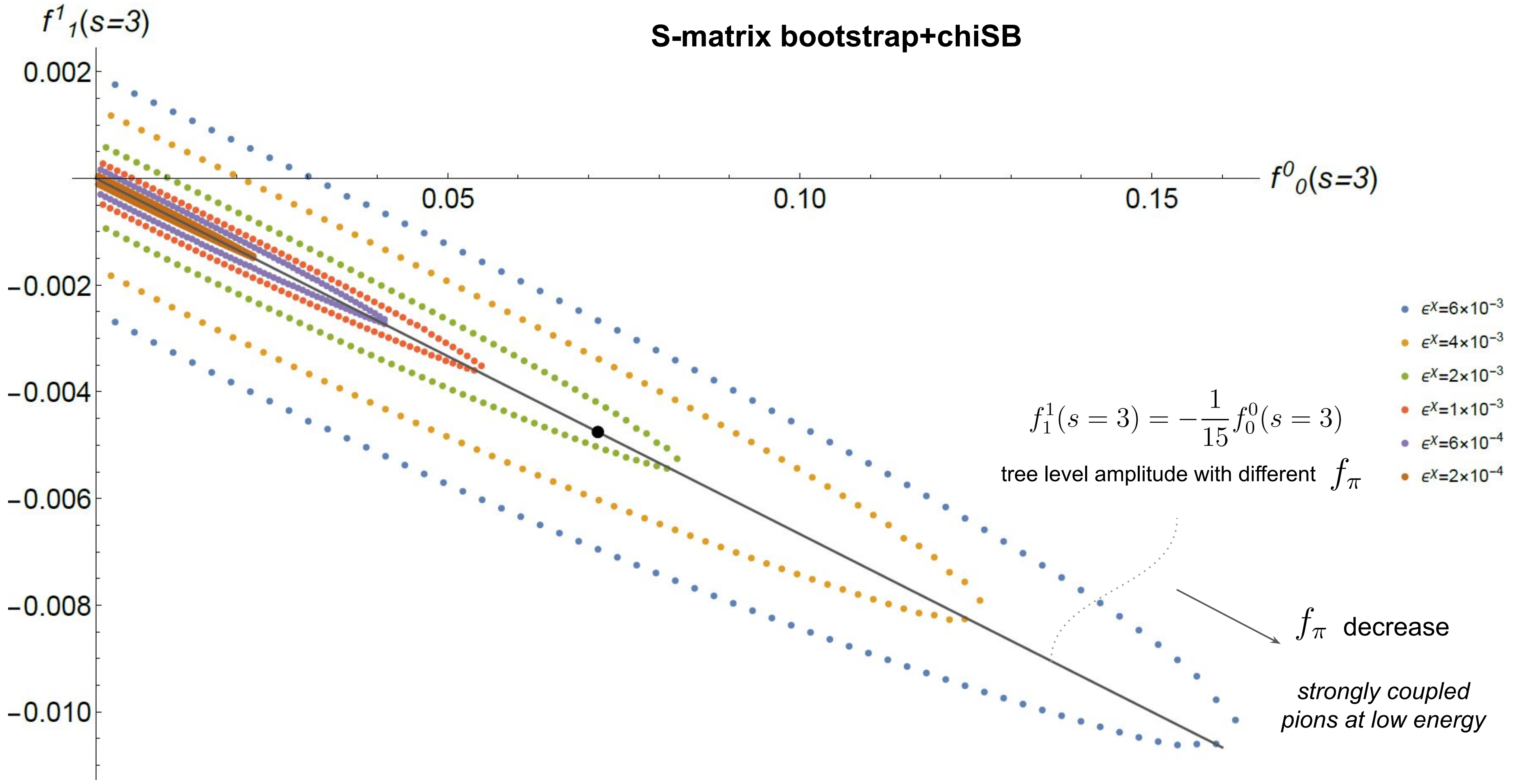
too tight: exclude the desired theory

a series of tolerance

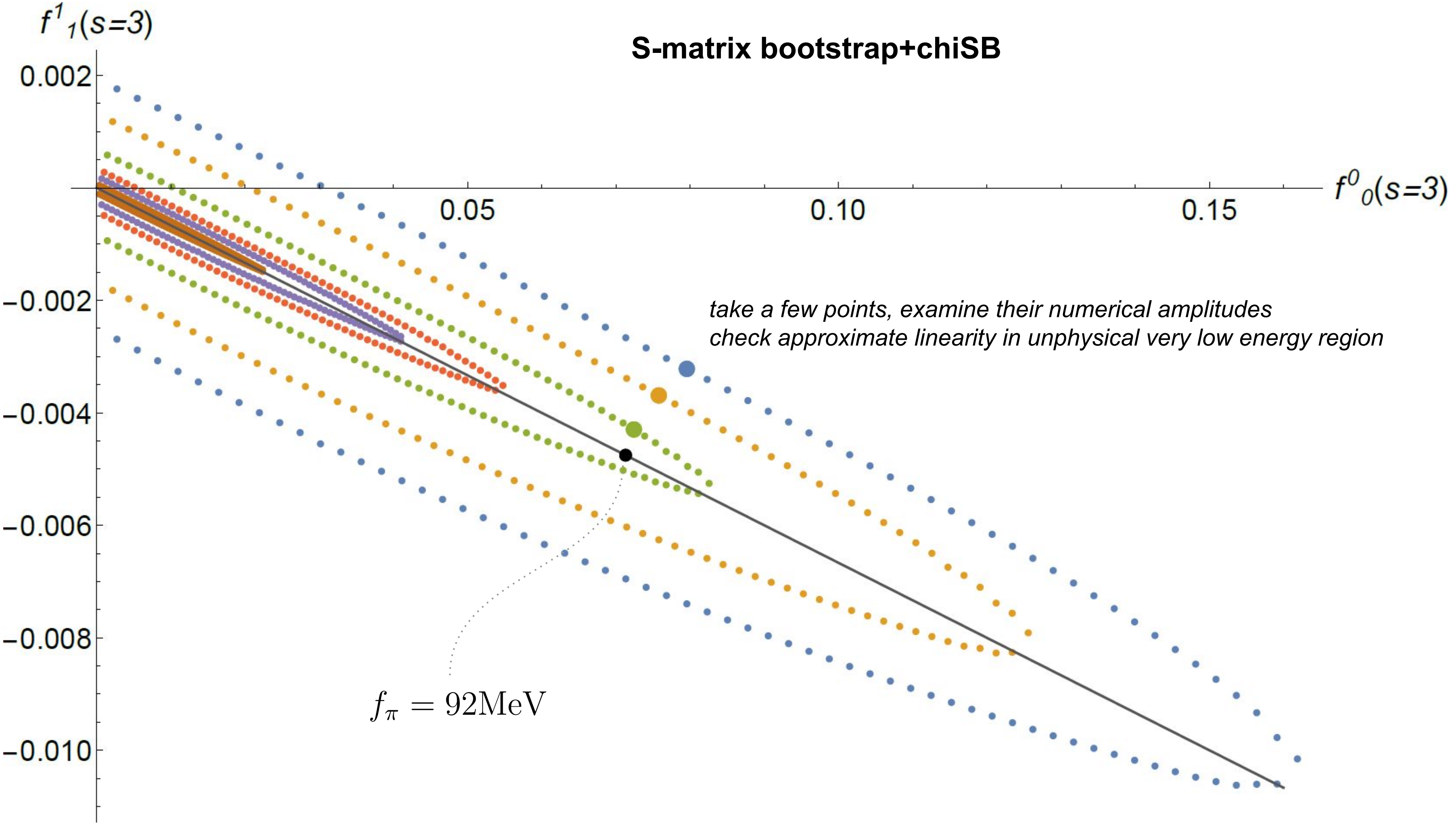
use $f_\pi = 92\mathrm{MeV}$ to select appropriate tolerance



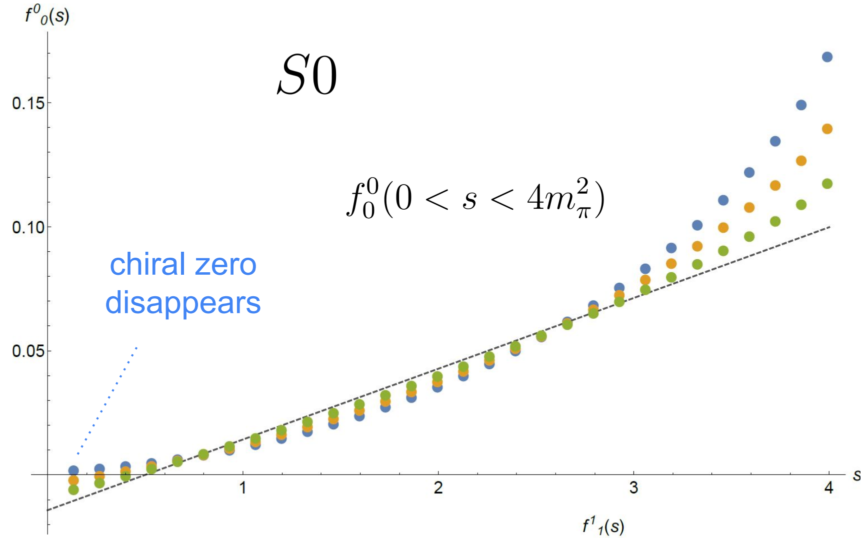
f¹₁(s=3)
f⁰₀(s=3)
−3
−2
−1
1
2
−0.2
−0.4
−0.6
tiny region of weakly coupled
Goldstone boson scattering

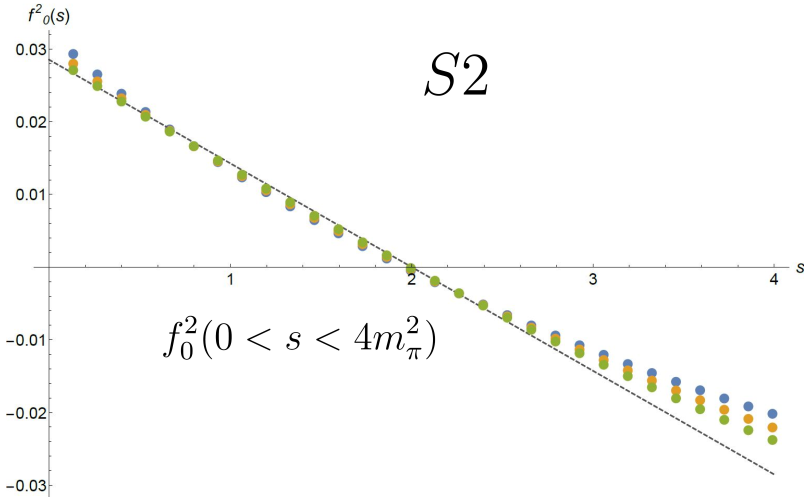


S-matrix bootstrap+chiSB
$f^1{}_1(s=3)$
$f^0{}_0(s=3)$
0.002
−0.002
−0.004
−0.006
−0.008
−0.010
0.05
0.10
0.15
$\epsilon^\chi=6\times10^{-3}$
$\epsilon^\chi=4\times10^{-3}$
$\epsilon^\chi=2\times10^{-3}$
$\epsilon^\chi=1\times10^{-3}$
$\epsilon^\chi=6\times10^{-4}$
$\epsilon^\chi=2\times10^{-4}$
$f_1^1(s=3) = -\frac{1}{15} f_0^0(s=3)$
tree level amplitude with different f_π
f_π decrease
strongly coupled pions at low energy



S-matrix bootstrap+chiSB
$f^1_1(s=3)$
$f^0_0(s=3)$
0.002
−0.002
−0.004
−0.006
−0.008
−0.010
0.05
0.10
0.15
take a few points, examine their numerical amplitudes
check approximate linearity in unphysical very low energy region
$f_\pi = 92\mathrm{MeV}$



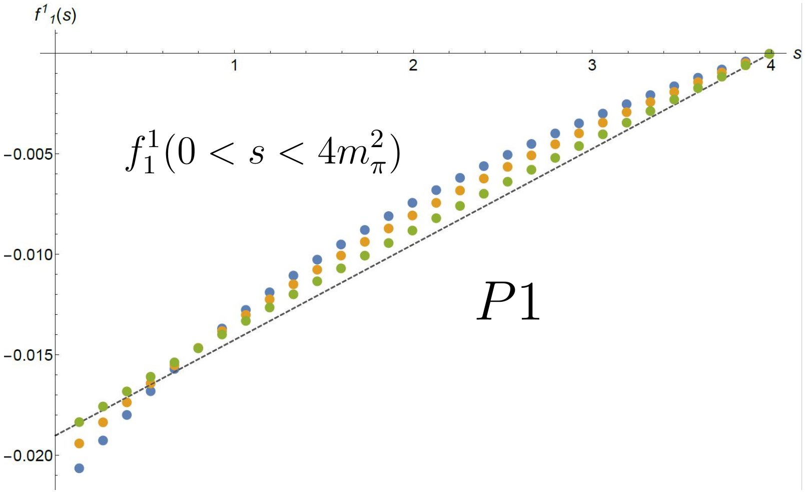


dashed line:

$$f_0^0(s) = \frac{2}{\pi}\frac{2s - m_\pi^2}{32\pi f_\pi^2} \qquad f_0^2(s) = \frac{2}{\pi}\frac{2m_\pi^2 - s}{32\pi f_\pi^2}$$

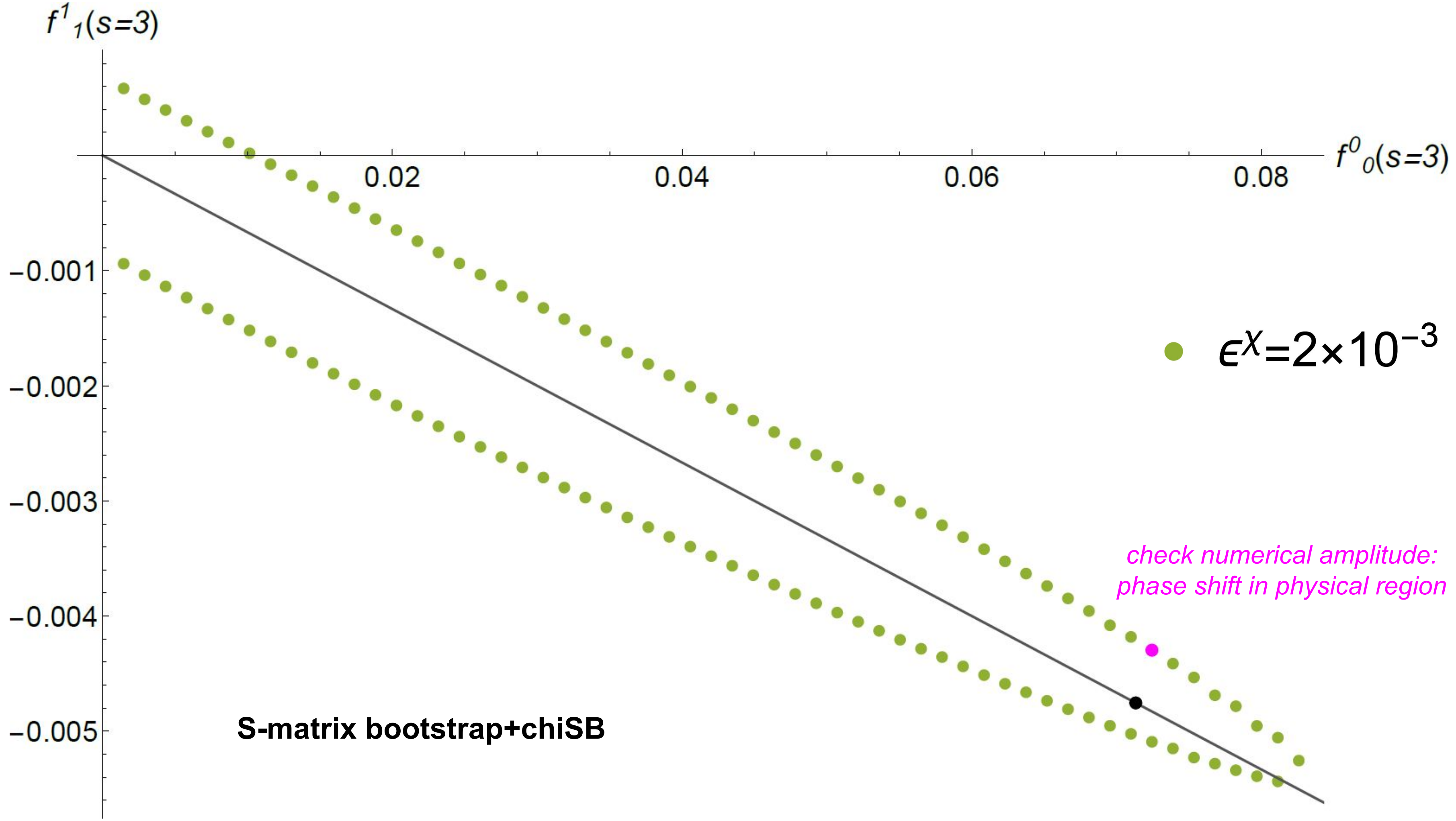
$$f_1^1(s) = \frac{2}{\pi}\frac{s - 4m_\pi^2}{96\pi f_\pi^2}$$

$$f_\pi = 92\mathrm{MeV}$$



- $\epsilon^\chi = 6\times10^{-3}$
- $\epsilon^\chi = 4\times10^{-3}$
- $\epsilon^\chi = 2\times10^{-3}$

$$0 < s < 4m_\pi^2$$



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−0.001
−0.002
−0.003
−0.004
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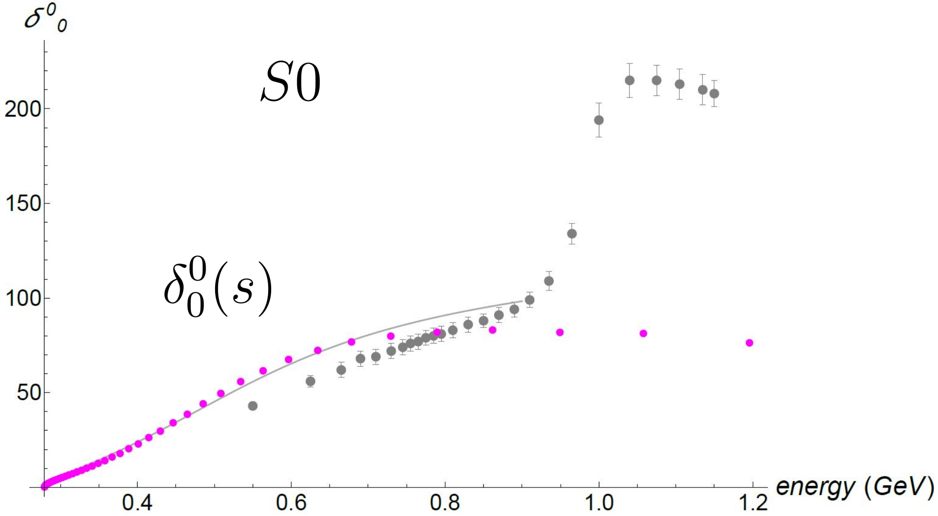
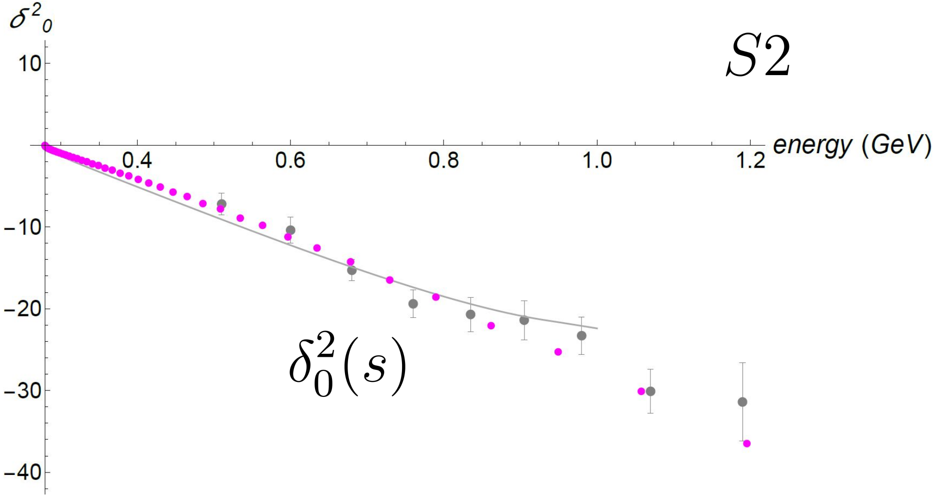
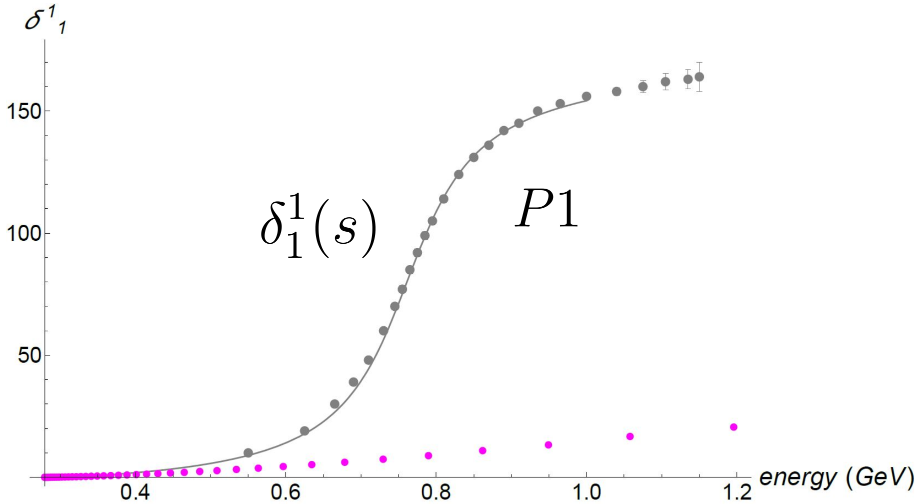
$\epsilon^\chi=2\times10^{-3}$
check numerical amplitude:
phase shift in physical region
S-matrix bootstrap+chiSB



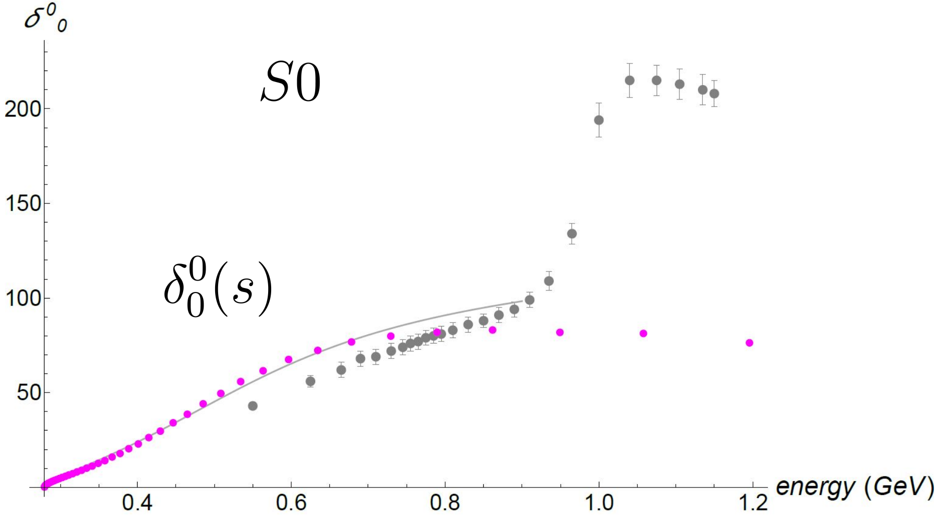


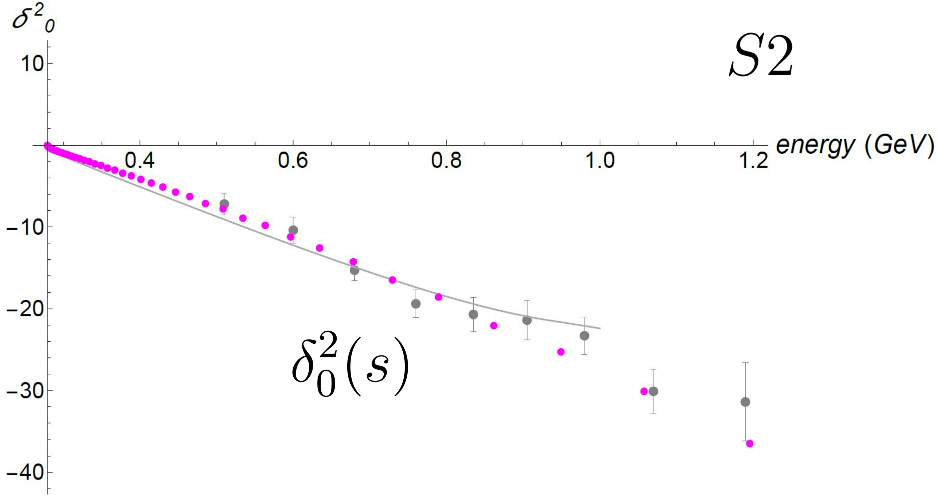
p.w. with only chiral symmetry breaking (EFT) input



experimental data (gray dots)
[Protopopescu et al, 1973]
[Losty et al, 1974]

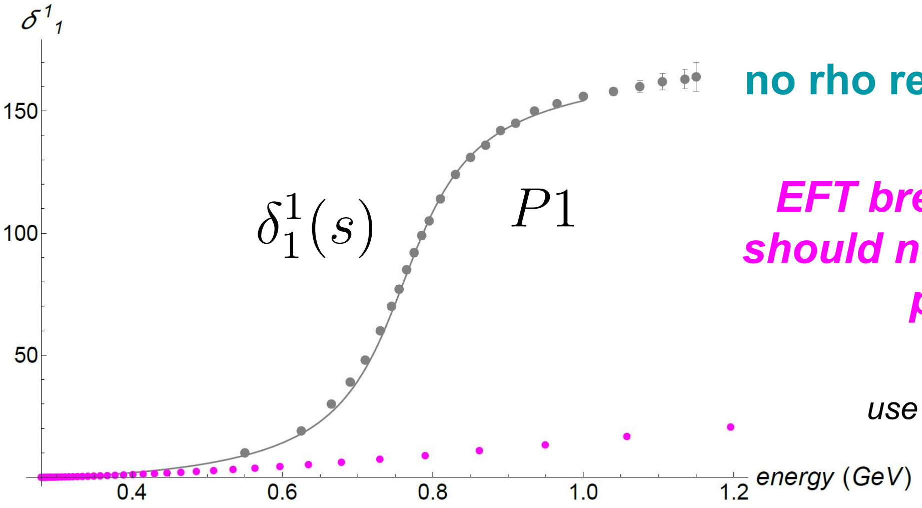
pheno fit (gray line)
[Pelaez, Yndurain, 2005]





good S0 S2 waves, from EFT+unitarity

p.w. with only chiral symmetry breaking (EFT) input



no rho resonance without UV info

makes sense
EFT breaks down at rho mass
should not expect to get rho with purely EFT input

experimental data (gray dots)
[Protopopescu et al, 1973]
[Losty et al, 1974]

pheno fit (gray line)
[Pelaez, Yndurain, 2005]

use same ϵ^{χ} *for imposing SR next*

Form factor bootstrap + SVZ sum rules

form factor bootstrap problem parameterized by:

$$T_0,\ \underbrace{\sigma_{\alpha=1,2}(x),\ \rho_{\alpha=1,2}}_{\text{amplitude part}}(x,y),\ \mathrm{Im}F_\ell^I(x),\ \rho_\ell^I(x) \qquad F_\ell^I(s) = 1 + \frac{1}{\pi}\int_4^\infty dx\left(\frac{1}{x-s} - \frac{1}{x}\right)\mathrm{Im}F_\ell^I(x)$$

discretize

$$\{T_0, \sigma_{\alpha,i}, \rho_{\alpha,ij}, \mathrm{Im}F_{\ell,i}^I, \rho_{\ell,i}^I\}$$

impose positive semidefinite:

$$\begin{pmatrix} 1 & S_{\ell,i}^I & \mathcal{F}_{\ell,i}^I \\ S_{\ell,i}^{I*} & 1 & \mathcal{F}_{\ell,i}^{I*} \\ \mathcal{F}_{\ell,i}^{I*} & \mathcal{F}_{\ell,i}^I & \rho_{\ell,i}^I \end{pmatrix} \succeq 0$$

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inputting QCD parameters in the FESR for S0, P1:

$$\frac{1}{s_0^{n+2}}\int_4^{s_0} \rho_0^0(x) x^n dx \simeq 3.09 \times 10^{-8} \left\{\frac{27.38}{n+2} + 0.61\ \delta_n\right\}$$

$$\frac{1}{s_0^{n+2}}\int_4^{s_0} \rho_1^1(x) x^n dx \simeq -4.34 \times 10^{-6} \left\{-\frac{13.26}{n+2} + 0.41\ \delta_n\right\}$$

can be done for higher pw in general

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can be done for higher pw in general

discretize integral
2 sum rules/p.w.
impose with tolerance ϵ^{SR}

- too loose: uv info does not enter
- too tight: infeasible

numerically: tune down before bootstrap becomes infeasible

Asymptotic behavior of form factor

need control of asymptotic behavior of form factors

at large s $$|F_\pi(s)| \sim \frac{|q|}{|s| R_\pi^2}$$

e.g. more precisely for electromagnetic FF from pQCD

$$F_\pi(s) \simeq -\frac{16\pi\alpha_s(s) f_\pi^2}{s}$$

[Peter Lepage, Brodsky, 1979]

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in practical numerical implementation, only require smallness above $s = s_0$

factor due to charges

S0, P1: $$||\mathcal{F}_0^0(s_i)||^2 \lesssim 2m_q^2\,\epsilon^{FF}, \qquad ||\mathcal{F}_1^1(s_i)||^2 \lesssim \frac{1}{2}\epsilon^{FF}, \quad s_i > s_0$$

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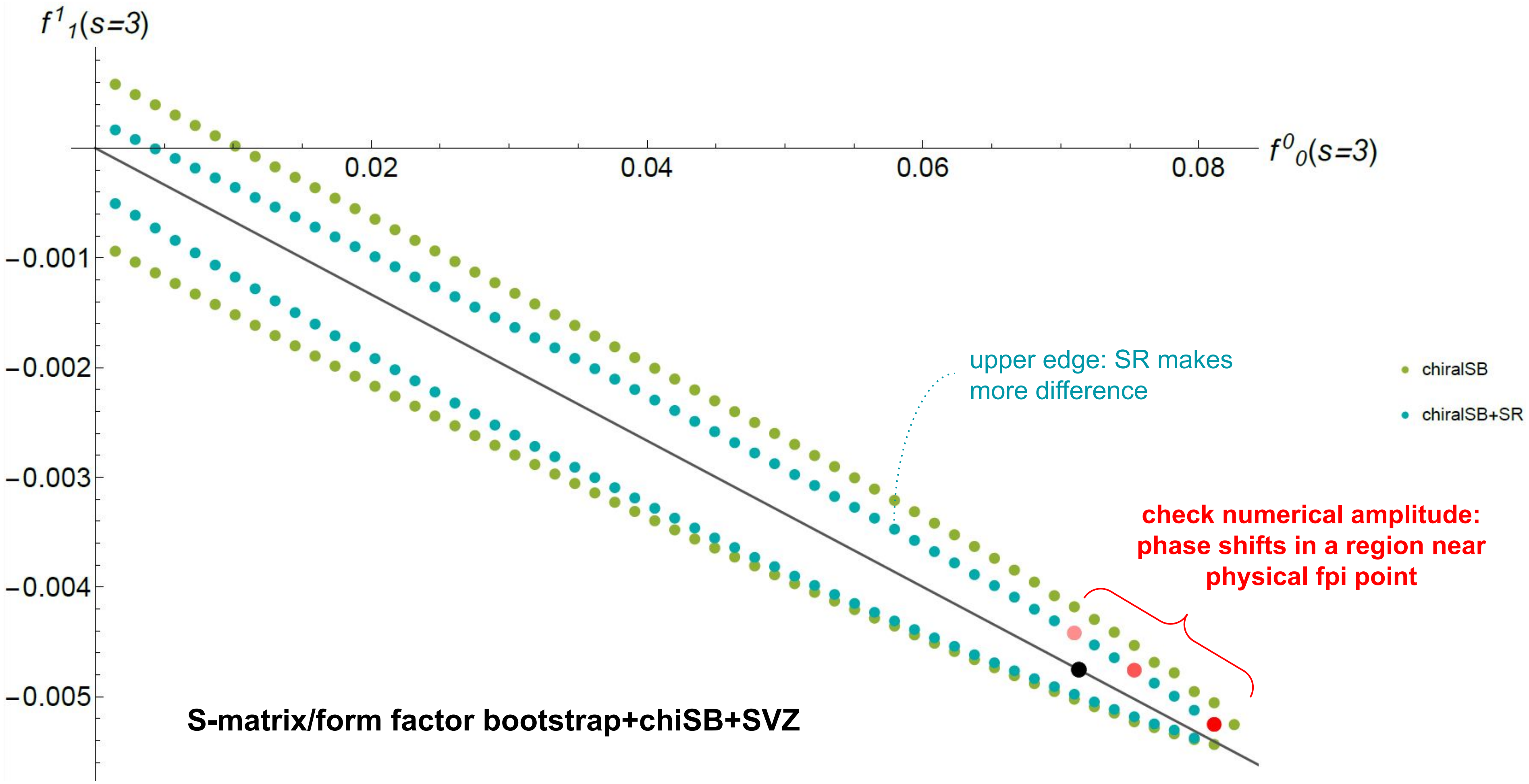
order of magnitude can be estimated

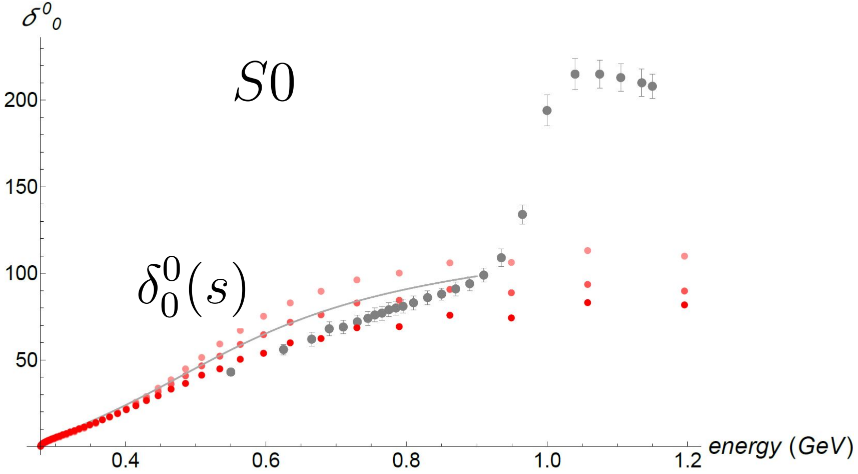
$$\mathcal{O}(10^{-2}) \lesssim |F_\pi(s \geq s_0)| \lesssim \mathcal{O}(10^0)$$

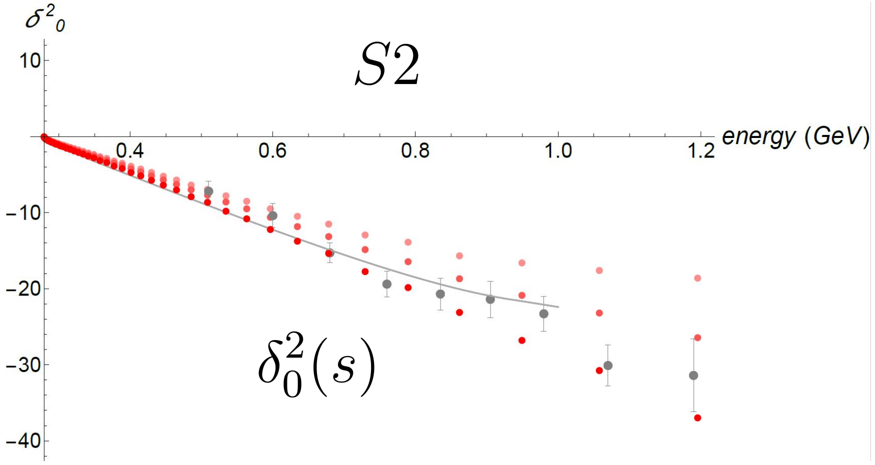
$|\mathcal{F}|^2 \leq \rho$

underestimate from asymptotics FF

overestimate from spectral density


$f^1_1(s=3)$
$f^0_0(s=3)$
0.02
0.04
0.06
0.08
−0.001
−0.002
−0.003
−0.004
−0.005
upper edge: SR makes more difference
chiralSB
chiralSB+SR
check numerical amplitude: phase shifts in a region near physical fpi point
S-matrix/form factor bootstrap+chiSB+SVZ





S-matrix bootstrap with chiSB + SVZ

experimental data (gray dots)
[Protopopescu et al, 1973]
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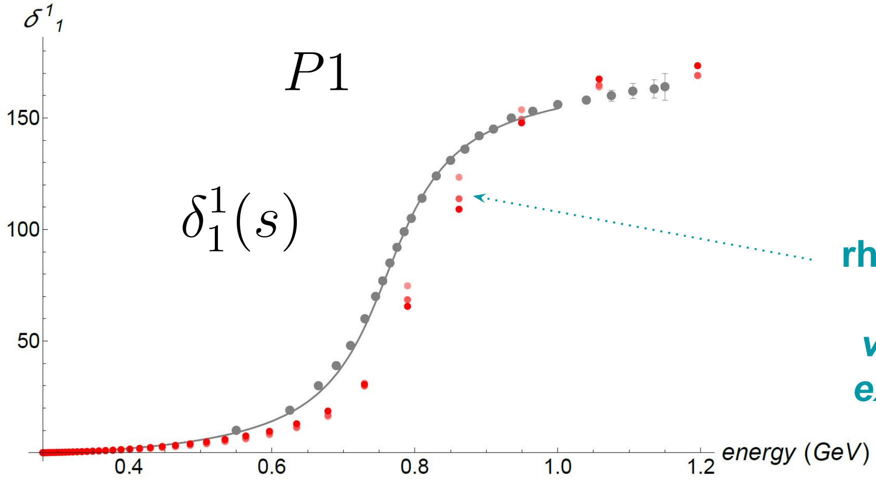
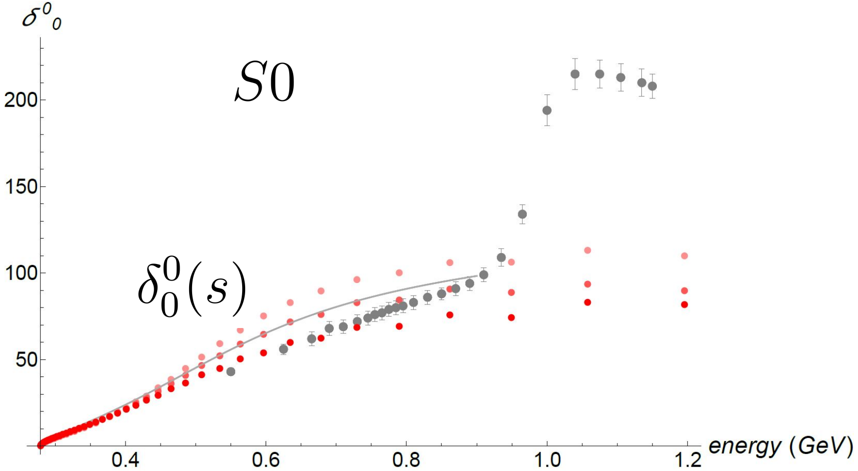
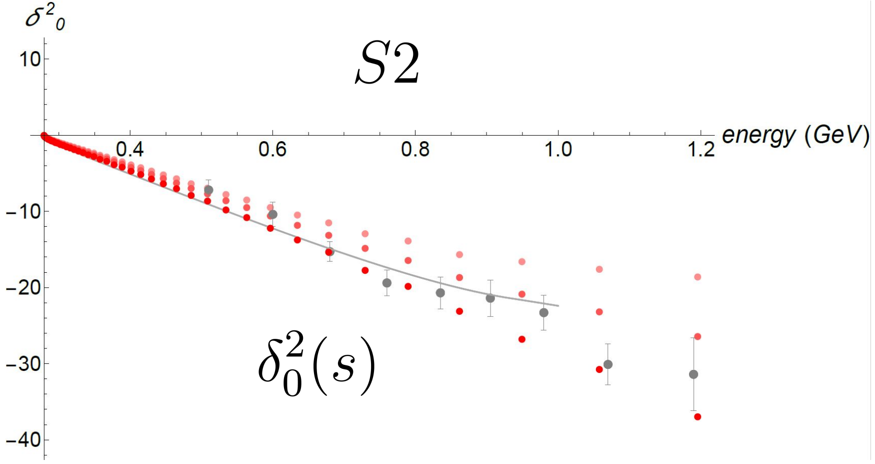


rho resonance, info from UV

very good agreement with experiment given the small amount of input





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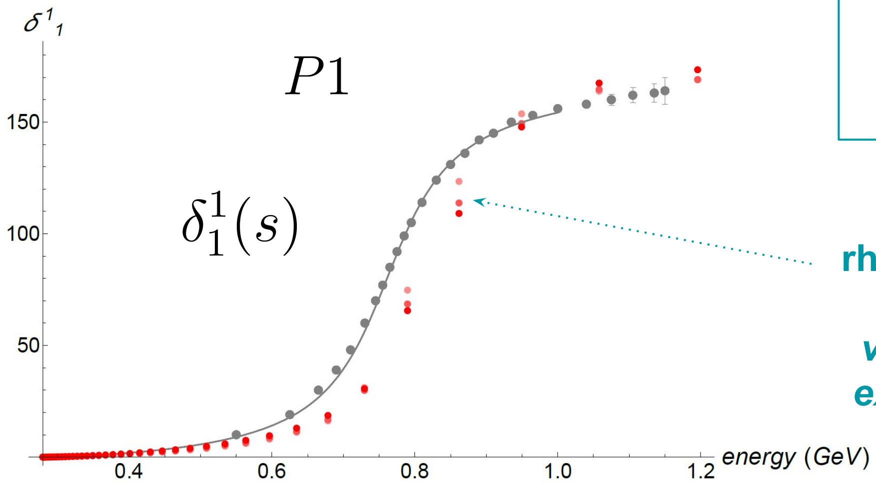

fit to check the rho mass shifted less than 6%
770 MeV vs 800–814 MeV

experimental data (gray dots)
[Protopopescu et al, 1973]
[Losty et al, 1974]
pheno fit (gray line)
[Pelaez, Yndurain, 2005]

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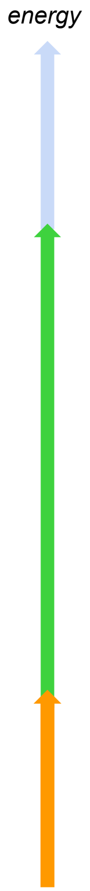
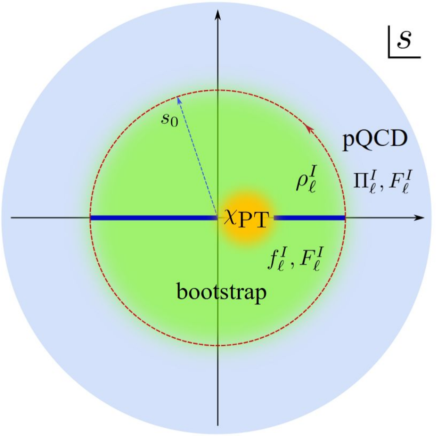
Why does it work?



gauge theory info

pQCD $\Pi_\ell^I(s)$ $F_\ell^I(s)$



subthreshold linear behavior $f_0^0(s) = \frac{2}{\pi}\frac{2s - m_\pi^2}{32\pi f_\pi^2}, \quad f_1^1(s) = \frac{2}{\pi}\frac{s - 4m_\pi^2}{96\pi f_\pi^2}, \quad f_0^2(s) = \frac{2}{\pi}\frac{2m_\pi^2 - s}{32\pi f_\pi^2}$

chiPT (EFT)

Why does it work?

energy

gauge theory info

pQCD $\Pi_\ell^I(s)$ $F_\ell^I(s)$

FESR

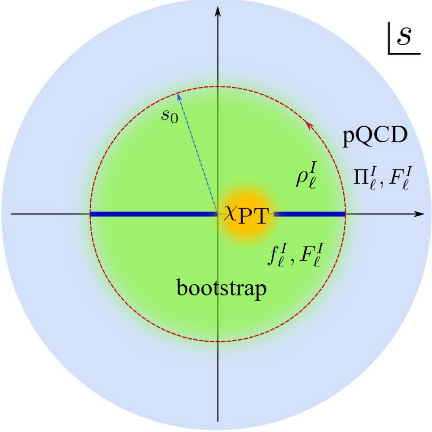
$$\int_4^{s_0} \rho(x) x^n dx = -s_0^{n+1} \int_0^{2\pi} e^{i(n+1)\varphi} \Pi(s_0 e^{i\varphi}) d\varphi$$

$$\rho(s) = |\mathcal{F}(s)|^2$$

$$\mathcal{F}(s) = \sqrt{\rho(s)} e^{i\alpha(s)} \quad \ln \mathcal{F}(s) = \frac{1}{2} \ln \rho(s) + i\alpha(s)$$

dispersion relation

bootstrap



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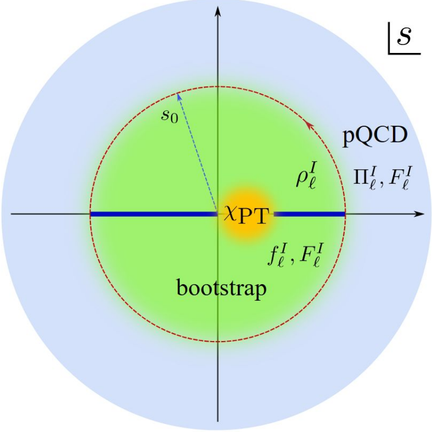

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$$|S(s)| = 1 \qquad e^{2i\alpha(s)} = \frac{\mathcal{F}(s)}{\mathcal{F}^*(s)} = S(s) = e^{2i\delta(s)}$$

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chiPT (EFT)

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FESR

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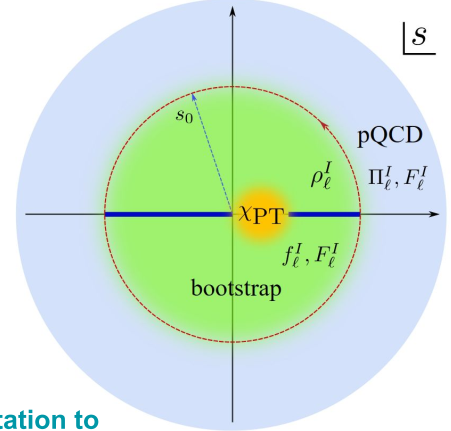

amplitude computation to compare with experiments

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chiPT (EFT)

Conclusions

- Combining old and new techniques: using only $\underbrace{N_c\ N_f\ m_q\ \Lambda_{\rm QCD}}_{\text{gauge theory parameters}}\ \underbrace{f_\pi\ m_\pi}_{\text{low energy parameters}}$

  computed the pion S-matrix in the strongly coupled regime of QCD

  Numerical test find good agreement with experiments

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  Numerical test find good agreement with experiments

- Results show:

  strongly coupled QCD is computable with bootstrap

- Further developments ⟶ deep understanding of low energy QCD

Thank you!