Special Functions for Five-Point One-Mass Scattering in QCD

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based on

Chicherin, VS, Zoia [arXiv:2110.10111]

Abreu, Chicherin, Ita, Page, VS, Tschernow, Zoia [arXiv:2306.15431]

QCD meets gravity 2023,
CERN

15th December 2023

Introduction

Motivation

[A. Huss's talk]

- LHC today is a precision machine
- 20x more data to be taken at HL-LHC, future colliders
- Theoretical understanding of SM predictions is key to interpret data
- At least NNLO QCD and NLO EW corrections (⊕ parton shower, resummation, . . .) must be included to achieve percent level theory uncertainties

"Hard" partonic cross section

$$d\sigma_{h_1 h_2 \to X}(p_1, p_2) = \sum_{i,j} \int \mathrm{d}x_1 \, \mathrm{d}x_2 \, f_i(x_1, \mu) f_j(x_2, \mu) \boxed{\mathrm{d}\hat{\sigma}_{ij \to X}(x_1 p_1, x_2 p_2, \mu)} \qquad + \mathcal{O}(\Lambda_{\mathsf{QCD}}/Q)$$

Intrinsic uncertainty

$$\mathrm{d}\hat{\sigma}_0 \left(1 + \alpha_s \sigma^{(1,0)} + \alpha_s^2 \sigma^{(2,0)} + \alpha \sigma^{(0,1)} + \alpha_s^3 \sigma^{(3,0)} + \alpha \alpha_s \sigma^{(1,1)} + \ldots \right) \qquad \begin{array}{c} \alpha_s(M_Z) \sim 0.1 \\ \alpha(M_Z) \sim 0.01 \end{array}$$

NNLO QCD multiplicity frontier

[S. Zoia's talk]

- NLO QCD, EW conceptually solved, in practice $\lesssim 8$ partons
- NNLO QCD beyond $2 \to 2$ remarakably challenging

both technical and conceptual

What kind of functions loop integrals evaluate to?

$$\sigma^{F+X}_{\text{NNLO}} = \sigma^{F+X}_{\text{NLO}} + \int_{\Phi_{F\boxed{+2}}} \mathrm{d}\sigma_{\text{RR}} + \int_{\Phi_{F\boxed{+1}}} \mathrm{d}\sigma_{\text{RV}} + \boxed{\int_{\Phi_F} \mathrm{d}\sigma_{\text{VV}}}$$

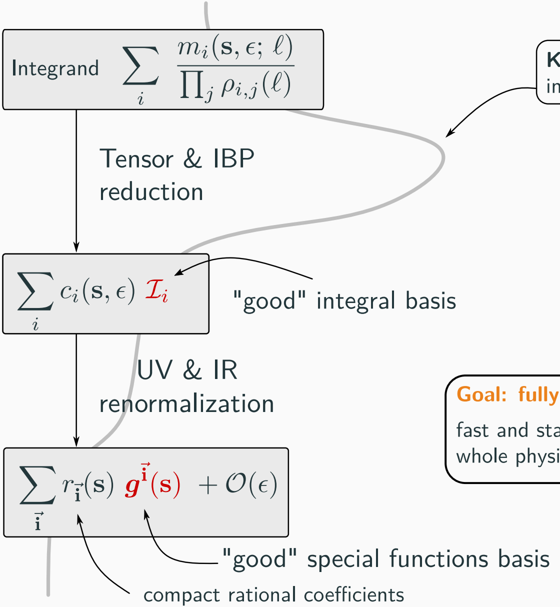
IR divergences

Two-loop amplitudes

This talk:

$2 \to 3$ processes with one external mass
$pp \to Vjj$, $pp \to Hjj$, $e^+e^- \to 4j$

Analytic multi-loop amplitude calculations



Integrand $\sum_i \frac{m_i(\mathbf{s}, \epsilon;\ \ell)}{\prod_j \rho_{i,j}(\ell)}$

Key bottleneck
intermediate expression swell

Tensor & IBP reduction

$$\sum_i c_i(\mathbf{s}, \epsilon)\ \mathcal{I}_i$$

"good" integral basis

UV & IR renormalization

$$\sum_{\vec{\mathbf{i}}} r_{\vec{\mathbf{i}}}(\mathbf{s})\ \boldsymbol{g}^{\vec{\mathbf{i}}}(\mathbf{s})\ + \mathcal{O}(\epsilon)$$

"good" special functions basis

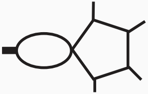
compact rational coefficients

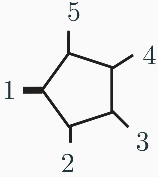
Goal: fully differential cross sections

fast and stable evaluation over
whole physical phase space

Setup of the computation

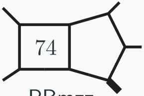
Integral topologies & kinematics



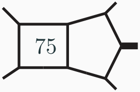


Variables

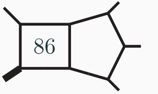
$p_1^2,\ s_{12},\ s_{23},\ s_{34},\ s_{45},\ s_{15}$



PBmzz



PBzmz



PBzzz

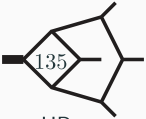
Gram determinants:

$$\Delta_5 = 16\ G(p_1, p_2, p_3, p_4)$$

$$\Delta_3^{(1)} = -4\ G(p_1, p_2 + p_3)$$

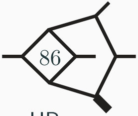
$$\Delta_3^{(2)} = -4\ G(p_1, p_2 + p_4)$$

$$\Delta_3^{(3)} = -4\ G(p_1, p_3 + p_4)$$



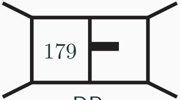
HBzzz

HBzmz



HBmzz

DPmz



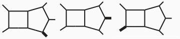
DPzz

+ permutations

Previous work

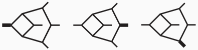
Canonical DE

Planar



[Abreu, Ita, Moriello, Page, Tschernow, Zeng '20]

Hexa-box



[Abreu, Ita, Page, Tschernow '21]

Results through generalized polylogarithms

[Papadopoulos, Tommasini, Wever '15]
[Canko, Papadopoulos, Syrrakos '20] [Syrrakos '20]
[Kardos, Papadopoulos, Smirnov, Syrrakos, Wever '22]

Function basis (planar)

[Badger, Hartanto, Zoia '21] color-ordered

planar one-mass pentagon functions [Chicherin, VS, Zoia '21]

Semi-numerical DE solution

`DiffExp` [Moriello '19] [Hidding '20]
`AMFLow` [Liu, Ma, Wang '17] [Liu, Ma '21]
`SeaSyde` [Armadillo, Bonciani, Devoto, Rana, Vicini '22]
[Hidding, Usovitsch '22]

Precious numerical data:

- Initial (boundary) conditions
- Validation

Canonical differential equations

Pure integrals and canonical differential equations

Pure Feynman integrals
[Henn '13]

$$\boxed{\begin{aligned} \mathrm{d}\vec{g} &= \epsilon\, A\, \vec{g} \\ A &= \sum_i \mathrm{d}\log W_i(\mathbf{s})\; A_i \end{aligned}}$$

Letters of symbol alphabet

Encode singularity structure
and branch cuts

Rational matrix

⚠ Canonical DE very challenging to obtain for multi-scale integrals

other cutting edge examples:
[Febres Cordero, Figueiredo, Kraus, Page, Reina '23]
[Badger, Becchetti, Chaubey, Marzucca '22]
[Henn, Peraro, Xu, Zhang '21]

✓ We succeed in deriving this form for complete set of integrals

The alphabet

Hexa-box alphabet [Abreu, Ita, Page, Tschernow '21], no new letters from DP

$$A = \sum_{i=1}^{204} \mathrm{d}\log W_i \;\, A_i$$

c.f. 31 letters,
1 square root
for massless!

Degree	Letters	
linear	27	117 rational
quadratic	66	
qubic	24	

Root	Degree	Perms./ Letters	
$\sqrt{\Delta_5}$	4	1	10 square roots
$\sqrt{\Delta_3^{(i)}}$	2	3	
$\sqrt{\Sigma_5^{(i)}}$	4	6	

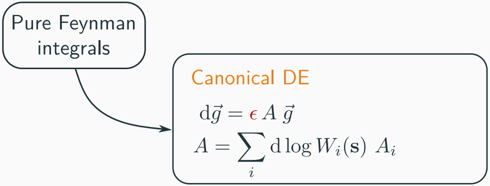
Algebraic letters, odd under root sign flip

$$\text{e.g.} \quad W_{118} = \frac{p_1^2 - s_{23} + s_{45} + \sqrt{\Delta_3^{(1)}}}{p_1^2 - s_{23} + s_{45} - \sqrt{\Delta_3^{(1)}}}$$

Roots	Letters	
$\sqrt{\Delta_5}$	32	68 one square root
$\sqrt{\Delta_3^{(i)}}$	12	
$\sqrt{\Sigma_5^{(i)}}$	24	
$\sqrt{\Delta_5}, \sqrt{\Delta_3^{(i)}}$	3	9 two square roots
$\sqrt{\Delta_5}, \sqrt{\Sigma_5^{(i)}}$	6	

$$\Sigma_5^{(1)} = (s_{12}s_{15} - s_{12}s_{23} - s_{15}s_{45} + s_{34}s_{45} + s_{23}s_{34})^2 \\ - 4 s_{23}s_{34}s_{45}(s_{34} - s_{12} - s_{15})$$

How to solve DE?



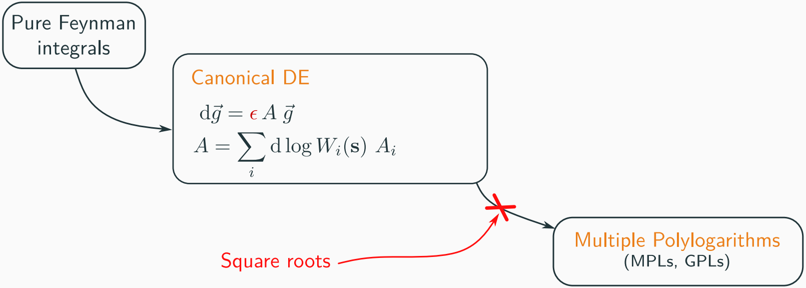
Pure Feynman integrals

Canonical DE

$$\mathrm{d}\vec{g} = \epsilon \, A \, \vec{g}$$

$$A = \sum_i \mathrm{d} \log W_i(\mathbf{s}) \; A_i$$

How to solve DE?



Pure Feynman integrals

Canonical DE

$$\mathrm{d}\vec{g} = \epsilon\, A\, \vec{g}$$

$$A = \sum_i \mathrm{d}\log W_i(\mathbf{s})\; A_i$$

Square roots

Multiple Polylogarithms
(MPLs, GPLs)

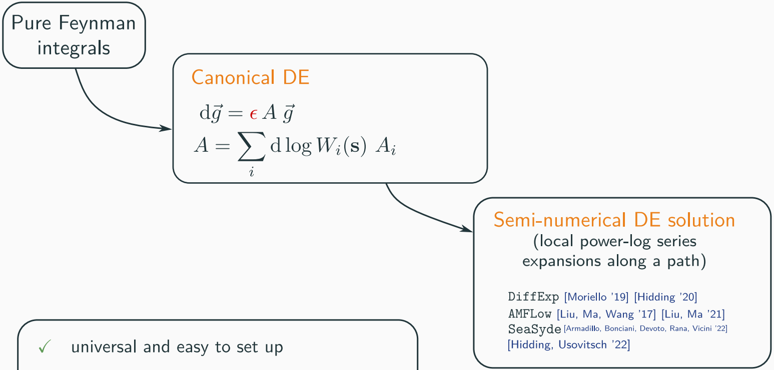
⚠ Map may not exist [Duhr, Brown '20]

Sometimes still possible

[Heller, von Manteuffel, Schabinger '19][Heller '21]
[Bonetti, Panzer, Smirnov, Tancredi '20]
[Kreer, Weinzierl '21][Duhr, Smirnov, Tancredi '21]
[Papadopoulos, Tommasini, Wever '15] [Papadopoulos '14]
[Canko, Papadopoulos, Syrrakos '20]

⚠ but “good” representation
even more challenging

How to solve DE?



Pure Feynman integrals

Canonical DE

$$\mathrm{d}\vec{g} = \epsilon\, A\, \vec{g}$$

$$A = \sum_i \mathrm{d}\log W_i(\mathbf{s})\; A_i$$

Semi-numerical DE solution
(local power-log series
expansions along a path)

`DiffExp` [Moriello '19] [Hidding '20]
`AMFLow` [Liu, Ma, Wang '17] [Liu, Ma '21]
`SeaSyde` [Armadillo, Bonciani, Devoto, Rana, Vicini '22]
[Hidding, Usovitsch '22]

✓ universal and easy to set up

⚠ evaluation of integrals (no function basis)

- hidden cancellations
- prohibitive run times
- stability over phase space?

Function basis

Chen's iterated integrals

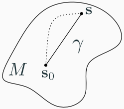
[Chen '77] (see also *"Iterated integrals in QFT"* [Brown '11])

ϵ-factorized DE can be readily solved through Chen's iterated integrals [Chen '77], along a path $\gamma \in \mathcal{P}$ (phase space) connecting $\mathbf{s}_0$ and $\mathbf{s}$,

$$\vec{g}(\mathbf{s}) = \mathbb{P}\exp\left[\epsilon \int_\gamma A\right] \vec{g}(\mathbf{s}_0) \;=\; \vec{g}^{(0)}(\mathbf{s}_0) + \sum_i \epsilon^i \vec{g}^{(i)}(\mathbf{s}),$$

$\vec{g}^{(n)}$ are linear combinations of $g_i^{(n_1)}(\mathbf{s}_0) \cdot [\omega_{i_1}, \ldots, \omega_{i_{n_2}}]_\gamma$, $n_1 + n_2 = n$.

Let $\omega_1, \ldots, \omega_n$ be differential 1-forms on $\mathcal{P}$, and path $\gamma : [0,1] \to \mathcal{P}$. Pull the forms back on the path $\omega_i(\mathbf{s}) \xrightarrow{\gamma^\star} w_i(t)\,\mathrm{d}t$. Iterated integrals are



$$[\omega_1, \ldots, \omega_n]_\gamma = \int_0^1 w_n(t_n)\,\mathrm{d}t_n \ldots \int_0^{t_2} w_1(t_1)\,\mathrm{d}t_1 \qquad \text{(ii)}$$

Here we need only logarithmic forms $\omega_i = \mathrm{d}\log W_i$.

Weight = number of integrations = order of ϵ

Properties of iterated integrals

Linear independence

Iterated integrals with distinct words $[\omega_1, \ldots, \omega_n]$ are $\mathbb{Q}$-linear independent (if $\{\omega_i\}$ are).
$\implies$ graded vector space

Shuffle product

$$[\omega_1, \ldots, \omega_r]_\gamma \; [\omega_{r+1}, \ldots, \omega_n]_\gamma = \sum_{\boldsymbol{i} \in \{1,\ldots,r\} ⧢ \{r+1,n\}} [\omega_{i_1}, \ldots, \omega_{i_n}]_\gamma \tag{⧢}$$

$\implies$ graded algebra

Symbol map S

Let $\mathrm{d}F = \sum_i \mathrm{d}\log W_i \;\; F_i$

(recall $\mathrm{d}[\omega_1, \ldots, \omega_n]_\gamma = \omega_n \; [\omega_1, \ldots, \omega_{n-1}]_\gamma$)

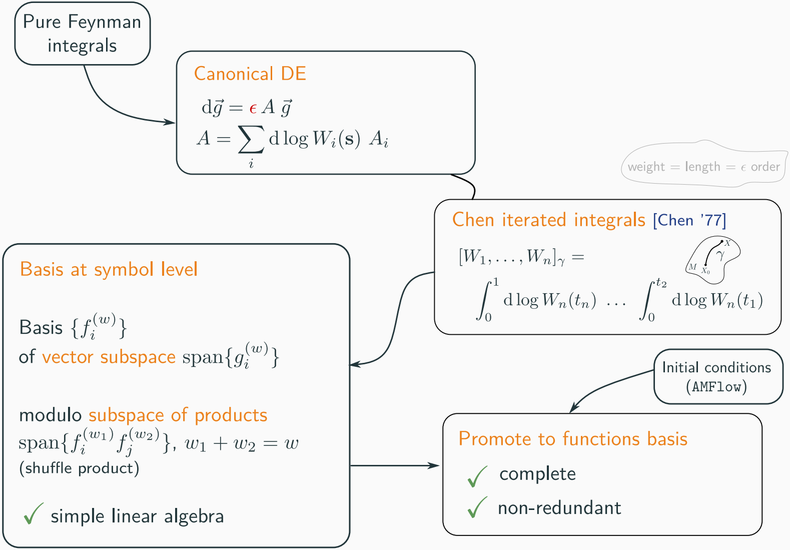
$$S(F) = \sum_i S(F_i) \otimes W_i,$$

$$S([\omega_i]_\gamma) = W_i.$$

Effectively discards initial values and path in DE solution.

Pentagon functions construction

[Abreu, Chicherin, Ita, Page, VS, Tschernow, Zoia '23] (see also [Chicherin, VS '20] [Badger, Hartanto, Zoia '21] [Chicherin, VS, Zoia '21])



Pure Feynman integrals

Canonical DE

$$\mathrm{d}\vec{g} = \epsilon\, A\, \vec{g}$$

$$A = \sum_i \mathrm{d}\log W_i(\mathbf{s})\ A_i$$

weight = length = ϵ order

Chen iterated integrals [Chen '77]

$$[W_1, \dots, W_n]_\gamma = \int_0^1 \mathrm{d}\log W_n(t_n)\ \dots\ \int_0^{t_2} \mathrm{d}\log W_n(t_1)$$

Basis at symbol level

Basis $\{f_i^{(w)}\}$
of vector subspace $\mathrm{span}\{g_i^{(w)}\}$

modulo subspace of products
$\mathrm{span}\{f_i^{(w_1)} f_j^{(w_2)}\}$, $w_1 + w_2 = w$
(shuffle product)

✓ simple linear algebra

Initial conditions
(`AMFlow`)

Promote to functions basis

✓ complete

✓ non-redundant

Promoting symbol-level basis

Full functions are defined by iterated integrals, including terms with transcendental constants (initial values $\vec{\chi} \coloneqq \vec{g}(\mathbf{s}_0)$)

$$\vec{g}^{(w)} = \underbrace{\int_\gamma A \cdots \int_\gamma A \; \vec{\chi}^{(0)}}_{w \text{ integrations (symbol)}} \quad + \quad \sum_{w'=0}^{w-1} \underbrace{\int_\gamma A \cdots \int_\gamma A}_{w' \text{ integrations}} \; \vec{\chi}^{(w-w')}$$

We want to work in a vector space $\implies$ we must know all algebraic identities between $\vec{\chi}^{(w-w')}$.
This turns out to be difficult [Chicherin, VS, Zoia '21] (need very high precision $\vec{\chi}$ for PSLQ).

New approach

1. Use relation of $f_i^{(w)}$ to $\vec{g}$ to define lifts of $f_i^{(w)}$ to functions (as iterated integrals).
2. Insist that symbol-level decomposition of $\vec{g}$ through polynomials of $f_i^{(w)}$ also holds as functions, modulo ζ values.
3. This is possible if $\vec{\chi}^{(w-w')}$ satisfy certain algebraic identities.
   We check (numerically) that they indeed do!

Basis construction: a toy example

$$d\vec{g} = \epsilon A \vec{g}$$

$$\vec{g} = \vec{g}^{(0)} + \epsilon \vec{g}^{(1)} + \epsilon^{2} \vec{g}^{(2)} + \ldots$$

$$\vec{x} = \vec{g}(S_0)$$

$$- A = \begin{pmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \omega_2 & 0 & 0 \\ \omega_1 & 0 & \omega_2 & \omega_1 \\ \omega_1 & \omega & 0 & \omega_2 \end{pmatrix}$$

$$\vec{x}^{(0)} = \vec{g}^{(0)} = \begin{pmatrix} 1 \\ 2 \\ -5 \\ -2 \end{pmatrix}$$

$$\vec{g}^{(1)} = \int_{\delta} A \, \vec{x}^{(0)} + \vec{x}^{(1)}$$

$$\vec{g}^{(2)} = \int_{\delta} A A \, \vec{x}^{(0)} + \int_{\delta} A \, \vec{x}^{(1)} + \vec{x}^{(2)}$$

Basis construction: a toy example

Weight 1

$$\vec{g}^{(1)} = \begin{pmatrix} [w_1] \\ 2[w_2] \\ -[w_1] - 5[w_2] \\ [w_1] \end{pmatrix} + \vec{x}^{(1)}$$

symbol level basis

$$f_1^{(1)} := g_1^{(1)}$$

$$f_2^{(1)} := g_2^{(1)}$$

$$\Rightarrow \quad \vec{g}^{(1)} = \begin{pmatrix} f_1^{(1)} \\ f_2^{(1)} \\ -f_1^{(1)} - \frac{5}{2} f_2^{(2)} \\ f_1^{(1)} \end{pmatrix}$$

lift possible if

$$\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \\ x_4^{(1)} \end{pmatrix} \overset{!}{=} \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ -x_1^{(1)} - \frac{5}{2} x_2^{(2)} \\ x_1^{(1)} \end{pmatrix}$$

Basis construction: a toy example

Weight 2

$$\vec{g}^{(2)} = \begin{pmatrix} [\omega_1, \omega_1] + x_1^{(1)}[\omega_1] \\ 2[\omega_2, \omega_2] + x_2^{(1)}[\omega_2] \\ 2[\omega_1, \omega_1] - [\omega_1, \omega_2] - 5[\omega_2, \omega_2] + x_1^{(1)}[\omega_1] + x_3^{(1)}[\omega_2] + x_4^{(1)}[\omega_1] \\ [\omega_1, \omega_1] + [\omega_1, \omega_2] + 2[\omega_2, \omega_2] + x_1^{(1)}[\omega_1] + x_2^{(1)}[\omega_2] + x_4^{(1)}[\omega_2] \end{pmatrix} + \vec{x}^{(2)}$$

$$h_{1,2,3}^{(2)} := g_{1,2,3}^{(2)}, \qquad g_4^{(2)} = 3h_1^{(2)} - \frac{3}{2}h_2^{(2)} - h_3^{(2)}$$

Basis construction: a toy example

shuffle product

$$\left(f_1^{(1)}\right)^2 = 2\,[\omega_1,\omega_1] + 2\,\chi_1^{(1)}[\omega_1] + \left(\chi_1^{(1)}\right)^2$$

$$\left(f_2^{(1)}\right)^2 = 8\,[\omega_2,\omega_2] + 4\,\chi_2^{(2)}[\omega_2] + \left(\chi_2^{(1)}\right)^2$$

$$f_1^{(1)}f_2^{(1)} = 2\,[\omega_2,\omega_1] + 2\,[\omega_1,\omega_2] + 2\,\chi_1^{(1)}[\omega_2] + \chi^{(2)}[\omega_1] + \chi_1^{(1)}\chi_2^{(1)}$$

Now consider span $\left\{\left(f_1^{(1)}\right)^2, \left(f_2^{(1)}\right)^2, f_1^{(1)}f_2^{(1)}, h_1^{(2)}, h_2^{(2)}, h_3^{(2)}\right\}$

$\xrightarrow{\text{row reduce}}$ only $f_1^{(2)} := h_3^{(2)} = g_3^{(2)}$ is new!

Basis construction: a toy example

$$\vec{g}^{(2)} = \begin{pmatrix} \frac{1}{2}\left(f_1^{(1)}\right)^2 \\ \frac{1}{4}\left(f_2^{(1)}\right)^2 \\ f_1^{(2)} \\ \frac{3}{2}\left(f_1^{(1)}\right)^2 - \frac{3}{8}\left(f_2^{(1)}\right)^2 - f_1^{(2)} \end{pmatrix}$$

symbol level basis

lift to functions possible if

$$\chi_1^{(2)} \overset{!}{=} \frac{1}{2}\left(\chi_1^{(1)}\right)^2 + c\,\zeta_2$$

$$\chi_4^{(2)} \overset{!}{=} \frac{3}{2}\left(\chi_1^{(1)}\right)^2 - \frac{3}{8}\left(\chi_2^{(1)}\right)^2 - \chi_3^{(2)} + c\,\zeta_2$$

We find these are always satisfied!

Features of the function space

Anomalous thresholds in massless scattering?

Consider iterated integral along $\gamma : t \in [0,1] \to \mathcal{P}_{\text{phys}}$, and $W_i(t^\star) = 0$,

$$\int_\gamma \mathrm{d}\log W_i \;\; h = \int_\gamma \frac{\mathrm{d}W_i}{W_i} h \;\; \xrightarrow{t \to t^\star} \frac{W'(t)}{t - t^\star} \left(h^{(0)} + h^{(1)}(t - t^\star) + \mathcal{O}\Big((t - t^\star)^2 \Big) \right)$$

Planar scattering

Only linear or quadratic letters vanish in $\mathcal{P}_{\text{phys}}$, poles always canceled, i.e. $h^{(0)} = 0$

Anomalous thresholds in massless scattering?

Consider iterated integral along $\gamma : t \in [0,1] \to \mathcal{P}_{\text{phys}}$, and $W_i(t^\star) = 0$,

$$\int_\gamma \mathrm{d}\log W_i \;\; h = \int_\gamma \frac{\mathrm{d}W_i}{W_i} h \;\; \xrightarrow{t\to t^\star} \frac{W'(t)}{t-t^\star}\left(h^{(0)} + h^{(1)}(t-t^\star) + \mathcal{O}\Big((t-t^\star)^2\Big)\right)$$

Planar scattering

Only linear or quadratic letters vanish in $\mathcal{P}_{\text{phys}}$, poles always canceled, i.e. $h^{(0)} = 0$

New feature of nonplanar scattering

Square roots of quartic polynomials $\sqrt{\Sigma_5}$ can vanish in $\mathcal{P}_{\text{phys}}$ $\implies$ new types of divergences

1. Integrable square root: $\mathrm{d}\log\frac{a+\sqrt{\Sigma_5}}{a-\sqrt{\Sigma_5}} \xrightarrow{\Sigma_5\to 0} \frac{\mathrm{d}\Sigma_5}{a\sqrt{\Sigma_5}} \xrightarrow{t\to t^\star} \frac{C}{\sqrt{t-t^\star}} + \ldots$
2. Uncompensated poles: $\mathrm{d}\log\sqrt{\Sigma_5} \xrightarrow{\Sigma_5\to 0} \frac{\mathrm{d}\Sigma_5}{2\Sigma_5} \xrightarrow{t\to t^\star} \frac{C}{t-t^\star} + \ldots \implies$ log divergence!

- Choose basis functions to localize non-analytic behavior
- Functions with type 2 divergence cancel out in physical results?
- Numerical evaluation more challenging

Basis structure

Weight	P ∪ PB	+HB	+DP	Total
1	11	0	0	11
2	25	10	0	35
3	145	72	0	217
4	675	305	48	1028
#MIs	1361	542	345	2248

Permutation closed
$\sigma\left(f_i^{(w)}\right) \to \sum_j c_{ij} f_j^{(w)} + \dots$

Basis structure

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1	11	0	0	11
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Permutation closed
$\sigma\left(f_i^{(w)}\right) \to \sum_j c_{ij} f_j^{(w)} + \ldots$

$\mathrm{d}\log\sqrt{\Delta_5}$ likely cancel in finite remainders
[Chicherin, Henn, Papathanasiou '20]

$\mathrm{d}\log\sqrt{\Sigma_5^{(i)}}$ also cancel?

7 functions diverge at $\Sigma_5^{(3)} = 0$

Weight 3

$$1,\ldots,145,\,146,\ldots,207,\overbrace{208,\ldots,213}^{\mathrm{d}\log\sqrt{\Sigma_5^{(i)}}},\underbrace{214,\ldots,217}_{\mathrm{d}\log\sqrt{\Delta_5}}$$

Weight 4

$$1,\ldots,112,\underbrace{113}_{\mathrm{d}\log\sqrt{\Delta_5}},114,\ldots,664,\underbrace{665,\ldots,675}_{\mathrm{d}\log\sqrt{\Delta_5}},$$

$$676,\ldots,920,\overbrace{921,\ldots,945,946,\ldots,950}^{\mathrm{d}\log\sqrt{\Sigma_5^{(i)}}},\underbrace{951,\ldots,974,\overbrace{975,\ldots,980}^{\mathrm{d}\log\sqrt{\Sigma_5^{(i)}}}}_{\mathrm{d}\log\sqrt{\Delta_5}},981,\ldots,1028$$

Numerical evaluation

Numerical evaluation

Weights 1 and 2

Well-defined combinations of $\log$, Li_2 functions

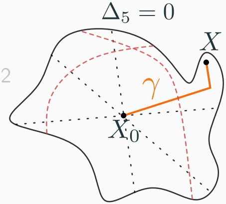
$$f_{13}^{(2)} = \mathrm{Li}_2\left(1 - \underbrace{\frac{s_{15} - s_{23} - s_{34}}{s_{15}}}_{>0}\right)$$

Weights 3 and 4

$$f_i^{(3)} = \int_0^1 \sum_j \frac{\partial \log W_j(t)}{\partial t} \boxed{h_{i,j}^{(2)}}\, \mathrm{d}t$$

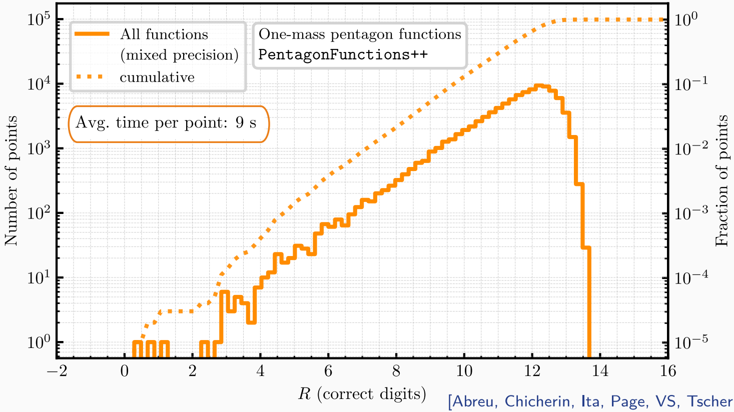
weight 2

$$f_i^{(4)} = \int_0^1 \sum_{j,k} \frac{\partial \log W_j(t)}{\partial t} \log \frac{W_k(1)}{W_k(t)} \boxed{h_{i,jk}^{(2)}}\, \mathrm{d}t$$



- Numerical one-fold integration [Caron-Huot, Henn '14] of analytic integrands $\implies$ exponential convergence [Takahasi, Mori '73]
- No crossing of physical thresholds $\implies$ no analytic continuation needed
- Dedicated series expansions around (spurious) singularities

Numerical performance



[Abreu, Chicherin, Ita, Page, VS, Tschernow, Zoia '23]

- Sample over physical phase space for NLO Wjj production at the LHC
- Evaluate all functions on each point, plot the worst accuracy per point
- Timing for all functions on one CPU

Available in `PentagonFunctions++`

gitlab.com/pentagon-functions/PentagonFunctions-cpp

Conclusions

Conclusions & Outlook

Conclusions

- Basis of special functions for two-loop five-point one-mass processes is available.
  Hopefully exciting phenomenology in near future!
- Existence of the basis "mysteriously" implies algebraic identities between initial values.
- Interesting feature discovered: anomalous thresholds in nonplanar massless scattering.
- All we need is iterated integrals with nice kernels.

Outlook

- Ideas generally useful for multi-scale problems with many square roots,
  e.g. EW corrections and (quantum) gravity amplitudes.
- Dream: extension beyond $\mathrm{d}\log$ forms.

Acknowledgments

This work has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme grant agreement 101019620 (ERC Advanced Grant TOPUP).

erc

European Research Council

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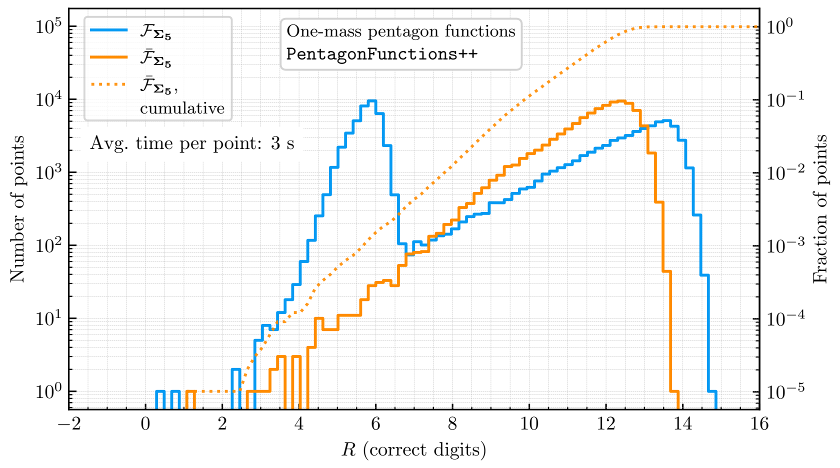
Backup

Numerical performance: Σ_5 split



Weight 3: one-fold integral representation

Weight 3 functions are one-fold integrals of weight 2 functions by definition

$$f_i^{(3)}(X) = \sum_{j,k} c_{i,j,k} \int_0^1 \mathrm{d}\log W_j(t)\, h_k^{(2)}(t) + \tau_i^{(3)}$$

Integrands analytic on the integration domain $\implies$ integration well-defined.

- Efficient numerical integration possible
- Some care to avoid numerical cancellations if $\mathrm{d}\log(W_j)$ can vanish along the path

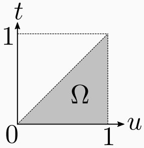
Weight 4: one-fold integral representation

Change order of integration [Caron-Huot, Henn '14]

$$I_\gamma[\omega_1,\dots,\omega_n] =$$

$$\int_0^1 (\gamma^\star \circ \omega_n)(t) \int_0^t (\gamma^\star \circ \omega_{n-1})(u)\; I_{\gamma(u)}[\omega_1,\dots,\omega_{n-2}]$$

$$= \int_0^1 (\gamma^\star \circ \omega_{n-1})(u) \left(\int_u^1 (\gamma^\star \circ \omega_n)(t) \right)\; I_{\gamma(u)}[\omega_1,\dots,\omega_{n-2}]$$

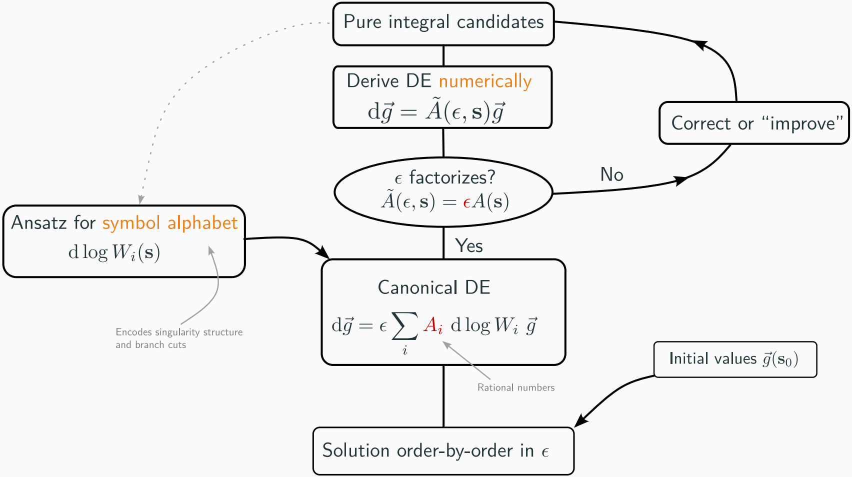


For logarithmic forms the last integration is trivial

$$\int_u^1 (\gamma^\star \circ \omega_n)(t) = \int_u^1 \mathrm{d}\log\left(W_n(t)\right) = \log(W_n(1)) - \log(W_n(u))$$

DE reconstruction strategy

Hope: the result is pure functions of uniform transcendentality [Henn '13]



Pure integral candidates

Derive DE numerically

$$d\vec{g} = \tilde{A}(\epsilon, \mathbf{s})\vec{g}$$

ϵ factorizes?

$$\tilde{A}(\epsilon, \mathbf{s}) = \epsilon A(\mathbf{s})$$

No

Correct or "improve"

Yes

Ansatz for symbol alphabet

$$d \log W_i(\mathbf{s})$$

Encodes singularity structure and branch cuts

Canonical DE

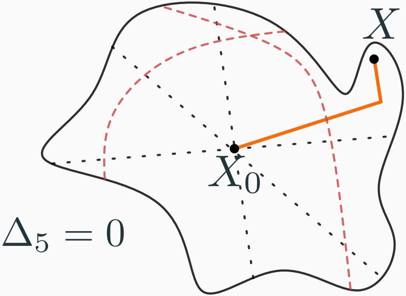
$$d\vec{g} = \epsilon \sum_i A_i \; d \log W_i \; \vec{g}$$

Rational numbers

Initial values $\vec{g}(\mathbf{s}_0)$

Solution order-by-order in ϵ

Physical region geometry



- Nontrivial geometry due to degree 4 polynomial boundary $\Delta_5 = 0$
- Not convex
- Not star shaped

- Positivity properties of the alphabet important for deriving well-defined functions in physical region
- Can be established by expressing through Gram determinants
- Non-sign-definite letters $\implies$ spurious singularities

Initial values identities: the old way

[Chicherin, VS, Zoia '21]

We choose an initial point $X_0 \in \mathcal{P}_{45}^{+}$,

$$X_0 := \left(p_1^2 = 1\,, s_{12} = 3\,, s_{23} = 2\,, s_{34} = -2\,, s_{45} = 7\,, s_{15} = -2\right)\,,$$

which satisfies the following requirements:

1. X_0 introduces the minimal number of distinct prime factors.
2. X_0 is invariant under the exchanges of momenta $2 \leftrightarrow 3$ and $4 \leftrightarrow 5$ (automorphisms of $\mathcal{P}_{45}$).
3. The four linear letters which have indefinite sign vanish at X_0.

Algebraic relations between initial values required.

- Numerical evaluation of available GPL expressions [Canko, Papadopoulos, Syrrakos '20], [Syrrakos '20] to 3000 digits
- Relations from PSLQ $\Longrightarrow$ generating set

	Linear span ($\oplus$ products)		Irreducible	
Weight	Re	Im	Re	Im
1	4^*	1	4^*	1
2	12	4	5	0
3	67	23	23	7
4	305	135	90	40

Pushing the limits of most advanced PSLQ algorithms [Bailey, Broadhurst '01] [Bailey, Borwein, Kimberley, Ladd '17]