More on bootstrapping matrix quantum mechanics

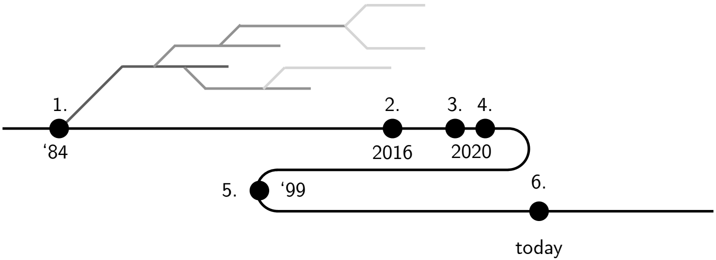
Henry Lin

CERN workshop on Matrix Quantum Mechanics

This talk is based on:
arXiv:2302.04416 [HL '23]
work in progress w/ Gauri Batra
work in progress w/ Zechuan Zheng

See also:
[Han, Harnoll, Kruthoff '20] (reviewed in David Berenstein's talk)
[Kazakov & Zheng '21] [Anderson & Kruczenski '16, HL '20, ...]

Bootstrap: a timeline



1. CFT bootstrap [Ferrara '73], [Polyakov '74], [Belavin, Polyakov, Zamolodchikov '84]
2. Lattice Yang Mills bootstrap [Anderson & Kruczenski '16, Kazakov & Zheng '22]
3. Matrix bootstrap [HL '20]
4. Quantum mechanical bootstrap [Han, Hartnoll, Kruthoff '20]
5. Virial bound [Polchinski '99]
6. BFSS [today]

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- Solve black holes. Two ingredients:
  - Large N: large semi-classical entropy
  - Strong coupling: maximal chaos/sub-AdS locality
- Strong coupling makes analytical methods hard. Large N makes numerics hard.

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  - MC ❤ bootstrap: complementary tools

Matrix model

$$Z = \lim_{N\to\infty} \int \mathrm{d}M \, e^{-N^2 \operatorname{tr} V(M)}$$

$$\langle \operatorname{tr} M^2 \rangle = \lim_{N\to\infty} Z^{-1} \int \mathrm{d}M \, e^{-N^2 \operatorname{tr} V(M)} \operatorname{tr} M^2$$

0. Does it exist?
1. Determine its values as a function of couplings

Bootstrapping matrices

1. Guess the value of some simple correlator, e.g. $\langle \operatorname{tr} M^2 \rangle$
2. Feed it through the loop eqns to generate more correlators
3. Demand that $\langle \operatorname{tr} \mathcal{O}^\dagger \mathcal{O} \geq 0 \rangle$. E.g., $\langle \operatorname{tr} M^{16} \rangle < 0$ would rule out the guess.

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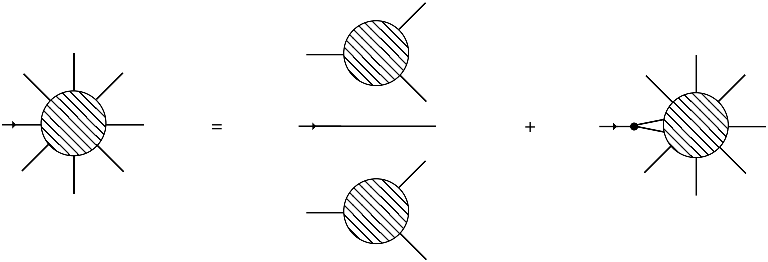
3. Demand that $\langle \operatorname{tr} \mathcal{O}^\dagger \mathcal{O} \geq 0 \rangle$. E.g., $\langle \operatorname{tr} M^{16} \rangle < 0$ would rule out the guess. More systematically, assemble all the correlators into a big matrix $\mathcal{M}$ and test if $\mathcal{M} \succeq 0$.

$$\mathcal{M} = \begin{pmatrix} N & \operatorname{Tr} A & \operatorname{Tr} B \\ \operatorname{Tr} A & \operatorname{Tr} A^2 & \operatorname{Tr} AB \\ \operatorname{Tr} B & \operatorname{Tr} BA & \operatorname{Tr} B^2 \end{pmatrix}$$

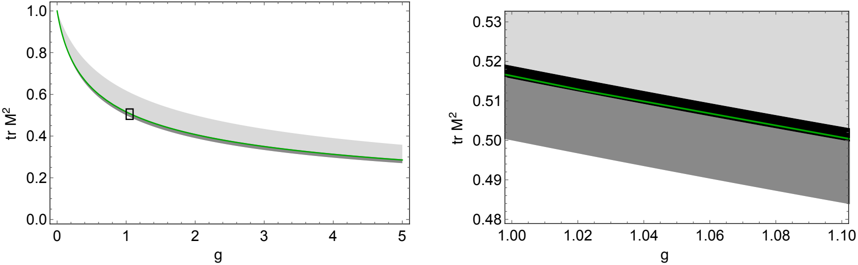
For a single matrix model $A = M, B = M^2, \cdots$.

Loop equations



- relates lower-pt correlators to higher-pt correlators
- uses large N factorization ('t Hooft)

Review of the matrix bootstrap



$$V(M) = \frac{1}{2}M^2 + \frac{g}{4}M^4$$

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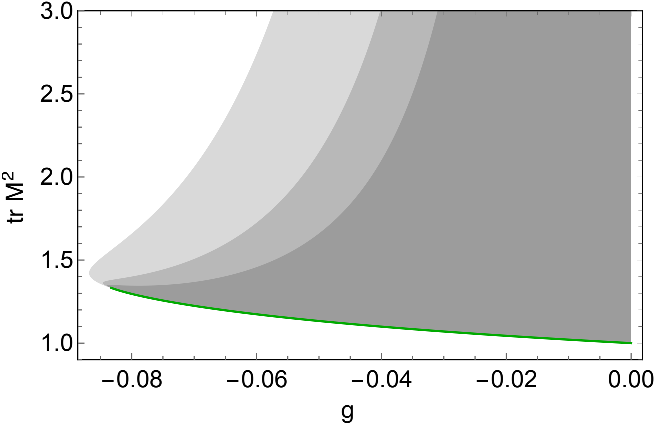

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For $-g_* < g < 0$ the model still makes sense at $N = \infty$

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  3. handily solves single particle QM and single matrix QM. Strong constraints on 2-matrix QM with $\operatorname{tr}[A, B]^2$ interaction.

D0-brane quantum mechanics

Hilbert space: 9 bosonic matrices and 16 fermionic matrices.
Transform as a fundamental and spinor of $SO(9)$.

$$H = \frac{1}{2}\mathrm{Tr}\left(g^2 P_I^2 - \frac{1}{2g^2}\left[X_I, X_J\right]^2 - \psi_\alpha \gamma^I_{\alpha\beta}\left[X_I, \psi_\beta\right]\right)$$

Most of what we know due to heroic Monte Carlo simulations [Kabat *et al.*, Anagnostopoulos *et al.*, Hanada *et al.*, ..., Berkowitz *et al.*, Pateloudis *et al.*]

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Metastable black hole in Type IIA [Itzhaki, Maldacena, Sonneschein, Yankielowicz]:

$$\frac{\mathrm{d}s^2}{\alpha'} = -f(r) r_c^2 \,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r) r_c^2} + \left(\frac{r}{r_c}\right)^{-3/2} \mathrm{d}\Omega_8^2$$

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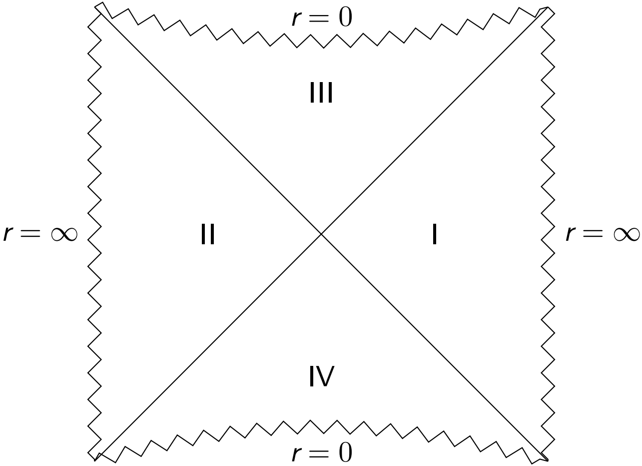
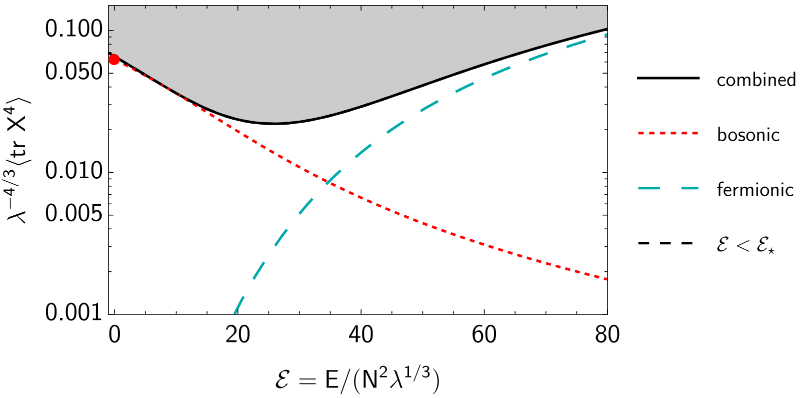
S_8 *shrinks* with r. At $r \sim \lambda^{1/3} \Rightarrow$ string scale curvature.



Euclidean cigar $r > r_H \propto T^{2/5}$. At $E/N^2 \sim \lambda^{1/3}$ geometry is nowhere reliable.

Lower bounds on $\langle \mathrm{tr}\, X^4 \rangle$



$\rightarrow$ first explain the red curve that extends the Polchinski point.

Lower bounds on $\langle \mathrm{tr}\, X^4 \rangle$

$\langle [H, XP] \rangle = 0, \langle H \rangle = 0 \rightarrow$ first explain the red curve that extends the Polchinski point.

Bosonic constraints: round 1

Commutator constraints:
$\left\langle [H, \mathrm{Tr}\, X^2] \right\rangle = 0 \Rightarrow \left\langle \mathrm{Tr}\, X^I P_I + P^I X_I \right\rangle = 0.$

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Positivity:

$$\mathcal{M} = \begin{pmatrix} \operatorname{Tr} X^2 & \operatorname{Tr} XP \\ \operatorname{Tr} PX & \operatorname{Tr} P^2 \end{pmatrix} \succeq 0$$

$$\Rightarrow \quad \sum_I \left\langle \operatorname{Tr} X^2 \right\rangle \left\langle \operatorname{Tr}(P^I P_I) \right\rangle \geq \frac{9}{4} N^4.$$

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Next: replace $\mathrm{Tr}\, P^2$ (kinetic energy) with potential energy.

Bosonic constraints: round 2

Commutator constraints:

$\langle [H, \mathrm{Tr}\, XP] \rangle = 0, \quad \langle H \rangle = E$

$-2\langle K \rangle + 4\langle V \rangle + \langle F \rangle = 0, \quad \langle K \rangle + \langle V \rangle + \langle F \rangle = E$

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Positivity:

Recall $V = -\frac{1}{4g^2}\,\mathrm{Tr}\big[X^I, X^J\big]^2$. Relate to $\mathrm{Tr}\, X^4$ using

$$\begin{pmatrix} \mathrm{Tr}\, X^4 & \mathrm{Tr}\, X^2Y^2 \\ \mathrm{Tr}\, X^2Y^2 & \mathrm{Tr}\, X^4 \end{pmatrix} \succeq 0, \quad \begin{pmatrix} \mathrm{Tr}\, X^2Y^2 & \mathrm{Tr}\, XYXY \\ \mathrm{Tr}\, XYXY & \mathrm{Tr}\, X^2Y^2 \end{pmatrix} \succeq 0$$

$$\left\langle \mathrm{Tr}\, X^2 \right\rangle \left(\frac{144}{g^2}\left\langle \mathrm{Tr}\, X^4 \right\rangle + \frac{2E}{3} \right) \geq \frac{9}{4} g^2 N^4$$

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$$\sqrt{N \langle \mathrm{Tr}\, X^4 \rangle} \left(\frac{144}{g^2} \langle \mathrm{Tr}\, X^4 \rangle + \frac{2E}{3} \right) \geq \frac{9}{4} g^2 N^4$$

Comments:

- Setting $E = 0$ recovers Polchinski point. Assuming parametric saturation of the bd implies that ``typical eigenvalue'' $r \sim \lambda^{1/3}$, which is the size of the gravity region.
- Scale at which the bd varies is $E/N^2 \sim \lambda^{1/3}$, regime of validity of gravity.
- No good bound on $\langle \mathrm{Tr}\, X^2 \rangle$.

Fermionic constraints

Had 2 eqns:

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In addition to solving for V, can solve for F:

$$\langle F\rangle = 2\left(\tfrac{1}{3}\langle E\rangle - \langle V\rangle\right)$$

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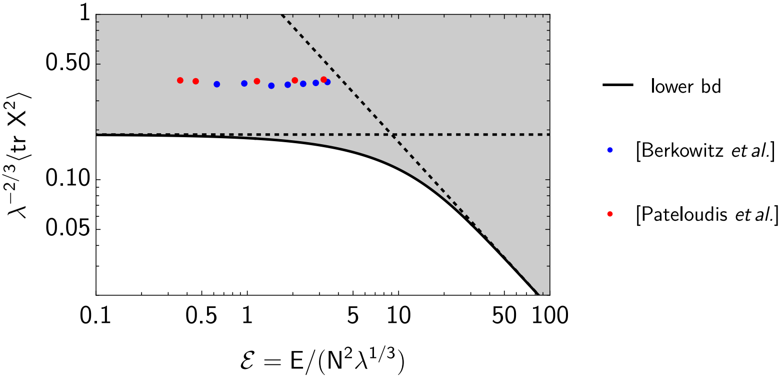
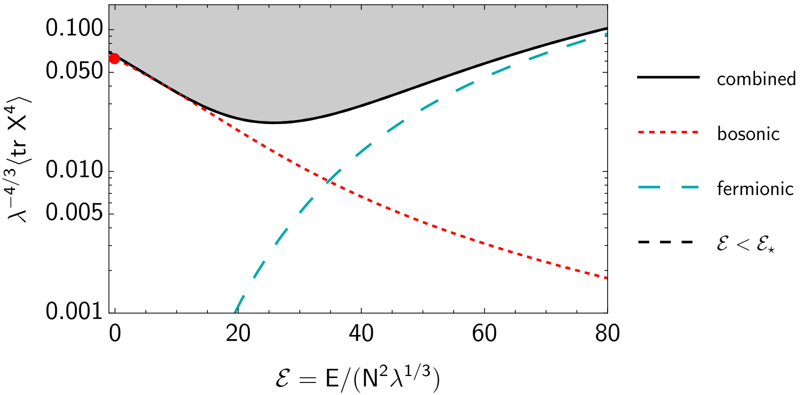
Fermionic term $F \sim \psi\psi X$. The operator $\psi\psi$ is bounded because it is made of Majorana fermions $\psi^2 = 1$. Therefore, if $F > 0$, X cannot be too small.

Constraints on $\langle \mathrm{tr}\, X^2 \rangle$



Large N extrapolation of Monte Carlo simulations [Pateloudis *et al.*] are $\sim 1/2$ from the lower bound.

Lower bounds on $\langle \mathrm{tr}\, X^4 \rangle$



In the remainder of the talk, I will comment on 2 questions:

1. Is there hope that numerics will lead to precision estimates?
2. What could we hope to learn by measuring $\langle \mathrm{tr}\, X^2 \rangle$ precisely?

w/ Zechuan Zheng, we are redoing the bootstrap for the much simpler case

$$H = N\left(\frac{1}{2}\,\mathrm{Tr}\,P^2 + \frac{1}{2}\,\mathrm{Tr}\,X^2 + \frac{g}{4}\,\mathrm{Tr}\,X^4\right).$$

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Improved the HHK method by using non-linear relaxation [Kazakov & Zheng '20].

Basic point: the constraints involve double traces, e.g.,

$$\langle \operatorname{tr} XP^3 \rangle = \langle \operatorname{tr} P^3 X \rangle + 2\mathrm{i}N \langle \operatorname{tr} P^2 \rangle + \mathrm{i}\langle \operatorname{tr} P \rangle \langle \operatorname{tr} P \rangle.$$

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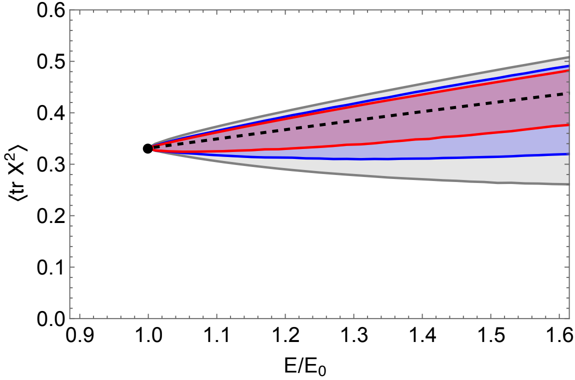
Using large N factorization, we can rewrite these double traces as products of single traces $\Rightarrow$ quadratic relations amongst correlation functions.

Introduce new variable $y = p^2$. Relax this to $y \leq p^2$ which can be written as $\begin{pmatrix} y & p \\ p & 1 \end{pmatrix} \succeq 0$.



Dashed line is the exact solution $g = 1$. Excellent convergence near the ground state. With more constraints, we expect rapid convergence $E > E_0$.

Suppose that one day we have high precision measurements of 1-pt functions like $\langle \mathrm{Tr}\, X^n \rangle$. What can we learn?

The semiclassical BH geometry and its stringy corrections

In principle, this includes properties that are currently inaccessible by worldsheet methods. See [Hanada *et al.*, Berkowitz *et al.*, Pateloudis, *et al.*] for similar discussions involving the BH thermodynamics.

A generic SO(9) singlet

$$\langle \mathrm{tr}\, X^n \rangle \sim a_{0,n} \langle 1 \rangle + a_{1,n} \langle H \rangle + a_{2,n} \langle T_{--} \rangle + b_{i,n} \langle \text{stringy}_i \rangle + \cdots$$

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The first 3 operators are the only single trace SO(9) supergravity singlets in this IIA background. Dual to h_{++}, h_{+-}, h_{--} in the M-theory picture.

The mode $\chi = h_{--}$ has scaling dimension $\Delta = 28/5$. [Sekino & Yoneya '00, Biggs & Maldacena '23]

To estimate $\langle T_{--} \rangle$ at low energies, in principle we need the leading α'^3 corrections to supergravity. Schematically of the form

$$\frac{(\alpha')^3}{G_N} \int \sqrt{g} e^{-2\phi} \chi \left(\#_1 R^4 + \#_2 e^{2\phi} R^3 F^2 + \cdots + \right)$$

Using that χ is an operator with dimension $\Delta = 28/5$ we find that such terms give

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We also estimated the stringy contribution $\Rightarrow$

$$\left\langle \mathrm{tr}\, X^2 \right\rangle \sim a_0 + a_1 T^{14/5} + c a_1 T^{23/5} + a_2 T^{46/5}$$
$$+ b_m T^{\nu} \exp\left\{ -2\sqrt{m} \gamma T^{-3/10} \right\} + \cdots$$

[WIP w/ Gauri Batra]

If we were willing to measure $\langle T_{--} \rangle$ directly using MC/bootstrap, we could learn about the α'^3 corrections to IIA SUGRA.

Matrix model expression for T_{--} can be obtained by expanding the DBI action of D0 branes [Van Raamsdonk and Taylor] in a weak background. Schematically,

$$T_{--} \sim \mathrm{Tr}\, P^I P^I P^J P^J + \mathrm{Tr}[X_I, X_J][X_J, X_K] P^K P^I + \cdots + \text{fermions}$$

More complicated but doable (in principle).
However, do to operator mixing we expect that T_{--} makes a contribution to $\langle \mathrm{Tr}\, X^2 \rangle$.

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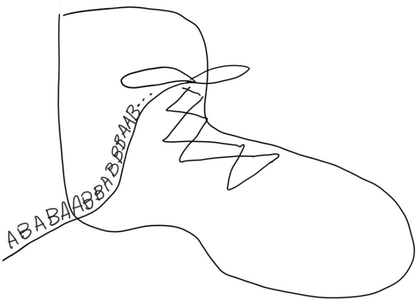

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Commutator constraint: $[H, F] = 0 \Rightarrow \mathrm{Tr}\, O_I P^I = 0$.
Positivity $\{O^I, X^I, P^I\}$:

$$\begin{bmatrix} \frac{1}{9}\langle \mathrm{Tr}\, O_I O_I \rangle & \frac{2}{9}\left(\frac{1}{3}E - \langle V \rangle\right) & 0 \\ \frac{2}{9}\left(\frac{1}{3}E - \langle V \rangle\right) & \left\langle \mathrm{Tr}\, X^2 \right\rangle & \mathrm{i}\frac{1}{2}N^2 \\ 0 & -\mathrm{i}\frac{1}{2}N^2 & \frac{2}{9}\left(\frac{1}{3}E + \langle V \rangle\right) \end{bmatrix} \succeq 0$$

Use $\frac{1}{9}\langle \mathrm{Tr}\, O_I O_I \rangle \le 16N^3$.