# *Integrability of high-energy QCD*

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*Based on the work with A. Belitsky, V. Braun, S. Derkachov, J. Drummond, L. Faddeev, J. Henn, L. Lipatov, A. Manashov, D. Müller, E. Sokatchev*

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# Very special year 2023

- ***50 years of discovery of asymptotic freedom in Quantum Chromodynamics***

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**Ultraviolet Behavior of Non-Abelian Gauge Theories***

David J. Gross† and Frank Wilczek
*Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540*
(Received 27 April 1973)

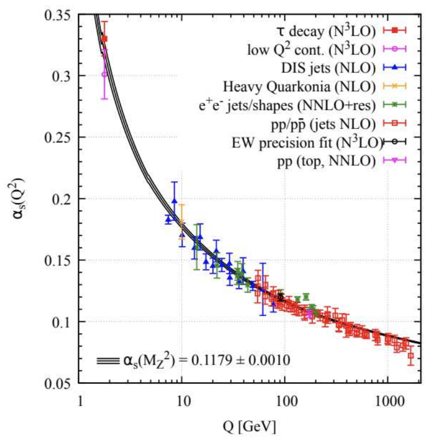
It is shown that a wide class of non-Abelian gauge theories have, up to calculable logarithmic corrections, free-field-theory asymptotic behavior. It is suggested that Bjorken scaling may be obtained from strong-interaction dynamics based on non-Abelian gauge symmetry.

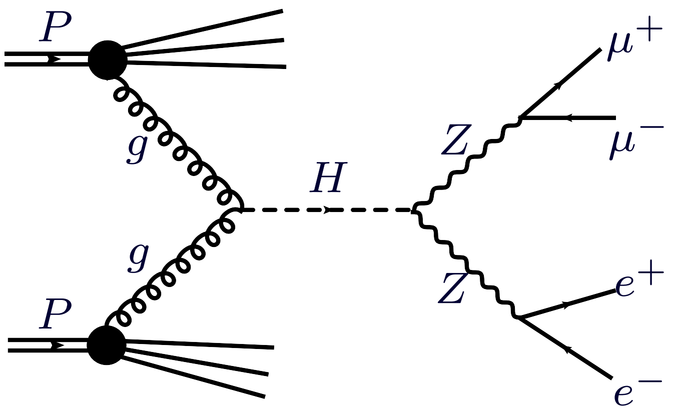
**Reliable Perturbative Results for Strong Interactions?***

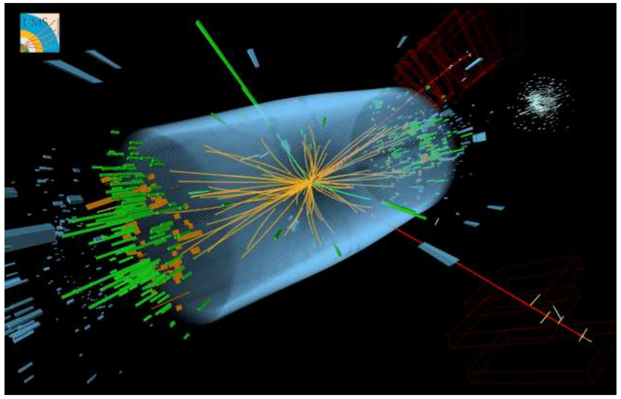
H. David Politzer
*Jefferson Physical Laboratories, Harvard University, Cambridge, Massachusetts 02138*
(Received 3 May 1973)

An explicit calculation shows perturbation theory to be arbitrarily good for the deep Euclidean Green's functions of any Yang-Mills theory and of many Yang-Mills theories with fermions. Under the hypothesis that spontaneous symmetry breakdown is of dynamical origin, these symmetric Green's functions are the asymptotic forms of the physically significant spontaneously broken solution, whose coupling could be strong.

- QCD has had tremendous success in describing the strong interaction at high energy



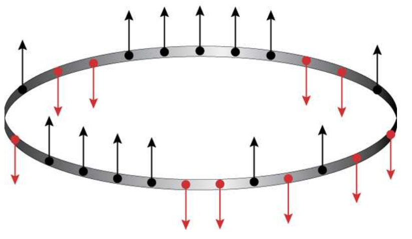
- Understanding quark confinement remains one of the most outstanding problem in QCD

  The structure of nuclear matter remains elusive

## 95 years of Heisenberg spin chain model

- ✔ Heisenberg antiferromagnetic XXX spin $1/2$ chain *Heisenberg, 1928*

$$H_{\rm XXX} = -\sum_{n=1}^{L} \vec{S}_n \cdot \vec{S}_{n+1}$$

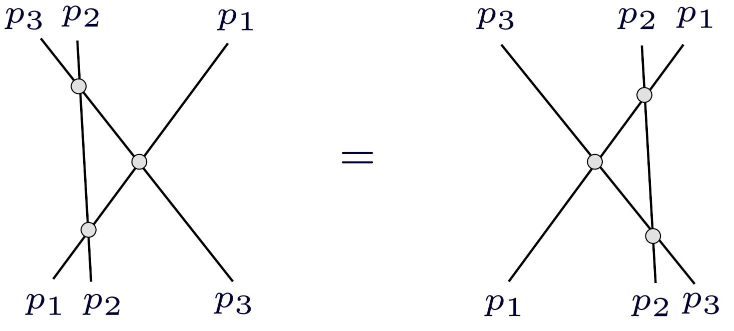


  Exact solution can be found using Bethe Ansatz *Bethe, 1931*

- ✔ ***Integrable models*** – family of *solvable* quantum field-theoretical models in 2 dimensions

  Infinitely many conserved charges $\to$ Elastic scattering $\to$ Factorizable S-matrices

- ✔ Two-particle S-matrix satisfies Yang-Baxter equation



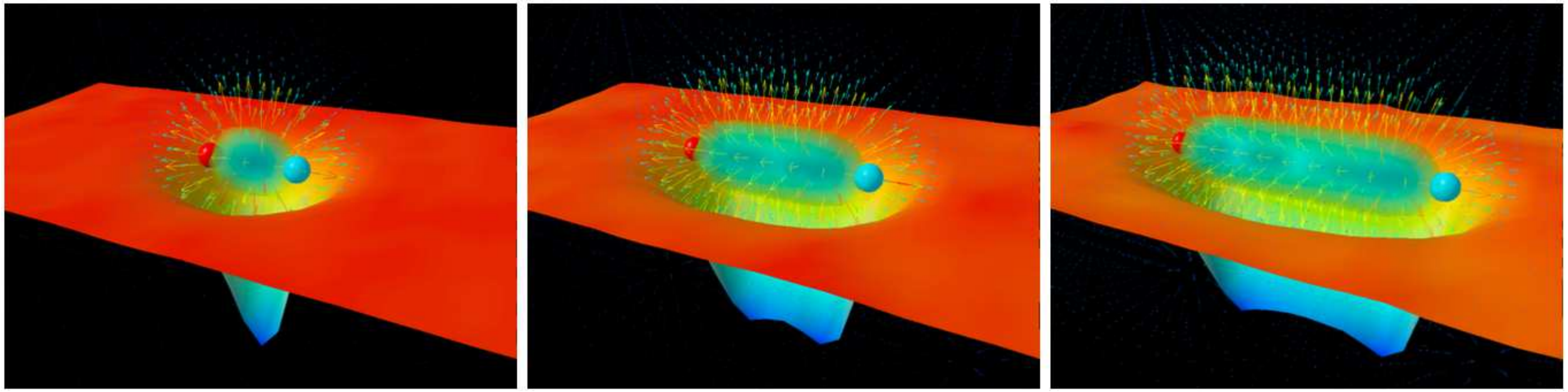

  Many-particle S–matrix is a product of 2-particle S–matrices

- ✔ What is the relation between 2-dim integrable models and QCD?

# Strings from Quantum Chromo Dynamics

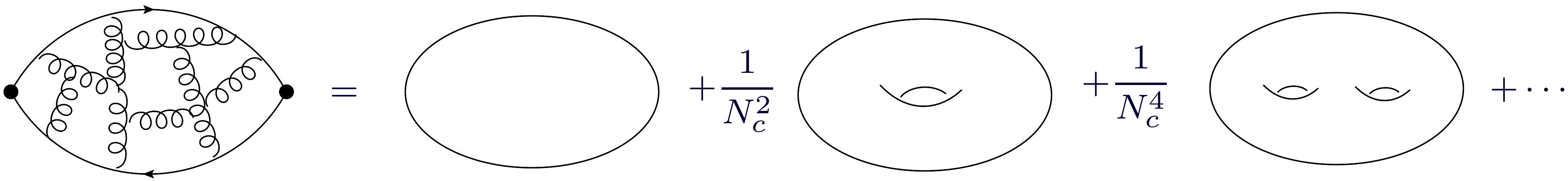
- ✔ The strength of the interaction decreases at short distances – asymptotic freedom

  At large distances quark and gluons are confined into hadrons (meson, baryons)



What is an effective string theory of QCD flux tubes? We don't know yet

- ✔ String description naturally appears in ***large*** $\boldsymbol{N_c}$ ***limit***




Dense Feynman diagrams = Sum over 2d Riemann surfaces (string world-sheet)

- ✔ If QCD at large distances is described by a string theory, this should have some manifestation at short distances $\Longrightarrow$ look for hidden symmetries

# What are the symmetries of QCD?

- ✔ QCD = (3+1)-dimensional Yang-Mills field theory with the $SU(N_c = 3)$ gauge group
- ✔ Symmetry of the *classical* theory:
  - ✗ gauge symmetry,
  - ✗ chiral symmetry,
  - ✗ conformal symmetry, . . .
- ✔ Many of classical symmetries are broken on the *quantum* level

**Q:** Could it be that QCD possesses some hidden symmetry which

(i) does *not* exhibit itself as a symmetry of the classical Lagrangian

(ii) is only revealed on the *quantum* level?

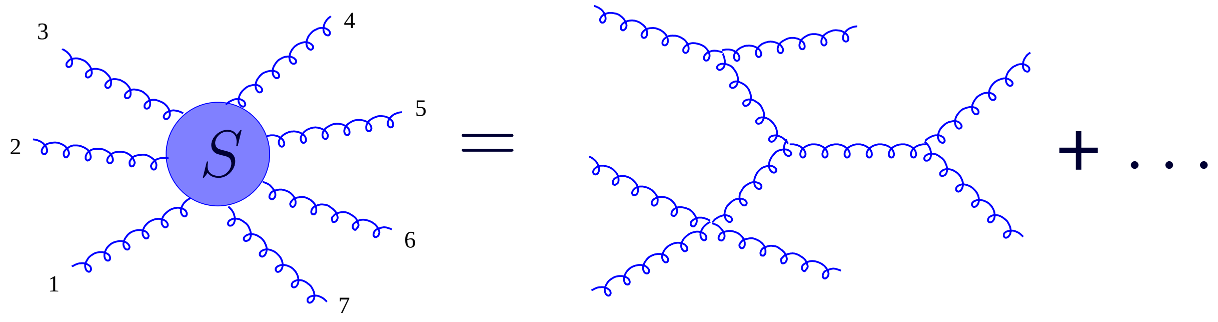
*Example: Integrability in AdS/CFT correspondence*

$\mathcal{N} = 4$ SYM at strong coupling $\iff$ type IIB string on the $AdS_5 \times S^5$ background

**A:** *Yes!* QCD at high energy is intrinsically related to *completely integrable models*

# Scattering amplitudes in QCD

Tree gluon scattering amplitudes




✔ The same in QCD and in maximally supersymmetric Yang-Mills theory

| Number of external gluons | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|
| Number of 'tree' diagrams | 4 | 25 | 220 | 2485 | 34300 | 559405 | 10525900 |

✔ Number of diagrams grows factorially for large number of external gluons

... but the final expression for tree amplitudes looks remarkably simple *Parke-Taylor'86*

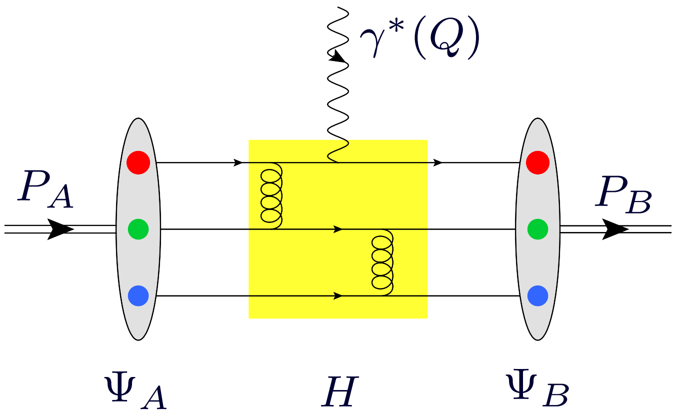
$$A_n^{\text{tree}}(\underbrace{1^+2^+3^-\dots n^-}_{\text{MHV amplitude}}) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\dots\langle n1\rangle}\,, \qquad \Big[\text{spinor notations: } \langle ij\rangle = \lambda^\alpha(p_i)\lambda_\alpha(p_j)\Big]$$

✔ What is the reason for remarkable simplicity of amplitudes? ***'Dual conformal'*** symmetry

It is powerful enough to fix all tree level gluon amplitudes

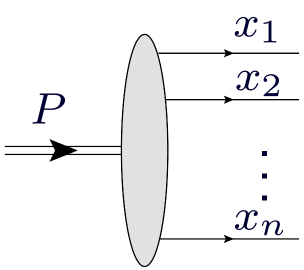
# Hard processes in QCD

Electromagnetic form factor of proton at large $Q^2$




$$= \Psi_A \otimes H \otimes \Psi_B$$

Hadrons in the infinite momentum frame $\approx$ system of quasi-free partons with virtuality $\mu^2$




$$= \Psi(x_1, \ldots, x_n; \mu^2) \quad , \quad \underbrace{0 \le x_k \le 1\,, \quad \sum_k x_k = 1}_{\text{momentum fractions}}$$

QCD factorization (scale separation)

$$F(Q^2) = \frac{1}{(Q^2)^{n-1}} \int_0^1 [dx][dy]\, \Psi_A(\{x\}; \mu^2) H(\{x, y\}, Q^2/\mu^2, \alpha_s(\mu^2)) \Psi_B(\{y\}; \mu^2)$$

Distribution amplitudes are nonperturbative, hard function is perturbative

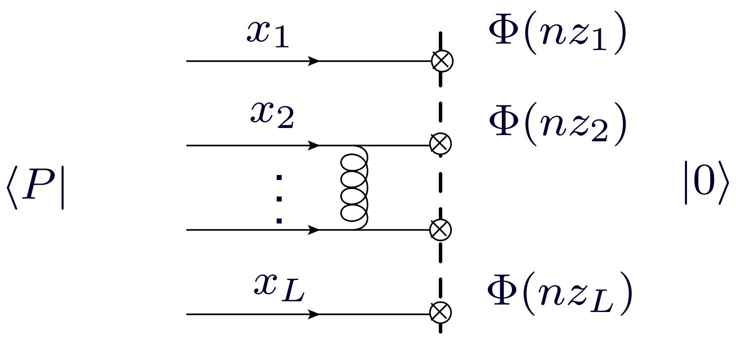
Perturbative QCD can be used to predict $Q^2-$dependence (= scaling violation)

# Hadron distribution amplitudes

- ✔ Nonperturbative definition *[Brodsky, Lepage'79],[Efremov,Radyushkin'79]*

$$\langle P|\Phi(nz_1)\ldots\Phi(nz_L)|0\rangle_{\mu^2} \overset{n^2=0}{=} \int_0^1 [dx]\, e^{-i(Pn)\sum_i z_i x_i}\,\Psi(x_1,\ldots,x_L;\mu^2)$$

Correlation functions of parton fields on the light front = Sum of plane waves




Parton fields $\Phi = \{$quark, gluon$\}$ connected by gauge links

- ✔ Moments of distribution amplitudes $\Longleftrightarrow$ local operators:

$$\widetilde{\Psi}_{k_1\ldots k_L} = \int [dx]\, x_1^{k_1}\ldots x_L^{k_L}\,\Psi(x_1,\ldots,x_L;\mu^2) = \langle P|(D_+^{k_1}\Phi)\ldots(D_+^{k_L}\Phi)|0\rangle_{\mu^2}$$

- ✔ Scale dependence of the distribution amplitudes

$$\mu\frac{d}{d\mu}\widetilde{\Psi}_{k_1\ldots k_L} = \sum_{m_j}\underbrace{V(k_i|m_j)}_{\text{mixing matrix}}\widetilde{\Psi}_{m_1\ldots m_L}$$

# Conventional QCD approach

✔ Diagonalize the mixing matrix and find the spectrum of anomalous dimensions

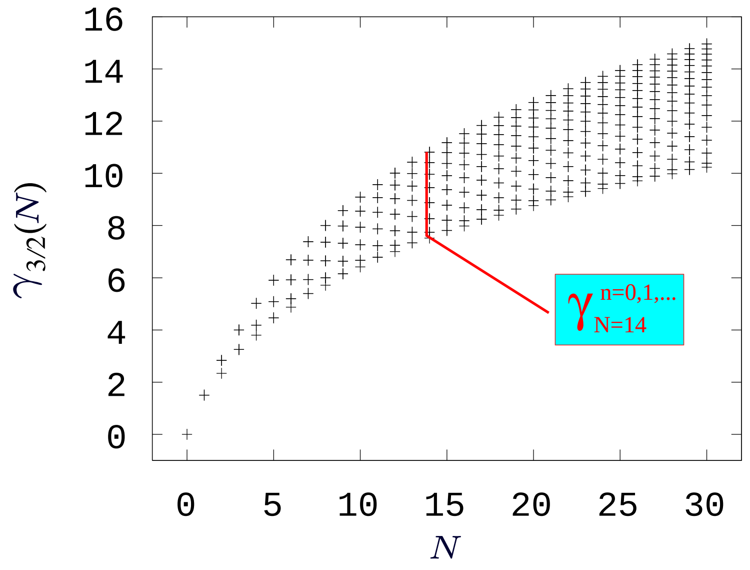
✔ Example: helicity$-3/2$ baryon distribution amplitude $[q = q^{\uparrow}(x) + q^{\downarrow}(x),\, q^{\uparrow(\downarrow)} = \frac{1\pm\gamma_5}{2}q]$

$$q^{\uparrow}(z_1 n)q^{\uparrow}(z_2 n)q^{\uparrow}(z_3 n) \longrightarrow (D_+^{k_1} q^{\uparrow})\,(D_+^{k_2} q^{\uparrow})\,q^{\uparrow}(0) + [\text{total derivatives}]$$

✗ Mixing matrix:

$$\sum_{n_1+n_2=N} V(k_1,k_2|n_1,n_2)\,\Psi^{(\ell)}_{n_1,n_2} = \gamma^{(\ell)}_{3/2}(N)\,\Psi^{(\ell)}_{k_1,k_2}\,, \qquad (\ell = 0,\ldots,N)$$

✗ Rich spectrum of anomalous dimensions:




- (Almost) all levels are double degenerate
- Where does this structure come from?

  *Conformal symmetry + Integrability!*

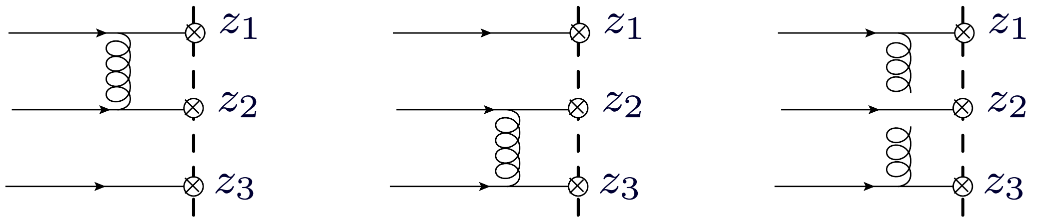
# Integrability on the light-cone

- Callan-Symanzik equation (helicity$-\frac{3}{2}$ baryon operator $B \equiv q^{\uparrow}(z_1 n) q^{\uparrow}(z_2 n) q^{\uparrow}(z_3 n)$)

$$\mu \frac{d}{d\mu} B(z_1, z_2, z_3) = [\mathbb{H} \cdot B](z_1, z_2, z_3),$$

- One-loop dilatation operator:

$\mathbb{H} \quad =$




$+ O(\alpha_s^2)$

- Two-particle structure:

$$\mathbb{H} = \frac{\alpha_s N_c}{2\pi} [\mathcal{H}_{12} + \mathcal{H}_{23} + \mathcal{H}_{13}] + O(\alpha_s^2)$$

Displaces quark fields along the light-cone

$$\mathcal{H}_{12} B(z_1, z_2, z_3) = \int_0^1 \frac{d\alpha\, \alpha}{1-\alpha} \big[ B(z_1 - \alpha z_{12}, z_2, z_3) + B(z_1, z_2 + \alpha z_{12}, z_3) - 2B(z_1, z_2, z_3) \big]$$

- QCD evolution = Quantum mechanics on the light cone

$\mathbb{H} =$ 3 particle Hamiltonian with nearest neighbour interaction

# Conformal symmetry on the light-cone

- QCD Lagrangian is invariant under conformal transformations
- It is broken in QCD but the conformal anomaly affects the anomalous dimensions starting from two loops only
- One-loop dilatation operator in QCD inherits conformal symmetry of the classical Lagrangian!
- Full conformal symmetry reduces on the light-cone $x_\mu = z n_\mu$ $(n^2 = 0)$ to its $SL(2)$ subgroup:

$$z \to z' = \frac{az+b}{cz+d}\,, \qquad q(zn) \to q'(zn) = q\left(\frac{az+b}{cz+d}\right)(cz+d)^{-2}$$

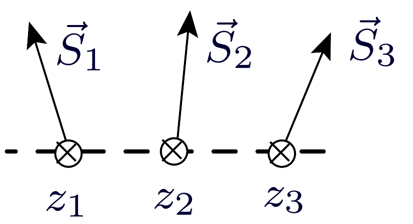
  Conformal generators:

$$S_- = -\frac{d}{dz}\,, \qquad S_+ = z^2\frac{d}{dz} + 2z\,, \qquad S_0 = z\frac{d}{dz} + 1$$

  Think about them as spin operators $[S_\alpha, S_\beta] = i\epsilon_{\alpha\beta\gamma}S_\gamma$

- Dilatation operator is the spin-chain Hamiltonian

$$\mathbb{H} = f(\vec{S}_1\cdot\vec{S}_2) + f(\vec{S}_2\cdot\vec{S}_3) + f(\vec{S}_3\cdot\vec{S}_1)$$




  Coincides with the Heisenberg $SL(2;\mathbb{R})$ spin chain Hamiltonian!

# Integrability on the light-cone (II)

- QCD anomalous dimensions are eigenvalues of the dilatation operator

$$\mathbb{H}\,\Psi_N(z_1,z_2,z_3) = \gamma_N\,\Psi_N(z_1,z_2,z_3)$$

- $SL(2)$ invariant form of the dilatation operator

$$\mathbb{H} = \frac{\alpha_s N_c}{\pi}\left[\mathcal{H}_{12}+\mathcal{H}_{23}+\mathcal{H}_{13}\right], \qquad \mathcal{H}_{12} = \underbrace{\psi(J_{12})}_{\text{Euler's } \psi-\text{function}} - \psi(1)$$

  Two-particle conformal spin

$$\mathbf{J}_{12}^2 = J_{12}(J_{12}-1) \equiv (\vec{S}_1+\vec{S}_2)^2$$

- One-loop dilatation operator $\equiv$ Hamiltonian of the $SL(2,\mathbb{R})$ Heisenberg spin chain
  - Number of sites = number of quark operators
  - Spin operators = Generators of the $SL(2,\mathbb{R})$ ‘collinear’ group
- The spectrum of anomalous dimensions can be found exactly using the Bethe Ansatz

$$\gamma_N = \frac{\alpha_s N_c}{\pi}\sum_{k=1}^{N}\frac{1}{\lambda_k^2+1}\,, \qquad \left(\frac{\lambda_k+i}{\lambda_k-i}\right)^3 = \prod_{j\neq k}^{N}\frac{\lambda_k-\lambda_j-i}{\lambda_k-\lambda_j+i}$$

  $\{\lambda_1,\ldots,\lambda_N\}$ = Bethe roots

# Integrable "zoo" in multi-color QCD

- ✔ Interaction between partons with the *aligned* helicities (quarks $q^\uparrow$, gluons $G^\uparrow$) is integrable

  One-loop dilatation operator $\mathbb{H} =$ Hamiltonian of a noncompact $SL(2,\mathbb{R})$ Heisenberg magnet:

  - ✗ Three-quark states:

$$[q^\uparrow(z_1)q^\uparrow(z_2)q^\uparrow(z_3)] \Longrightarrow \text{ \textit{closed} spin } j_q = 1 \text{ chain}$$

  - ✗ Multi-gluon states:

$$[G^\uparrow(z_1)G^\uparrow(z_2)...G^\uparrow(z_L)] \Longrightarrow \text{ \textit{closed} spin } j_g = 3/2 \text{ chain}$$

  - ✗ Antiquark-Glue-Quark states:

$$[\bar{q}(z_1)\, G^\uparrow(z_2)...G^\uparrow(z_{L-1})q(z_L)] \Longrightarrow \text{ \textit{open inhomogeneous} spin chain}$$

- ✔ Integrability is broken in the 'mixed' helicity sectors (ex: helicity$-1/2$ states $[q^\uparrow q^\downarrow q^\uparrow]$)

  - ✗ Symmetry breaking terms generate a mass gap in the spectrum of $\gamma(N)$ [scalar diquarks?]
  - ✗ ... but they do not affect large $N$ asymptotics

$$\gamma(N) = 2\underbrace{\Gamma_{\text{cusp}}(\alpha_s)}_{\text{cusp anom.dim.}} \ln N + N^0 \times \text{(nonintegrable terms)}$$

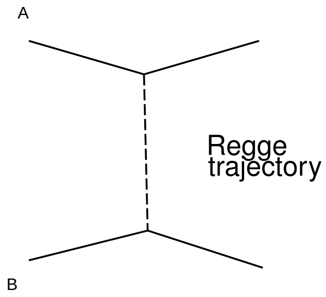
- ✔ What happens in supersymmetric cousins of QCD?

  Supersymmetry enhances QCD integrability and extends it to a larger class of Wilson operators as one goes from $\mathcal{N} = 0$ to $\mathcal{N} = 4$ SYM

# Scattering in QCD at high-energy (Regge asymptotics)

✔ Regge phenomena in strong interactions (since 60's):

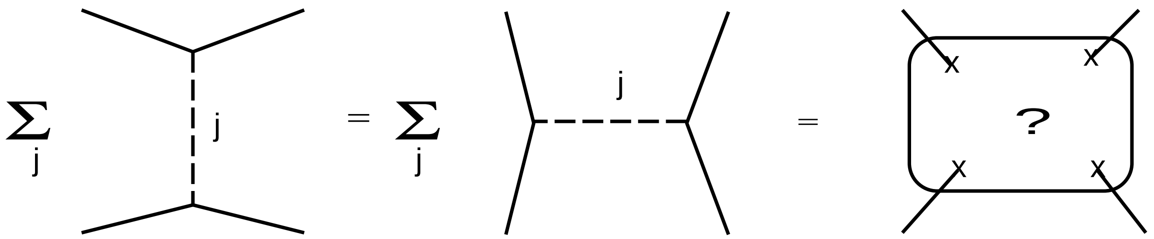
$$\sigma_{\mathrm{AB}}(s) =$$




$$= \sum_j \beta_A^j(t)\beta_B^j(t)\, s^{\alpha_j(t)-1}$$

Scattering amplitudes grow at high energy $s$ as a power $\sim s^{\alpha_j(t)}$

✔ Dual model:




Regge trajectories + duality condition = Hadronic string (?)

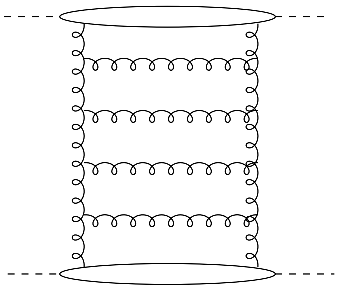
✔ High-energy asymptotics in QCD: interaction induces large corrections which need to be resummed to all order of perturbation theory *Balitsky-Fadin-Kuraev-Lipatov '78*

$$\sigma_{\mathrm{AB}}(s) = \sum_{n=0,1,\ldots} w_n \left(g_s^2 \ln s\right)^n \sim s^{\alpha_{\mathbb{P}}-1}$$

# BFKL Pomeron + Unitarity

- Leading contribution: BFKL Pomeron ($\lambda = g_s^2 N_c/(4\pi^2)$)

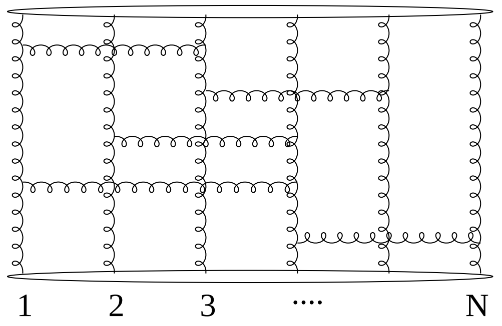
$$\sigma_{\rm LO} = \sum_{\rm rungs}$$



$$\sim \lambda^2 \, \frac{\exp(4\ln 2 \cdot \lambda \ln s)}{\sqrt{\lambda \ln s}} \sim \underbrace{s^{4\ln 2\cdot\lambda}}_{\text{violates unitarity}}$$

- BFKL Pomeron + Unitarity $\Longrightarrow$ generalized ladder diagrams

$$\sum_{N=2,3,..}$$




- Multi-Regge kinematics: $\int d^4k = \int dk_+ dk_- \int d^2k_\perp$

– strong ordering in the longitudinal momenta $y = \ln \frac{k_+}{k_-}$

$$y_1 \gg y_2 \gg y_3 \gg \ldots = \text{“evolution time”} \text{ in the } t\text{−channel}$$

– “random walk” in the transverse momenta

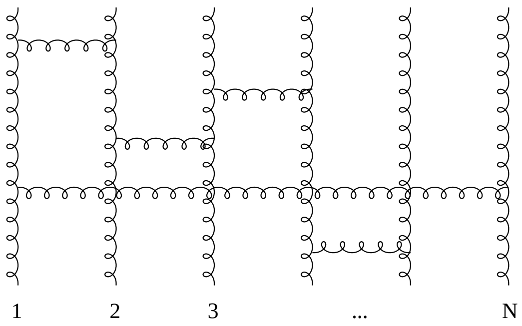
$$k_{1,\perp} \sim k_{2,\perp} \sim k_{3,\perp} \sim \ldots$$

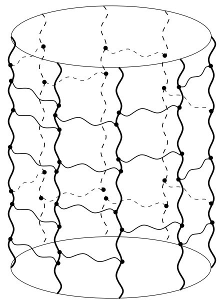
✌ Elastic pair-wise interaction of $N = 2, 3, \ldots$ particles “living” on the two-dimensional $k_\perp$−plane and propagating in the “time” $y = \ln s$.

✌ Nontrivial QCD dynamics occurs on the two-dimensional transverse space

# Color-singlet compound gluonic states

✌ The effective QCD Hamiltonian $\mathcal{H}_N$ has remarkable properties in the multi-color limit:

$\mathcal{A}(s,t) =$ 

=  $\sim \exp\left(y\mathcal{H}_N\right) = s^{\mathcal{H}_N}$

$\hookrightarrow$ Elastic scattering of $N$ reggeized gluons

✔ The Bartels-Kwiecinski-Praszalowicz equation $\equiv$ 2-dim Schrödinger equation

$$\mathcal{H}_N \Psi\underbrace{(\vec{z}_1, \vec{z}_2, ..., \vec{z}_N)}_{2-\text{dim coordinates}} = E_N \Psi(\vec{z}_1, \vec{z}_2, ..., \vec{z}_N)$$

✔ $\Psi(\vec{z}_1, \vec{z}_2, ..., \vec{z}_N) =$ colour-singlet compound states built from $N$ reggeized gluons

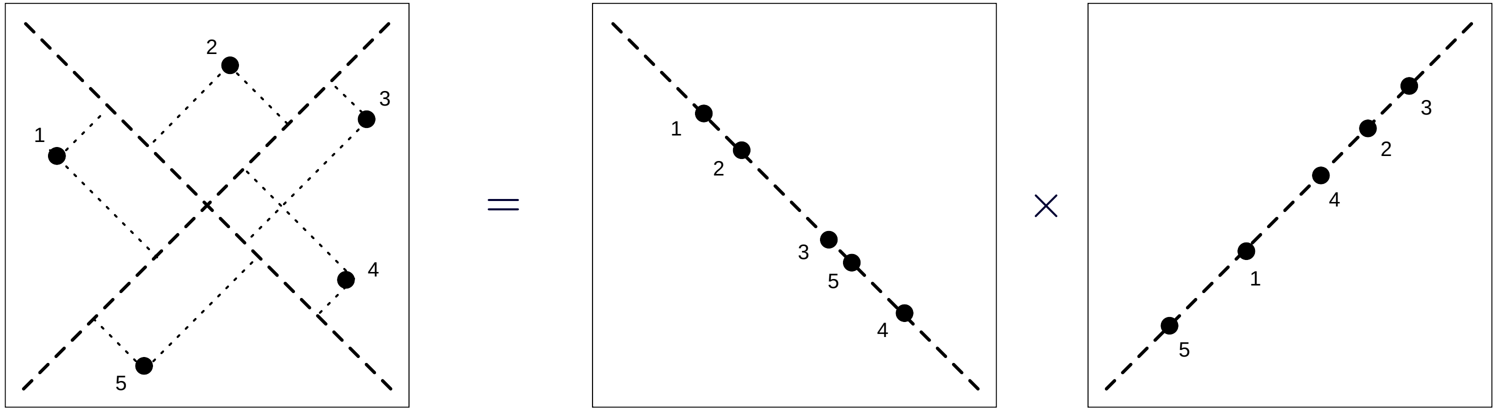
✔ High-energy asymptotics of the scattering amplitudes is governed by the contribution of these states

$$\mathcal{A}(s,t) \sim -is \sum_{\substack{N-\text{gluon}\\ \text{states}}} (i\lambda)^N \underbrace{s^{\lambda E_N}}_{\text{Regge behaviour}} \beta_N(t)$$

✔ Intercept = maximal energy $E_N$

# Reggeon Hamiltonian

✔ QM system of $N$ interacting particles "living" on a two-dimensional transverse $(x, y)-$plane




✔ Dynamics in (anti)holomoprhic coordinates $z_k = x_k + iy_k$ and $\bar{z}_k = z_k^*$ is independent on each other

$$\mathcal{H}_N = \mathbb{H}_z + \mathbb{H}_{\bar{z}} \,, \qquad [\mathbb{H}_z, \mathbb{H}_{\bar{z}}] = 0$$

✔ One-dimensional Hamiltonians describe spin chains

$$S_0^{(k)} = z_k \partial_{z_k} \,, \quad S_-^{(k)} = -\partial_{z_k} \,, \quad S_+^{(k)} = z_k^2 \partial_{z_k}$$

with nearest neighbour interaction

$$\mathbb{H}_z = \sum_{k=1}^{N} H(J_{k,k+1}) \,, \qquad H(x) = \psi(x+1) - \psi(1)$$

Coincides with the Heisenberg $SL(2;\mathbb{C})$ spin chain Hamiltonian!

# Integrability of high-energy QCD

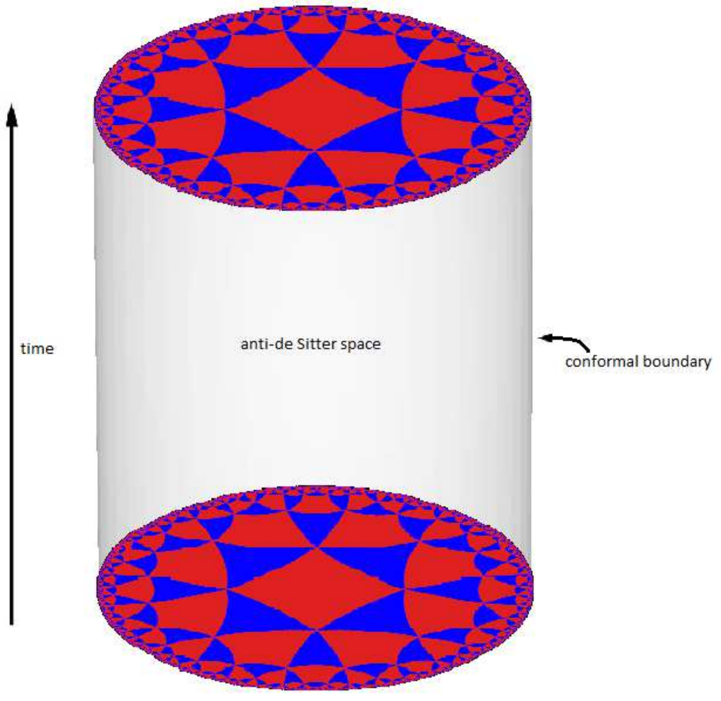
- ✔ Integrable models are QM systems with a *finite* number of degrees of freedom and the same number of conserved charges.
- ✔ QCD is a complex system with *infinite* number of degrees of freedom which are not integrable *per se*.
- ✔ Integrability emerges as a hidden symmetry of *effective* QCD dynamics in two *different* limits:
  - ✗ Scale dependence of multi-particle distribution amplitudes (=dilatation operator)
  - ✗ High-energy (Regge) behaviour of scattering amplitudes
- ✔ Does the $SL(2,\mathbb{R})$ integrability hold beyond one-loop in which case the conformal symmetry *is broken*? Yes, it does!
- ✔ Integrability is *not* tied to the conformal symmetry but is connected with the *multi-color limit* !
- ✔ Integrability is a general feature of (super) Yang-Mills theories in four dimensions

***What is the origin of integrability phenomenon in QCD and 4d gauge theories?***

- ✔ In maximally supersymmetric Yang-Mills theory it can be understood using AdS/CFT correspondence

# Gauge/string duality

Maximally supersymmetric $\mathcal{N}=4$ SYM is dual to string theory on $\mathrm{AdS}_5 \times S^5$ background




- ✔ Strings propagate inside the cylinder
- ✔ Yang-MIlls theory “lives” at the boundary of the cylinder

| | | |
|---|---|---|
| Large $N_c$ limit | $\Leftrightarrow$ | Free strings |
| Local operators | $\Leftrightarrow$ | String states |
| Scaling dimensions | $\Leftrightarrow$ | Energies of strings |

$$\langle O(x)O(0)\rangle = \frac{1}{(x^2)^\Delta}$$

Surprising correspondence of strongly interacting quantum field theories with gravitational theories:

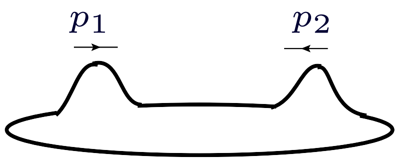
| | | |
|---|---|---|
| Strongly coupled maximally supersymmetric gauge theory | $\Longleftrightarrow$ | Weakly coupled ‘dual’ string theory on anti-de Sitter space |

# Integrability in AdS/CFT correspondence

***Gauge theory side:*** $\langle O(x)O(0)\rangle \sim 1/(x^2)^{\Delta(\lambda)}$

- ✔ Operators $O(x)$ $\Leftrightarrow$ States of one-dimensional spin chain
- ✔ Scaling dimensions $\Delta(\lambda)$ $\Leftrightarrow$ Energies of the spin-chain Hamiltonian

***String theory side:***




- ✔ Excitations of the strings scatter through integrable S-matrix

***Scaling dimensions can be found for any 't Hooft coupling by Bethe Ansatz !***

- ✔ Important example: Twist-2 operators with large spin

Govern scale dependence of quark and gluon distributions (DGLAP evolution)

AdS/CFT prediction

$$O_S = \mathrm{tr}\left[F_{+\mu} D_+^S F_{+\mu}\right] + \ldots \quad \Leftrightarrow \quad \text{Folded string spinning on AdS}$$

$$\Delta_S = \text{Energy of spinning string} = S + 2\,\Gamma_{\text{cusp}}(\lambda)\,\log S$$

$\Gamma_{\text{cusp}}(\lambda)$ is known in $\mathcal{N}=4$ SYM for any coupling

The same behaviour in QCD (but the cusp anom. dim. is different)

# Summary

- ✔ High-energy QCD possesses a hidden symmetry which does not exhibit itself as a symmetry of classical Lagrangian, but is only revealed on the quantum level
- ✔ The effective QCD dynamics in several important limits is described by completely integrable systems that prove to be related to the celebrated Heisenberg spin chain and its generalizations
- ✔ Integrability naturally appears in the AdS/CFT correspondence through symmetries of a string theory on the AdS background
- ✔ What is the origin of integrability phenomenon in QCD?
- ✔ 50 years later QCD remains an active area of ongoing research, with many unsolved problems and many aspects we would like to understand better

*Thank you for your attention!*