High dimension operators for high dimension CFTs

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Precision Holography Conference
June 9, 2023

Based on work to appear with:
Jaeha Lee, Hirosi Ooguri, and David Simmons-Duffin

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In 2d, there is a universal formula for **entropy** called Cardy's formula

$$\rho^{d=2}(\Delta, j) \sim \exp\left[\sqrt{\frac{c}{3}}\pi\left(\sqrt{\Delta + j - \frac{c}{12}} + \sqrt{\Delta - j - \frac{c}{12}}\right)\right], \quad \Delta \gg c.$$

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Derived from modular invariance of the torus partition function

$$Z(\beta) = Z(\beta^{-1})$$

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Valid for **all** 2d CFTs but for holographic theories it has a beautiful interpretation as black hole entropy

(Strominger, 1997)

Looking at modular invariance of the genus 2 partition function leads to a similar formula for three-point functions

$$C_{HHH}^2 \sim \left(\frac{27}{16}\right)^{3\Delta} e^{-6\pi\sqrt{\frac{c-1}{24}\Delta}} \Delta^{\frac{5c-11}{36}}, \quad \Delta \gg c.$$

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Similar formulas exist for HHL and HLL three-point functions, with interesting connections to Liouville theory

(Collier, Maloney, Maxfield, Tsiares, 2019)

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The partition function on $S^{d-1} \times S^1$ no longer has modular invariance! So the math is different

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2. There is no Virasoro symmetry in >2 d, so the formula is more analogous to “global primaries”

3. There is no RG monotonicity properties for the coefficient like c

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The gapped theory is kind of a higher dimensional "modular dual"

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Partition function of CFT on this geometry is the captured by the gapped (d-1)-dim theory coupled to (d-1)-dim background fields

$$Z_{\rm CFT}(G) = Z_{\rm gapped}(g_{ij}, A_i, \phi).$$

Lore of massive QFT: Z_{gapped} can be captured by local effective action for (d-1)-dim fields

$$Z_{\text{gapped}}(g_{ij}, A_i, \phi) \sim e^{-S_{\text{th}}[g_{ij}, A_i, \phi]}$$

(Bhattacharya, Minwalla, et al…)

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2. Weyl invariance of original theory

$$Z_{\text{CFT}}(e^{2\sigma}G) = Z_{\text{CFT}}(G)e^{-S_{\text{anom}}[G,\sigma]},$$

forces S_{th} to be a function of the gauge field and of Weyl-invariant metric $\widehat{g}_{ij} \equiv e^{-2\phi}g_{ij}$,

$$S_{\rm th} = \int d^{d-1}x \sqrt{\widehat{g}} \left(-f + c_1 \widehat{R} + c_2 F^2 + \ldots \right) + S_{\rm Weyl}.$$

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Moreover f is the Casimir energy of the CFT on a circle, so in 2d f is related to the central charge c

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Now we just need to compute $S[\hat{g}, A]$ in this geometry. Put manifold in KK form, plug in $\hat{g}, A$ into thermal effective action

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$$S_{\rm th} = \frac{\mathrm{vol}\, S^{d-1}}{\prod_{i=1}^n (1+\Omega_i^2)} \left[-f T^{d-1} + (d-2)\left((d-1)c_1 + (2c_1 + \tfrac{8}{d}c_2) \sum_{i=1}^n \Omega_i^2 \right) T^{d-3} + \ldots \right]$$

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Einstein term

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From this we can read off the partition function (at large T) and take an inverse Laplace transform to read off entropy as a function of Δ, J

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Some nice features —

1. In 2d, identifying f with c, we get

$$Z = \exp\left(\frac{4\pi^2 cT}{12(1+\Omega^2)} \right) + \text{non-pert}$$

as expected from S-transform of vacuum-dominance at low T

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Some nice features —

2. $S_{\rm th}$ diverges at $\Omega = \pm\, i$. This is the unitarity bound

$$Z(\beta, \vec{\Omega}) = \mathrm{Tr}\left[e^{-\beta(D+\varepsilon_0)+i\beta\vec{\Omega}\cdot\vec{M}} \right] \sim e^{-S[\widehat{g},A]}.$$

At $\Omega = \pm\, i$, states with $E \pm J$ constant get no Boltzmann suppression, which leads to a divergence in Z. Forbidden to have poles before there

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Some nice features —

3. Already this makes nontrivial predictions from the functional form. For example: power expansion in $1/T^2$, not $1/T$, and Ω dependence completely fixed

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In 2d, this of course reproduces the usual Cardy formula

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Note that by expanding in terms of characters instead of exponentials, we can also read off the density of **primaries** instead of states

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For example here is the result in d=3:

$$\log \rho_{d=3}^{\text{primaries}}(\Delta, J) = 3\pi^{1/3} f^{1/3} (\Delta + J)^{1/3} (\Delta - J)^{1/3} + \log\left(\frac{\Delta(2J+1)}{(\Delta^2 - J^2)^{7/3}}\right) + \log\left(\frac{16\pi^{2/3} f^{5/3}}{\sqrt{3}}\right) - 8\pi c_1 + \frac{32 c_2 J^2 \pi}{3(\Delta^2 - J^2)} + O(\Delta^{-1/3}).$$

(See also Shaghoulian, 2015, for leading term)

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Statistical Distribution of Spin, aka “Spin-Statistics Theorem” for CFT! :-)

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Example 1: From weak to strong coupling in N=4 SYM (marginal flow), f changes by a factor of 3/4 instead of staying constant

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Example 2: At large N, the d=3 O(N) model flowing to N-1 free scalar fields has f increasing by a factor of 5/4 instead of decreasing

(Sachdev, 1993)

What is the regime of validity for our entropy?

$$\log \rho^{\text{primaries}}_{d=3}(\Delta, J) = 3\pi^{1/3} f^{1/3} (\Delta + J)^{1/3} (\Delta - J)^{1/3} + \log\left(\frac{\Delta(2J+1)}{(\Delta^2 - J^2)^{7/3}}\right)$$
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d=2: $\Delta - |J| \gg c$

d>2: $\Delta - |J| \gg \sqrt{f\Delta}.$

(Aside: d>2 formula is for one fugacity turned on; for more fugacities exponent changes)

Examples

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Free theories: We can compute $Z(T,\Omega)$ exactly and expand in large T to verify it takes the functional form we predict

Holographic theories: We can approximate $Z(T,\Omega)$ by computing the area of a Kerr-AdS black hole and verifying it takes the functional form

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Free scalar

$$f = \frac{2^{\lfloor \frac{d}{2} \rfloor}\left(1-\frac{1}{2^{d-1}}\right)\zeta(d)}{\mathrm{vol}S^{d-1}}$$

$$c_1 = \frac{2^{\lfloor \frac{d}{2} \rfloor}\left(1-\frac{1}{2^{d-3}}\right)\zeta(d-2)}{24(d-2)\mathrm{vol}S^{d-1}}$$

$$c_2 = -\frac{2^{\lfloor \frac{d}{2} \rfloor}\left(1-\frac{1}{2^{d-3}}\right)\zeta(d-2)}{96(d-2)\mathrm{vol}S^{d-1}}$$

Free fermion

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Leading order in G_N we have:

$$f=\frac{(4\pi)^{d-1}\ell_{\mathrm{AdS}}^{d-1}}{4d^dG_N}$$

$$c_1=\frac{(4\pi)^{d-3}\ell_{\mathrm{AdS}}^{d-1}}{4(d-2)d^{d-2}G_N}$$

$$c_2=-\frac{(4\pi)^{d-3}\ell_{\mathrm{AdS}}^{d-1}}{32(d-2)d^{d-3}G_N}.$$

Kerr black holes in AdS for D>3 suffer from instability. They are only stable if (with one fugacity turned on):

$$E - J/\ell > \# \sqrt{E} \ell^{\frac{D-3}{2}} G_N^{-1/2}$$

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(See also Kim et al, 2023!)

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Similar analogy in $\mathrm{AdS}_3/\mathrm{CFT}_2$

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If light states are “sparse” then $\Delta - |J| \gtrsim \sqrt{f\Delta},$ is enough??

(c.f. Mefford, Shaghoulian, Shyani, 2017)

3d Ising model

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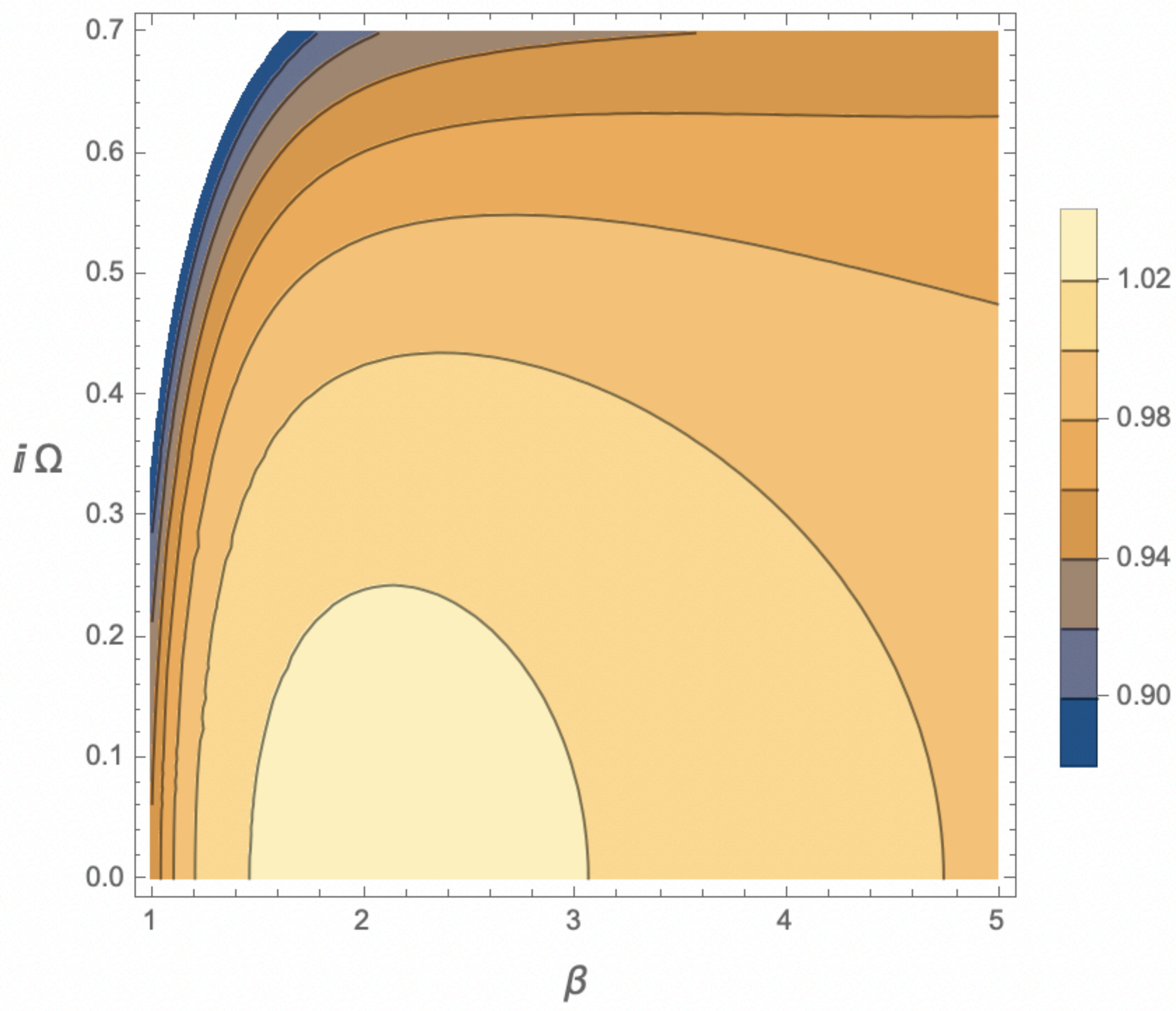
Can take bootstrapped operators and explicitly build partition function at plot at “medium” temperature

(Simmons-Duffin, et al)

3d Ising model

$$Z^{\text{3d Ising}}(\beta,\Omega)/\exp\left(4\pi\frac{0.153}{\beta^2(1+\Omega^2)}\right)$$



Three-point functions

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The other observable is in **correlation functions**, specifically the three-point functions

Three-point functions

So far we've discussed universal formulas for the entropy of local operators in d-dim CFT

The other observable is in **correlation functions**, specifically the three-point functions

Are there universal formulas for the three point functions of three heavy operators?

Let’s first ask how this problem is solved in d=2 CFT

(Cardy, Maloney, Maxfield 2017)

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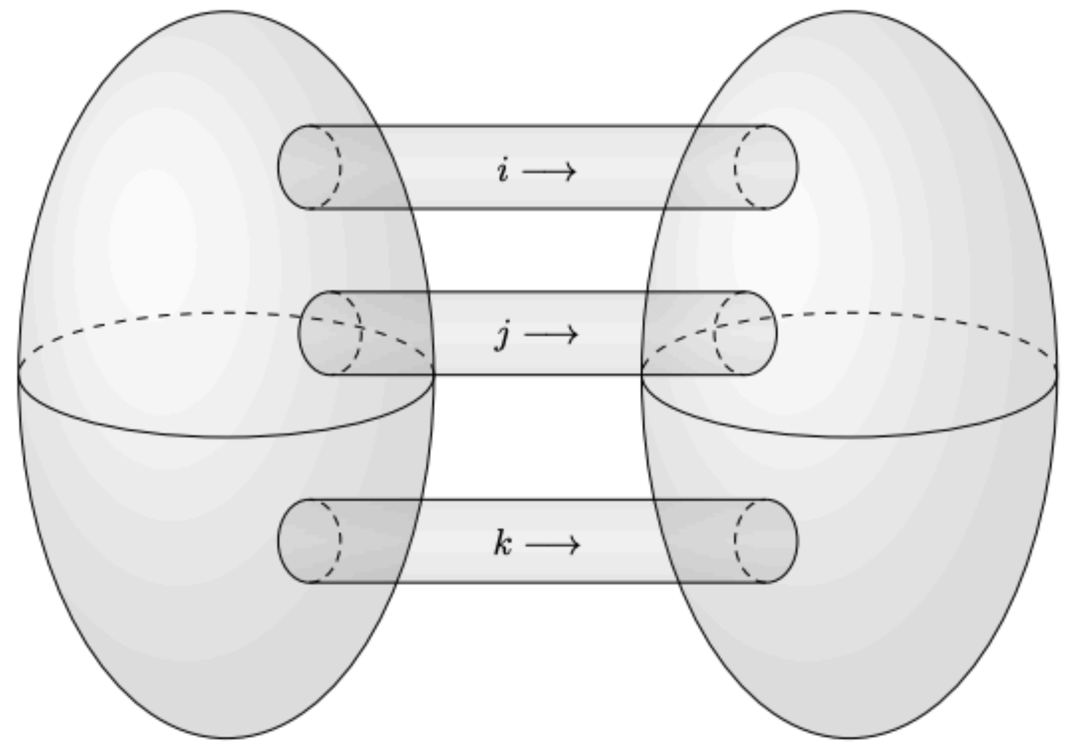

Genus two

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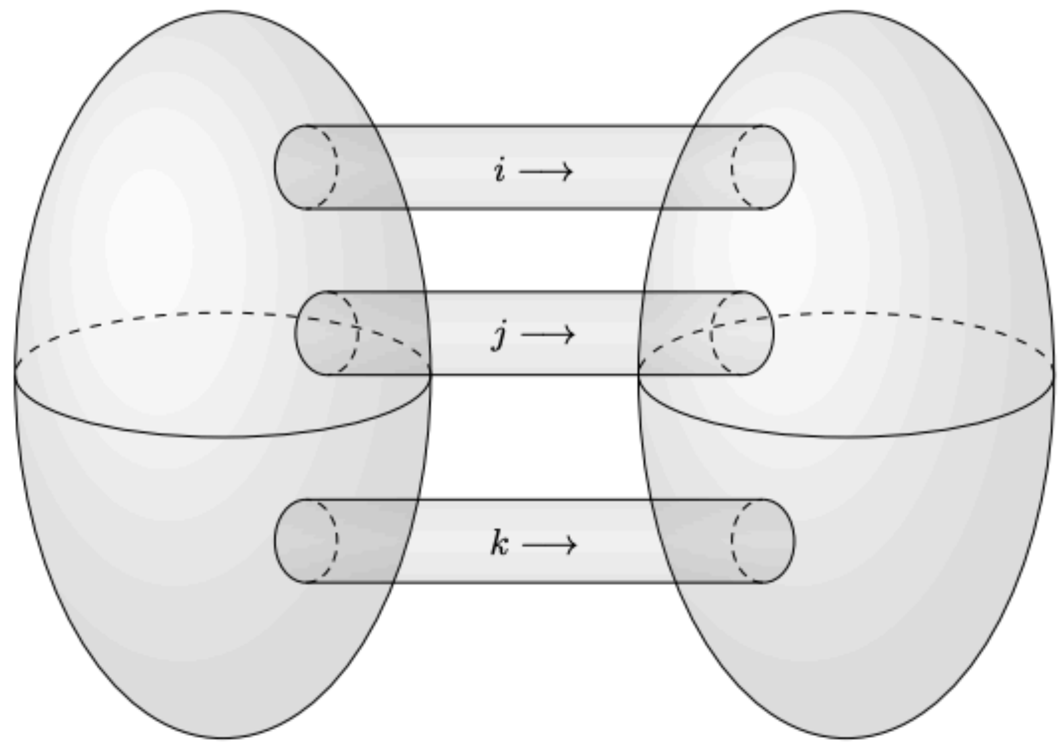

Genus two

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(Note that to get the **primary** operators instead of **states** we need a “conformal block” — conceptually similar to characters)

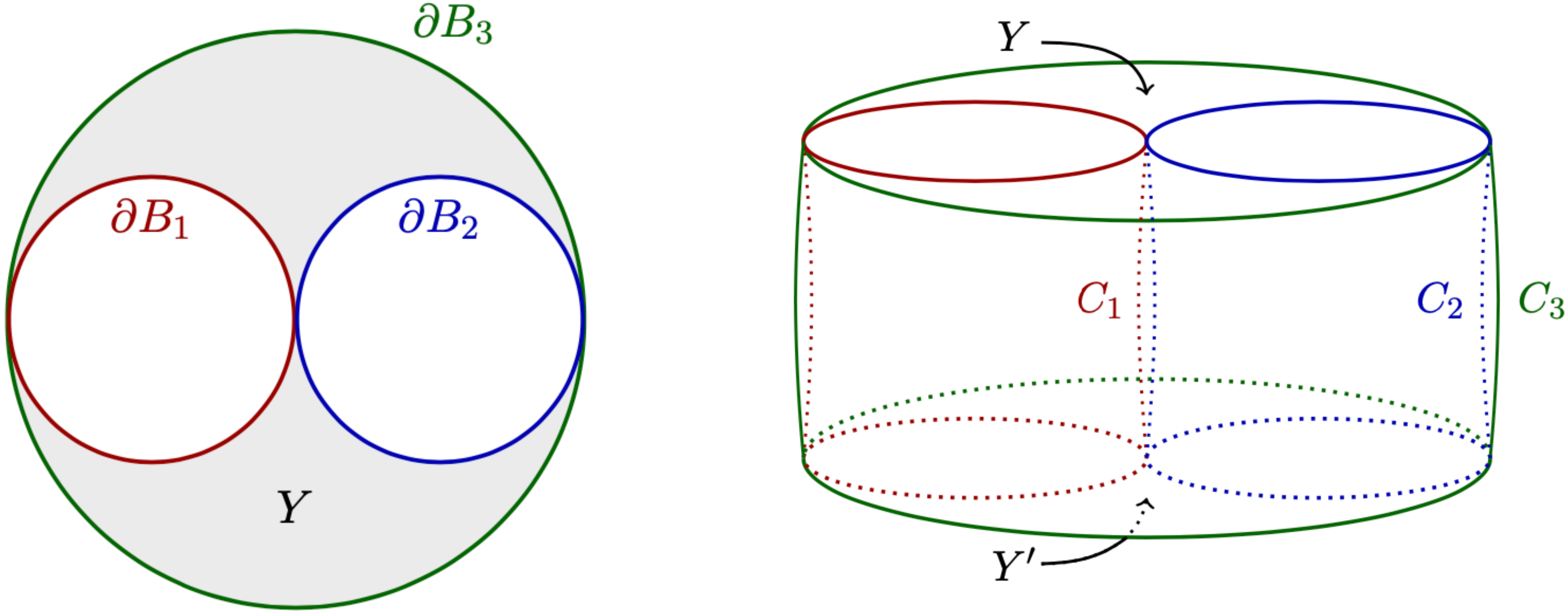
So to summarize, the key points were:

1. Find a geometry whose partition function computes C^2_{ijk}
2. Take a limit where that partition function is computable
3. Invert to find C^2_{ijk} (essentially an inverse Laplace transform, if the blocks are complicated, some souped up version of that)

So to summarize, the key points were:

1. Find a geometry whose partition function computes C_{ijk}^2
2. Take a limit where that partition function is computable
3. Invert to find C_{ijk}^2 (essentially an inverse Laplace transform, if the blocks are complicated, some souped up version of that)

The generalization of a genus 2 surface will be to take two copies of $\mathbb{R}^d$, cut out three mutually-tangent balls, and glue the boundaries of the balls (S^{d-1}'s) together with cylinders



We consider the geometry on the following manifold. This computes

$$Z = \sum_{O_1,O_2,O_3} |c_{123}|^2 e^{-\beta_1\Delta_1-\beta_2\Delta_2-\beta_3\Delta_3}$$

where β_i are the heights of the cylinders

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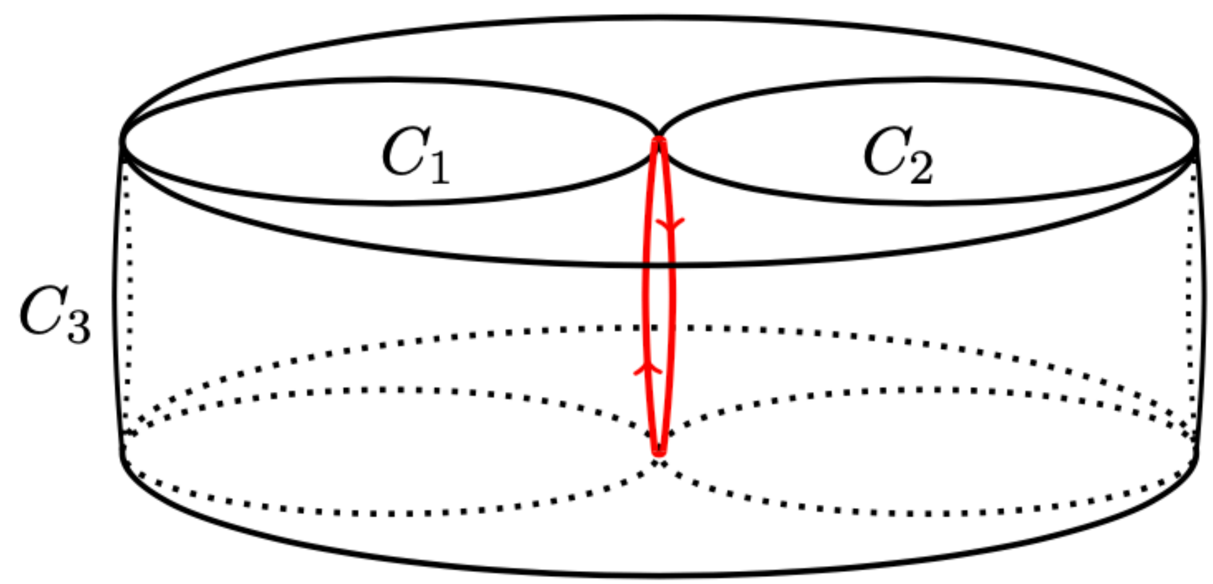

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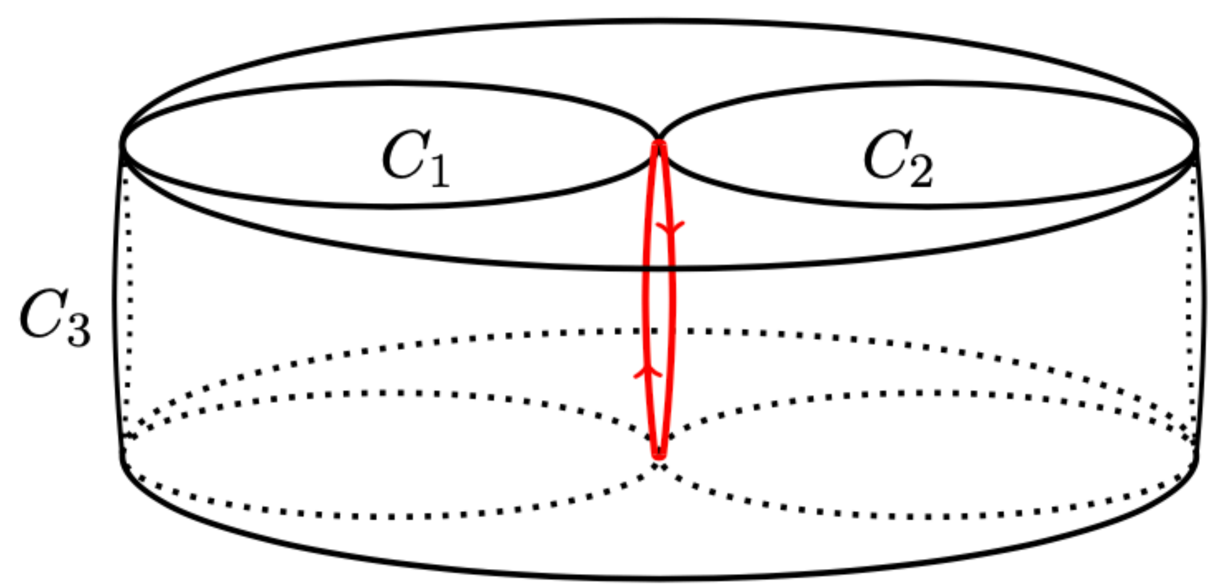

In the limit of high temperature, the geometry looks like a circle fibration in the red region

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There is still more work to read off the three point functions by decomposing into **conformal blocks** but it's kinematical — I will just write the results

In d=3, three heavy operators of dimension Δ :

$$c^2_{\Delta\Delta\Delta} \sim \left(\frac{3}{2}\right)^{6\Delta} \frac{2^{\frac{49}{4}} f^{\frac{9}{4}} e^{3\sqrt{2\pi f \Delta}}}{3^{\frac{19}{2}} \pi^{\frac{1}{4}} \Delta^{\frac{31}{4}}}$$

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Is there an interpretation for our formula? Maybe not because we do not have “Virasoro” structure. Is there a way to “upgrade” our formula to be related to wormhole actions??

Summary

We described a technique called the thermal effective action to systematically study CFT data at large dimension

This encodes the spectrum of local CFT operators as a function of dimension and spin at large dimension, checked against free and holographic theories

Built “genus-two-like” partition function in higher d to also encode three-point-functions