EQUIVARIANT LOCALIZATION AND HOLOGRAPHY

CERN 8 JUNE 2023

PRECISION HOLOGRAPHY WORKSHOP

BASED ON D. MARTELLI & A.Z 2306.03891

MANY SUCCESSFUL STORIES OF EXTREMIZATION PROBLEMS IN HOLOGRAPHY

- ENTROPY FUNCTIONS HAVE BEEN USED TO STUDY AND COUNT BLACK HOLE MICROSTATES

- "TRIAL" CENTRAL CHARGES TO COMPUTE "EXACT" CENTRAL CHARGES

$$\boxed{\mathcal{F}(\Delta) \xrightarrow[\text{EXTREMIZE}]{} S_{BH} \,/\, c_{CFT}}$$

Δ PARAMETERIZE

GLOBAL SYMMETRIES IN CFT

(CHEMICAL POTENTIALS)

ISOMETRIES IN GRAVITY

(EQUIVARIANT PARAMETERS)

A TALE OF FOUR EXTREMIZATIONS

I a - maximization

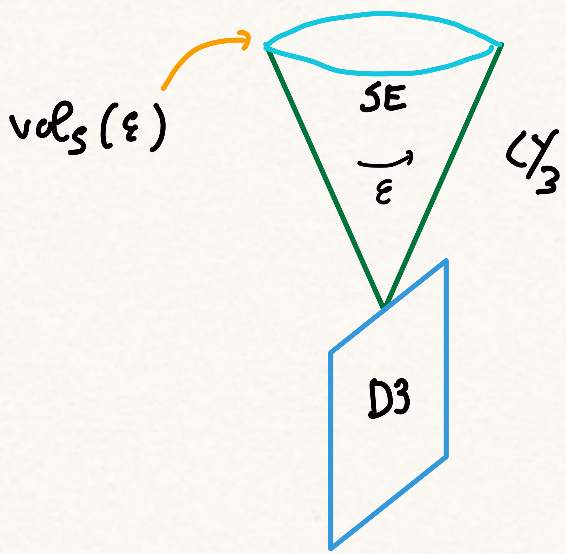
[INTRILIGATOR-WECHT 03]

4d SCFT $\quad a(\Delta) = \mathrm{tr}\, R^3(\Delta)$

VARYING R-SYMMETRY

volume minimization

[MARTELLI-SPARKS-YAU 05]



$AdS_5 \times SE_5$ solution

MORE EXPLICITLY FOR TORIC CY_3 WITH TORIC DATA v^I

$$\Delta_I \implies \text{global symmetries of the } CFT_4$$

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3) \implies \text{isometries of the } CY_3$$

$$R(\Delta) = \sum \Delta_I T_I$$

$$a(\Delta) = \sum C_{IJK} \Delta_I \Delta_J \Delta_K$$

$$C_{IJK} = \left| \det(v^I, v^J, v^K) \right|$$

$$\mathrm{Vol}_S(\varepsilon) = \frac{1}{\varepsilon_1} \sum_I \frac{\det(v^{I-1}, v^I, v^{I+1})}{\det(\varepsilon, v^{I-1}, v^I)\,\det(\varepsilon, v^I, v^{I+1})}$$

$$\Delta \implies \Delta(\varepsilon)$$

$$\boxed{a(\Delta(\varepsilon)) \equiv \mathrm{Vol}_S(\varepsilon)}$$

Proof a mess, but true [BUTTI-AZ 05]

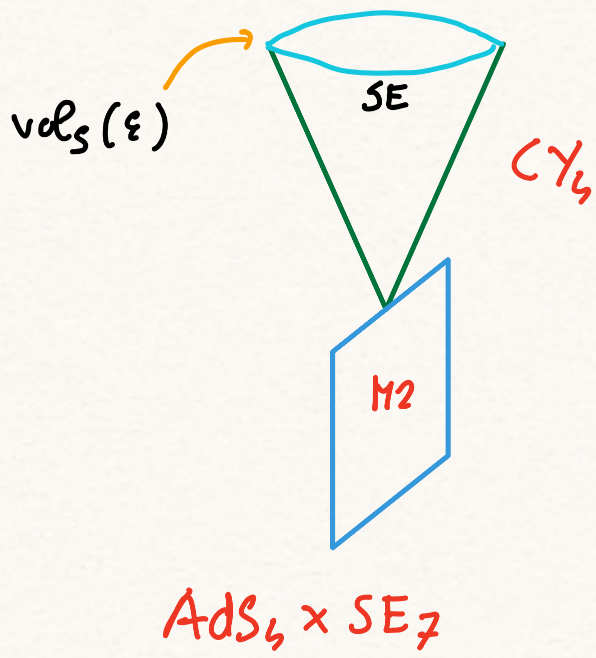
SAME STORY FOR $SCFT_3$

② **F-extremization** — **volume minimization**

S_3-free energy $F_{S_3}(\Delta)$

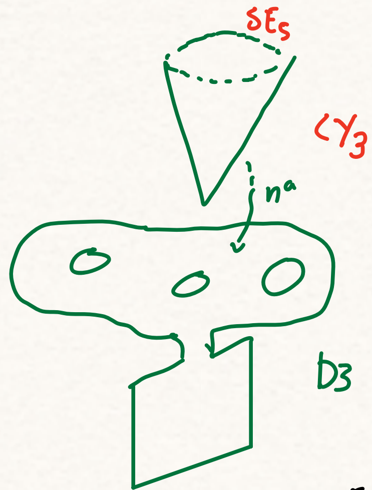
VARYING R-SYMMETRY

[JAFFERIS ; JAFFERIS - KLEBANOV - PUFU - SAFDI 11]



III c-extremization

D3 BRANES COMPACTIFIED ON RIEMANN SURFACES [BENINI-BOBEV 13]



$$CFT_4 \text{ on } \Sigma_g \longrightarrow CFT_2$$

$$AdS_3 \times \Sigma_g \times SE_5$$

$$c(\Delta) = \operatorname{tr} \gamma_3 R^2(\Delta)$$

EXTREMAL FUNCTION

$$Vol(\varepsilon, \lambda) \to S(\varepsilon, \lambda)$$

"master volume"

[COUZENS-GAUNTLETT-MARTELLI-SPARKS 18]
[GAUNTLETT-MARTELLI-SPARKS 18]

$$\lambda \to \lambda(\varepsilon, n)$$
$$\Delta \to \Delta(\varepsilon, n)$$

$$\boxed{c(\Delta(\varepsilon, n)) \equiv S(\varepsilon, \lambda)}$$

[HOSSEINI-AZ 19]

(IV) $\mathcal{I}$ - extremization

M2 BRANES COMPACTIFIED ON RIEMANN SURFACES [BENINI, AZ 15]

BLACK HOLE $\quad AdS_2 \times \Sigma_g \times SE_5$

microstates $\quad$ QM

$S(\Delta)$ entropy function

$\mathrm{Vol}(\varepsilon, \lambda) \Rightarrow S(\varepsilon, \lambda)$

"master volume"

[COUZENS-GAUNTLETT-MARTELLI-SPARKS 18]

$\lambda \to \lambda(\varepsilon, n)$

$\Delta \to \Delta(\varepsilon, n)$

$$\boxed{c(\Delta(\varepsilon, n)) \equiv S(\varepsilon, \lambda)}$$

[HOSSEINI-AZ 19]

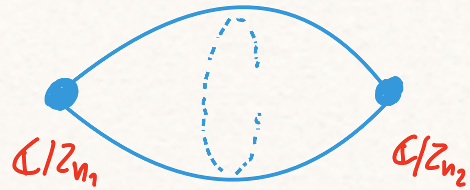
[GAUNTLETT-MARTELLI-SPARKS 19]

ALL THIS RECENTLY EXTENDED TO ORBIFOLDS

- LOCAL SINGULARITIES, ALSO IN CODIMENSION LESS THAN TWO

PROTOTYPE: THE SPINDLE $\mathbb{WP}_{[n_1, n_2]}$

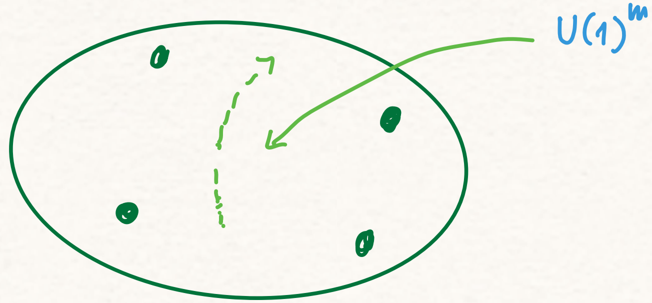
$$\boxed{(z_1, z_2) \sim (\lambda^{n_1} z_1, \lambda^{n_2} z_2) \qquad \lambda \in \mathbb{C}^*}$$



- DESPITE OBSCURE INTERPRETATION OF SINGULARITIES, QFT ON ORBIFOLDS EXIST ACCORDING TO HOLOGRAPHY AND DEFINE IR FIXED POINTS

[FERRERO - GAUNTLETT - IPIÑA - MARTELLI - SPARKS 11]
and many others

ALL THESE EXTREMAL FUNCTIONS FACTORIZE ON TORIC ORBIFOLDS



CFT COMPACTIFIED ON TORIC M

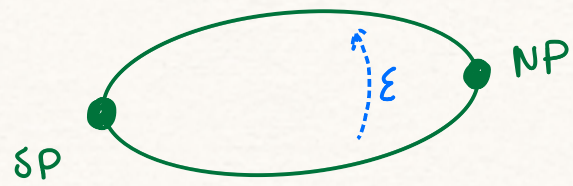
$$\boxed{F(\Delta, \varepsilon_1) = \sum_{\substack{\text{fixed} \\ \text{points}}} \frac{F_m(\Delta + \varepsilon_i^{(a)} p_i)}{\varepsilon_1^{(a)} \cdots\cdots \varepsilon_m^{(a)}}}$$

- GRAVITATIONAL BLOCKS F_m universal given the CFT
- gluing depends on details of the compactification

SMOKING GUN FOR EQUIVARIANT LOCALIZATION

ALL THESE EXTREMAL FUNCTIONS FACTORIZE ON TORIC ORBIFOLDS

CFT COMPACTIFIED ON SPINDLE



$$\boxed{F(\Delta,\varepsilon) = \frac{F_m(\Delta + \varepsilon p)}{\varepsilon} \pm \frac{F_m(\Delta - \varepsilon p)}{\varepsilon}}$$

- GRAVITATIONAL BLOCKS universal given the CFT
- gluing depends on details of the compactification

SMOKING GUN FOR EQUIVARIANT LOCALIZATION

Gravitational blocks

Blocks are universal:

	$\mathcal{B}(\Delta_i, \omega_a)$	universal $\mathcal{F}(\Delta_i)$	QFT interpretation (large N)
$\mathrm{AdS}_4 \times S^7$	$-\frac{\mathcal{F}(\Delta_a)}{\epsilon_1}$	$N^{3/2}\sqrt{\Delta_1\Delta_2\Delta_3\Delta_4}$	S^3-free energy
$\mathrm{AdS}_5 \times S^5$	$-\frac{\mathcal{F}(\Delta_a)}{\epsilon_1\epsilon_2}$	$N^2\Delta_1\Delta_2\Delta_3$	4d anomaly polynomial
$\mathrm{AdS}_6 \times_w S^4$	$-\frac{\mathcal{F}(\Delta_a)}{\epsilon_1\epsilon_2}$	$N^{5/3}(\Delta_1\Delta_2)^{3/2}$	S^5-free energy
$\mathrm{AdS}_7 \times S^4$	$-\frac{\mathcal{F}(\Delta_a)}{\epsilon_1\epsilon_2\epsilon_3}$	$N^3(\Delta_1\Delta_2)^2$	6d anomaly polynomial

[HOSSEINI–HRISTOV–AZ 09]

for orbifolds see

MARTELLI – FAEDO – FONTANAROSSA 21
BOIDO – GAUNTLETT – MARTELLI – SPARKS 22
GENOLINI – GAUNTLETT – SPARKS 23

EXTREMAL FUNCTIONS RELATED TO ANOMALIES ETC

- THEY SHOULD BE INDEPENDENT OF METRIC DETAILS, TOPOLOGICAL IN NATURE

- THEY SHOULD BE EQUIVARIANT

IS THERE ANY SIMPLE UNIVERSAL GEOMETRICAL OBJECT THAT CAPTURES ALL THESE PROPERTIES ?

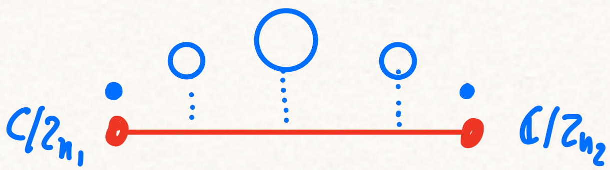
CONSIDER TORIC ORBIFOLDS

$$(M_{2m}, \omega) \qquad \omega = \sum_{i=1}^{m} dy_i \wedge d\phi_i \qquad \phi_i \to U(1)^m$$

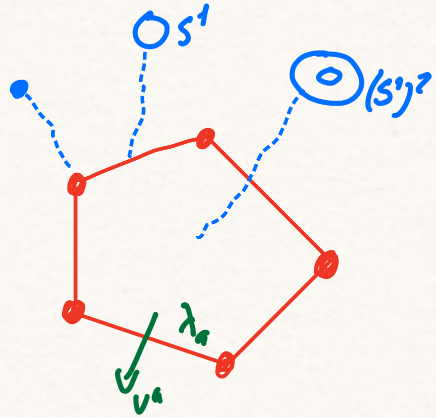
DELZANT THEOREM, generalized by LERMAN-TOLMAN

$$\boxed{M_{2m} \xrightarrow[y_i]{\qquad\qquad} P = \left\{ y \in \mathbb{R}^m \;\middle|\; v_i^a y_i - \lambda^a \geq 0 \right\}}$$

← CONVEX POLYTOPE



SPINDLE



INTRODUCE EQUIVARIANT PARAMETERS ε_i

$U(1)^m$ acts as $\xi = \varepsilon_i \frac{\partial}{\partial \phi_i}$ with Hamiltonian $H = \varepsilon_i y_i$

$(i_\xi \omega = -dH)$

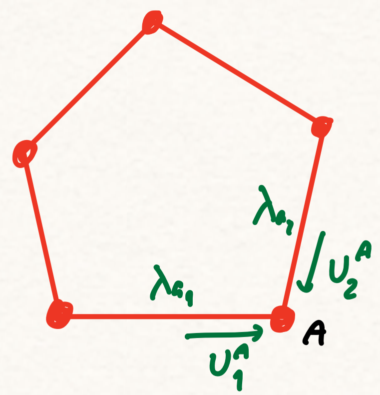
THE EQUIVARIANT VOLUME IS

$$\boxed{V(\lambda_a, \varepsilon_i) = \frac{1}{(2\pi)^m} \int_M e^{-H} \frac{\omega^m}{m!} = \int_P e^{-\varepsilon_i y_i} dy_1 \dots dy_m .}$$

TWO WAYS OF COMPUTING IT: EQUIVARIANT LOCALIZATION

Fixed point formula [BERLINE-VERGNE; DUISTERMAAT-ECKMAN; ATIYAH-BOTT, ...]

$$\boxed{V(\lambda,\varepsilon) = \sum_{\text{fixed}} \frac{e^{-\lambda_{a_i}\frac{\varepsilon U_A^i}{d_A}}}{d_A \prod_{i=1}^{m} \frac{\varepsilon \cdot U_A^i}{d_A}}}$$



d_A = order of orbifold sing. at fixed point A

U_A^i = orthogonal to V^a — along the edges

TWO WAYS OF COMPUTING IT: MOLIEN-WEYL FORMULA

symplectic quotient description: $M = \mathbb{C}^d /\!/ U(1)^{d-m}$

$$\sum_{a=1}^{d} Q_a^A v_i^a = 0$$

$$\boxed{V_{MW}(t,\bar{\epsilon}) = \int \prod_{A=1}^{d-m} \frac{d\phi_A}{2\pi} \, \frac{e^{\phi_A t_A}}{\prod_{a=1}^{d} \left(\bar{\epsilon}_a + \sum_A Q_a^A \phi_A\right)}}$$

used recently by [CASSIA-NEKRASOV-PIAZZALUNGA-ZABZINE 21]

RELATION AMONG THE TWO

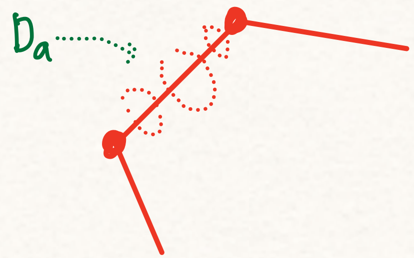
$$\boxed{V_{MW}\left(t_A = -\sum Q_A^a \lambda_a \,,\, \bar{\varepsilon}_a\right) = e^{\lambda_a \bar{\varepsilon}_a}\, V\left(\lambda_a ,\, \varepsilon_I = V_I^a \bar{\varepsilon}_a\right)}$$

- V_{MW} depends on $d-m$ t_A and d $\bar{\varepsilon}_a$
- V depends on d λ_a and m ε_i

- **COMPACT CASE**

$$\boxed{[\omega] = -2\pi \sum_{a=1}^{d} \lambda_a c_1(L_a)}$$

TORIC DIVISORS D_a WITH LINE BUNDLE L_a



$$\boxed{V(\lambda, \varepsilon) = \sum_{p} \frac{1}{p!} \sum_{a_1, \dots a_p} \lambda_{a_1} \cdots \lambda_{a_p} \int_M \underbrace{c_1^{eq}(L_{a_1}) \cdots c_1^{eq}(L_{a_p})}_{\text{equivariant intersections numbers}}}$$

polynomials in ε_i

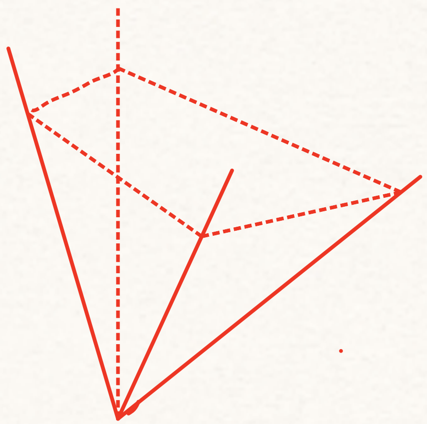
- **NON COMPACT CASE:** $V(\lambda,\varepsilon)$ is a rational function of ε_i

$$V(\lambda,\varepsilon) = \int_P e^{-\varepsilon_i y_i}\, dy_1 \cdots dy_n \neq 0$$

also for $\lambda = 0$

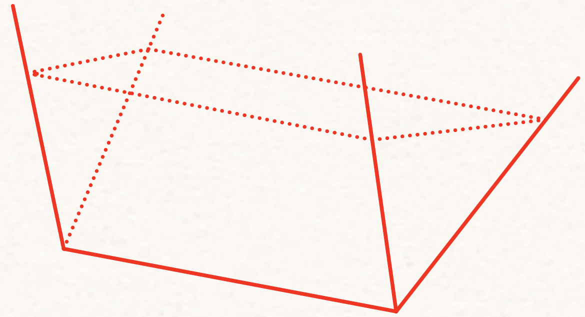
FOR CALABI-YAU CONES

P:



CONIFOLD

$\lambda = 0$



RESOLVED CONIFOLD

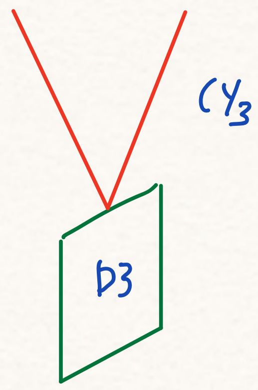
$\lambda \neq 0$

- INCORPORATE ALL EXTREMAL FUNCTION FOR AdS BRANE SOLUTIONS WITH CY_m

- $\boxed{V(\lambda=0,\varepsilon) = \text{SASAKIAN VOLUME}}$

(a-maximization)

$$a(\Delta(\varepsilon)) = V(0,\varepsilon)$$

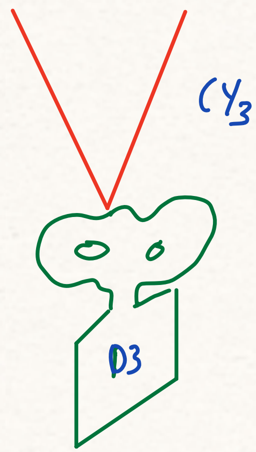


- $\boxed{V^{(m-1)}(\lambda,\varepsilon) = \text{MASTER VOLUME}}$

PIECE OF DEGREE $m-1$ IN λ_a

(c-extremization)

$$c(\Delta(\varepsilon,\lambda)) = V^{(m-1)}(\lambda,\varepsilon)$$



- FOR EXAMPLE, FOR ALL KNOWN D2, M2, D3, D4, M5 BRANE SOLUTIONS COMPACTIFIED ON S^2 OR A SPINDLE

$$AdS_d \times \underbrace{M}_{\text{toric geometry}} \qquad \text{with fluxes } M_a$$

$$\boxed{\begin{aligned} F(\varepsilon) &= V^{(p)}(\xi, \lambda) \\ M_a &= \frac{\partial V^{(q)}(\xi, \lambda)}{\partial \lambda_a} \end{aligned}}$$

- MOREOVER FOR FIBRATIONS

$$X = CY_m \longrightarrow M_{2n}$$

$$(\varepsilon_1, \dots, \varepsilon_{m+n}) \equiv (\varepsilon_1, \dots, \varepsilon_n) \oplus (\tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_n)$$

$$\boxed{V(\lambda, \varepsilon) = \sum_{\substack{\text{fixed}\\ \text{points}\\ \text{of } M}} \frac{V_{CY_m}\left(\varepsilon_i + \alpha_i^j \tilde{\varepsilon}_j^{(p)}, \lambda_a + \beta_a^b \lambda_b^{(p)}\right)}{\tilde{\varepsilon}_1^{(p)} \dots \dots \tilde{\varepsilon}_n^{(p)}}}$$

thus explaining the ubiquitous factorization in blocks

CONCLUSIONS

- THE EQUIVARIANT VOLUME IS THE KEY OBJECT FOR ALL EXTREMAL PROBLEMS FOR SUPERSYMMETRIC GEOMETRIES WITH AN HOLOGRAPHIC DUAL

- IT IS TOPOLOGICAL IN NATURE: NO NEED OF EXPLICIT METRIC OR DETAILS: IN TORIC CASE, WE JUST NEED TORIC DATA

- ALL COMPUTATIONS OF INTEGRATED ANOMALIES ON ORBIFOLDS APPEARED IN THE LITERATURE CAN BE REFORMULATED IN TERMS OF EQUIVARIANT LOCALIZATION

- EQUIVARIANT NATURE EXPLAINS ALL FACTORIZATION PROPERTIES FOR EXTREMAL/ENTROPY FUNCTIONS