Black hole cohomologies in $\mathcal{N}=4$ SYM

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Talk based on collaborations with

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"The shape of non-graviton operators for SU(2)" arXiv:2209.12696.

"Towards quantum black hole microstates" arXiv.2304.10155.

See also:

- Chi-Ming Chang, Ying-Hsuan Lin,
  "Words to describe a black hole" arXiv:2209.06728.

Introduction

Better understanding black hole microstates:

- Enumeration: $S_{BH} = A/4G = \log(\text{microstates})$
- Constructing & better characterizing the individual microstates?

AdS black hole microstates from CFT:

- Requires strong coupling QFT calculations: Hard in general
- BPS black holes: Easier, but still very hard to construct exact BPS operators.

I will explain a modest version of constructing BPS black hole microstates.

- 4d maximal SYM, in terms of certain classical cohomologies.
- Want to eventually study $SU(N \gg 1)$. $\leftrightarrow$ But today, will report SU(2) (& perhaps SU(3)).
- Explore qualitative features & rough comparison with the “gravity dual”

The operators I present should have more general lessons beyond black holes.

- If you are familiar with chiral rings, SQCD & mesons/baryons, etc., try to compare them with our new ones and find similarities/differences.

N=4 Yang-Mills & BPS operators

SU(N) maximal SYM on R^4:

- Fields: adjoint representation, i.e. $N \times N$ matrices (written in N=1 language)

  3 chiral multiplets: $\phi_m(x), \bar{\phi}^m(x)$ and $\psi_{m\alpha}, \bar{\psi}^m_{\dot{\alpha}}$ $(m = 1,2,3)$

  vector multiplet: $A_\mu(x) \sim A_{\alpha\dot{\beta}}$ and $\lambda_\alpha, \bar{\lambda}_{\dot{\alpha}}$ $(\mu = 1, \cdots, 4)$ $(\alpha = \pm, \dot{\alpha} = \dot{\pm})$

- Supercharges: Poincare $Q^i_\alpha, \bar{Q}_{i\dot{\alpha}}$ & conformal $S_i^\alpha = (Q^i_\alpha)^\dagger, \bar{S}^{i\dot{\alpha}} = (\bar{Q}_{i\dot{\alpha}})^\dagger$ $(i = 1, \cdots, 4)$

Gauge-invariant local BPS operators: (at $x^\mu = 0$ on R^4)

- Pick $Q \equiv Q^4_-$, $S \equiv S_4^- = Q^\dagger$: Invariant operators satisfy $[Q, O(0)\} = [Q^\dagger, O(0)\} = 0$.
- Generally hard to construct. Easier at weak coupling.

- Free limit ($g_{YM} \to 0$): Trivially constructed with invariant fields under Q, S :

  $\bar{\phi}^m \equiv \bar{\phi}^m,\ \psi_{m+},\ \bar{\lambda}_{\dot{\alpha}},\ f_{++} \equiv F_{1+i2\,,\,3+i4}$ & derivatives $\partial_{1+i2} \equiv \partial_1 - i\partial_2$, $\partial_{3+i4} \equiv \partial_3 - i\partial_4$ acting on them

- Not all of them are invariant when $g_{YM} \neq 0$: At small $g_{YM} \ll 1$,

  $Q\,\bar{\phi}^m = 0\ ,\ Q\psi_{m+} \sim g_{YM}\epsilon_{mnp}[\bar{\phi}^n, \bar{\phi}^p]\ ,\ Qf_{++} \sim g_{YM}\sum_m [\psi_{m+}, \bar{\phi}^m]\ ,\ Q\,\bar{\lambda}_{\dot{\alpha}} = 0\ ,\ [Q, D_{+\dot{\alpha}}] \sim g_{YM}[\,\bar{\lambda}_{\dot{\alpha}}\,,\ \}$

  $\to$ Q & S at ½-loop → Anomalous dimension $QQ^\dagger + Q^\dagger Q \sim E - E_{BPS}$ at 1-loop, $O(g_{YM}^2)$.

The cohomology problem

The supercharges are nilpotent, $Q^2 = 0$, $(Q^\dagger)^2 = 0$

→ The equation $[QQ^\dagger + Q^\dagger Q, O(0)] = 0$ is formally like that for the harmonic form

1-to-1 map: harmonic forms ↔ Q-cohomology class:

- Local operator $\tilde{O}(0)$ satisfying $Q\tilde{O}(0) = 0$, with equivalence $\tilde{O} \sim \tilde{O} + Q\Lambda$.

This is generally NOT the physical BPS state. (addition of Q-exact terms)

- Apparently, just tells us the information on the BPS spectrum.
- Still, it provides more information than the index.
- Perhaps there may be more information insensitive to the Q-exact terms…?

Classical (weak-coupling) problem vs. black holes (strong-coupling) ?

- Perturbative non-renormalization proven (w/ certain assumptions) [Chang, Lin] (2022)
- The index counts cohomologies & captures black holes. [Cabo Bizet, Cassani, Martelli, Murthy] [Choi, J. Kim, SK, Nahmgoong] [Benini, Milan] (2018) → At least some of them are protected.

Gravitons vs. black holes

Two different classes of cohomologies:

- Gravitons & all the rest: The latter could possibly be "black hole" type.
- "Gravitons" in practice: (well-defined even at finite N)
  1) Construct single-trace (~single-particle) cohomologies:

     → Chiral primaries $\mathrm{tr}[\bar{\phi}^{(m_1} \cdots \bar{\phi}^{m_n)}]$ & their superconformal descendants (in PSU(1,2|3))
  2) Construct multi-trace (~multi-particle) cohomologies by multiplying them.

True "harmonic forms" are not multiplicative, but cohomologies are.

- Mutually BPS objects are often "multiplied" or "superposed" (subject to further corrections).
- Cohomology realizes the "superpositions" of BPS multi-gravitons trivially. (More later)

"Gravitons at finite N" ?: trace relations in QFT ↔ giant gravitons in gravity

- Subtracting these, we wish to study "quantum" black hole operators for "quantum" gravity.
- Newton constant, controlling the quantumness of gravity: $G_N \sim (\text{radius of AdS})^3/N^2$

The problem & progress

The problem at finite N:

- Grade operators with a charge w/ lower bound: Like energy, or in our studies

$$j \equiv 6(R+J) = 2(R_1+R_2+R_3)+3(J_1+J_2) \geq 0.$$

- At fixed j, construct all "Q-closed", remove "Q-exact" & remove gravitons: ∃remainders?
- Increase j and repeat: E.g. has been performed till $j \leq 25$ for SU(2). [Chang, Lin] (2022)

SU(N ≥ 3) → No progress reported so far. (Some works in progress…)

SU(2) → Progress since last September. [Chang, Lin] [Choi, E. Lee, SK, Park] (2022)

- Streamlined studies [Choi, Eunwoo Lee, Siyul Lee, SK, Park] (2023) :

  Compute the index over black hole cohomologies to detect them first:

$$\begin{aligned} Z(t) &= 1+6t^4-6t^5-7t^6+18t^7+6t^8-36t^9+6t^{10}+84t^{11}-80t^{12}-132t^{13}+309t^{14}-18t^{15}-567t^{16} \\ &\quad +516t^{17}+613t^{18}-1392t^{19}-180t^{20}+2884t^{21}-1926t^{22}-4242t^{23}+7890t^{24}+792t^{25}-15876t^{26} \\ &\quad +13804t^{27}+15177t^{28}-37536t^{29}+7049t^{30}+57522t^{31}-58704t^{32}+\cdots \\ Z_{\rm grav}(t) &= 1+6t^4-6t^5-7t^6+18t^7+6t^8-36t^9+6t^{10}+84t^{11}-80t^{12}-132t^{13}+309t^{14}-18t^{15}-567t^{16} \\ &\quad +516t^{17}+613t^{18}-1392t^{19}-180t^{20}+2884t^{21}-1926t^{22}-4242t^{23}+7891t^{24}+786t^{25}-15864t^{26} \\ &\quad +13804t^{27}+15138t^{28}-37476t^{29}+7048t^{30}+57414t^{31}-58566t^{32}+\cdots \end{aligned}$$

$$\boxed{Z-Z_{\rm grav} = -t^{24}+6t^{25}-12t^{26}+0t^{27}+39t^{28}-60t^{29}+t^{30}+108t^{31}-138t^{32}+\cdots}$$

The threshold operator

A representative of the first non-graviton cohomology at $j = 24$.

- The “threshold” cohomology [Chang, Lin] [Choi, SK, E. Lee, Park] [Choi, SK, E. Lee, S. Lee, Park]:

$$O_0 \equiv \epsilon^{p_1 p_2 p_3} v^m{}_{p_1} v^n{}_{p_2} (\psi_m \cdot \psi_n \times \psi_{p_3})$$

$$v^m{}_n \equiv (\phi^m \cdot \psi_n) - \frac{1}{3}\delta^m_n (\phi^p \cdot \psi_p)$$

[Used 3d vector notation for SU(2) adjoints: $A \cdot B \sim \mathrm{tr}(AB)$ and $A \times B \sim [A, B]$.]

One may speculate it as the “smallest black hole” in the “most quantum AdS/CFT”

- Entropy is $S = \log 1 = 0$. Not like semi-classical black holes at all.
- Unclear to what extent it behaves like a black hole, if any.
- Not all aspects of semi-classical black holes are respected, but some seem to be.

To better appreciate the last point, helpful to study the higher order terms:

- It apparently looks like there are many non-graviton states at $j > 24$.
- But most of them below are superconformal descendants of O_0.

$$Z - Z_{\mathrm{grav}} = -t^{24} + 6t^{25} - 12t^{26} + 0t^{27} + 39t^{28} - 60t^{29} + t^{30} + 108t^{31} - 138t^{32} + \cdots$$

A no-hair theorem?

Superconformal representation of the threshold operator:

- Cohomology problem has $PSU(1,2|3) \subset PSU(2,2|4)$ symmetry, after picking Q, S.
- O_0 at $j = 24$ is the primary of a PSU(1,2|3) rep.
- The index over this rep. & the remainder:

$$\chi_0(t) = -t^{24} + 6t^{25} - 12t^{26} + 0t^{27} + 39t^{28} - 60t^{29} + t^{30} + 108t^{31} - 135t^{32} + \cdots$$

$$Z - Z_{\text{grav}} - \chi_0(t) = -3t^{32} + \cdots$$

There is a "boring" range $25 \leq j \leq 31$, which in fact is quite novel.

- O_0 x (graviton) may yield new cohomologies. But most of them are not seen in the index.
- Simplest possibility: All Q-exact (i.e. absent) ← Checked explicitly for many (next slide).
- Signals a black hole no-hair theorem: "No extra graviton hairs can dress a black hole."

A "partial no-hair theorem" in the index

- "$-3\ t^{32}$" is the product $tr\left(2\bar{\phi}^m f + \epsilon^{mnp}\psi_n\psi_p\right) O_0$: limited "hairy BH operators".
- Conformal primaries of gravitons: 29 of 32 dressing O_0 do not appear in the index.
- Conformal descendants…? (More later)

Illustration: Q-exactness

t^{28}:

$$O_0(\bar\phi^{(m}\cdot\bar\phi^{n)}) = -\frac{1}{14}Q[20\epsilon^{rs(m}(\bar\phi^{n)}\cdot\psi_{p+})(\bar\phi^p\cdot\psi_{r+})(\bar\phi^q\cdot\psi_{q+})(f_{++}\cdot\psi_{s+})$$
$$-20\epsilon^{prs}(\bar\phi^{(m}\cdot\psi_{p+})(\bar\phi^{n)}\cdot\psi_{r+})(\bar\phi^q\cdot\psi_{q+})(f_{++}\cdot\psi_{s+})$$
$$+30\epsilon^{prs}(\bar\phi^{(m}\cdot\psi_{p+})(\bar\phi^{n)}\cdot\psi_{r+})(\bar\phi^q\cdot\psi_{s+})(f_{++}\cdot\psi_{q+})$$
$$-7\epsilon^{a_1a_2p}\epsilon^{b_1b_2(m}(\bar\phi^{n)}\cdot\psi_{p+})(\bar\phi^q\cdot\psi_{q+})(\psi_{a_1+}\cdot\psi_{a_2+})(\psi_{b_1+}\cdot\psi_{b_2+})$$
$$+18\epsilon^{a_1a_2p}\epsilon^{b_1b_2(m}(\bar\phi^{n)}\cdot\psi_{q+})(\bar\phi^q\cdot\psi_{p+})(\psi_{a_1+}\cdot\psi_{a_2+})(\psi_{b_1+}\cdot\psi_{b_2+})]$$

t^{29}:

$$O_0(\bar\phi^m\cdot\bar\lambda_{\dot\alpha}) = \frac{1}{8}Q[40\epsilon^{mnp}(f_{++}\cdot\psi_{q+})(\bar\lambda_{\dot\alpha}\cdot\psi_{r+})(\bar\phi^q\cdot\psi_{n+})(\bar\phi^r\cdot\psi_{p+})$$
$$-4\epsilon^{ma_1a_2}\epsilon^{nb_1b_2}(\bar\lambda_{\dot\alpha}\cdot\psi_{n+})(\bar\phi^p\cdot\psi_{p+})(\psi_{a_1+}\cdot\psi_{a_2+})(\psi_{b_1+}\cdot\psi_{b_2+})$$
$$+6\epsilon^{ma_1a_2}\epsilon^{nb_1b_2}(\bar\lambda_{\dot\alpha}\cdot\psi_{p+})(\bar\phi^p\cdot\psi_{n+})(\psi_{a_1+}\cdot\psi_{a_2+})(\psi_{b_1+}\cdot\psi_{b_2+})$$
$$+\epsilon^{na_1a_2}\epsilon^{pb_1b_2}(\bar\lambda_{\dot\alpha}\cdot\psi_{n+})(\bar\phi^m\cdot\psi_{p+})(\psi_{a_1+}\cdot\psi_{a_2+})(\psi_{b_1+}\cdot\psi_{b_2+})]$$

t^{30}:

$$O_0\left(\bar\phi^m\cdot\psi_{n+}-\tfrac{1}{3}\delta^m_n\bar\phi^p\cdot\psi_{p+}\right)$$
$$=\frac{1}{4}Q\left[\epsilon_{npq}\epsilon^{ra_1a_2}\epsilon^{qb_1b_2}\epsilon^{mc_1c_2}(\bar\phi^p\cdot\psi_{r+})(\psi_{a_1+}\cdot\psi_{a_2+})(\psi_{b_1+}\cdot\psi_{b_2+})(\psi_{c_1+}\cdot\psi_{c_2+})\right]$$

The BMN subsector

Even for SU(2), computations take long time (especially Z_{grav}).

∃subsector containing $\bar{\phi}^m, \psi_m, f$ (no derivatives and gauginos):

- $Q\bar{\phi}^m = 0$, $Q\psi_m \sim \epsilon_{mnp}[\bar{\phi}^n, \bar{\phi}^p]$, $Qf \sim \sum_m [\bar{\phi}^m, \psi_m]$.
- BMN matrix model truncation of SYM. [Berenstein, Maldacena, Nastase] [Plefka, N. Kim, Klose]

Result in this sector: [Choi, E. Lee, S. Lee, SK, Park]

$$[Z(t) - Z_{grav}(t)]_{BMN} = -\frac{t^{24}}{1-t^{12}} \cdot (1-t^2)^3 \cdot \frac{1}{(1-t^8)^3}$$

dressing by gravitons $tr(2\bar{\phi}^m f + \epsilon^{mnp}\psi_n\psi_p)$ (only 3 out of 17 gravitons in BMN sector) — for $\frac{1}{(1-t^8)^3}$

series of "core black hole" primary operators — for $\frac{t^{24}}{1-t^{12}}$

superconformal descendants within BMN — for $(1-t^2)^3$

- The ∞-tower of "core" primaries (not of the "BH x graviton" form)

$$\begin{aligned} O_n &= (f\cdot f)^n \epsilon^{c_1c_2c_3}(\phi^a\cdot\psi_{c_1})(\phi^b\cdot\psi_{c_2})(\psi_a\cdot\psi_b\times\psi_{c_3}) \\ &+n(f\cdot f)^{n-1}\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}(f\cdot\psi_{b_1})(\phi^a\cdot\psi_{c_1})(\psi_{b_2}\cdot\psi_{c_2})(\psi_a\cdot\psi_{b_3}\times\psi_{c_3}) \\ &-\left(\frac{n}{72}+\frac{n(n-1)}{108}\right)(f\cdot f)^{n-1}\epsilon^{a_1a_2a_3}\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}(\psi_{a_1}\cdot\psi_{b_1}\times\psi_{c_1})(\psi_{a_2}\cdot\psi_{b_2}\times\psi_{c_2})(\psi_{a_3}\cdot\psi_{b_3}\times\psi_{c_3}) \end{aligned}$$

- Entropically not that many, but they all respect partial no-hair behaviors in the index

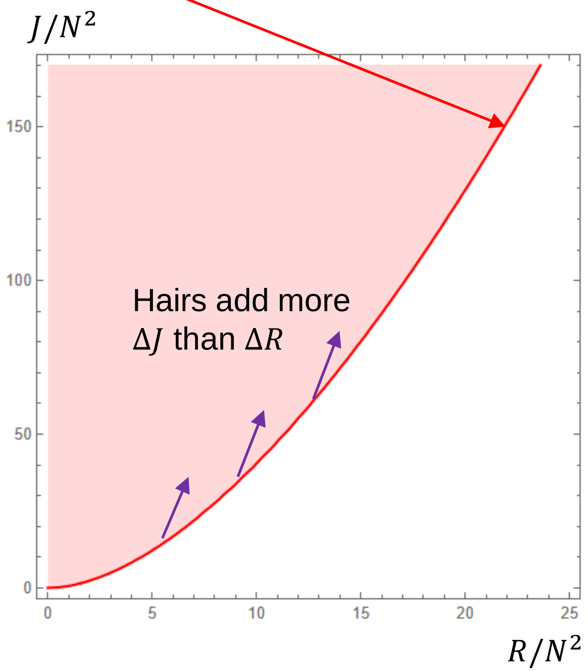
The "gravity dual"

Now, instead of $N = 2$ that we studied so far, we study $N = \infty$.

BPS black hole solutions in $AdS_5 \times S^5$: [Gutowski, Reall] (2004)

- Exists only when a charge relation is met.

$$R^3 + \frac{N^2}{2}J^2 - \left(3R + \frac{N^2}{2}\right)(3R^2 - N^2 J) = 0$$



Scalar hair: Φ dual to $tr(X^2 + Y^2 + Z^2)$:

- We found no-hair behavior for this operator in QFT (s-wave)
- Can we turn on small hair, $\Phi(x) \sim \varepsilon \ll 1$, without substantially changing the background at leading order in ε?
- In other words, we try to "multiply" these gravitons to BH.
- Solution to BPS equation:

$$\Phi(x, \theta, \phi, \psi) = \varepsilon x^{\frac{m-2q/\ell^2}{1+3q/\ell^2}} \left(1 + \frac{3q}{\ell^2} + \frac{x}{\ell^2}\right)^{-\frac{1+m+q/\ell^2}{1+3q/\ell^2}} \left(\cos\frac{\theta}{2} e^{i\phi_1}\right)^{m_1} \left(\sin\frac{\theta}{2} e^{i\phi_2}\right)^{m_2}$$

$$m_1 + m_2 = 2m \qquad m_1, m_2 = 0, 1, 2, \cdots$$

- Singular at event horizon $x = 0$ for $m < 2q/\ell^2$.
- Including "s-wave" (~conformal primary) at $m = 0$.
- Regular perturbative hairs allowed only for conformal descendants.

Hairy BPS black holes

With Φ, hairy BPS black holes are studied. [Markeviciute, Santos] [Markeviciute] (2018)

- Studied "s-wave" sector. Φ at s-wave always back-reacts heavily to BH, even at $\varepsilon \ll 1$.
- Induces (mild) singularity at the horizon.
- Doesn't look like "superposing" or "multiplying" gravitons to BH.

Very crude comparisons & lessons

1) Over-rotating hairs:

- "Dress" black holes in the traditional spirit of "hairs"

  Similar to what we found in SU(2). (Except the partial hair at $-3t^{32}$ and O_1 at t^{36}, all the rest till $j \leq 38$ can be explained as O_0 times conformal descendant gravitons.)
- Over-rotating hairy solutions can be constructed even beyond BPS limit: Hairs back-react weakly, basically "multiplied" or "superposed". [SK, Kundu, E. Lee, J. Lee, Minwalla, Patel] (2023)

2) Under-rotating hairs:

- Want to "back-react" substantially to the background BH.
- Not admitting small graviton hairs dressing the BH. More studies needed.

Conclusion

Recent progress on AdS black holes from exact QFT observable.

Today, I explained a tangential program of "constructing" individual microstates.

- Weak-coupling cohomology problem
- Technical strategies: First count finite N gravitons & subtract from the index

  BMN matrix model subsector
- Higher $SU(N)$? Higher charges? Partial progress for SU(3):

  Use of Groebner basis to count gravitons;

  Identified BH threshold level [work in progress → by my students Jae Hyeok Choi & Jehyun Lee]
- Ideas/techniques from: computer science, algebraic geometry, quantum information, …
- Insights from the emergent structures in the twisted sector? [Costello, Gaiotto], ……

Difference of over-/under-rotating hairy BH's & similarities with $SU(2)$ cohomologies.

Some challenging questions on black holes may be better addressed.

- We already see a hint of the black hole "no-hair" behaviors.
- Black hole interior? Quantum complexity? ……