Worldsheet instantons in holography

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Precision holography
June 6th, 2023

Based on [2304.12340] with Valentina Giangreco M. Puletti and Jesse van Muiden

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Motivation

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Let me review two of these formulae as a motivation for the holographic study we will be carrying out for the rest of the talk.

MOTIVATION

First is the 1/2 BPS (circular) Wilson loop in planar $\mathcal{N} = 4$ SYM

Erickson, Semenoff, Zarembo (2000)
Pestun (2007)

$$\langle \mathcal{W} \rangle = \frac{2N}{\sqrt{\lambda}} I_1\big(\sqrt{\lambda}\big) + \mathcal{O}(1/N)\,,$$

where λ is the 't Hooft coupling and N is the rank of the gauge group.

Motivation

Similarly a "simple" answer exists for the S^3 partition function of the ABJM theory

Fuji, Hirano, Moriyama (2011), Mariño, Putrov (2011)

$$Z_{S^3} = C^{-1/3} \mathrm{e}^{A} \mathrm{Ai}\Big[C^{-1/3} (N - \frac{k}{24} - \frac{1}{3k}) \Big] \,,$$

where $C = 2/(k\pi^2)$, and A is a complicated function of k. This is the exact perturbative answer.

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The Wilson loop (and other observables) can similarly expressed in terms of the Airy function.

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Is there some analogous tool to localization on the string side?

Outline

- String saddle point expansion.
- Worldsheet instantons in 5D SYM.
- Worldsheet instantons in ABJ(M).
- Summary.

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The integration over the zero-modes with the contribution of the non-zero-modes results in the 10D supergravity action

Fradkin-Tseytlin ('85), Tseytlin ('88,'89,'07)

$$\mathcal{Z}_{\text{string}} \approx -S_{\text{sugra}} + \sum_{\text{instantons}} \mathrm{e}^{-S_{\text{cl}}} Z_{\text{1-loop}} .$$

WORLDSHEET INSTANTONS

The leading term in this equation reproduces (after regularization) the holographic free energy $F_{\rm QFT} = -\log Z_{\rm QFT}$ which suggests the general expression

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It should be noted that there are other (pertubative) corrections to this formula in both α' and g_s. These are responsible e.g. for higher derivative corrections to the supergravity action. Furthermore, we will only focus on the genus 0 partition function.

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Instead we will focus on the two canonical M-theory examples: ABJM and the (2,0) theory in the appropriate type IIA limit.

5D maximal SYM

LOCALIZATION

Consider the Euclidean maximal SYM in five dimensions on the sphere. This theory can be localized to a matrix model a la Pestun which allows us to evaluate the free energy in the large N limit.

Kim, Kim (2012)

$$F = -N^2\left(\frac{\xi}{6} - \frac{\pi^2}{6\xi} + \frac{\zeta(3)}{\xi^2} - \frac{\mathrm{Li}_3(\mathrm{e}^{-\xi})}{\xi^2}\right) + \mathcal{O}(N\log N)\,.$$

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These are dual to worldsheet instantons.

Holographic dual

Bobev, Bomans, FFG (2018)

The holographic dual is the backreaction of D4 branes on S^5. It was found in six-dimensional supergravity and then uplifted to 10D. Can also be found by dimensionally reducing $\mathrm{AdS}_7 \times S^4$

$$\mathrm{d}s^2_{10} = \ell_s^2 (N\pi \mathrm{e}^{\Phi})^{2/3} \Big[\frac{4(\mathrm{d}\sigma^2 + \mathrm{d}\Omega_5^2)}{\sinh^2\sigma} + \mathrm{d}\theta^2 + \cos^2\theta \mathrm{d}\Omega_2^2 + \frac{\sin^2\theta}{1 - \frac{1}{4}\tanh^2\sigma \sin^2\theta} \mathrm{d}\phi^2 \Big] \,,$$

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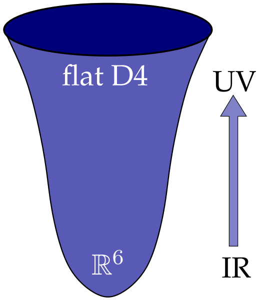
This background exhibits $\mathrm{SU}(4|2)$ symmetry just like the QFT.

Spherical D4 solution



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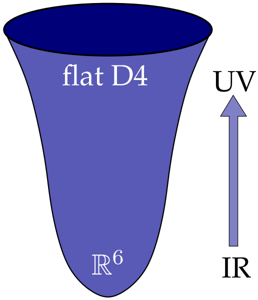

Evaluating the renormalized (6D) supergravity action we obtain a leading order match with the QFT answer for the free energy obtained by localization

Bobev, Bomans, FFG, Minahan, Nedelin (2019)

$$S^{\text{reg.}}_{\text{on-shell}} = -\frac{\xi N^2}{6}\,.$$

Worldsheet instantons

Have to extremize the string action

$$S_{\text{cl}} = \mathcal{A} \pm \mathcal{B} = \frac{1}{2\pi\ell_s^2} \int \sqrt{\gamma} \pm \frac{i}{2\pi\ell_s^2} \int B_2 \,,$$

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We have found a tower of worldsheet instantons whose exponential dependence matches the QFT expectation.

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$$S_{\text{FT}} = \frac{1}{4\pi} \int \sqrt{\gamma} \Phi R_\gamma \,.$$

The string ghosts are cancelled by longitudinal fluctuation of the string modes leaving a universal contribution. These together with other measure factors are collected into the universal factor $C(\chi)$.

Drukker, Gross, Tseytlin (2000), Giombi, Tseytlin (2020)

Physical fields

We get the following spectrum of fields

Field	Degeneracy	M^2L^2
scalars	6	$\frac{1}{4}$
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$$\mathrm{Sdet}'\mathbb{K} = \frac{\prod_b (\det' L^2\mathcal{K})}{\prod_f (\det' L\mathcal{D})},$$

where $\mathcal{K}$ and $\mathcal{D}$ are the standard Klein-Gordon and Dirac operators on S^2 with radius L.

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Using ζ-function regularization, we find

$$(\mathrm{Sdet}'\mathbb{K})^{-1/2} = (\mathrm{Sdet}\mathbb{K})^{-1/2} = -4 \,.$$

The dilaton

Recall that the 10D dilaton was

$$\mathrm{e}^{\Phi} = \frac{\xi^{3/2}}{N\pi}\big(\coth^2\sigma - \tfrac{1}{4}\sin^2\theta\big)^{3/4} \to \frac{\xi^{3/2}}{N\pi}\,.$$

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Since $\chi = 2$, we have

$$\mathrm{e}^{-S_{\text{FT}}} = \mathrm{e}^{-\chi\Phi} = \frac{N^2\pi^2}{\xi^3}\,.$$

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$$C(1) = \frac{\sqrt{-\mathcal{A}}}{2\pi} .$$

By a comparison to the QFT prediction we find

$$C(2) = \frac{\mathcal{A}}{8\pi^2} .$$

We will verify this answer by performing a separate check.

SUMMARY

If we collect all pieces we recover the field theory answer for the rank one instanton

$$\begin{aligned} Z_{\text{1-loop}} &= \mathrm{e}^{-S_{\text{FT}}} C(\chi) (\mathsf{Sdet}' \mathbb{K})^{-1/2} Z_{\text{zero-modes}} \\ &= \frac{N^2 \pi^2}{\xi^3} \frac{(2\xi)}{8\pi^2} (-4) \\ &= -\frac{N^2}{\xi^2} \,. \end{aligned}$$

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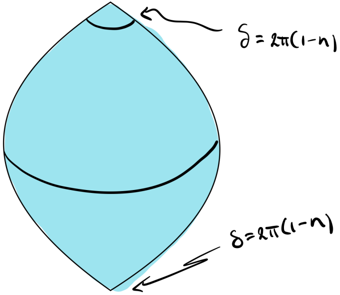
What about higher rank instantons?

Higher rank instantons

For rank n instanton at one-loop level we have to compute the partition function on an orbifold where the two poles of the S^2 now have angular excess $\delta = 2\pi(1-n)$.



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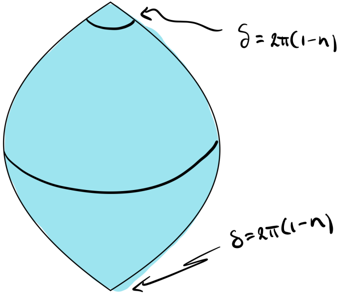
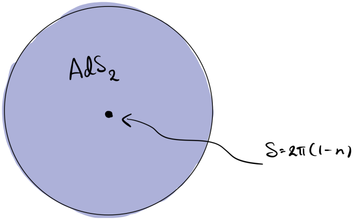


Most of the calculation should go through the same way since the spectrum of fields has not changed.

Higher rank instantons

A very similar story occurs for the higher rank Wilson loop in say AdS_5.

Kruczenski, Tirziu (2008), Buchbinder, Tseytlin (2014), Bergamin, Tseytlin (2015), Forini, Tseytlin, Vescovi (2017)



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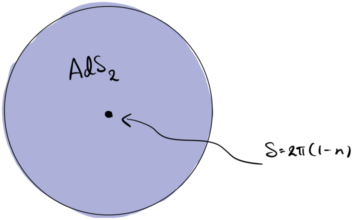

Despite many attempts, a match with the QFT has not been reached. Possibly because ghosts play an important role.

Higher rank instantons

The QFT answer is however very suggestive. It is as if the orbifold just affects the answer 'locally' by a multiplicative factor:

$$Z^{(n)}_{\text{1-loop}} = Z^{(1)}_{\text{1-loop}} z_n = \frac{Z^{(1)}_{\text{1-loop}}}{n^{3/2}} ,$$

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For a spherical worldsheet we should then get

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We have checked this conjecture for a number of examples, but we do not have a proof.

Summary, part 2

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Summing over the instantons we find a perfect, non-perturbative match with the QFT

$$\sum_{n=1}^{\infty} Z^{(n)}_{\text{string}} = -\frac{N^2}{\xi^2}\mathrm{Li}_3(\mathrm{e}^{-\xi})\,.$$

ABJ(M)

ABJ(M) AND TOPOLOGICAL STRING

Kapustin, Willet, Yaakov, Marino, Putrov, Drukker, Hatsuda, Moriyama, Okuyama, Grassi, Kallen, ... (many many papers)

In the remaining time, we will take a look at instanton corrections to the ABJ(M) theory ($\mathrm{U}(N+l)_k \times \mathrm{U}(N)_{-k}$ Chern-Simons theory in 3D) in the type IIA limit ($N \gg 1$, $k \gg 1$, $\lambda = N/k =$fixed)

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The perturbative free energy is given by the Airy function, but there is an infinite series of non-perturbative corrections:

$$F = F_{\text{pert.}} + \sum c_{m,n} \mathrm{e}^{-\frac{nN2\pi}{\sqrt{2\lambda}} - 2\pi m\sqrt{2\lambda}}$$

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These are dual to D2 and fundamental string instantons in the dual geometry.

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The explicit coefficients $c_{m,n}$ can be computed using the localization matrix model, in particular for (a single) worldsheet instanton:

$$F_{\mathrm{inst}}^{(1)} = \frac{N^2}{(2\pi\lambda)^2} \cos\left(\frac{2\pi l}{k}\right) \mathrm{e}^{-2\pi\sqrt{2\lambda}}\,.$$

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Higher rank instantons are given in terms of Gopakumar-Vafa invariants on $\mathbb{P}^1 \times \mathbb{P}^1$.

Worldsheet instantons in $\mathrm{AdS}_4 \times \mathbf{C}P_3$

The holographic dual to ABJ(M) in the type IIA limit is

$$\mathrm{d}s_{10}^2 = L^2\left(\mathrm{d}s_{\mathrm{AdS}_4}^2 + 4\mathrm{d}s_{\mathbf{C}P_3}^2\right),$$

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Note: Both string and antistring is allowed, we should sum over both.

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spectrum of fluctuations

Field	Degeneracy	M^2L^2	q
scalars	4	0	0
	2	$-\frac{1}{2}$	1
	2	$-\frac{1}{2}$	-1
fermions	4	-1	0
	2	0	1
	2	0	-1

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There is an exact cancellation between the non-zero-modes

$$(\mathrm{Sdet}'\mathbb{K})^{-1/2} = 1\,,$$

Mass deformed ABJ(M)

In order to lift the zero-modes we deform the ABJ(M) theory on S^3 by modifying the R-charge assignment of the bifundamental chiral multiplets:

$$R[X^1] = \frac{1}{2}+2c\,, \quad R[X^2] = \frac{1}{2}-2c\,, \quad R[Y_1] = \frac{1}{2}-2c\,, \quad R[Y_2] = \frac{1}{2}+2c\,.$$

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Putting all together (with the measure factor $C(2)$!) we find an exact match with the QFT prediction.

SUMMARY AND OUTLOOK

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- We have been able to reproduce a large part of the QFT answer for the S^5 free energy:

$$F_{S^5} = -N^2\left(\frac{\xi}{6} - \frac{\pi^2}{6\xi} + \frac{\zeta(3)}{\xi^2} - \frac{\mathrm{Li}_3(\mathrm{e}^{-\xi})}{\xi^2}\right) + \mathcal{O}(N\log N)$$

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- The other two terms should be reproduced by higher derivative terms in supergravity.
- We fixed the measure factor for genus zero string partition function using holography. We checked this by comparing with ABJM, but is there a derivation (see Giombi, Tseytlin (2023))?
- We also conjectured the contribution of orbifolds to our answer. It passes many tests but we are lacking a proof. A very similar result is available for the topological string.

Summary and Outlook

- For ABJM, we managed to recover exactly the rank 1 worldsheet instanton answer.
- For higher rank instantons we have a partial answer, there seem to be more intricate solutions at higher rank involving strings and antistrings.
- We have looked at two generalizations of ABJM, mass-deformed and orbifolded. There are partial answers known in the literature, but we plan to return to this.
- Other observables should also receive worldsheet instanton corrections, in particular AdS_4 BH entropy.

Thank you

5D MAXIMAL SYM ON S^5

Consider the Euclidean maximal SYM in five dimensions:

$$\mathcal{L} = -\tfrac{1}{2g_{\rm YM}^2}{\rm Tr}\left(|F|^2 - |D\Phi_m|^2 + \bar{\Psi}\not{D}\Psi - \frac{1}{2}[\Phi_m,\Phi_n]^2 + \bar{\Psi}\Gamma^m[\Phi_m,\Psi]\right).$$

We are using 10D language to write down the 5D fermions (Ψ has 16 components but should be decomposed into a pair of 5D spinors). The indices are $m = 0, 1, \ldots, 4$ and R-symmetry is $\mathrm{SO}(1,4)$.

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When we place euclidean SYM on S^5, we can preserve SUSY by adding terms to the Lagrangian

Blau (2000)

$$\delta\mathcal{L} = -\frac{1}{\mathcal{R}^2}\text{Tr}\left(3\Phi_m\Phi^m + \Phi_a\Phi^a\right) + \frac{1}{2\mathcal{R}}\text{Tr}\left(\bar{\Psi}\Gamma_{012}\Psi - 8\Phi_0\left[\Phi_1, \Phi_2\right]\right),$$

where $a = 0, 1, 2$. The radius of S^5 is $\mathcal{R}$. R-symmetry is broken to $\text{SU}(1,1) \times \text{U}(1) \in \text{SU}(4|1,1)$.

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Consider a theory invariant under some Grassmann odd charge Q and a Grassmann even charge B with

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One can show

Witten

$$\langle \mathcal{O}_{\text{BPS}} \rangle = \int [\mathcal{D}\varphi] \mathcal{O}_{\text{BPS}} e^{-S[\varphi] - tQP_F[\varphi]}\,,$$

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This integral is independent of t and one can take the limit $t\to\infty$ to *localise* the path integral to the saddle points of $QP_F[\varphi]$.

Wilson loop in 5D SYM

The Wilson loop in this theory is arguably the simplest expression so far,

Kim, Kim, Kim (2012)

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Notice that in this example there are only few terms of the perturbative expansion (at leading order in N) and then non-perturbative corrections.

Holographic dual in flat space

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The holographic dual to 5D SYM is given by the gravitational geometry around D4 branes

$$\mathrm{d}s_{10}^2 = H^{-1/2}\mathrm{d}s_{||}^2 + H^{1/2}\mathrm{d}s_{\perp}^2 \,.$$

where $\mathrm{d}s_{||}^2$, $\mathrm{d}s_{\perp}^2$ are the metrics on flat 5D spacetimes and H is harmonic on $\mathrm{d}s_{\perp}^2$.

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Since we are interested in Euclidean 5D SYM on S^5, we want $\mathrm{d}s_{||}^2 = \mathrm{d}\Omega_5^2$. We need a *spherical brane* solution.

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For the case at hand there is a quick way to obtain this solution by uplifting the near-horizon metric around flat D4s to 11D where one obtaines $\mathrm{AdS}_7 \times S^4$. Then we can change coordinates and reduce back to 10D carefully making sure supersymmetry is not broken.

MASS DEFORMED ABJ(M) IN 10D

Uplifting the $\mathrm{SO}(4) \times \mathrm{U}(1)$ invariant solution of Freedman and Pufu gives

$$\begin{aligned}
\mathrm{d}s_{10}^2 &= L^2(\mathrm{d}s^2_{\mathrm{AdS}_4} + 4\mathrm{d}s_6^2)\,,\\
\mathrm{d}s^2_{\mathrm{AdS}_4} &= \mathrm{d}\rho^2 + \sinh^2\rho\,\mathrm{d}\Omega_3^2\,,\\
\mathrm{d}s_6^2 &= \mathrm{d}\theta^2 + \frac{\cos^2\theta}{4Y_1}\mathrm{d}\Omega_1^2 + \frac{\sin^2\theta}{4Y_2}\mathrm{d}\Omega_2^2 + \sin^2\theta\cos^2\theta\,\Sigma^2\\
\mathrm{e}^{2\Phi} &= \frac{\pi(2\lambda)^{5/2}}{N^2 Y_1 Y_2}\,,\\
\Sigma &= \mathrm{d}\varphi + \frac{1}{2}\cos\theta_1\,\mathrm{d}\phi_1 - \frac{1}{2}\cos\theta_2\,\mathrm{d}\phi_2\,.
\end{aligned}$$

The two functions Y_1 and Y_2 implement the squashing of the internal space and take the form

$$Y_1 = 1 + c\frac{\cos^2\theta}{\cosh^2(\rho/2)}\,, \qquad Y_2 = 1 - c\frac{\sin^2\theta}{\cosh^2(\rho/2)}\,.$$

HIGHER RANK IN ABJ(M)

string theory gives (all strings/antistrings wrapping $\mathbf{C}P^1 \subset \mathbf{C}P^3$)

$$\sum_{n=1}^{\infty} Z_{\text{inst}}^{(n)} = \frac{N^2}{2(2\pi\lambda)^2} \left(\text{Li}_3(-\beta \mathrm{e}^{-2\pi\sqrt{2\lambda}}) + \text{Li}_3(-\beta^{-1} \mathrm{e}^{-2\pi\sqrt{2\lambda}}) \right) ,$$

where $\beta = \mathrm{e}^{-2\pi i l/k}$. Matrix model result is

$$F_{\text{inst}} = \frac{N^2}{(4\pi\lambda)^2} \sum_{dm=n} \sum_{d_1+d_2=d} \frac{(-1)^n}{n^3} n_0^{d_1,d_2} \beta^{\frac{d_2-d_1}{d} n} \mathrm{e}^{-2\pi n\sqrt{2\lambda}} ,$$

where $n_0^{i,j}$ are Gopakumar-Vafa invariants on $\mathbb{P}^1 \times \mathbb{P}^1$. Restricting to the $d_1 + d_2 = 1$ sector of the sum, the two answers agree.

Clearly more configurations of strings should be possible at higher rank.