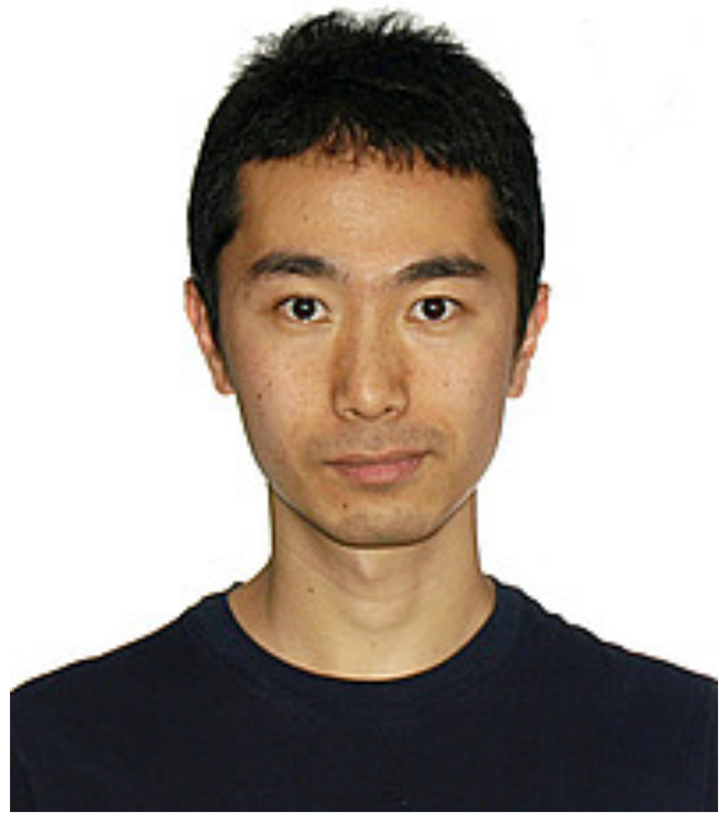
Boundary Correlators and the Reparametrization Mode on the AdS_2 String

Bendeguz Offertaler, Princeton University

CERN, Workshop on "Precision Holography"

June 7th, 2023

based on arXiv:2212.14842, arXiv:2306.XXXXX with



Simone Giombi
(Princeton)

Shota Komatsu
(CERN)

Jieru Shan
(Princeton)

Outline

Part I: AdS_2/CFT_1 holography on the open string / Wilson loop

Part II: Conformal gauge and the boundary reparametrization mode

Part I

AdS_2/CFT_1 holography on the open string / Wilson loop

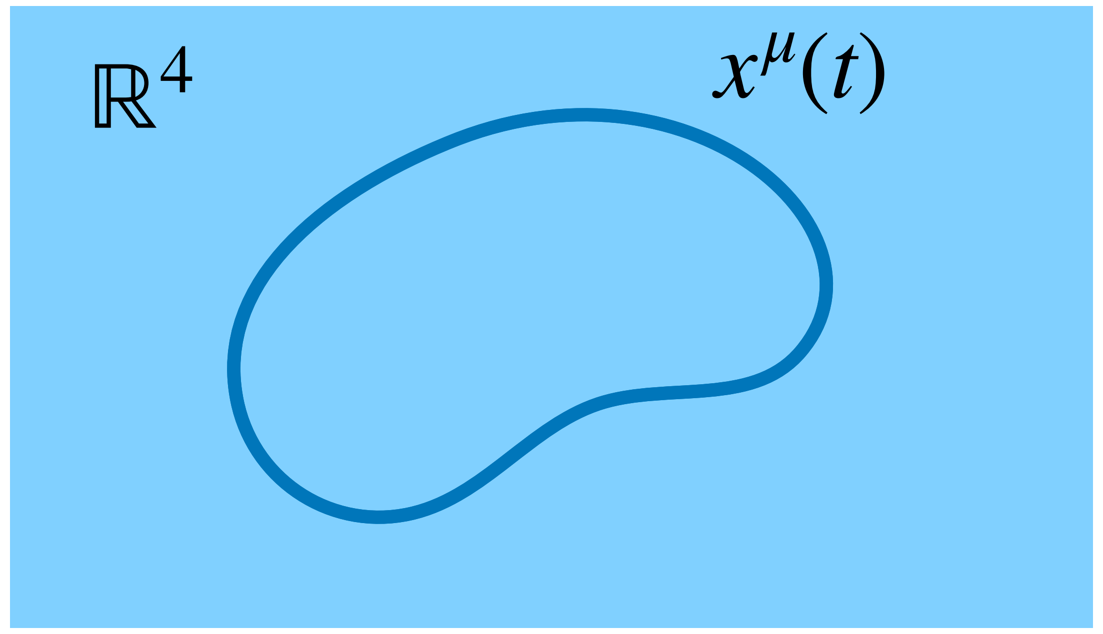
Wilson loops in gauge theory

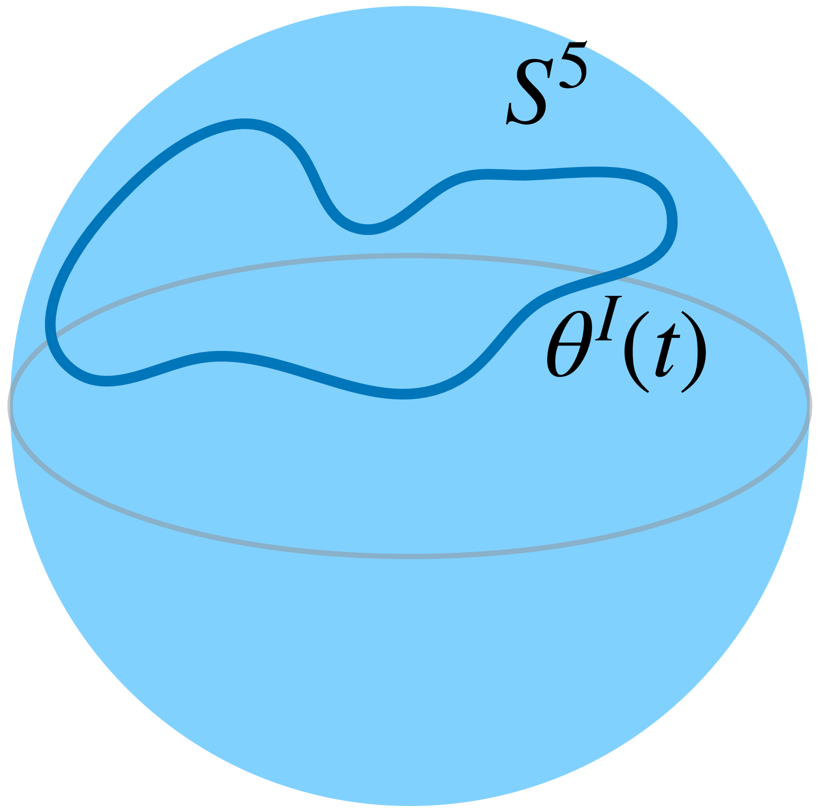
$$\mathscr{W} = \mathrm{P}\exp\left(i\int A_\mu dx^\mu\right)$$

Fundamental, gauge-invariant, non-local observables.

Wilson loops in $\mathcal{N} = 4$ Super Yang-Mills

$$\mathscr{W} = \mathrm{P}\exp\left(\int\left(iA_\mu \dot{x}^\mu + |\dot{x}|\theta^I\Phi^I\right)dt\right)$$





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Amenable to exact computations.

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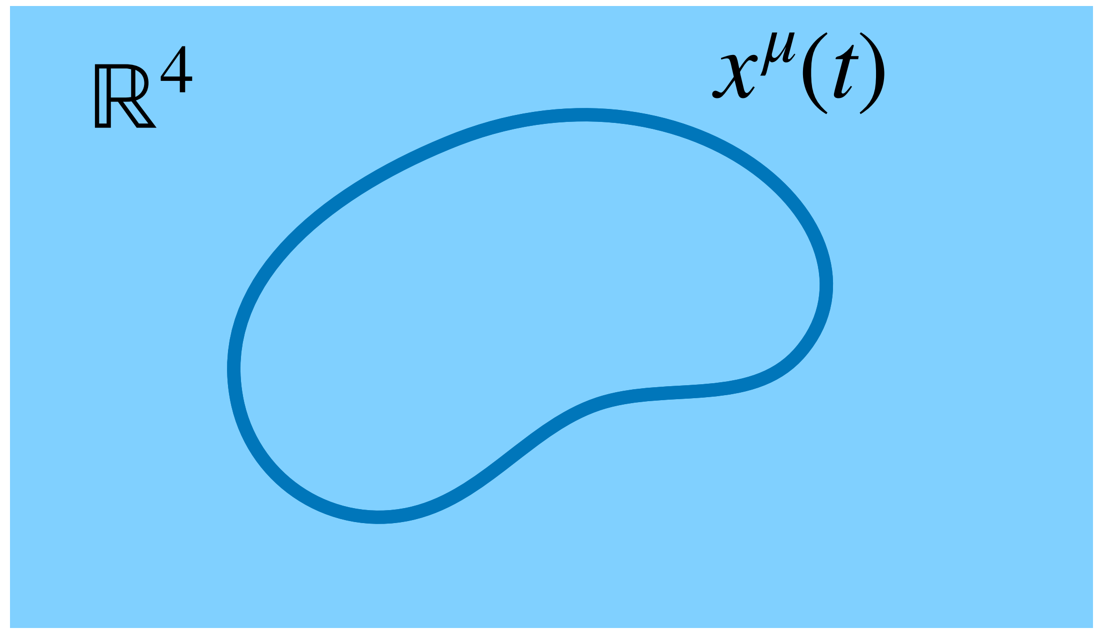

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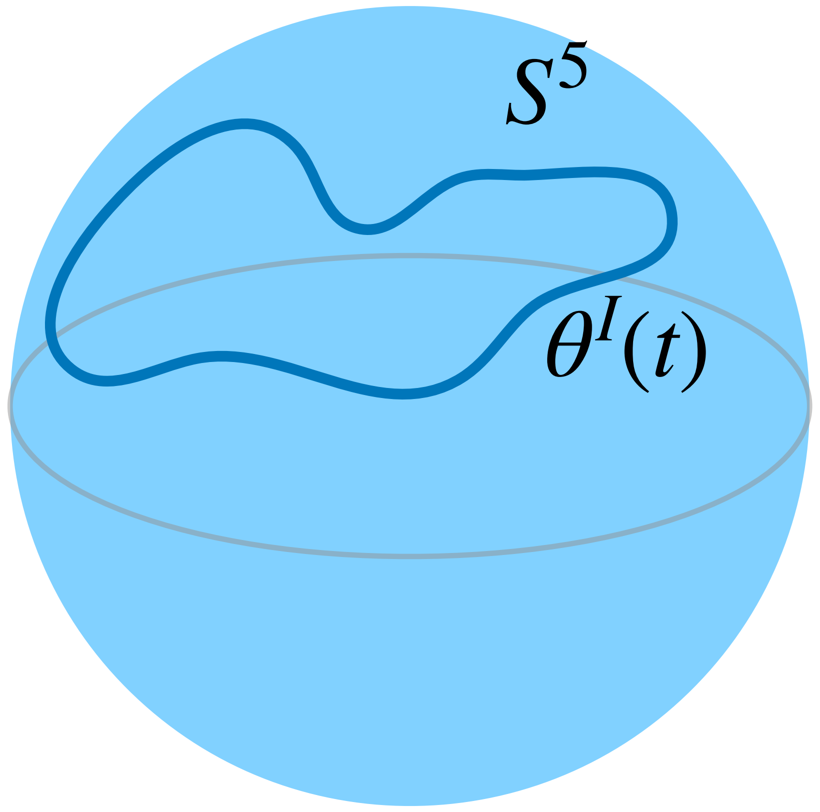

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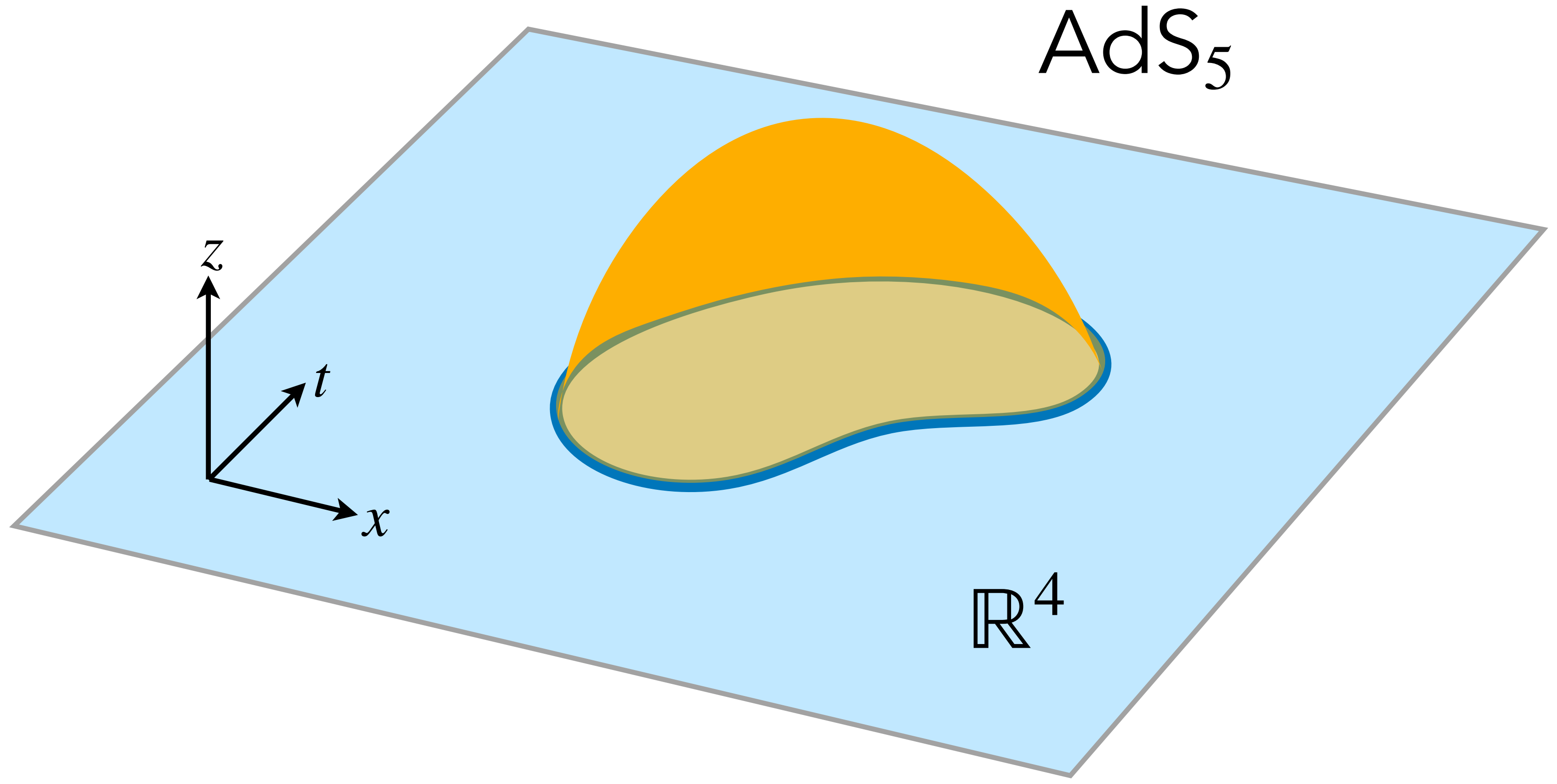
Dual to the open string in $AdS_5 \times S^5$ incident on the contour on the boundary: [Maldacena '98; Rey, Yee' 98]

$$\langle \mathscr{W} \rangle = Z_{\text{string}} \approx e^{-S_{\text{cl}}}$$



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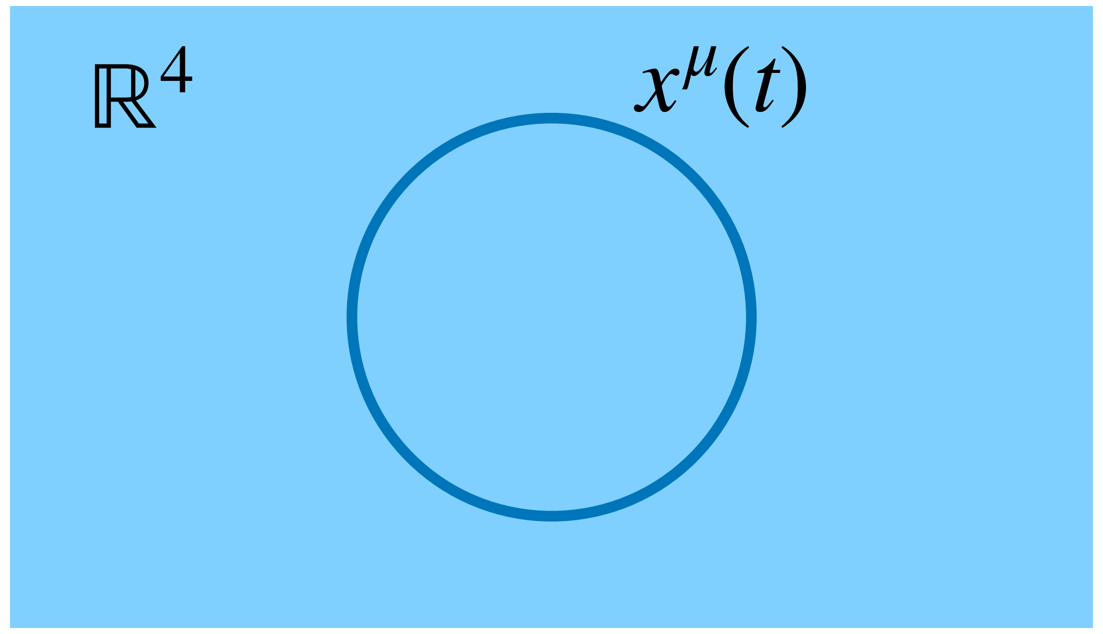
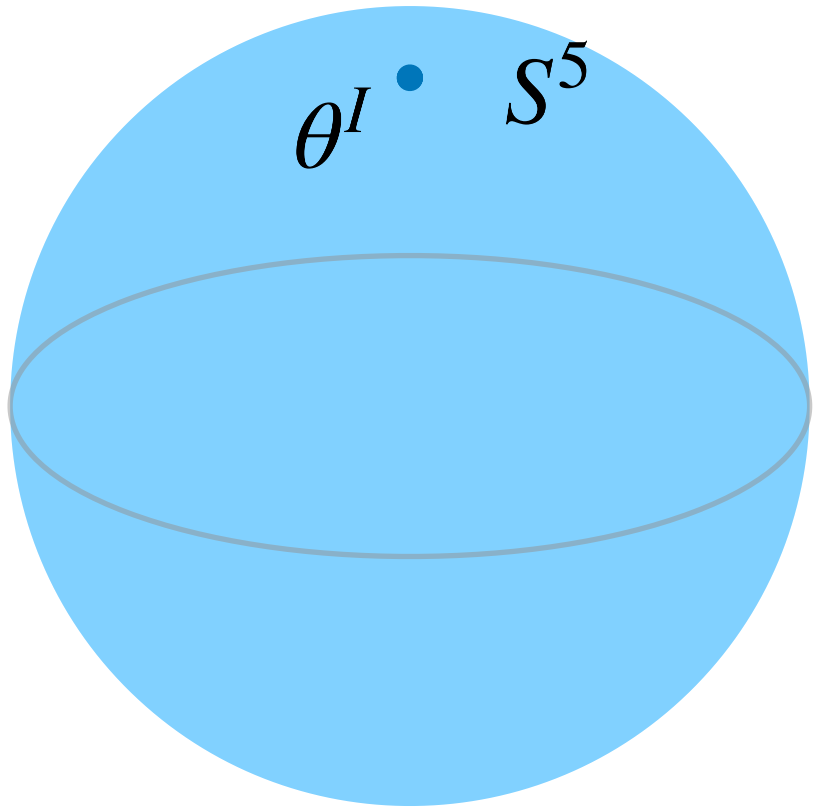


Circular spacetime contour: $x^\mu(t) = R(\cos(t), \sin(t), 0, 0)$. Point on S^5: $\theta^I = \delta^{I6}$.



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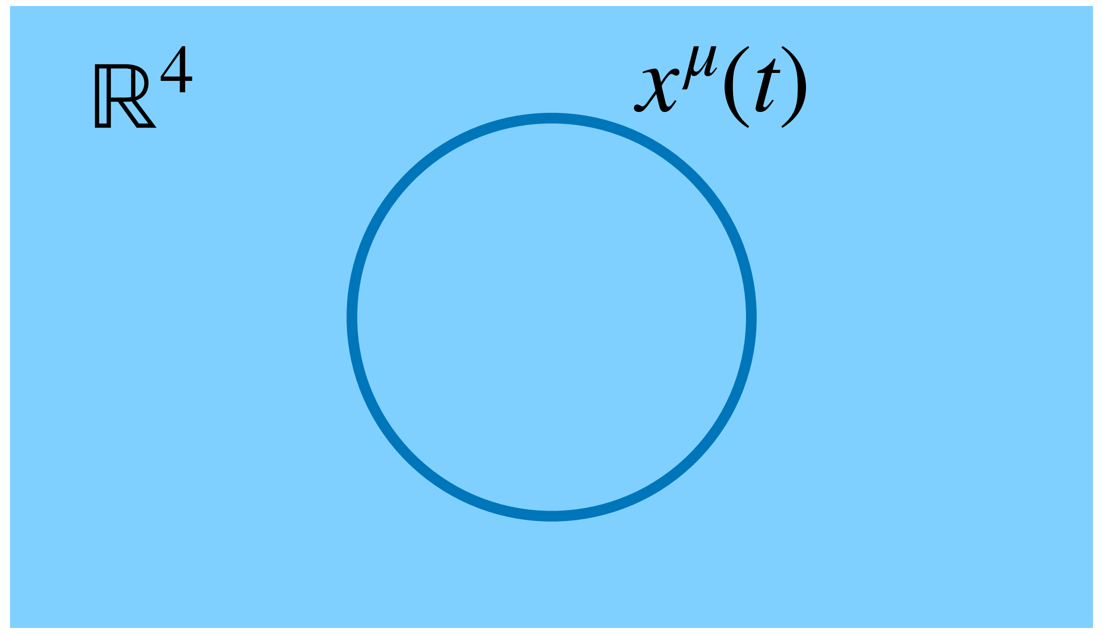

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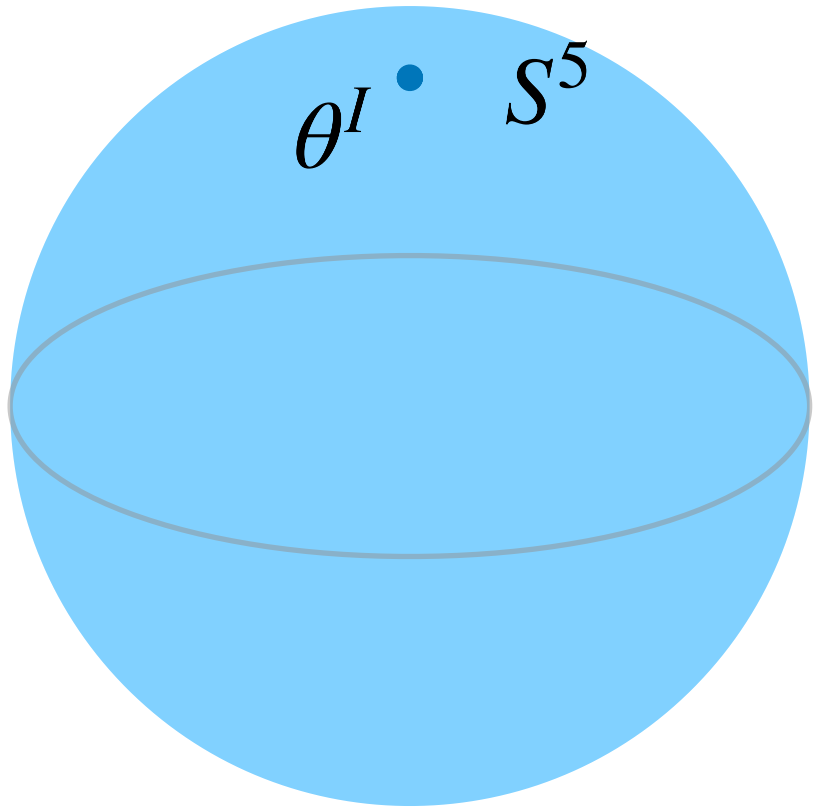

$$\mathcal{W} = L^1_{N-1}\left(-\frac{g^2_{YM}}{4}\right)e^{\frac{g^2_{YM}}{8}} \overset{\text{planar}}{\rightarrow} \frac{2}{\sqrt{\lambda}}I_1\left(\sqrt{\lambda}\right) \overset{\lambda\gg1}{\sim} \sqrt{\frac{2}{\pi}}\lambda^{-\frac{3}{4}}e^{\sqrt{\lambda}}$$

[Erickson, Semenoff, Zarembo ’00; Drukker, Gross ’01; Pestun ’07]

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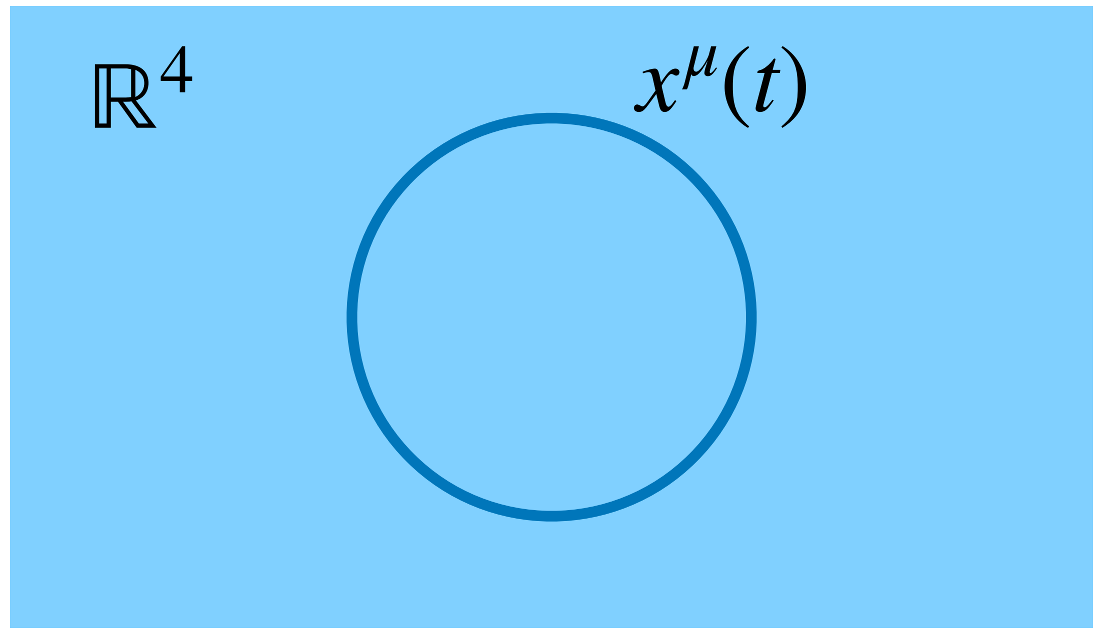

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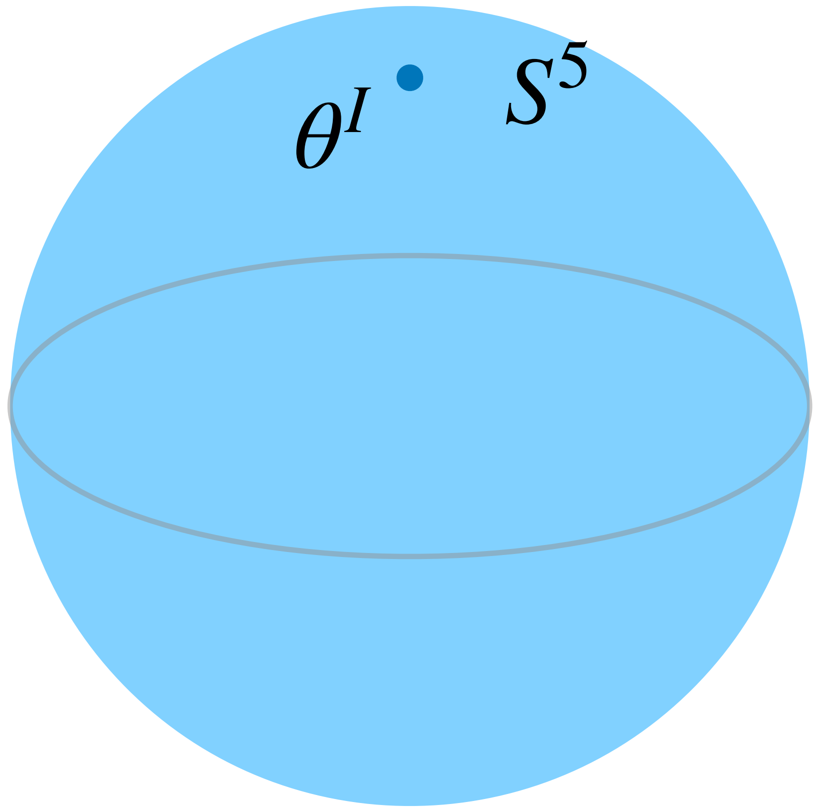

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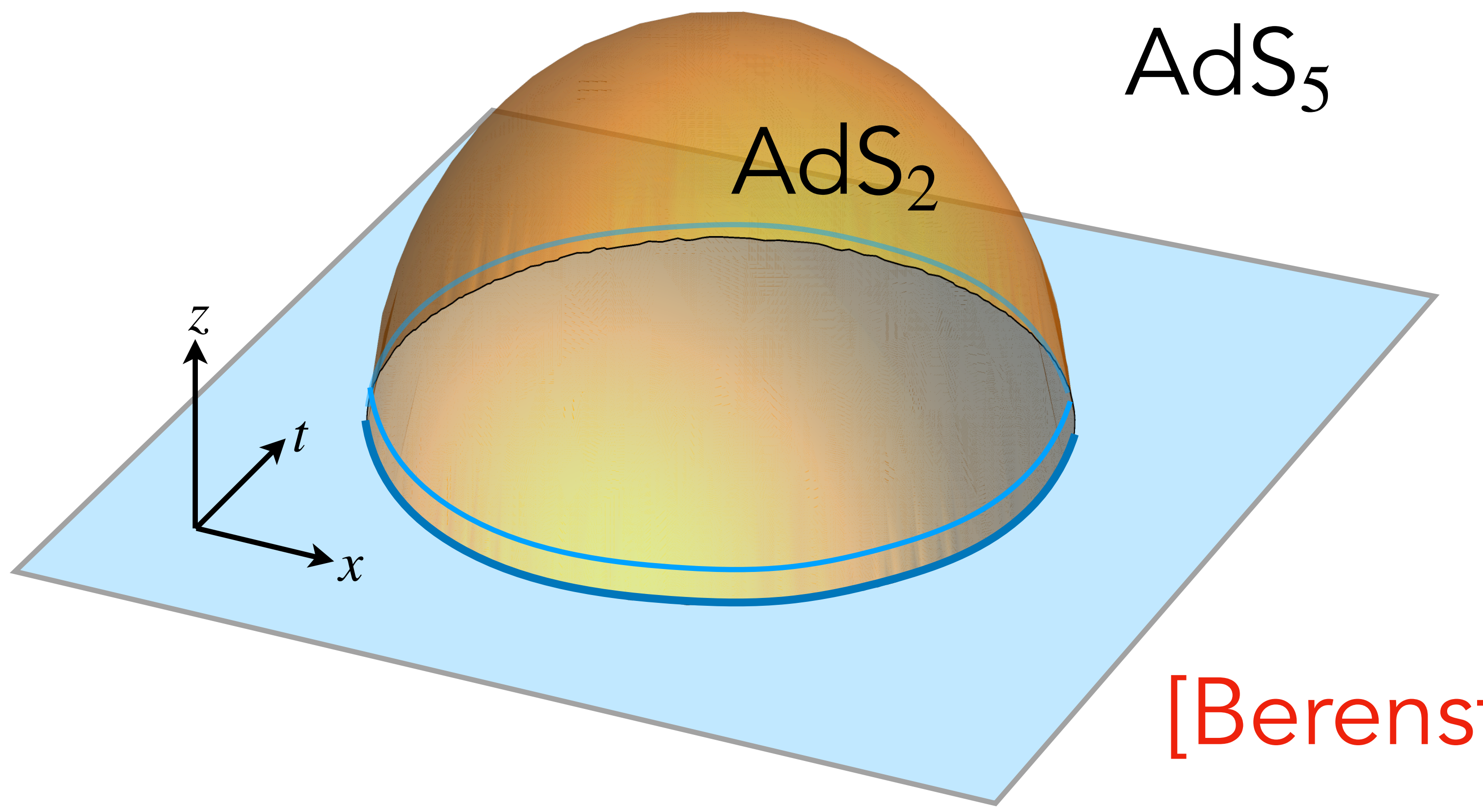

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[Berenstein, Corrado, Fischler, Maldacena 98;
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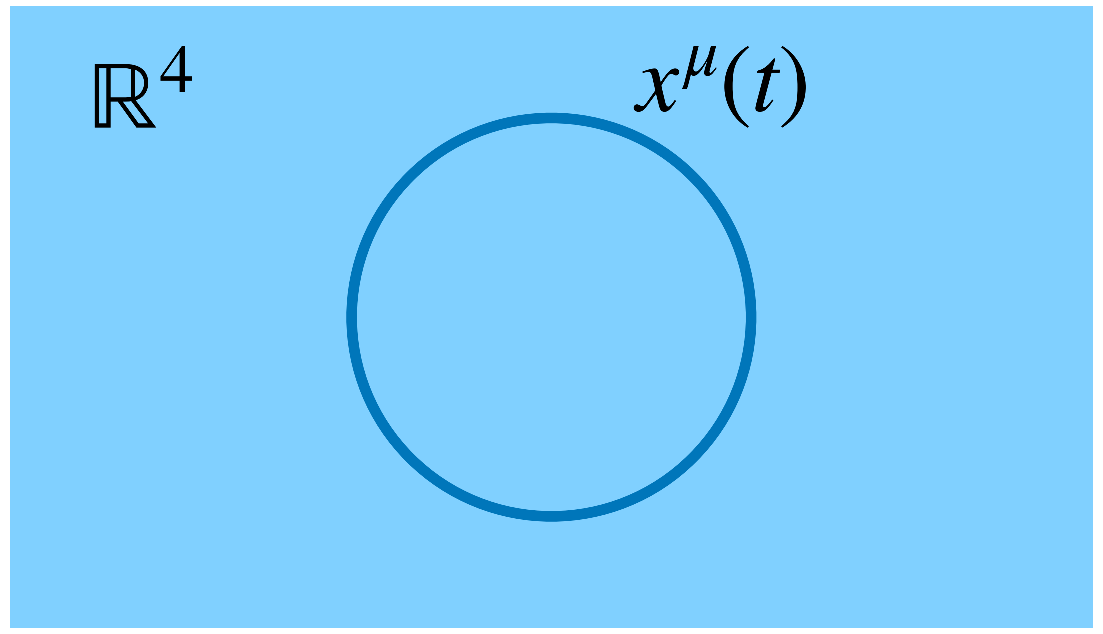

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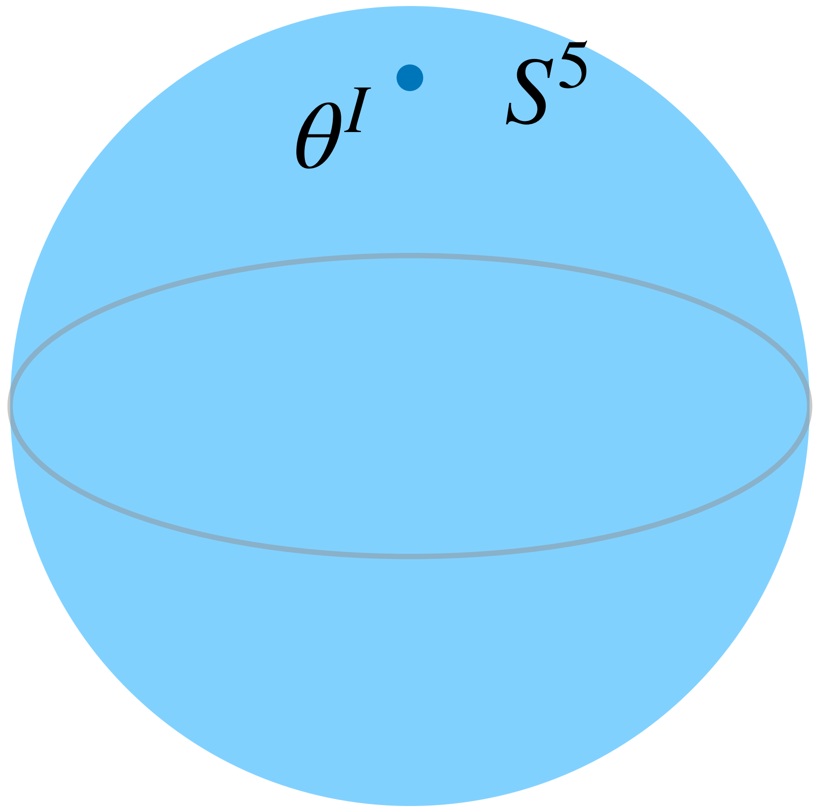

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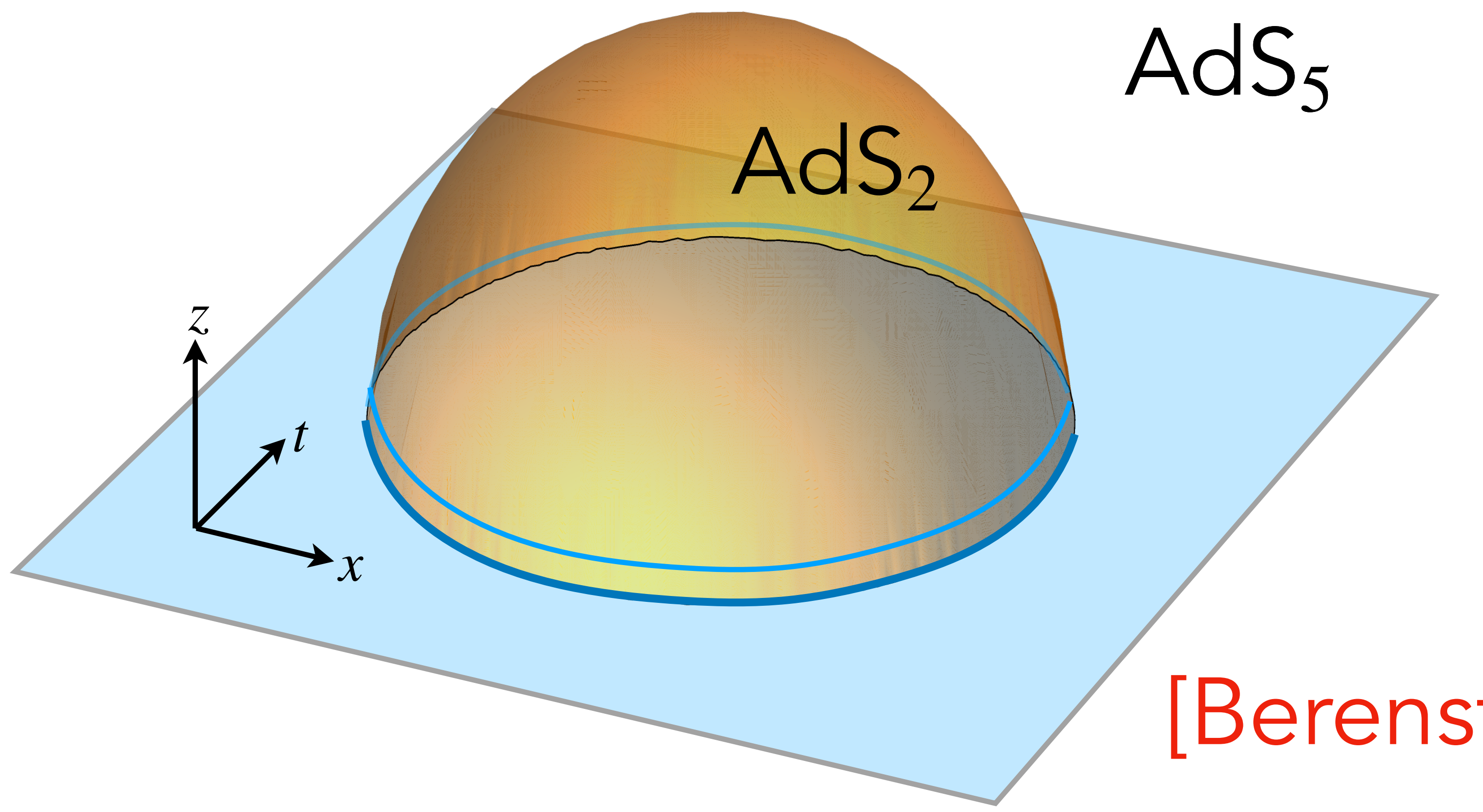

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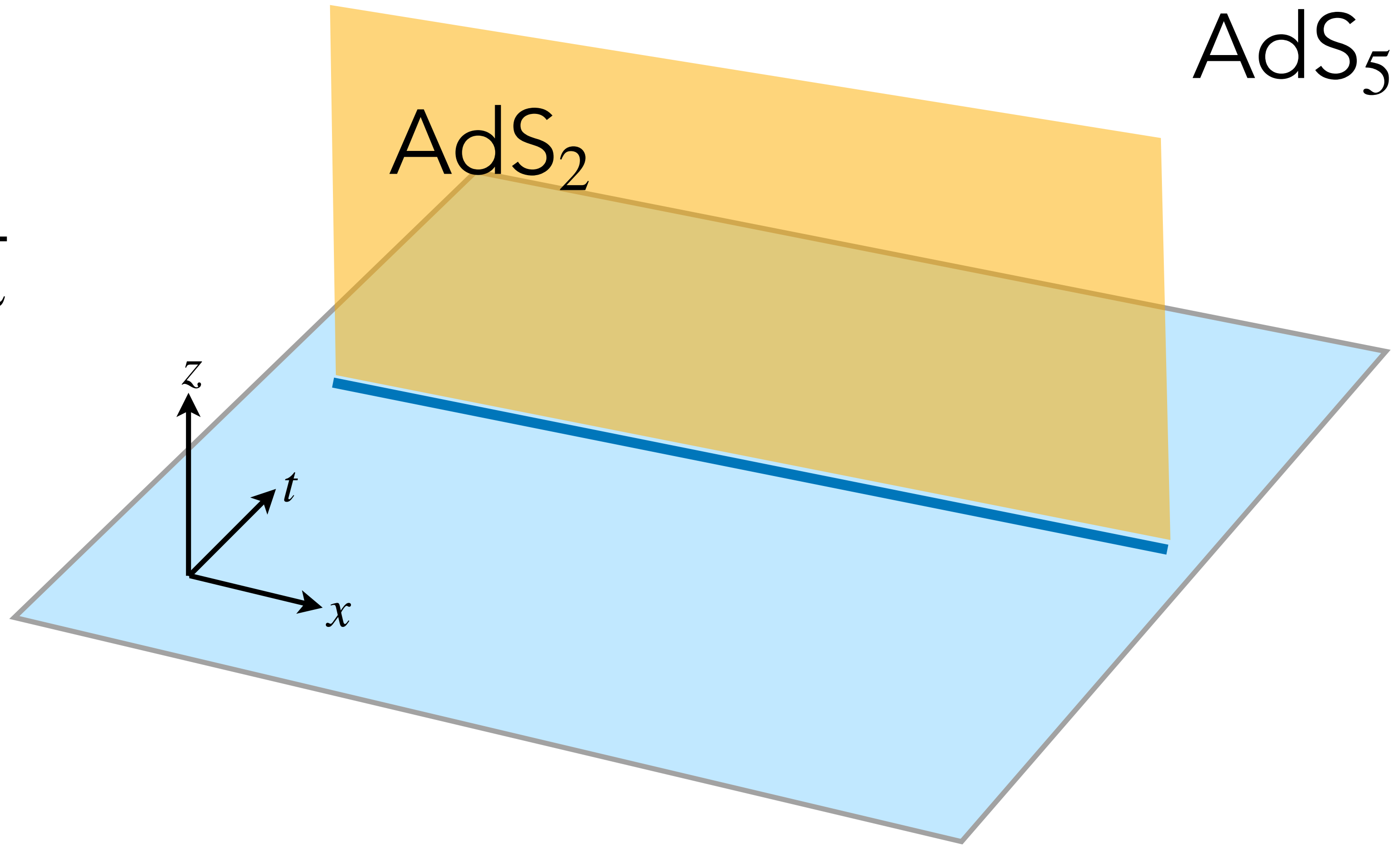

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Conformal defects

The Wilson line (or circle) defines a 1d conformal defect.

Defects in QFT have been studied a lot in recent years for a variety of reasons:

- Boundaries, interfaces, and impurities in physical systems
- Generalized symmetries
- Entanglement entropy
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In d-dim. CFT, one can study p-dim. *conformal defects*, which preserve $SO(p+1,1) \subset SO(d+1,1)$

One can then study the interplay of operators in the bulk and on the defect.

Wilson line defect CFT

The Wilson line (or circle) breaks $PSU(2,2|4) \ni SO(5,1) \times SO(6)$ to the 1d superconformal group.
$OSp(4^*|4) \ni SL(2,\mathbb{R}) \times SO(3) \times SO(5)$.

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We can define correlators of operators on the Wilson line

$$\langle O(t_1)\ldots O(t_n)\rangle \equiv \frac{1}{\langle \mathscr{W} \rangle}\Big\langle \mathrm{P}[O(t_1)O(t_2)\ldots O(t_n)e^{\int_{-\infty}^{\infty} dt(iA_0+\Phi_6)}]\Big\rangle_{\mathcal{N}=4\ \mathrm{SYM}}$$

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Defect correlators arise naturally when considering small fluctuations of the contour. [Polyakov, Rychkov '00; Drukker, Kawamoto '06, …]:

$$\langle \mathbb{D}_i(t_1)\Phi^I(t_2)\dots\rangle = \frac{1}{\langle \mathscr{W} \rangle}\frac{\delta}{\delta x^i(t_1)}\frac{\delta}{\delta \theta^I(t_2)}\dots\langle \mathscr{W}[x^i,\theta^I]\rangle :$$

Where $\mathbb{D}_i = iF_{0i} + D_i\Phi^6$, $i = 1,2,3$, is the displacement operator and Φ^I, $I = 1,\dots,5$, are the orthogonal scalars.

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Where $\mathbb{D}_i = iF_{0i} + D_i\Phi^6$, $i = 1,2,3$, is the displacement operator and Φ^I, $I = 1,\ldots,5$, are the orthogonal scalars.

These are dual to the 3 transverse fluctuations of the string in AdS_5 and the 5 transverse fluctuations in S^5

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Defect correlators are constrained by 1d conformal symmetry.

Operators are labelled by scaling dimension Δ under $SL(2,\mathbb{R})$, and the representation under the "global group" $SO(3) \times SO(5)$.

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The $SL(2,\mathbb{R})$ fixes the two and three-point functions:

$$\langle O_i(t_1) O_j(t_2) \rangle = \frac{\mathcal{N}_i \delta_{ij}}{t_{12}^{2\Delta_i}}, \qquad \langle O_i(t_1) O_j(t_2) O_k(t_3) \rangle = \frac{C_{ijk}}{|t_{12}|^{\Delta_{ij|k}} |t_{13}|^{\Delta_{ik|j}} |t_{23}|^{\Delta_{jk|i}}},$$

Four point functions:

$$\langle O(t_1) \dots O(t_4) \rangle = \frac{1}{t_{12}^{2\Delta} x_{34}^{2\Delta}} G(\chi),$$

Where $\chi = \dfrac{t_{12} t_{34}}{t_{13} t_{24}}$.

Wilson line defect CFT

This defect CFT has been studied extensively:

- At weak coupling using perturbation theory [Cooke, Dekel, Drukker ’17; Komatsu, Kiryu ’18; Barrat, Liendo, Peveri, Plefka ’21 …]
- At strong coupling by considering fluctuations of the dual string [Giombi, Roiban, Tseytlin ‘17]
- Using localization [Correa, Henn, Maldacena, Sever ‘12; Giombi, Komatsu ‘18]
- Using integrability [Drukker ’12; Correa, Maldacena, Sever ‘12; Cavaglia, Gromov, Julius, Preti ’21 ’22, …]
- Using the analytic conformal bootstrap [Liendo, Meneghelli, Mitev ‘18; Ferrero, Meneghelli ‘21]
- …

Correlators on the string

Our focus will be on the boundary correlators at strong coupling in the planar limit (i.e. $N \to \infty, \lambda = g_{YM}^2 N$: fixed, and $\lambda \gg 1$) involving a single scalar

In the dual string, this corresponds to considering weakly interacting fluctuations of the string (i.e. $g_s = 0$, $T_s = \ell_s^{-1} = \dfrac{\sqrt{\lambda}}{2\pi} \gg 1$) in a subspace $\mathrm{AdS}_2 \times S^1 \subset \mathrm{AdS}_5 \times S^5$.

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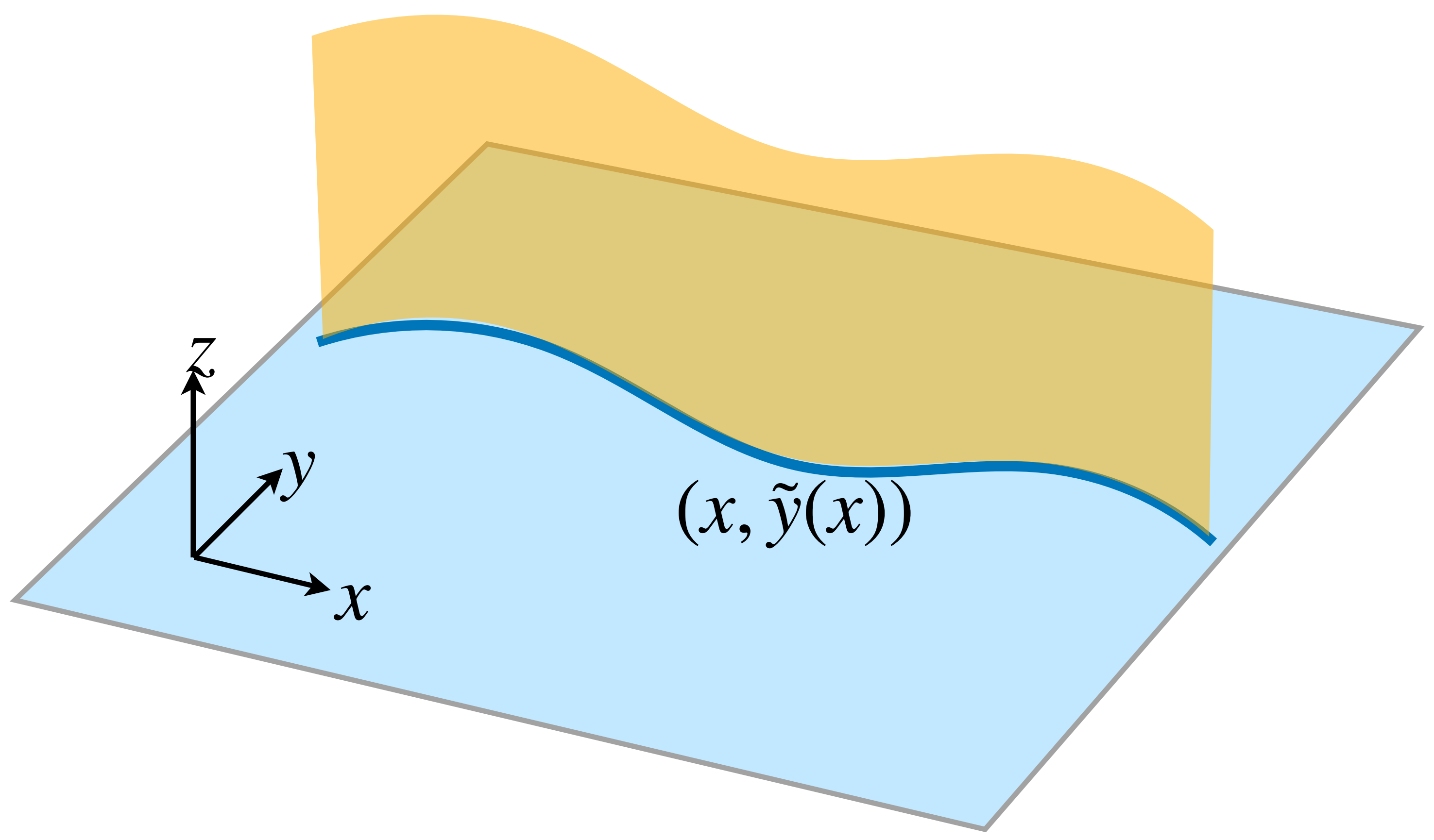
We work with Poincare coordinates $x \in \mathbb{R}$, $z \in [0,\infty)$ on AdS_2 and polar coordinate $y \in [0,2\pi]$ on S^1:

$$ds^2 = \frac{dz^2 + dx^2}{z^2} + dy^2.$$



We represent the boundary curve as $(x, \tilde{y}(x))$.

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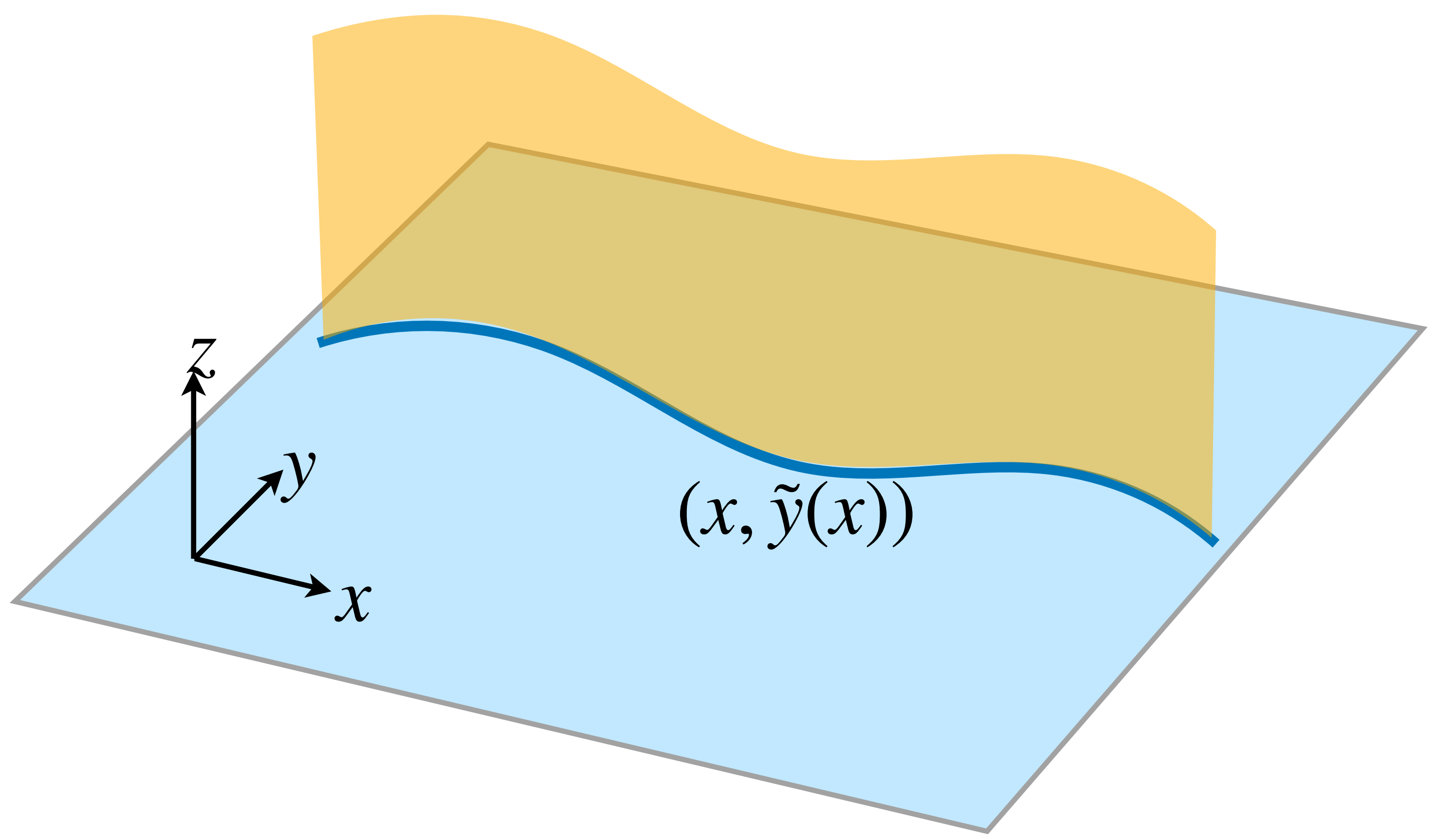

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The Wilson loop / string is then a function of $\tilde{y}(x)$ only and the correlators are:

$$\langle \Phi(x_1)\ldots\Phi(x_n)\rangle = \langle y(x_1)\ldots y(x_n)\rangle = \frac{1}{Z_{\mathrm{string}}}\frac{\delta}{\delta\tilde{y}(x_1)}\cdots\frac{\delta}{\delta\tilde{y}(x_n)}Z_{\mathrm{string}}[\tilde{y}]\Big|_{\tilde{y}=0.} \approx e^{S_{\mathrm{cl}}[\tilde{y}]}\frac{\delta}{\delta\tilde{y}(x_1)}\cdots\frac{\delta}{\delta\tilde{y}(x_n)}e^{-S_{\mathrm{cl}}[\tilde{y}]}\Big|_{\tilde{y}=0}.$$

Correlators on the string in static gauge

[Giombi, Roiban, Tseytlin '17]

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With boundary condition: $y(0,t) = \tilde{y}(t)$. The induced metric is $g_{\alpha\beta} = \frac{1}{s^2}\delta_{\alpha\beta}$.

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Where:

$$L_0 = 1, \qquad L_2 = \frac{1}{2} g^{\alpha\beta}\partial_\alpha y \partial_\beta y, \qquad L_4 = -\frac{1}{8}(g^{\alpha\beta}\partial_\alpha y \partial_\beta y)^2.$$

We see that $y(s,t)$ is a massless field in AdS_2 with a tower of interactions. Recall that $m^2 = \Delta(\Delta - 1)$.

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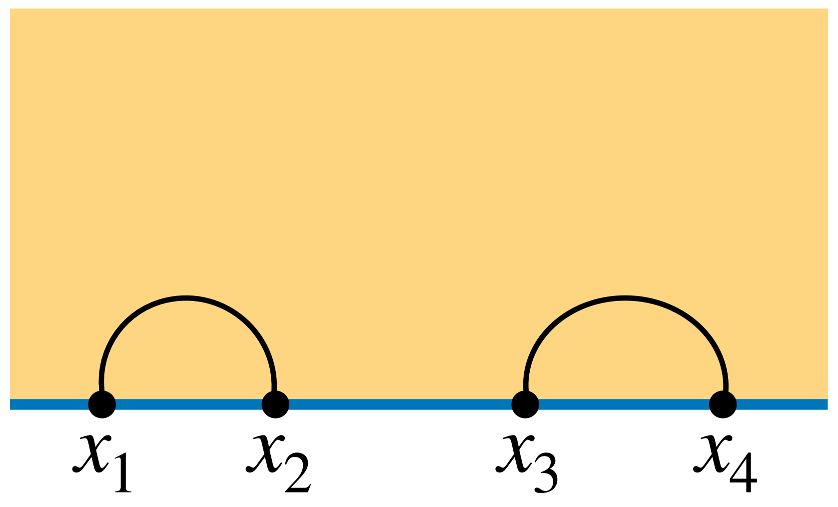

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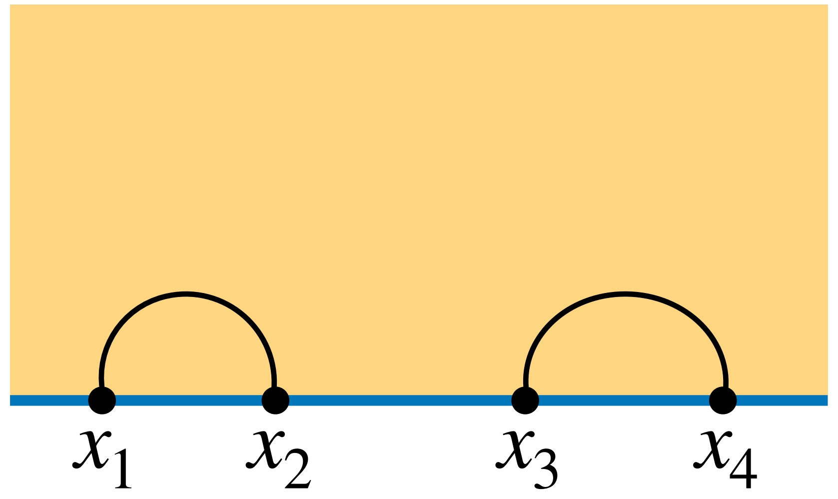

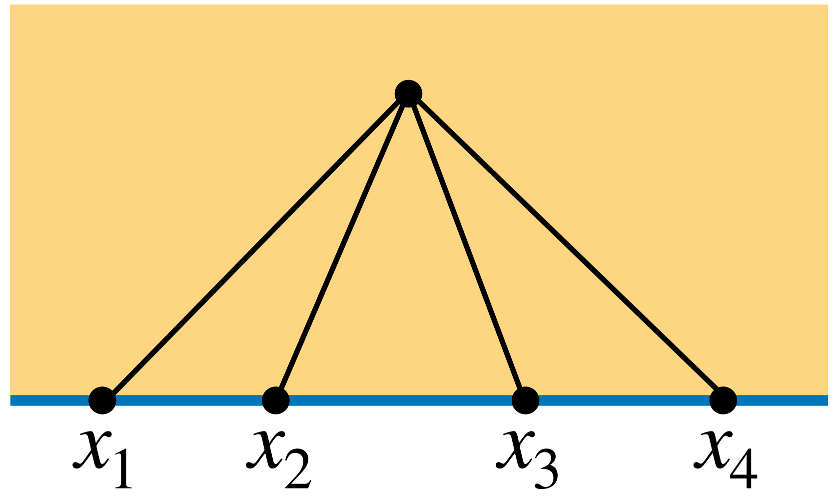

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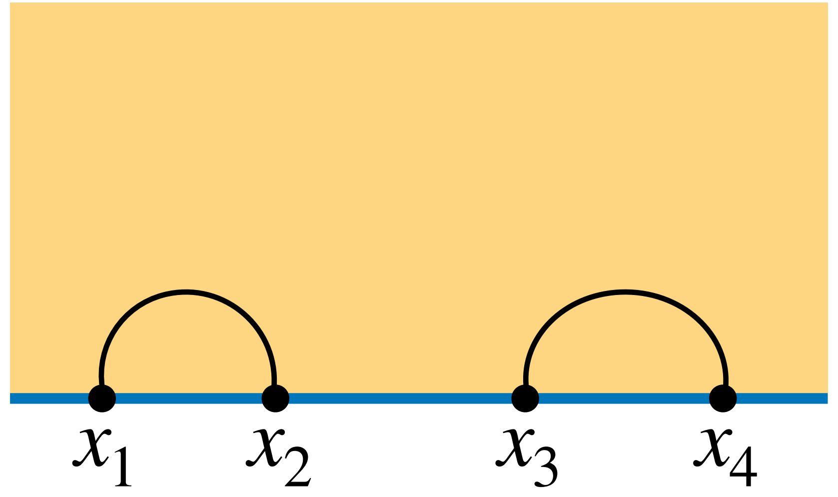

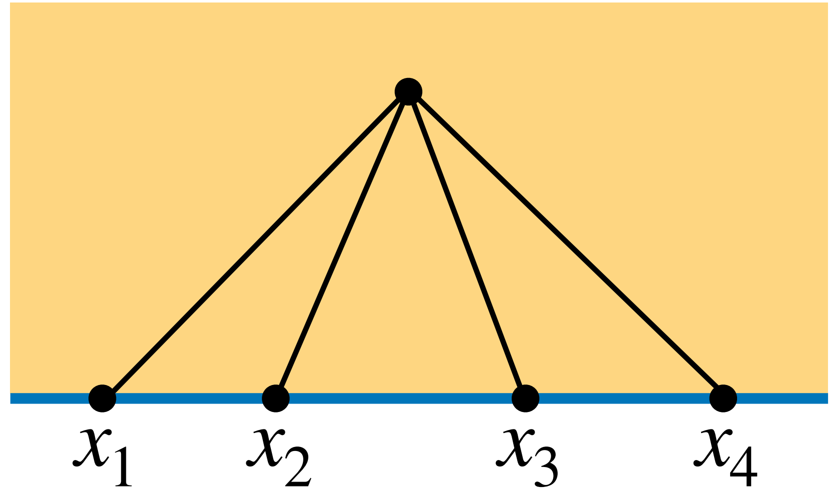

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We find the normalized four-point function takes the form:

$$\frac{\langle y_1 y_2 y_3 y_4 \rangle}{\langle y_1 y_2 \rangle \langle y_3 y_4 \rangle} = G_{\text{free}}(\chi) + \frac{1}{2\pi T_s} G_{\text{tree}}(\chi) + \dots,$$

Where

$$G_{\text{free}}(\chi) = 1 + \chi^2 + \frac{\chi^2}{(1-\chi)^2},$$

$$G_{\text{tree}}(\chi) = -\frac{2(\chi^2 - \chi + 1)^2}{(1-\chi)^2} + \frac{-2 + \chi + \chi^3 - 2\chi^4}{2\chi} \log((1-\chi)^2) + \frac{\chi^2(2 - 4\chi + 9\chi^2 - 7\chi^3 + 2\chi^4)}{2(\chi - 1)^3} \log \chi^2$$

Part II

Computing boundary correlators in conformal gauge

via the reparametrization mode

String action in conformal gauge

We start with the Polyakov action:

$$S = \frac{T_s}{2} \int d^2\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X)$$

Here $X^\mu = (x, z, y)$, $\quad G_{\mu\nu}(X) dX^\mu dX^\nu = \dfrac{dx^2 + dz^2}{z^2} + dy^2$, $\quad \sigma^\alpha = (s, t)$.

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Where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.

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The condition for the string $X^\mu(s, t)$ to be incident on the curve $\tilde{X}^\mu(\lambda)$ is:

$$X^\mu(0,t) = \tilde{X}^\mu(\alpha(t)),$$

where $\alpha(t)$ is a reparametrization of the boundary

String action in conformal gauge

Equations of motion:

$$0 = \frac{1}{T_s}\frac{\delta S}{\delta X^\mu} = \quad -\partial_\alpha(G_{\mu\nu}\partial^\alpha X^\nu) + \frac{1}{2}\partial_\mu G_{\nu\rho}\partial_\alpha X^\nu \partial^\alpha X^\rho$$

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Classical action:

$$S_{\text{cl}}[\tilde{X}] = S[X]\Big|_{\text{Virasoro}}$$

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Virasoro constraint = extremization over reparametrizations

Let $X^\mu(\sigma)$ solve the EOM with BC $X^\mu(0,t) = \tilde{X}^\mu(\alpha(t))$. If $\alpha(t) \to \alpha(t) + \delta\alpha(t)$, the variation in the on-shell action is:

$$\delta S = \int dt T_{st}(0,t) \frac{\delta\alpha(t)}{\dot{\alpha}(t)}.$$

with $\delta s = 0$ and $\delta t = \dfrac{\delta\alpha(t)}{\dot{\alpha}(t)}$. This is somewhat analogous to Hamilton-Jacobi: $\delta S = p(t_f)\delta q(t_f) - p(t_i)\delta q(t_i)$.

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Recall that $T \equiv T_{tt} + iT_{st}$ satisfies $\bar{\partial} T = 0$.

If f is holomorphic in the UHP with zero imaginary part on the real axis, then $f = 0$. Thus, we conclude:

$$T(s,t) = T_{\alpha\beta}(s,t) = 0.$$

String action in conformal gauge

Working in conformal gauge gives rise to a dynamical boundary reparametrization mode

- In flat space [Polyakov '81, '86; Alvarez '83; Fradkin, Tseytlin '82; Cohen, Moore, Nelson, Polchinski '86, …]
- In AdS [Polyakov, Rychkov, Makeenko, Ambjorn, Oleson, …'00 - '10]
- In the Douglas integral formula for the area of a minimal surface in flat space [Douglas '31]:

$$A = \underset{\alpha}{\text{minimize}} \left[\frac{1}{4\pi} \int_0^{2\pi} d\tau \int_0^{2\pi} d\tau' \frac{[\vec{x}(\alpha(\tau)) - \vec{x}(\alpha(\tau'))]^2}{\left[2\sin\left(\frac{\tau-\tau'}{2}\right)\right]^2} \right]$$

The AdS$_2$ string in conformal gauge

Let's see how far we can get with the string in AdS$_2 \times S^1$:

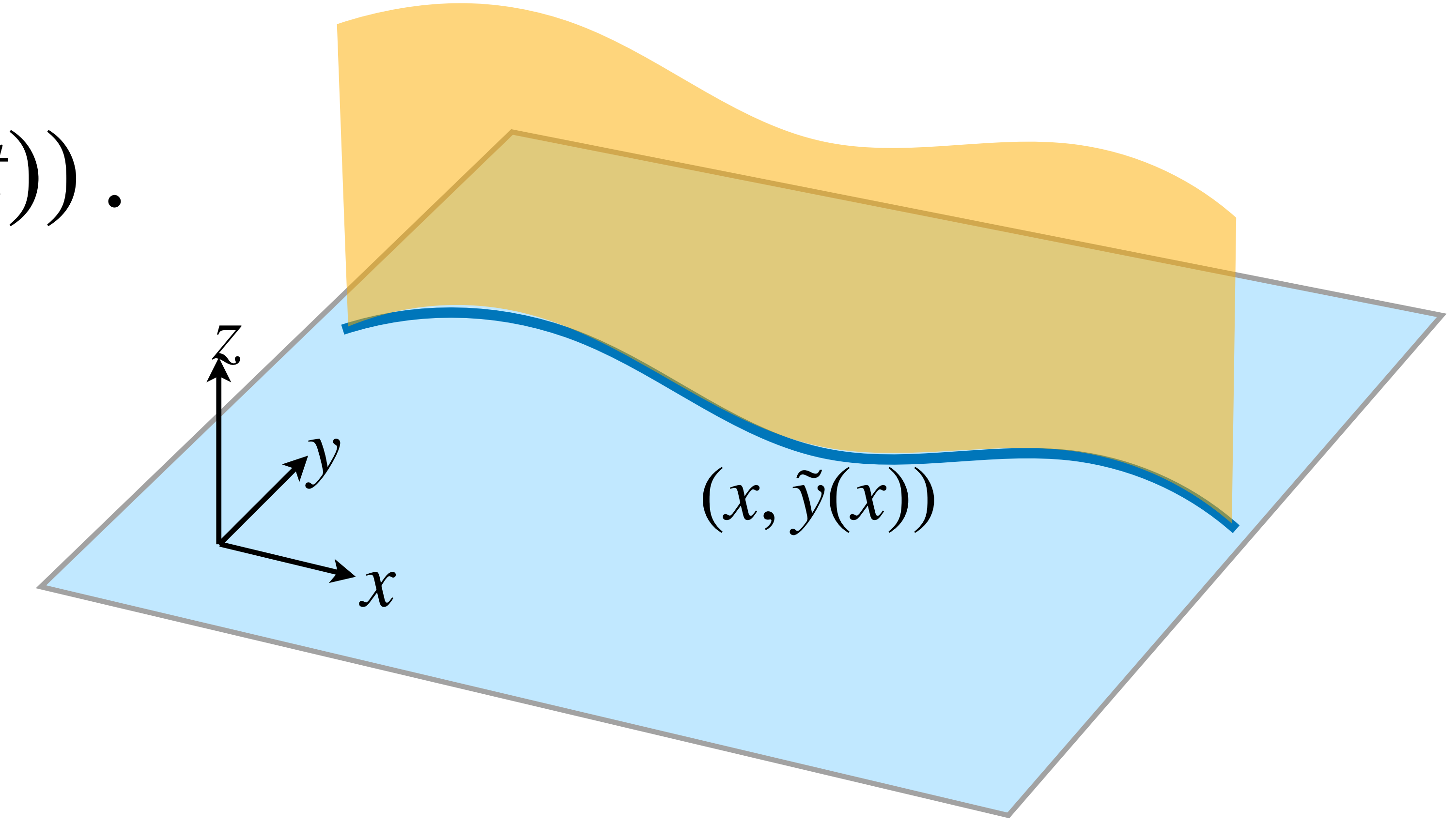
$$S = S_L[x, z] + S_T[y] + T_s A$$

Where

$$S_L[x, z] = \frac{T_s}{2}\int d^2\sigma \left[\frac{\partial_\alpha x \partial^\alpha x + \partial_\alpha z \partial^\alpha z}{z^2} - \frac{2}{s^2}\right], \qquad S_T[y] = \frac{T_s}{2}\int d^2\sigma \partial_\alpha y \partial^\alpha y, \qquad A = \int d^2\sigma \frac{1}{s^2}$$

And the BCs are:

$$x(0,t) = \alpha(t), \qquad z(0,t) = 0, \qquad y(0,t) = \tilde{y}(\alpha(t))\,.$$



Transverse action

$$S_T[y] = \frac{T_s}{2} \int d^2\sigma \partial_\alpha y \partial^\alpha y, \qquad y(0,t) = \tilde{y}(\alpha(t)).$$

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EOM and its solution is:

$$\partial_\alpha \partial^\alpha y = 0, \qquad y(s,t) = \int dt' K(s,t,t')\tilde{y}(\alpha(t')),$$

Where $K(s,t,t') = \dfrac{1}{\pi}\dfrac{s}{(t-t')^2}$.

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Where $K(s,t,t') = \dfrac{1}{\pi}\dfrac{s}{(t-t')^2}$. The action becomes:

$$S_T[\tilde{y}(\alpha(t))] = -\frac{T_s}{2\pi}\int dt dt' \frac{\tilde{y}(\alpha(t))\tilde{y}(\alpha(t))}{(t-t')^2}\ .$$

Longitudinal action

$$S_L[x, z] = \frac{T_s}{2} \int d^2\sigma \left[\frac{\partial_\alpha x \partial^\alpha x + \partial_\alpha z \partial^\alpha z}{z^2} - \frac{2}{s^2} \right], \qquad x(0,t) = \alpha(t), \quad z(0,t) = 0.$$

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EOM:

$$\partial_\alpha \left(\frac{1}{z^2}\partial^\alpha x\right) = 0, \qquad \partial_\alpha \left(\frac{1}{z^2}\partial^\alpha z\right) + \frac{1}{z^3}\left(\partial^\alpha x \partial_\alpha x + \partial^\alpha z \partial_\alpha z\right) = 0.$$

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Expanding $\alpha(t) = t + \epsilon(t)$, then to cubic order in $\epsilon(t)$: [Polyakov Rychkov '00; Rychkov '02; Makeenko, Ambjorn '11]

$$S_L = \frac{6T_s}{\pi}\int dt dt' \frac{\epsilon(t)\epsilon(t')}{(t-t')^4} - \frac{12T_s}{\pi}\int dt dt' \frac{\epsilon(t)^2\epsilon(t)}{|t-t'|^4(t-t')} + O(\epsilon^4)$$

Setting $\epsilon(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t}\epsilon(\omega)$, in Fourier space this becomes

$$S_L = \frac{T_s}{2\pi}\int d\omega \epsilon(\omega)\epsilon(-\omega)|\omega|^3 + \frac{iT_s}{8\pi^2}\int d\omega d\omega' \epsilon(\omega)\epsilon(\omega')\epsilon(-\omega-\omega')|\omega+\omega'|^4 \mathrm{sgn}(\omega+\omega').$$

From line to circle

We can map from the line to the circle by setting $t = \tan(\tau/2)$, $\alpha(t) = \tan(\tilde{\alpha}(\tau)/2)$.

Expanding $\alpha(t) = t + \epsilon(t)$, $\tilde{\alpha}(\tau) = \tau + \tilde{\epsilon}(\tau)$, we have $\tilde{\epsilon}(\tau) = 2\cos^2\left(\tau/2\right)\epsilon(t) + O(\epsilon^2)$.

This gives the transverse action:

$$S_T[\tilde{y}(\tilde{\alpha}(\tau))] = -\frac{T_s}{2\pi}\int d\tau d\tau' \frac{\tilde{y}(\tilde{\alpha}(\tau))\tilde{y}(\tilde{\alpha}(\tau))}{\left[2\sin\left(\frac{\tau-\tau'}{2}\right)\right]^2}.$$

And the longitudinal action:

$$S_L = \frac{6T_s}{\pi}\int d\tau d\tau' \frac{\tilde{\epsilon}(\tau)\tilde{\epsilon}(\tau')}{\left[2\sin\left(\frac{\tau-\tau'}{2}\right)\right]^4} - \frac{12T_s}{\pi}\int d\tau d\tau' \frac{\sin(\tau-\tau')}{|\,2\sin\left(\frac{\tau-\tau'}{2}\right)|^6}\epsilon(\tau)^2\epsilon(\tau').$$

Letting $\epsilon(\tau) = \sum_{n\in\mathbb{Z}} \epsilon_n e^{-in\tau}$, we find the longitudinal action in Fourier space:

$$S_L = 2\pi T_s \sum_{n\in\mathbb{Z}}^{\infty} |\,n\,|(n^2-1)\epsilon_n\epsilon_{-n} + i\pi T_s \sum_{n,m\in\mathbb{Z}} \epsilon_m\epsilon_n\epsilon_{-m-n}\mathrm{sgn}(m+n)(m+n)^2((m+n)^2-1).$$

Note: $n = 0, \pm 1$ are zero modes.

Reparametrization path integral

The result of the classical analysis of the string in conformal gauge is:

$$Z[\tilde{y}] \approx e^{-S_{\text{cl}}[\tilde{y}]} = \underset{\alpha}{\text{extremize}}\{e^{-S_L[\alpha]-S_T[\tilde{y}(\alpha)]}\}$$

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Now promote this to an integral over reparametrizations [Rychkov '02; Ambjorn, Makeenko '12, …]:

$$Z[\tilde{y}] \sim \int \mathscr{D}\alpha e^{-S_L[\alpha]-S_T[\tilde{y}(\alpha)]}$$

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$$Z_{\mathrm{JT}}[\tilde{y}] = \int \mathscr{D}\alpha\, e^{-S_{\mathrm{Schw.}}[\alpha]-S_{\mathrm{matter}}[\tilde{y},\alpha]}$$

[Kitaev; Almheiri, Polchinksi; Maldacena, Stanford, Yang; Jensen; Mertens, Engelsoy, Verlinde; Bagrets, Altland, Kamenev; Stanford, Witten; Mertens, Turiaci, Verlinde; Kitaev, Suh; …]

Correlators in the rep. path integral

Recall that $\langle y(x_1)\ldots\rangle = \frac{1}{Z}\left(\frac{\delta}{\delta\tilde{y}(x_1)}\ldots\right)Z[\tilde{y}]\Big|_{\tilde{y}=0}$.

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$\tilde{y}$ only appears in $S_T[\tilde{y}(\alpha)] = -\frac{T_s}{2\pi}\int dtdt'\frac{\tilde{y}(\tilde{\alpha}(t))\tilde{y}(\tilde{\alpha}(t'))}{(t-t')^2}$, and therefore:

$$-\frac{\delta^2 S_T}{\delta\tilde{y}(x_1)\delta\tilde{y}(x_2)} = \frac{T_s}{\pi}\frac{\dot{\beta}(x_1)\dot{\beta}(x_2)}{(\beta(x)-\beta(x'))^2} \equiv \frac{T_s}{\pi}B(x_1,x_2)$$

Where $\alpha(\beta(x)) = x$, $\beta(\alpha(t)) = t$.

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Where $\alpha(\beta(x)) = x$, $\beta(\alpha(t)) = t$. Thus, two, four, and six-point functions take the form:

$$\langle y(x_1)y(x_2)\rangle = \frac{T_s}{\pi}\int\mathscr{D}\alpha e^{-S_L[\alpha]}B(x_1,x_2), \qquad \langle y(x_1)\ldots y(x_4)\rangle = \frac{T_s^2}{\pi^2}\int\mathscr{D}\alpha e^{-S_L[\alpha]}\Big[B(x_1,x_2)B(x_3,x_4) + 2\text{ perms}\Big].$$

$$\langle y(x_1)\ldots y(x_6)\rangle = \frac{T_s^3}{\pi^3}\int\mathscr{D}\alpha e^{-S_L[\alpha]}\Big[B(x_1,x_2)B(x_3,x_4)B(x_5,x_6) + 14\text{ perms}\Big]$$

$SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ in rep. path integral

We have two $SL(2,\mathbb{R})$ symmetries:

(1) $t + is \to \dfrac{a(t+is)+b}{c(t+is)+d}, \qquad \alpha(t) \to \alpha\left(\dfrac{at+b}{ct+d}\right)$

This is a gauge symmetry, and needs to be gauge fixed in the path integral.

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(2) $x + iz \to \dfrac{a(x+iz)+b}{c(x+iz)+d}, \qquad \alpha(t) \to \dfrac{a\alpha(t)+b}{c\alpha(t)+d}$

This is a physical symmetry and leads to Ward identities of the form:

$$\langle y(x_1)y(x_2)\rangle = \dot{f}(x_1)\dot{f}(x_2)\langle y(f(x_1))y(f(x_2))\rangle.$$

Perturbation theory in the reparametrization path integral

Let $\alpha(t) = t + \epsilon(t)$. Then $\beta(x) = x - \epsilon(x) + \epsilon(x)\dot{\epsilon}(x) + O(\epsilon^3)$, and

$$B(x_1, x_2) = \frac{\dot{\beta}(x_1)\dot{\beta}(x_2)}{(\beta(x_1) - \beta(x_2))^2} = \frac{1}{x_{12}^2}\left(1 + \mathscr{B}_1(x_1, x_2) + \mathscr{B}_2(x_1, x_2) + O(\epsilon^3)\right).$$

Where

$$\mathscr{B}_1(x_1, x_2) = \dot{\epsilon}_1 + \dot{\epsilon}_2 - \frac{2}{x_{12}}\epsilon_{12},$$

$$\mathscr{B}_2(x_1, x_2) = \frac{3}{x_{12}^2}\epsilon_{12}^2 + \frac{2}{x_{12}}(-2\epsilon_1\dot{\epsilon}_1 + 2\epsilon_2\dot{\epsilon}_2 + \dot{\epsilon}_1\epsilon_2 - \dot{\epsilon}_2\epsilon_1) + \dot{\epsilon}_1^2 + \dot{\epsilon}_2^2 + \dot{\epsilon}_1\dot{\epsilon}_2 + \ddot{\epsilon}_1\epsilon_1 + \ddot{\epsilon}_2\epsilon_2$$

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The ϵ propagator in Fourier space and position space is given by:

$$\langle \epsilon(\omega)\epsilon(\omega') \rangle = \frac{\pi}{T_s}\delta(\omega + \omega')\frac{1}{|\omega|^3}, \qquad \langle \epsilon(x)\epsilon(0) \rangle = \frac{1}{4\pi T_s}\int d\omega \frac{e^{-i\omega x}}{|\omega|^3} = \frac{1}{T_s}\left[a + bx^2 + \frac{1}{8\pi}x^2\log(x^2)\right].$$

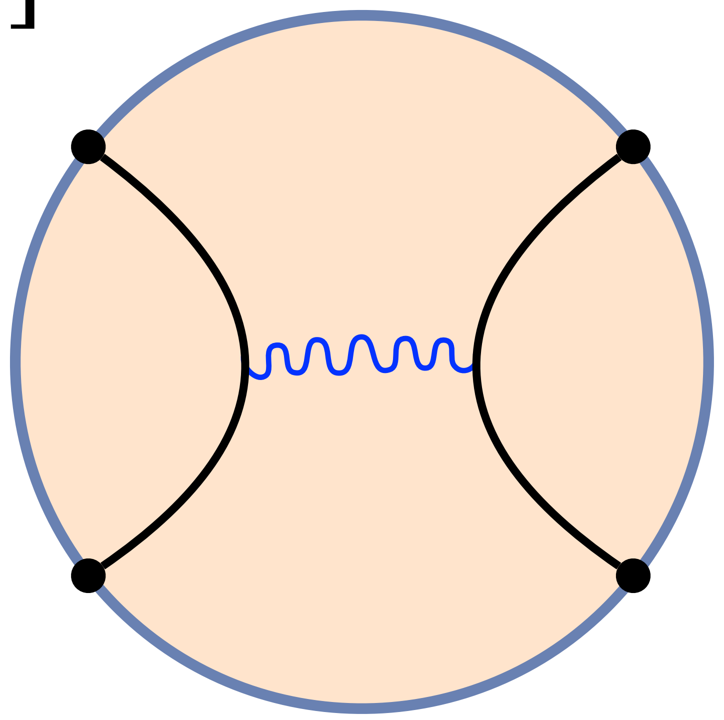
Four-point function

$$\langle y(x_1)y(x_2)y(x_3)y(x_4)\rangle_{\text{conn.}} = \frac{(T_s/\pi)^2}{x_{12}^2 x_{34}^2}\left(\langle \mathscr{B}_1(x_1,x_2)\mathscr{B}_1(x_3,x_4)\rangle + \text{perms} + O(1/T_s^2)\right),$$

We find

$$\langle \mathscr{B}_1(x_1,x_2)\mathscr{B}_1(x_3,x_4)\rangle = G(\xi_{1234}) \equiv -\frac{1}{4\pi T_s}\left[4 + \frac{1+\xi_{1234}}{1-\xi_{1234}}\log\left(\xi_{1234}^2\right)\right].$$

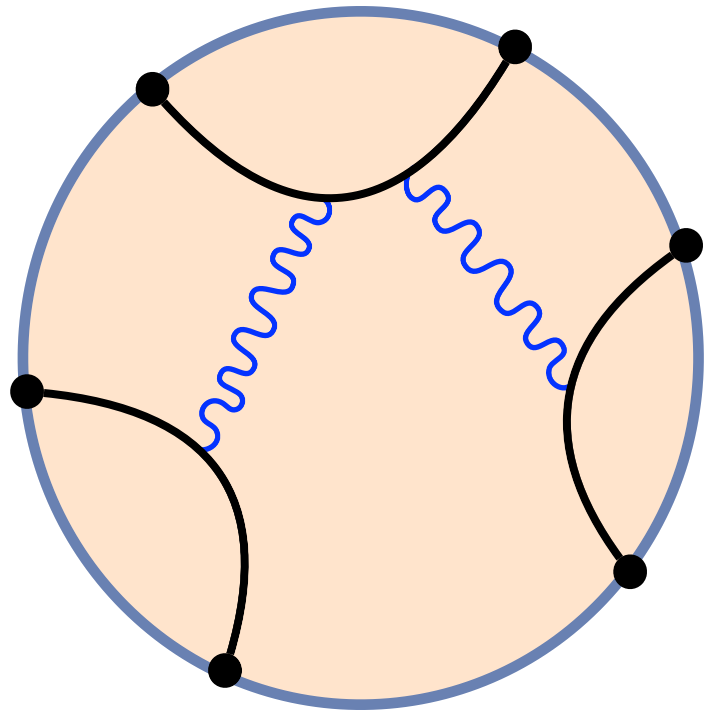
where $\xi_{ijkl} = \dfrac{x_{ik}x_{jl}}{x_{il}x_{jk}}$.

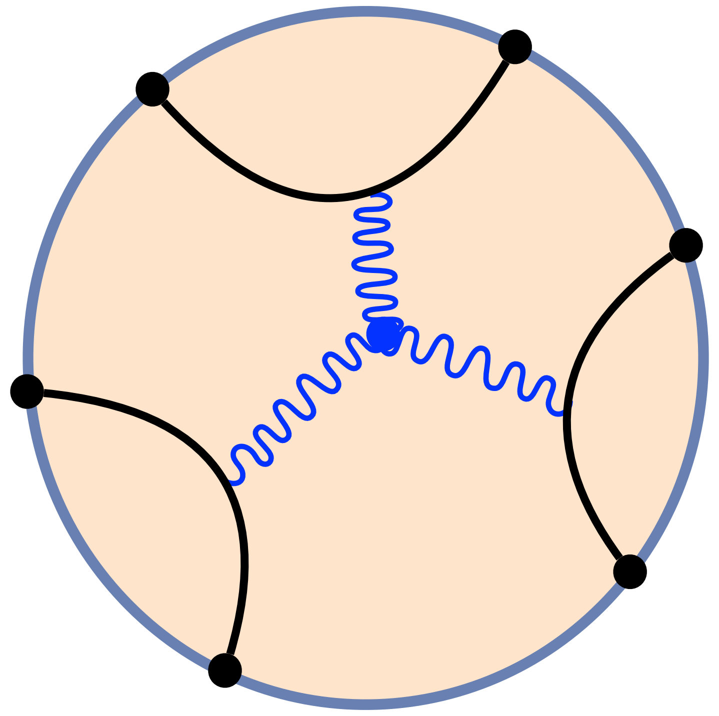


This reproduces the four-point contact diagram computed in static gauge.

Six-point function

$$\langle y(x_1)\ldots y(x_6)\rangle_{\text{conn.}} = \frac{(T_s/\pi)^3}{x_{12}^2 x_{34}^2 x_{56}^2}\Big(\langle \mathscr{B}_1(x_1,x_2)\mathscr{B}_1(x_3,x_4)\mathscr{B}_2(x_5,x_6)\rangle + \langle \mathscr{B}_1(x_1,x_2)\mathscr{B}_1(x_3,x_4)\mathscr{B}_1(x_5,x_6)S_{L,3}\rangle + \text{perms}\Big)$$



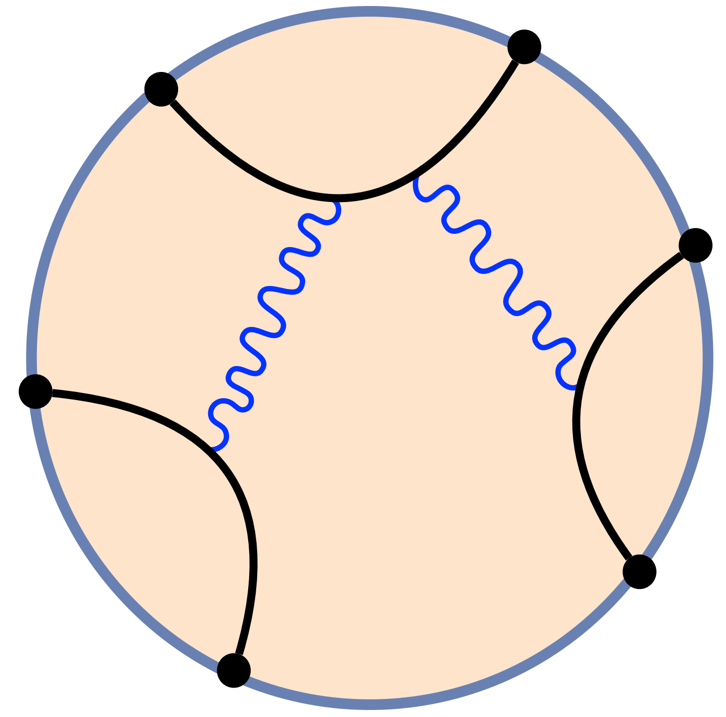


Six-point function

$$\langle y(x_1)\ldots y(x_6)\rangle_{\text{conn.}} = \frac{(T_s/\pi)^3}{x_{12}^2 x_{34}^2 x_{56}^2}\big(\langle \mathscr{B}_1(x_1,x_2)\mathscr{B}_1(x_3,x_4)\mathscr{B}_2(x_5,x_6)\rangle + \langle \mathscr{B}_1(x_1,x_2)\mathscr{B}_1(x_3,x_4)\mathscr{B}_1(x_5,x_6)S_{L,3}\rangle + \text{perms}\big)$$

The new ingredient is $\langle \epsilon(x_1)\epsilon(x_2)\epsilon(x_3)S_{L,3}\rangle$. Recall the Fourier space representation of the propagator:

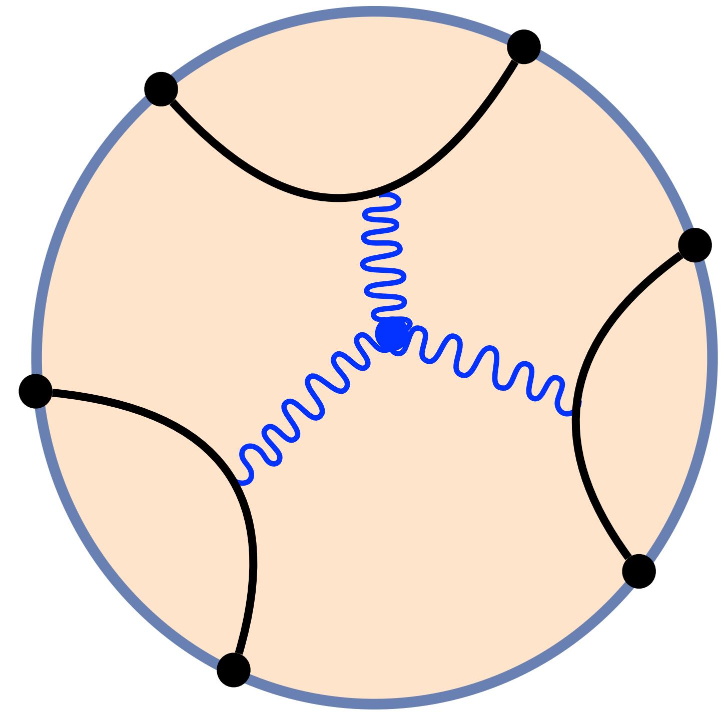
$$\langle \epsilon(\omega)\epsilon(\omega')\rangle = \frac{\pi}{T_s}\delta(\omega+\omega')\frac{1}{|\omega|^3}$$



and the cubic action

$$S_{L,3} = \frac{iT_s}{8\pi^2}\int d\omega d\omega' \epsilon(\omega)\epsilon(\omega')\epsilon(-\omega-\omega')\underbrace{|\omega+\omega'|^4 \,\text{sgn}(\omega+\omega')}_{|\omega+\omega'|^3(\omega+\omega')}$$

It follows that the interacting diagram effectively "factorizes:"



$$\langle \epsilon(x_1)\epsilon(x_2)\epsilon(x_3)S_{L,3}\rangle = -\frac{1}{2}\partial_{x_3}\left[\langle \epsilon(x_1)\epsilon(x_3)\rangle\langle \epsilon(x_2)\epsilon(x_3)\rangle\right] + \text{perms}$$

Six-point function

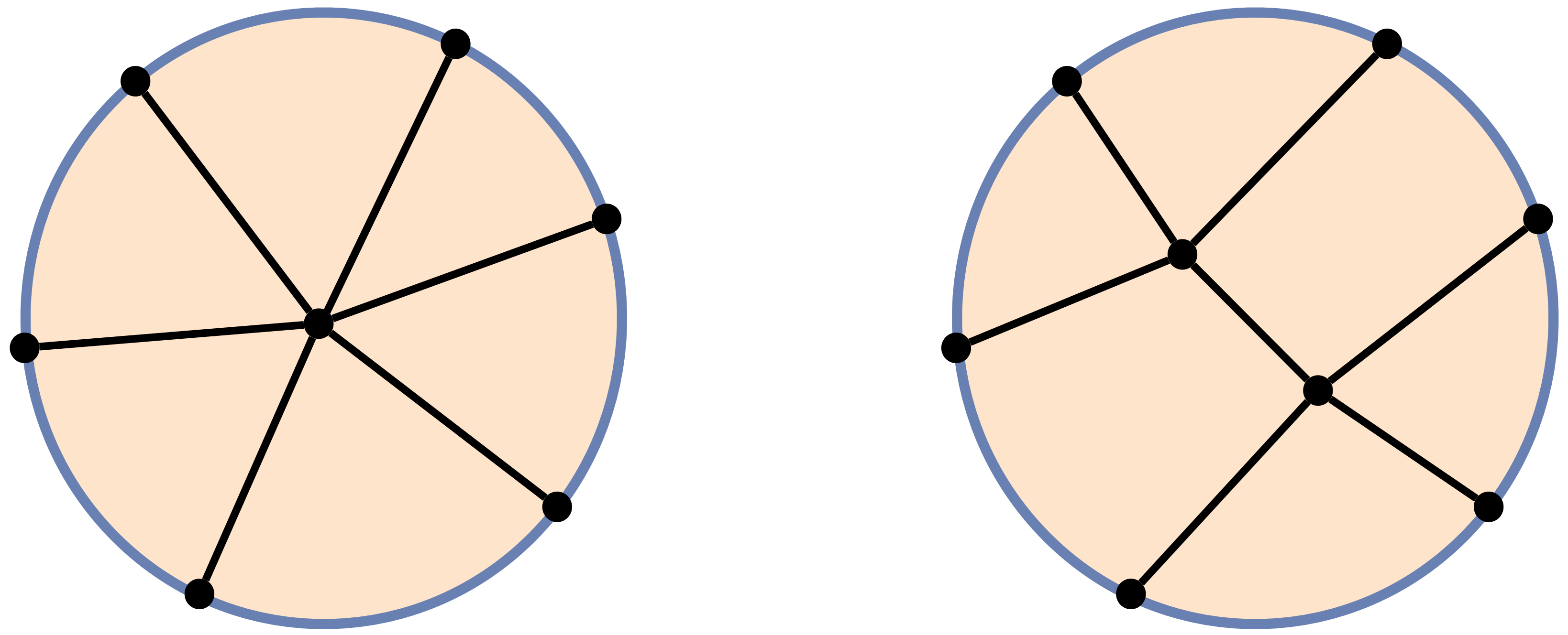
$$\frac{\langle y(x_1)\ldots y(x_6)\rangle_{\text{conn.}}}{\langle y(x_1)y(x_2)\rangle\langle y(x_3)y(x_4)\rangle\langle y(x_5)y(x_6)\rangle} = G(\xi_{1234})G(\xi_{1256}) + G(\xi_{1234})G(\xi_{3456}) + G(\xi_{1256})G(\xi_{3456})$$

$$+F_{12;3456}\xi_{3456}G'(\xi_{3456}) + F_{34;1256}\xi_{1256}G'(\xi_{1256}) + F_{56;1234}\xi_{1234}G'(\xi_{1234}) + \text{perms}$$

Where

$$F_{ij;kl;mn} = -\frac{1}{8\pi T_s}\left[\frac{x_{ki}x_{kj}x_{mn}}{x_{ij}x_{km}x_{kn}}\log\left(\frac{x_{ki}^2}{x_{kj}^2}\right) - \frac{x_{li}x_{lj}x_{mn}}{x_{ij}x_{lm}x_{ln}}\log\left(\frac{x_{li}^2}{x_{lj}^2}\right) + \frac{x_{im}x_{jm}x_{kl}}{x_{ij}x_{km}x_{lm}}\log\left(\frac{x_{im}^2}{x_{jm}^2}\right) - \frac{x_{in}x_{jn}x_{kl}}{x_{ij}x_{kn}x_{ln}}\log\left(\frac{x_{in}^2}{x_{jn}^2}\right)\right]$$

Six-point function



Compare with static gauge.

There are contributions from both contact and exchange diagrams, with the interaction vertices:

$$L_4 = -\frac{1}{8}(g^{\alpha\beta}\partial_\alpha y \partial_\beta y)^2, \qquad L_6 = \frac{1}{16}(g^{\alpha\beta}\partial_\alpha y \partial_\beta y)^3$$

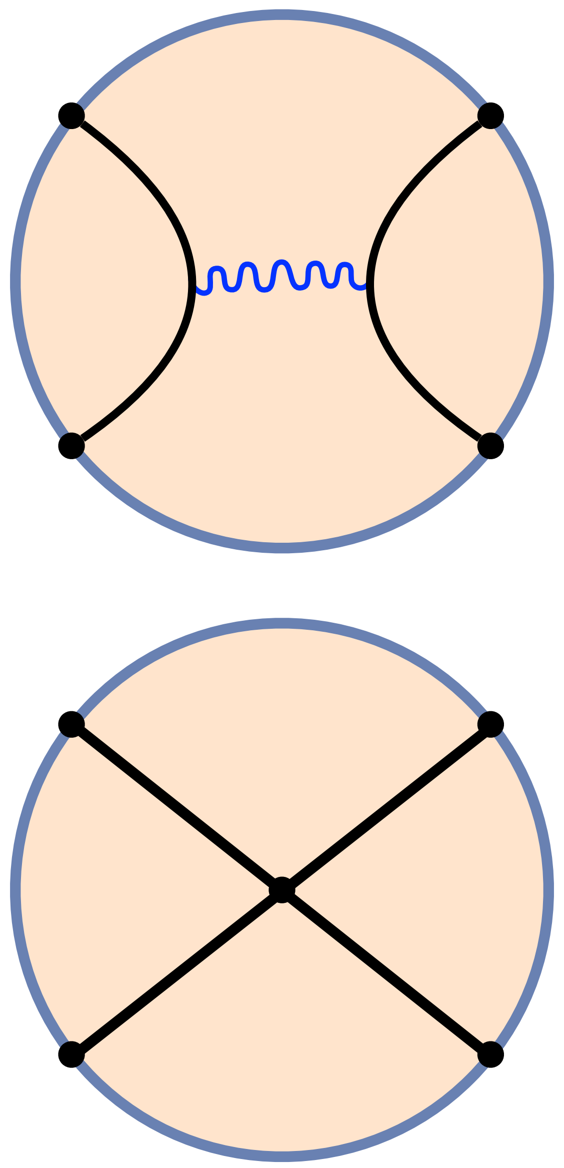
Contact diagrams are analytically tractable (but combinatorially a bit messy) [Bliard '22]. It seems that the exchange diagrams should be too.

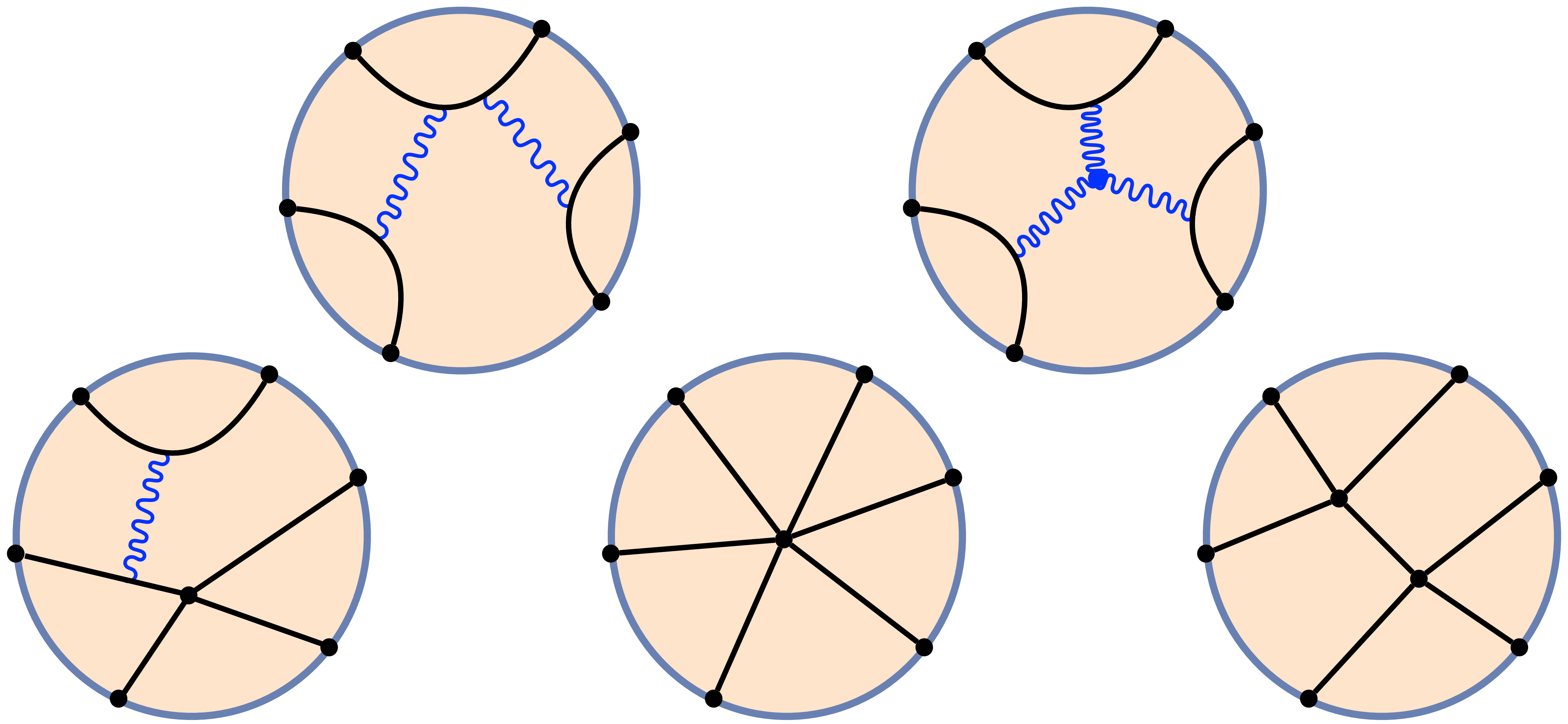
In a more pedestrian approach, we can evaluate the contact and exchange diagrams numerically and check the conformal gauge results.

Comments on the Boundary Reparametrization Method

It becomes more complicated if we try to compute correlators with different scalars, Φ^I, $I = 1,\ldots,5$. String in $\mathrm{AdS}_2 \times S^5$, action in conformal gauge:

$$S_T[y^I] = \frac{T_s}{2}\int d^2\sigma \frac{\partial_\alpha y^I \partial^\alpha y^I}{\left(1+\frac{1}{4}y^2\right)^2} = T_s \int d^2\sigma \Big[\frac{1}{2}\partial_\alpha y^I \partial^\alpha y^I - \frac{1}{2} y^J y^J (\partial_\alpha y^I \partial^\alpha y^I) + \frac{3}{16}(y^J y^J)^2 \partial_\alpha y^I \partial^\alpha y^I + \ldots\Big]$$





Comments on the Boundary Reparametrization Method

- Can we compute correlators of displacement operators? Is there a clean separation between longitudinal and transverse modes for the string in AdS_3?

  See also: [Kruczenski ’14; Dekel ’15; Kruczenski, He, Huang ’15-‘17]

- Can we compute boundary correctors on higher dimensional branes?

Summary

- We showed how to compute tree-level correlators of the transverse fluctuations of the AdS_2 string in conformal gauge. We saw that a key role is played by the reparametrizations of the boundary of the string. The boundary reparametrization mode is formally analogous to the Schwarzian mode in JT gravity.

- When applicable, the computation in conformal gauge appears easier than in static gauge.

- *One thing I did not discuss:* Four and six-point OTOCs on the AdS_2 string, which diagnose chaos. They correspond to scattering processes on the string worldsheet, which are controlled by the flat space scattering phase. They can be computed in the reparametrization path integral by doing an eikonal resummation of diagrams with reparametrization mode exchanges.

Open questions

What more can we do with the reparametrization path integral?

- Can it be defined more precisely and used to compute other observables? (E.g., loop corrections to two- or four-point functions? Partition function?)

- Is there a similar reparametrization action for the AdS$_2$ string in AdS$_3$?

- What about a non half-BPS string in AdS$_3$ or in AdS$_3 \times S^1$?

Can we compute the six-point function of arbitrary scalars? Can we extract useful OPE data or say something about the manifestation of the worldsheet integrability in the six-point function?

Can we connect the scattering analysis of the OTOC to flat space or bulk point limits in AdS?

Thank you for listening

Extra slides

OTOCs and scattering on the AdS_2 string

Out-of-time-order correlators (OTOCs)

OTOCs are correlators of the form $\langle y(0)y(t)y(0)y(t)\rangle_\beta$.

They are simple diagnostics of quantum chaos. [Larkin, Ovchinnikov '79; Kitaev; Shenker Stanford '14, Maldacena SS '15…] In chaotic semiclassical systems, they are characterized by a Lyapunov exponent λ_L:

$$\frac{\langle y(0)y(t)y(0)y(t)\rangle_\beta}{\langle yy\rangle_\beta\langle yy\rangle_\beta} = 1 - \epsilon e^{\lambda_L t} + O(\epsilon^2), \qquad \beta \ll t \ll \frac{1}{\lambda_L}\log\frac{1}{\epsilon}$$

The quantum Lyapunov exponent satisfies a bound [MSS '15]:

$$\lambda_L \leq \frac{2\pi}{\beta},$$

Which is saturated by systems holographically dual to gravity [SS '14], in which case the OTOC is related to a high-energy scattering process in the bulk.

OTOC on the string via static gauge

We were interested in studying OTOCs on the AdS2 string. To compute the OTOC, we can analytically continue the euclidean four-point function. Let $x_i = \tan\left(\frac{\theta_i}{2}\right)$ where (setting $\beta = 2\pi$):

$$\theta_1 = \frac{3\pi}{2}, \qquad \theta_2 = \frac{\pi}{2}, \qquad \theta_3 = \pi + it, \qquad \theta_4 = it.$$

The four point function computed in static gauge is:

$$\frac{\langle y_1 y_2 y_3 y_4 \rangle}{\langle y_1 y_2 \rangle \langle y_3 y_4 \rangle} = G_{\text{free}}(\chi) + \frac{1}{2\pi T_s} G_{\text{tree}}(\chi) + \ldots,$$

Where $G_{\text{free}}(\chi) = 1 + \ldots, \quad G_{\text{tree}}(\chi) = \frac{-2 + \ldots}{2\chi} \log((1-\chi)^2) + \ldots$. Letting $\chi(t) = \frac{2}{1 - i\sinh t}$:

$$\frac{\langle yy(t) yy(t) \rangle}{\langle yy \rangle \langle yy \rangle} = 1 - \frac{e^t}{4T_s} + \ldots$$

[Maldacena, Stanford, Yang '17; Murata '17; de Boer, Llabres, Pedraza, Vegh '17]

This saturates the chaos bound: $\lambda_L = \frac{2\pi}{\beta}$.

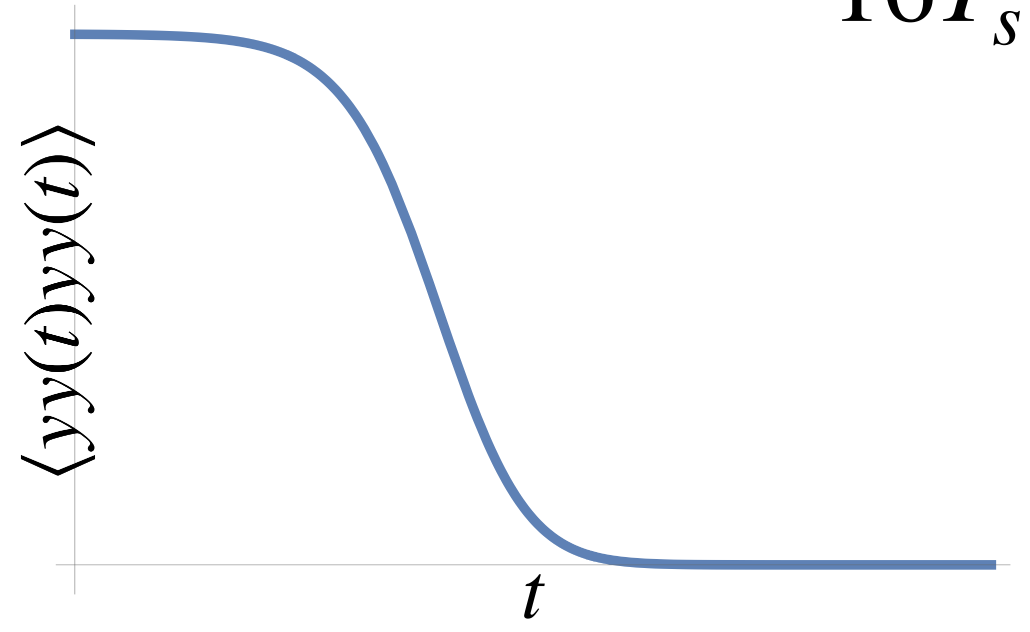
OTOC up to three-loops

[Ferrero, Meneghelli '21]

$$\frac{\langle y_1 y_2 y_3 y_4\rangle}{\langle y_1 y_2\rangle\langle y_3 y_4\rangle} = G_{\text{free}}(\chi) + \frac{1}{2\pi T_s} G_{\text{tree}}(\chi) + \frac{1}{(2\pi T_s)^2} G_{1\text{-loop}}(\chi) + \frac{1}{(2\pi T_s)^3} G_{2\text{-loop}}(\chi) + \frac{1}{(2\pi T_s)^4} G_{3\text{-loop}}(\chi) + \dots$$

$$\begin{aligned}\frac{\langle yy(t)yy(t)\rangle}{\langle yy\rangle\langle yy\rangle} &= 1 - \frac{e^t}{4T_s} + \frac{9e^{2t}}{128T_s^2} - \frac{3e^{3t}}{128T_s^3} + \frac{75e^{4t}}{8192T_s^4} + \dots \\ &= \frac{1}{\kappa^2} U(2,1,\kappa^{-1})\end{aligned}$$

taking $t, T_s \to \infty$ with $\kappa = \dfrac{e^t}{16T_s}$ fixed



This is the same result as in JT gravity(!)

[Maldacena, Stanford, Yang '16; Lam, Mertens, Turiaci, Verlinde '18]

An OTOC puzzle

This raises two questions:

- Can we derive the double scaled OTOC more simply?
- Why is the double scaled OTOC the same on the worldsheet and in JT gravity?

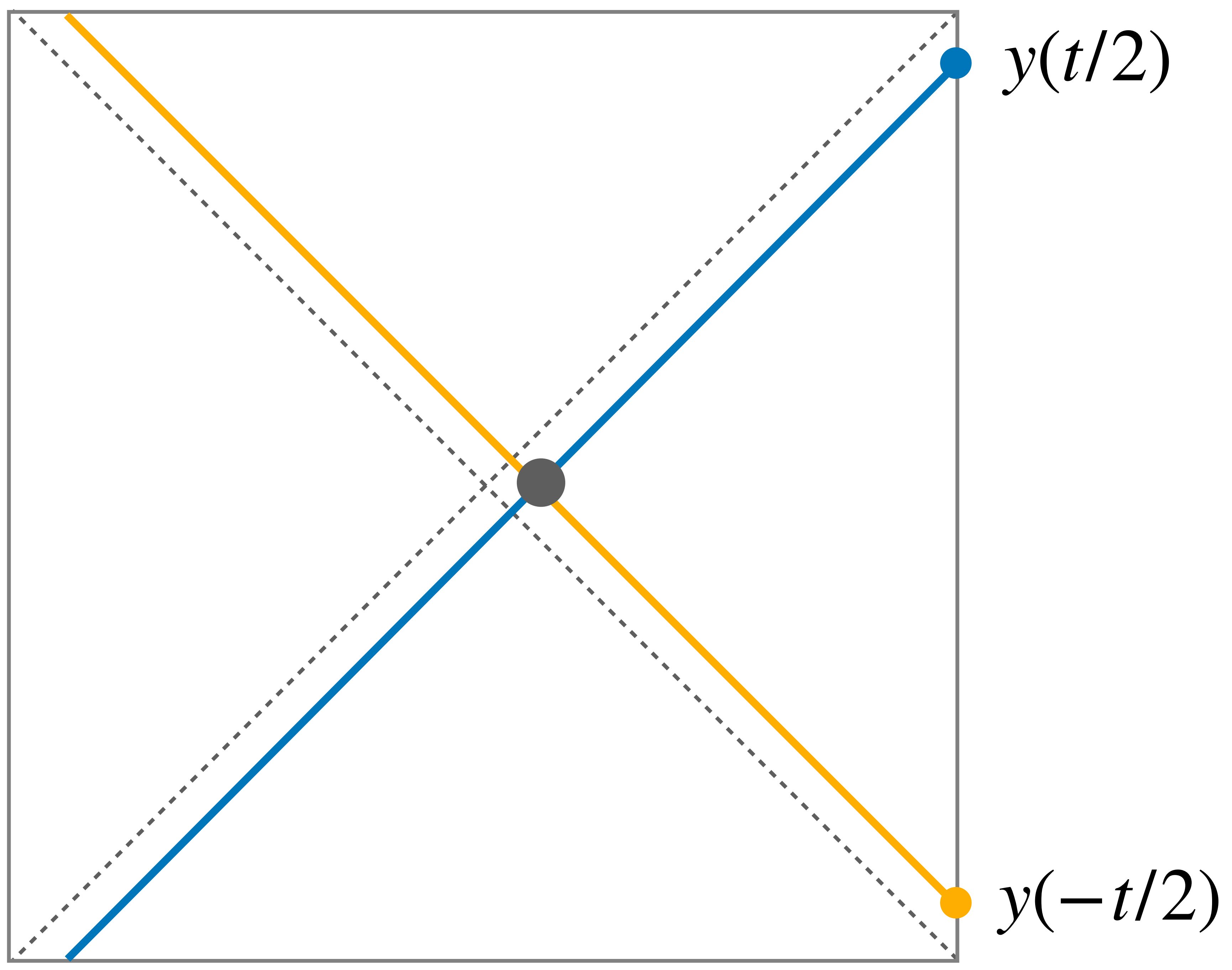
Two possible approaches:

- High-energy scattering process on the worldsheet
- “Eikonal resummation” in the reparametrization path integral <— analogous to eikonal resummation in the Schwarzian path integral in JT gravity

Scattering on AdS_2 worldsheet

[Shenker, Stanford '14;
de Boer, Llabres, Pedraza, Vegh '17]

$$\langle y(-t/2)y(t/2)y(-t/2)y(t/2)\rangle = \langle \mathrm{out}\,|\,\mathrm{in}\rangle$$



OTOC as a scattering amplitude

We write the OTOC as a scattering amplitude :

[Shenker, Stanford, '14; de Boer, Llabres, Pedraza, Vegh '17]

$$\langle y_1 y_3 y_2 y_4 \rangle = \int \prod_{i=1}^{4} dp_i \Psi(p_1^u, t_1)^* \Phi(p_3^v, t_3)^* \underbrace{\mathcal{S}(p_1^u, p_3^v; p_2^u, p_4^v)}_{p_1^u p_3^v \delta(p_1^u - p_2^u)\delta(p_3^v - p_4^v)\ e^{i\delta(s)}} \Psi(p_2^u, t_2)\Phi(p_4^v, t_4),$$

Where the wave functions are Fourier transforms of the bulk-boundary propagator (here $(u_i, v_i) = (-e^{-t_i}, e^{t_i})$):

$$\Psi(p^u, t_i) = \int dv e^{2ip^u v} K_\Delta(0, v, t_i), \qquad \Phi(p^v, t_i) = \int du e^{2ip^v u} K_\Delta(u, 0, t_i),$$

And $e^{i\delta(s)}$ is the scattering phase for excitations propagating on the infinitely long string in *flat space*:

$$e^{i\delta(s)} = e^{i\ell_s^2 p^u p^v}$$

[Dubovsky, Flauger, Gorbenko '12; Lam, Mertens, Turiaci, Verlinde '18]

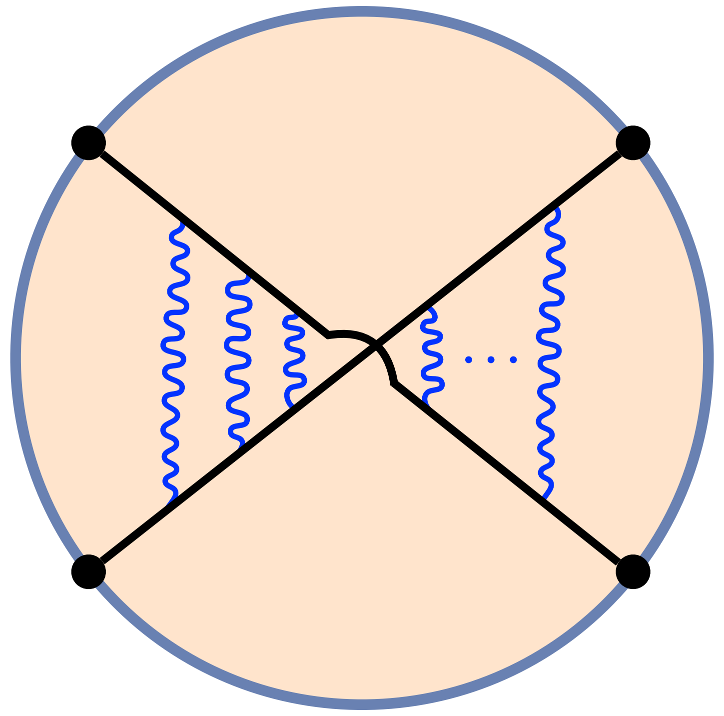
The end result is precisely:

$$\langle y_1 y_3 y_2 y_4 \rangle = \frac{1}{\kappa^2} U(2, 1, \kappa^{-1})\,.$$

OTOC from eikonal resummation

We can also reproduce the double scaled OTOC from the reparametrization path integral.

We do an "eikonal resummation" of the diagrams in which the reparametrizations are exchanged directly between the scattering particles.

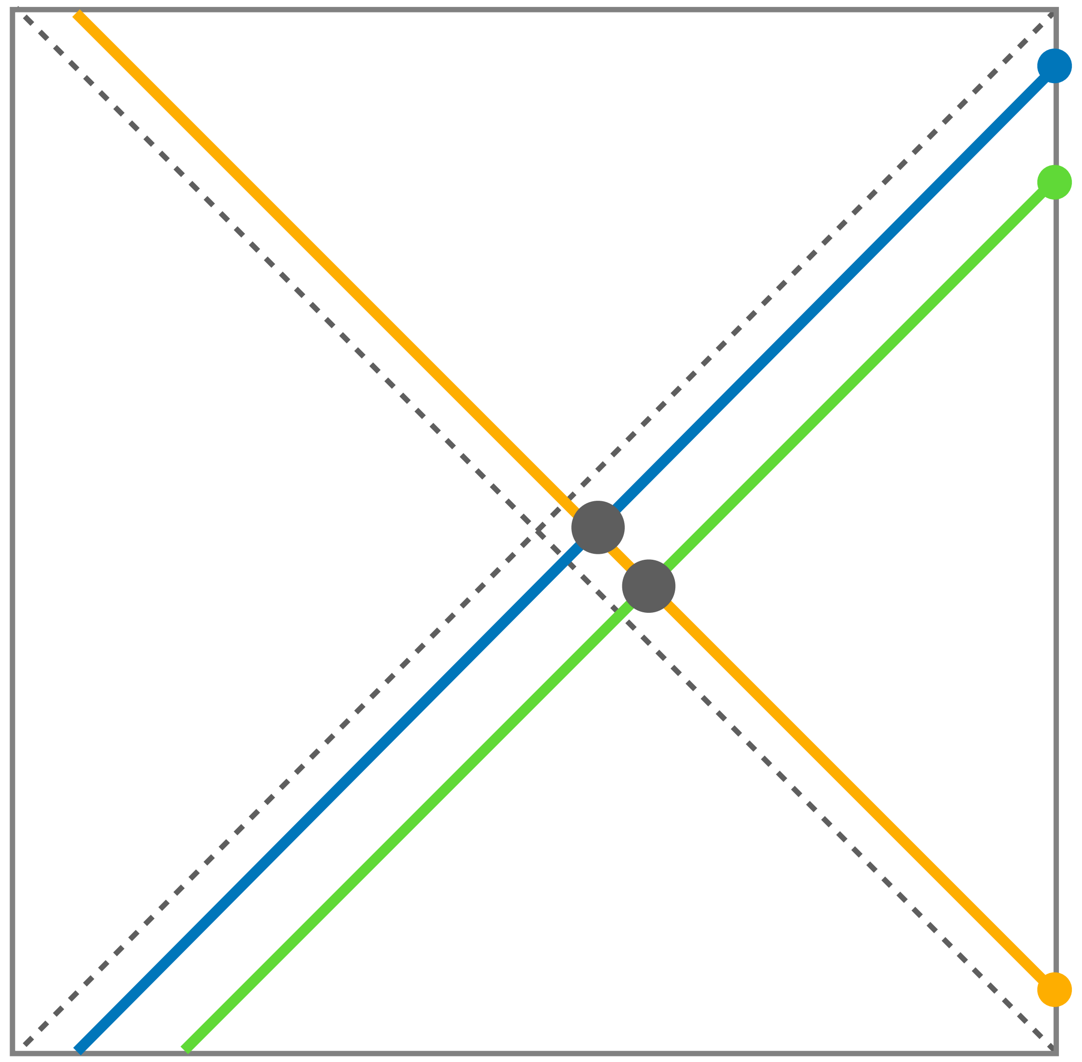


This corresponds to keeping only the quadratic part of the reparametrization action.

The end result is again:

$$\langle y_1 y_3 y_2 y_4 \rangle = \int \mathscr{D}\alpha e^{-S_2[\alpha]} B(x_1, x_2) B(x_3, x_4) = \frac{1}{\kappa^2} U(2,1,\kappa^{-1}) \,.$$

Six-point OTOC



$$\frac{\langle y_1 y_5 y_2 y_3 y_6 y_4 \rangle}{\langle y_1 y_2 \rangle \langle y_3 y_4 \rangle \langle y_5 y_6 \rangle} = \int dp^u dp^v dq^v \Psi(p^u, t_5)^* \Phi(p^v, t_2)^* \Phi(q^v, t_3)^* \; e^{i\ell_s^2 p^u p^v} e^{i\ell_s^2 p^u q^v} \; \Psi(p^u, t_6) \Phi(p^v, t_1) \Phi(q^v, t_4)$$

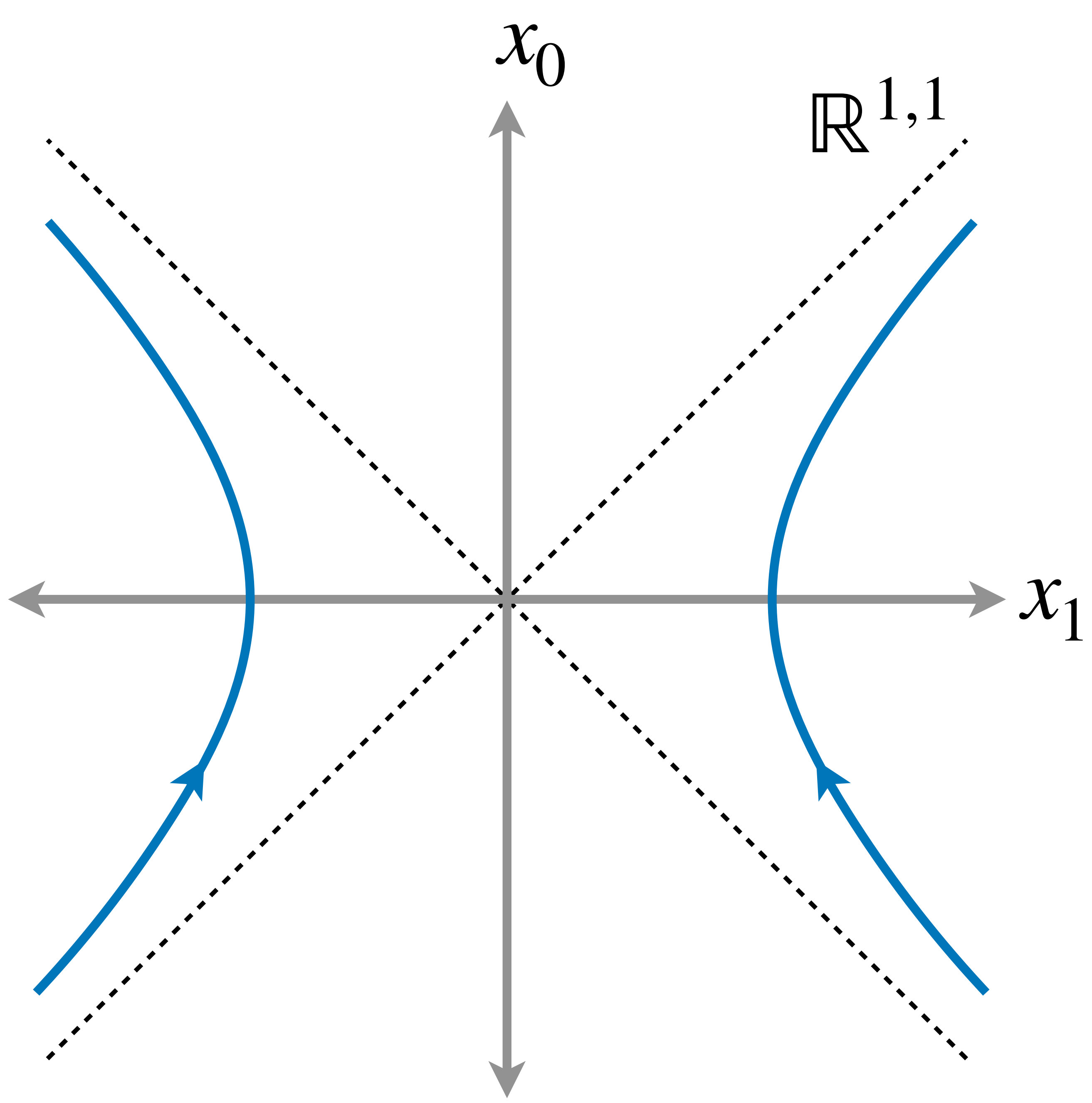
$$= \int \mathscr{D}\alpha e^{-S_2[\alpha]} B(x_1, x_2) B(x_3, x_4) B(x_5, x_6)$$

$$= 1 - \frac{e^{t_a - t_b}}{2\sqrt{3}T_s} - \frac{e^{t_b - t_c}}{2\sqrt{3}T_s} + \boxed{\frac{1}{8}\frac{e^{t_a - t_c} e^{t_b - t_c}}{T_s^2}} + \ldots$$

This term matches the analytic continuation of the connected six point function

More details about OTOCs and scattering on the AdS_2 string

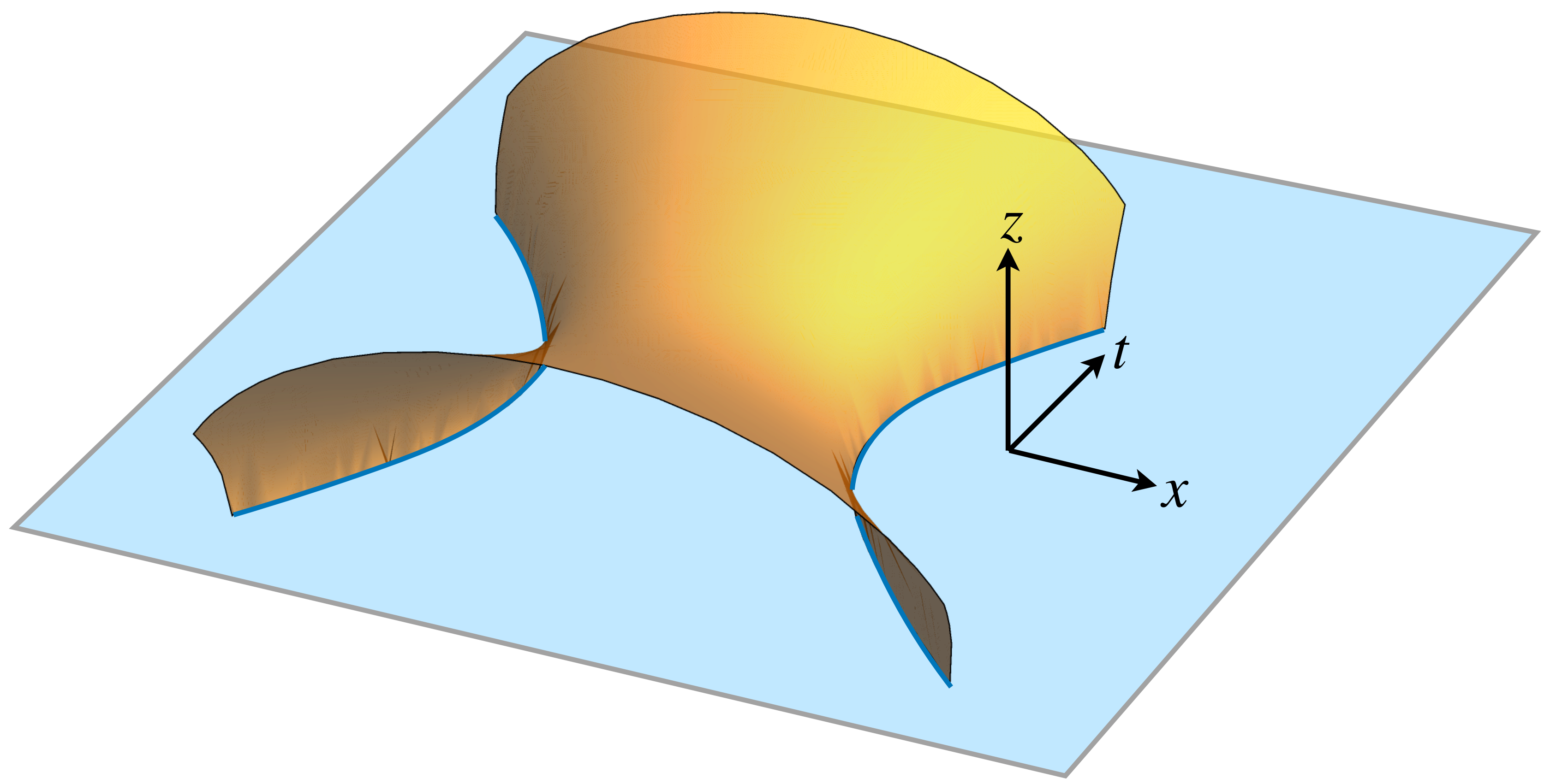
Scattering on AdS_2 worldsheet



$$x_0(t) = \sinh t, \quad x_1(t) = \pm \cosh t$$

$$x_1^2 - x_0^2 = 1$$

[Xiao ’08; Jensen, Karch ’13; Sonner ’13, …]

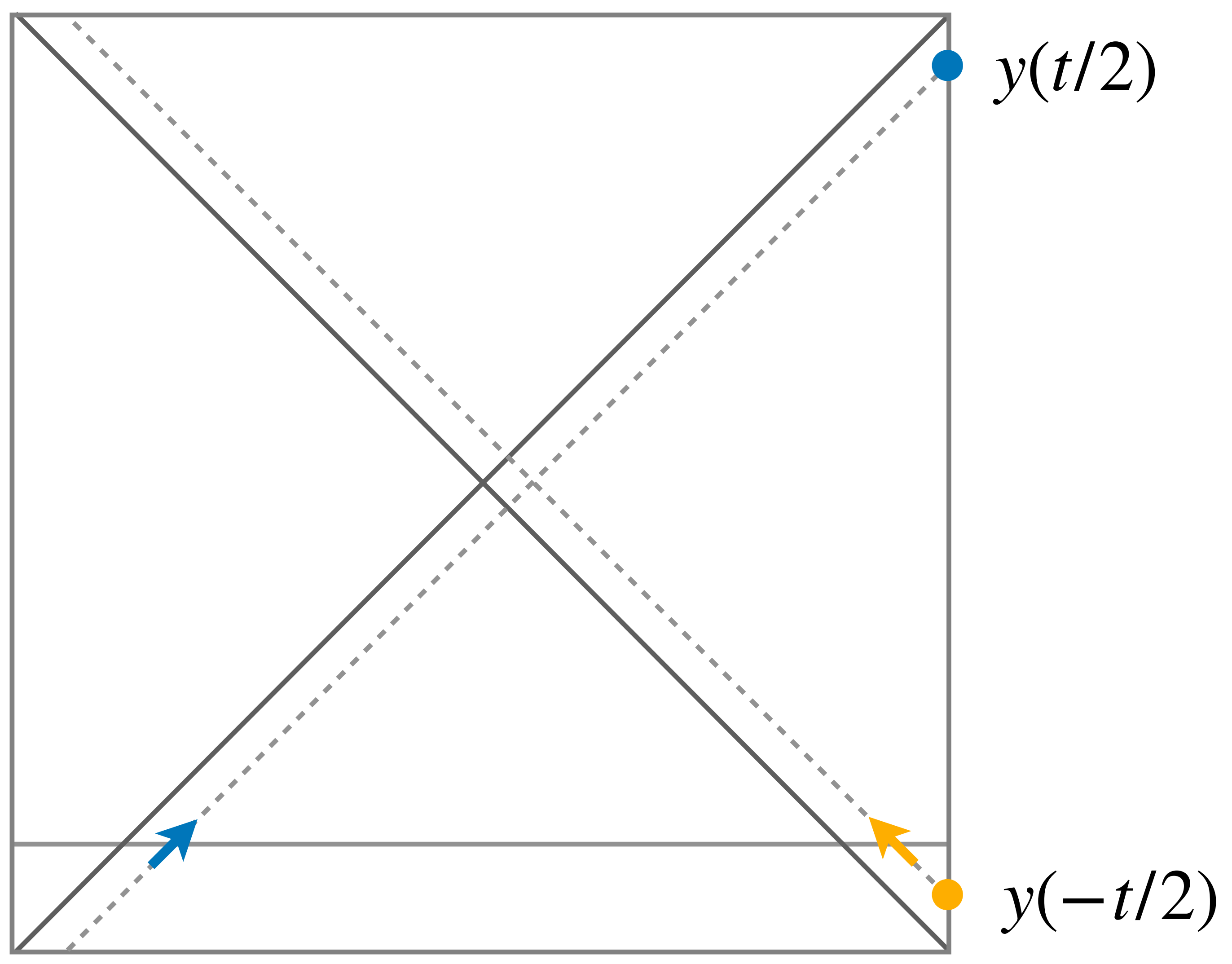


$$-x_0^2 + x_1^2 + z^2 = 1$$

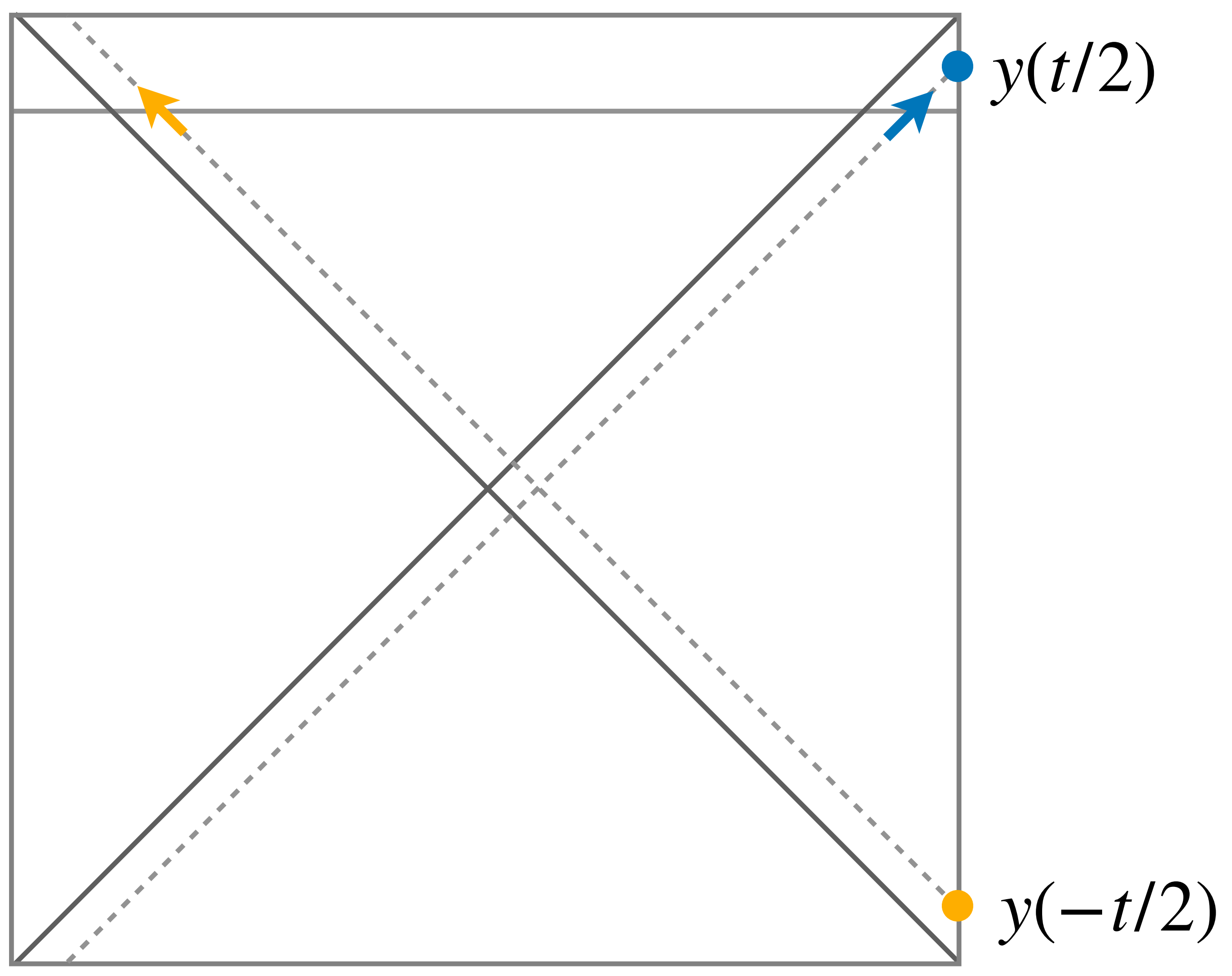
Scattering on AdS$_2$ worldsheet

[Shenker, Stanford '14;
de Boer, Llabres, Pedraza, Vegh '17]

$$\langle y(-t/2)y(t/2)y(-t/2)y(t/2)\rangle = \langle \mathrm{out}|\mathrm{in}\rangle$$



$$|\mathrm{in}\rangle = y(-t/2)y(t/2)|0\rangle$$



$$|\mathrm{out}\rangle = y(t/2)y(-t/2)|0\rangle$$

OTOC as a scattering amplitude

We write the OTOC as a scattering amplitude : [Shenker, Stanford, '14; de Boer, Llabres, Pedraza, Vegh '17]

$$\langle V_1 W_3 V_2 W_4 \rangle = \int \prod_{i=1}^{4} dp_i \Psi_{\Delta_W}(p_1^u, t_1)^* \Phi_{\Delta_W}(p_3^v, t_3)^* \mathcal{S}(p_1^u, p_3^v; p_2^u, p_4^v) \Psi_{\Delta_V}(p_2^u, t_2) \Phi_{\Delta_W}(p_4^v, t_4),$$

The wave functions are FTs of the bulk-boundary propagator, $K_\Delta = c_\Delta \left(\frac{(1+uv)}{(1+uv_i)(1+vu_i)} \right)^\Delta$:

$$\Psi_\Delta(p^u, t_i) = \int dv e^{2ip^u v} K_\Delta(0, v, t_i) = \theta(p^u) \frac{(2ip^u v_i)^\Delta}{\sqrt{\Gamma(2\Delta)} p^u},$$

$$\Phi_\Delta(p^v, t_i) = \int du e^{2ip^v u} K_\Delta(u, 0, t_i) = \theta(p^v) \frac{(2ip^v u_i)^\Delta}{\sqrt{\Gamma(2\Delta)} p^v}$$

Here, $(u_i, v_i) = (-e^{-t_i}, e^{t_i})$.

The S-matrix is diagonal: $\mathcal{S}(p_1^u, p_3^v; p_2^u, p_4^v) = p_1^u p_3^v \delta(p_1^u - p_2^u) \delta(p_3^v - p_4^v) e^{2i\delta(p_1^u, p_3^u)}$.

Scattering on the long flat string

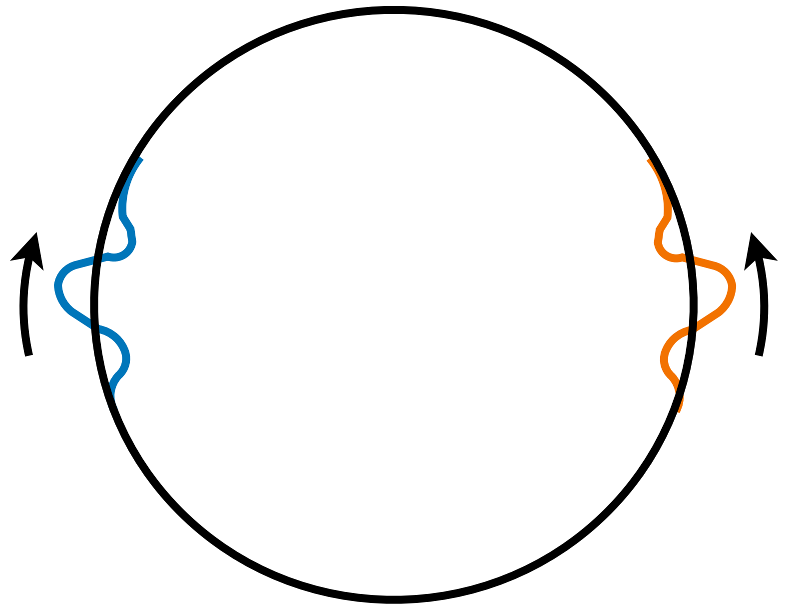
[Dubovsky, Flauger, Gorbenko '12]

$$\boxed{e^{2i\delta(p_1^u,p_3^u)} = e^{i\ell_s^2 p^u p^v} = e^{\frac{i}{4}s\ell_s^2}}$$

where $s = 4p^u p^v, \ell_s = T_s^{-1}$

$$e^{-iE_{2p}t} = e^{-i2E_{1p}t+2i\delta(s)\frac{2t}{R}}$$

$$p^v = \frac{N_R}{R}$$



$$p^u = \frac{N_L}{R}$$

$$E(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^4}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$

OTOC as a scattering amplitude

We write the OTOC as a scattering amplitude:

$$\langle V_1 W_3 V_2 W_4 \rangle = (4v_1v_2)^{\Delta_V}(4u_3u_4)^{\Delta_W}\int dp^u dp^v \frac{(p^u)^{2\Delta_V-1}(p^v)^{2\Delta_W-1}}{\Gamma(2\Delta_V)\Gamma(2\Delta_W)} e^{2ip^u(v_2-v_1)}e^{2ip^v(u_4-u_3)}e^{i\ell_s^2 p^u p^v}$$

$$= \langle V_1 V_2\rangle\langle W_3 W_4\rangle \kappa^{-2\Delta_V} U(2\Delta_V, 1+2\Delta_V-2\Delta_W, \kappa^{-1}), \qquad \kappa = \frac{i\ell_s^2}{4(v_1-v_2)(u_3-u_4)} \to \frac{e^t}{16T_s}.$$

Similar to the case of OTOCs in Einstein gravity [Shenker, Stanford '14]:

Boosts near the horizon: $s \propto e^t$, Gravitational interaction: $e^{i\delta_{\text{grav.}}(s)}$, with $\delta_{\text{grav.}}(s) \sim G_N s$

Essentially the same analysis holds in the case of OTOCs in JT gravity [Lam, Mertens, Turiaci, Verlinde '18]

Doubled-scaled OTOC

We put the component of $B(x_1, x_2)$ into the exponent: $(1+\dot{\epsilon}_i)^\Delta = \dfrac{\partial^\Delta}{\partial \alpha_i^\Delta} e^{\alpha_i(1+\dot{\epsilon}_i)}$, $\dfrac{1}{(1+\frac{\epsilon_{ij}}{x_{ij}})^{2\Delta}} = \dfrac{1}{\Gamma(2\Delta)} \displaystyle\int_0^\infty dp p^{2\Delta-1} e^{-p(1+\frac{\epsilon_{ij}}{x_{ij}})}$

Then:

$$\frac{\langle V_1 V_2 W_3 W_4 \rangle}{\langle V_1 V_2 \rangle \langle W_3 W_4 \rangle} = \prod_i \frac{\partial^{\Delta_i}}{\partial \alpha_i^{\Delta_i}} \left[\prod_i e^{\alpha_i} \int_0^\infty \frac{p^{2\Delta_V - 1} dp}{\Gamma(2\Delta_V)} e^{-p} \int_0^\infty \frac{q^{2\Delta_V - 1} dq}{\Gamma(2\Delta_W)} e^{-q} \times X \right]$$

Where

$$X = \left\langle \exp\left(-p\frac{\epsilon_{12}}{x_{12}} - q\frac{\epsilon_{34}}{x_{34}} + \sum_{i=1}^{4} \alpha_i \dot{\epsilon}_i \right) \right\rangle = \exp\left\langle \frac{1}{2}\left(-p\frac{\epsilon_{12}}{x_{12}} - q\frac{\epsilon_{34}}{x_{34}} + \sum_{i=1}^{4} \alpha_i \dot{\epsilon}_i \right)^2 \right\rangle.$$

In the double scaling limit, $\log X = \dfrac{pq}{x_{12}x_{34}} \langle \epsilon_{12}\epsilon_{34} \rangle + \ldots \to -\kappa pq$ where $\dfrac{e^t}{16 T_s}$. Therefore:

$$\frac{\langle V_1 V_2 W_3 W_4 \rangle}{\langle V_1 V_2 \rangle \langle W_3 W_4 \rangle} = \int_0^\infty \frac{p^{2\Delta_V - 1} dp}{\Gamma(2\Delta_V)} e^{-p} \int_0^\infty \frac{q^{2\Delta_W - 1} dq}{\Gamma(2\Delta_W)} e^{-q} e^{-\kappa pq}$$

$$= \kappa^{-2\Delta_V} U(2\Delta_V, 1 + 2\Delta_V - 2\Delta_W, \kappa^{-1}).$$

In agreement with the scattering result.

More details of the longitudinal action

Longitudinal action: perturbative analysis

With transverse modes turned off ($\tilde{y} = 0$), extremizing $\alpha(t)$ imposes:

$$\alpha(t) = \frac{at + b}{ct + d}, \qquad x + iz = \frac{a(t + is) + b}{c(t + is) + d}.$$

This is because the Virasoro constraint without transverse modes becomes:

$$0 = T^L_{\alpha\beta} = \frac{\partial_\alpha x \partial_\beta x + \partial_\alpha z \partial_\beta z}{z^2} - \frac{1}{2}\delta_{\alpha\beta}\frac{\partial^\gamma x \partial_\gamma x + \partial^\gamma z \partial_\gamma z}{z^2}$$

Letting $X = x + iz$, and $T^L = T^L_{tt} + iT^L_{st}$, this can be written as

$$0 = T^L = -\frac{8\partial X \partial \bar{X}}{(X - \bar{X})^2}.$$

So $\bar{\partial} X = 0$ (or $\partial X = 0$).

Longitudinal action: perturbative analysis

Let $\alpha(t) = t + \epsilon(t), \quad x(s,t) = t + x_1(s,t) + x_2(s,t) + \ldots, \quad z(s,t) = s + z_1(s,t) + z_2(s,t) + \ldots.$

The linear EOM becomes:

$$0 = s(\ddot{x}_1 + x_1'') - 2(x_1' + \dot{z}_1), \qquad 0 = s(\ddot{z}_1 + z_1'') + 2(\dot{x}_1 - z_1'),$$

With BCs: $x_1(0,t) = \epsilon(t), \quad z_1(0,t) = 0$. The solutions are:

$$x_1(s,t) = \int dt' K_x(s,t,t')\epsilon(t'), \qquad z_1(s,t) = \int dt' K_z(s,t,t')\epsilon(t'),$$

Where

$$K_x(s,t,t') = \frac{4}{\pi}\frac{s^3(s^2-(t-t')^2)}{(s^2+(t-t')^2)^3}, \qquad K_z(s,t,t') = -\frac{8}{\pi}\frac{s^4(t-t')}{(s^2+(t-t')^2)^3}.$$

Longitudinal action: perturbative analysis

Let $\alpha(t) = t + \epsilon(t)$, $\quad x(s,t) = t + x_1(s,t) + x_2(s,t) + \ldots$, $\quad z(s,t) = s + z_1(s,t) + z_2(s,t) + \ldots$.

The linear EOM becomes:

$$0 = s(\ddot{x}_1 + x_1'') - 2(x_1' + \dot{z}_1), \qquad 0 = s(\ddot{z}_1 + z_1'') + 2(\dot{x}_1 - z_1'),$$

With BCs: $x_1(0,t) = \epsilon(t)$, $\quad z_1(0,t) = 0$. The solutions are:

$$x_1(s,t) = \int dt' K_x(s,t,t')\epsilon(t'), \qquad z_1(s,t) = \int dt' K_z(s,t,t')\epsilon(t'),$$

Where

$$K_x(s,t,t') = \frac{4}{\pi}\frac{s^3(s^2 - (t-t')^2)}{(s^2 + (t-t')^2)^3}, \qquad K_z(s,t,t') = -\frac{8}{\pi}\frac{s^4(t-t')}{(s^2 + (t-t')^2)^3}.$$

Longitudinal action: perturbative analysis

Let $\alpha(t) = t + \epsilon(t)$, $\quad x(s,t) = t + x_1(s,t) + x_2(s,t) + \ldots$, $\quad z(s,t) = s + z_1(s,t) + z_2(s,t) + \ldots$.

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Longitudinal action: perturbative analysis

The quadratic order longitudinal action becomes:

$$S_L = T_s \int d^2\sigma \left[\mathscr{L}_{x_1^2,\ z_1^2,\ x_1 z_1} + \ldots \right]$$

$$\Bigg\downarrow$$

Integration by parts, e.o.m.

$$x(s,t) = \alpha(t) - \frac{\ddot{\alpha}(t)}{2} s^2 + O(s^3), \qquad z(s,t) = \dot{\alpha}(t) s + O(s^3).$$

$$= -\frac{T_s}{4} \int dt x_1(0,t) x_1'''(0,t) \qquad = -\frac{T_s}{4} \lim_{s \to 0^+} \int dt dt' \epsilon(t) \epsilon(t') K_x'''(s,t,t')$$

$$= \frac{6T_s}{\pi} \int dt dt' \frac{\epsilon(t)\epsilon(t')}{(t-t')^4}\,.$$

Longitudinal action: perturbative analysis

The cubic order longitudinal action becomes:

$$S_L = T_s \int d^2\sigma \left[\dots + \mathscr{L}_{x_1^3,\ z_1^3} + \cancel{\mathscr{L}_{x_2 x_1,\ z_2 z_1}} + \dots \right]$$

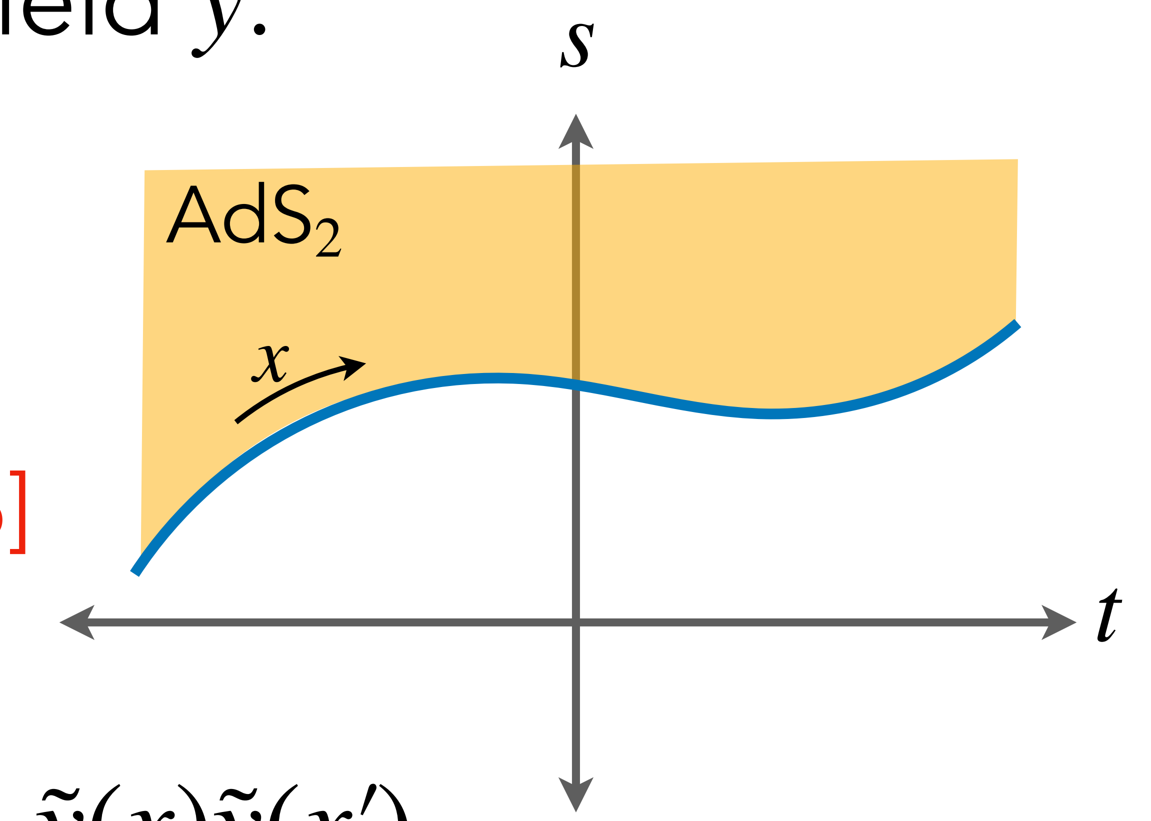
$$S_{L,3} = -\frac{12T_s}{\pi} \int dt_1 dt_2 \frac{\epsilon(t_1)^2 \epsilon(t_2)}{|t_{12}|^4 t_{12}}$$

Miscellaneous

Lightning review of JT Gravity

Toy model of 2D gravity [Teiltelboim '83, Jackiw '85]. Dilaton ϕ, metric h, matter field y:

$$S[\phi, h, y] = S_{\rm JT}[\phi, h] + S_{\rm m}[y, h]$$



Introduce a cut-off curve (s(x),t(x)) along the boundary [Maldacena, Stanford Yang '16]

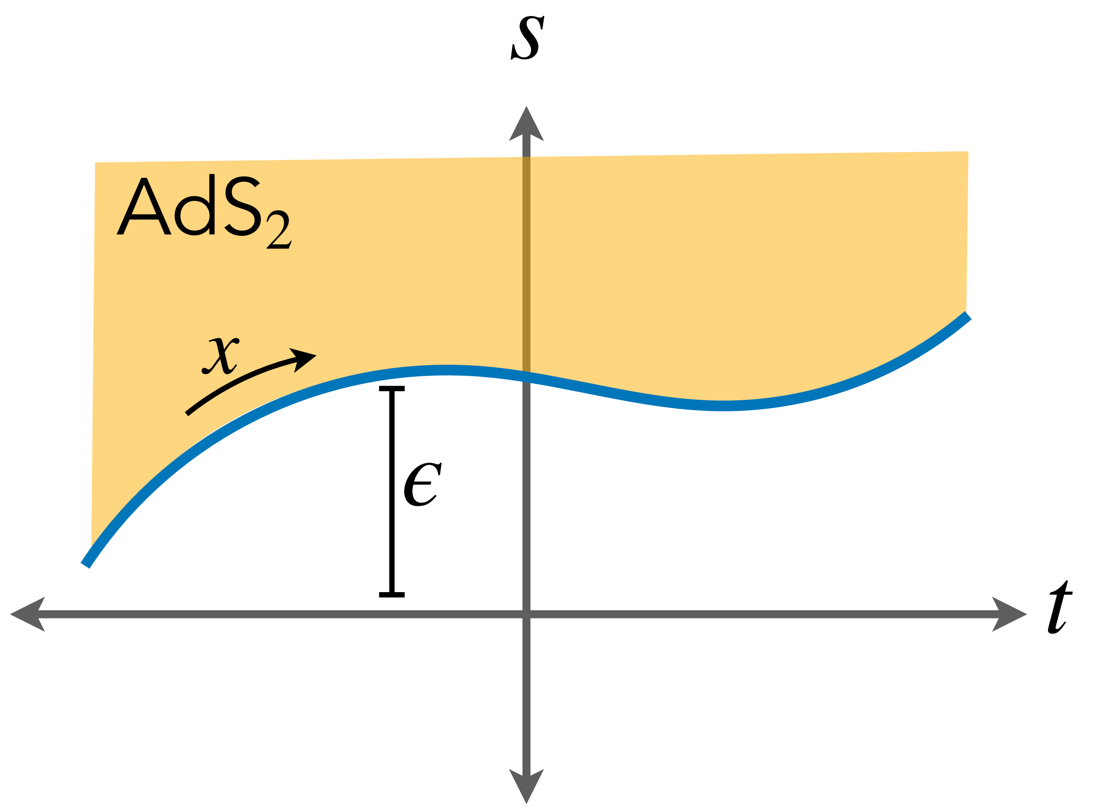
$$S_{\rm JT}[\tilde\phi, t] = -\frac{\tilde\phi}{8\pi G_N}\int dx\{t, x\}, \qquad S_{\rm m}[\tilde y, t] = -\frac{D}{2}\int dx dx' \frac{\dot t(x)\dot t(x')}{(t(x)-t(x'))^2}\tilde y(x)\tilde y(x')$$

Where $\{t, x\} = \dfrac{\dddot t}{\dot t} - \dfrac{3}{2}\dfrac{\ddot t^2}{\dot t^2}$. The partition function becomes:

$$Z[\tilde y] = \int \mathscr{D}h\mathscr{D}\phi\mathscr{D}y e^{-S[h,\phi,y]} = \int_{Diff(S^1)/SL(2,\mathbb{R})} \mathscr{D}t e^{-S_{\rm Schw.}[t(x)]-S_{\rm m}[\tilde y, t]}$$

[Kitaev; Almheiri, Polchinksi; Maldacena, Stanford, Yang; Jensen; Mertens, Engelsoy, Verlinde; Bagrets, Altland, Kamenev; Stanford, Witten; Mertens, Turiaci, Verlinde; Kitaev, Suh; …] [Mertens, Turiaci '22]

Lightning review of JT Gravity



Action:

$$S_{\mathrm{JT}}[\phi, h] = S_{\text{Gauss-Bonnet}} - \frac{1}{16\pi G_N}\left[\int_{\mathcal{M}} d^2\sigma\sqrt{h}\phi(R+2) + 2\int_B dx\sqrt{h_{xx}}\phi(K-1)\right],$$

$$S_{\mathrm{m}}[y, h] = \int_{\mathcal{M}} d^2\sigma \frac{\sqrt{h}}{2}\left(h^{\alpha\beta}\partial_\alpha y \partial_\beta y + m^2 y^2\right),$$

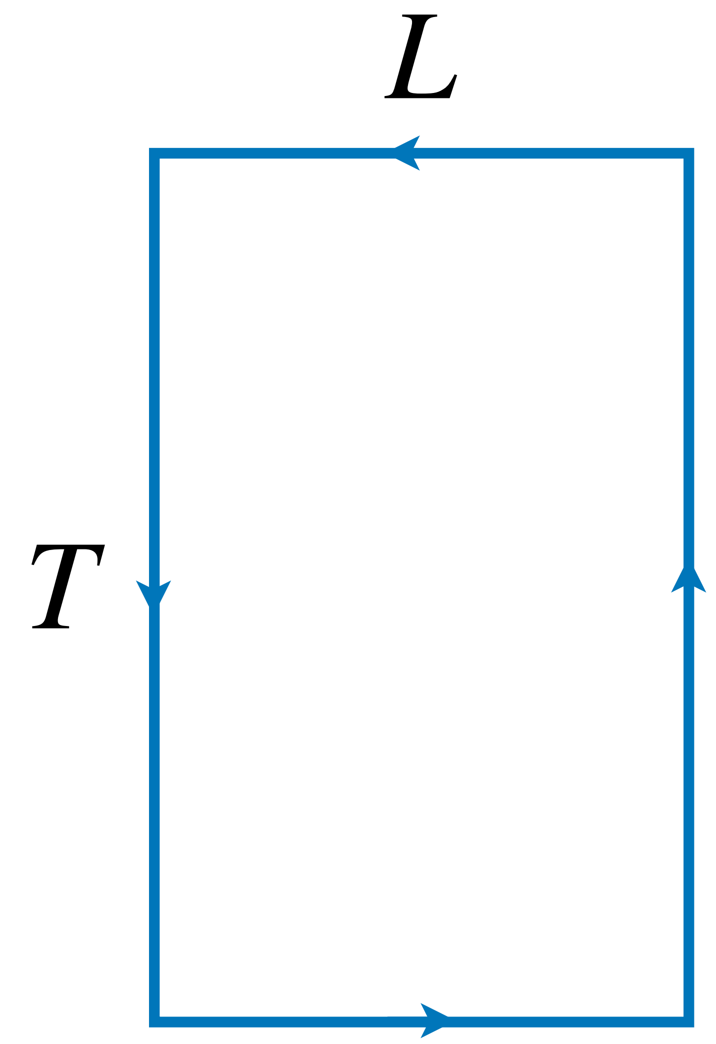
Boundary conditions along a boundary curve $(s(x), t(x))$:

$$h_{xx} = \frac{\dot{s}(x)^2 + \dot{t}(x)^2}{s^2(x)} = \frac{1}{\epsilon^2}, \qquad \phi(s(x), t(x)) = \frac{\tilde{\phi}}{\epsilon}, \qquad y(s(x), t(x)) = \frac{\tilde{y}(x)}{\epsilon^{\Delta-1}}$$

The metric BC implies: $s(x) = \epsilon\dot{t}(x) + O(\epsilon^2)$. Note also that: $R = -2$ and $K = 1 + \epsilon^2\{t, x\} + O(\epsilon^4)$

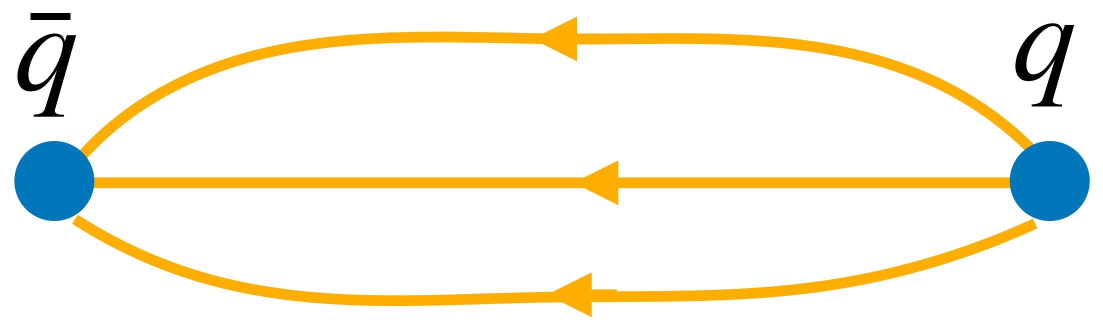
Wilson loops in gauge theory

$$\mathscr{W} = \mathrm{P}\exp\left(i\int A_\mu dx^\mu\right)$$



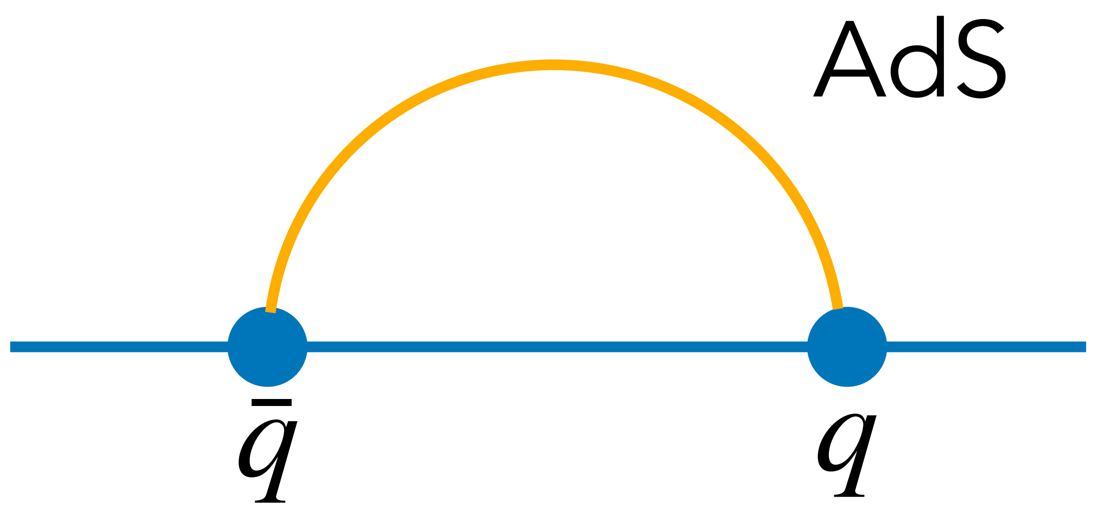
Fundamental, gauge-invariant, non-local observables.

$\langle \mathscr{W} \rangle \sim e^{-E(L)T}$ Confining: $E(L) \propto L$. Conformal: $E(L) \propto \dfrac{1}{L}$.



Effective QCD string

The area law is realized in AdS/CFT holographically.



Wilson loops as versatile observables

- Higher representations dual to D-branes [Drukker, Fill ’05; Yamaguchi ’06; Gomis, Passerini ’06, …]

- Different contours preserving different fractions of superconformal symmetry [Zarembo ’02; Drukker, Giombi, Ricci, Trancanelli ‘07]

- Can study the anomalous dimensions of cusps [Correa, Henn, Maldacena, Sever ’12; Correa, Maldacena, Sever ‘12; Drukker ’12; Gromov, Levkovich-Masyluk ‘15]

Wavy Wilson line

[Semenoff, Young '04] [Paraphrasing]: "Consider the contour $x^\mu(s) = (s, x^i(s))$ in $\mathbb{R}^4$ on the boundary of AdS$_5$. The area of the minimal surface incident on the contour is a functional of the geometry of the contour.

To quadratic order in $x^i(s)$, rotation and translation invariance dictate that it has the form:

$$A \propto \int ds ds' \dot{x}^i(s) K(s-s') \dot{x}_i(s').$$

Scale invariance indicates that $K(s-s')$ has dimension 1/distance^2 and is of the form $K(s-s') \sim \dfrac{1}{(s-s')^2}$.

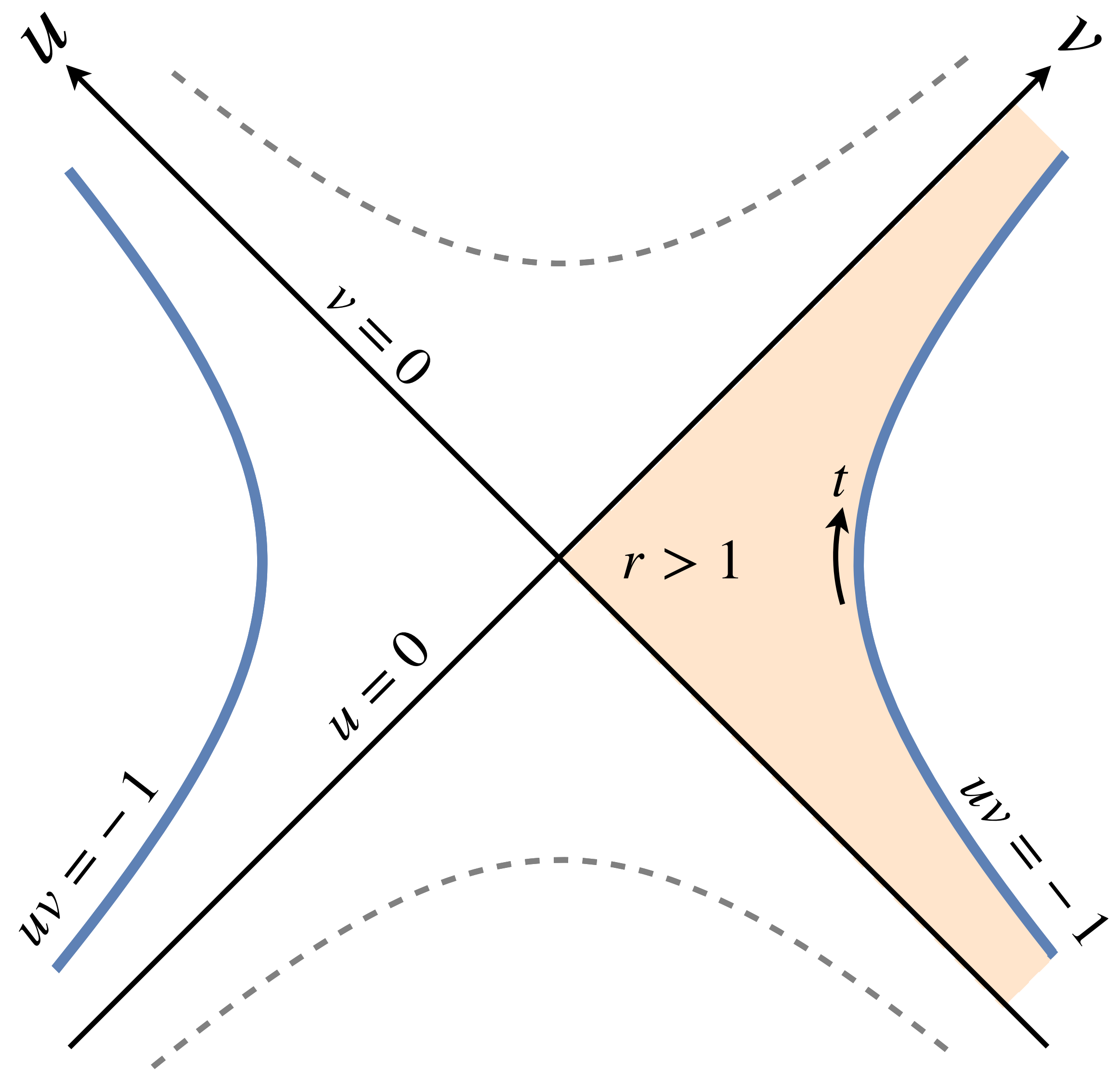
Treating the kernel more precisely as a distribution, one finds:

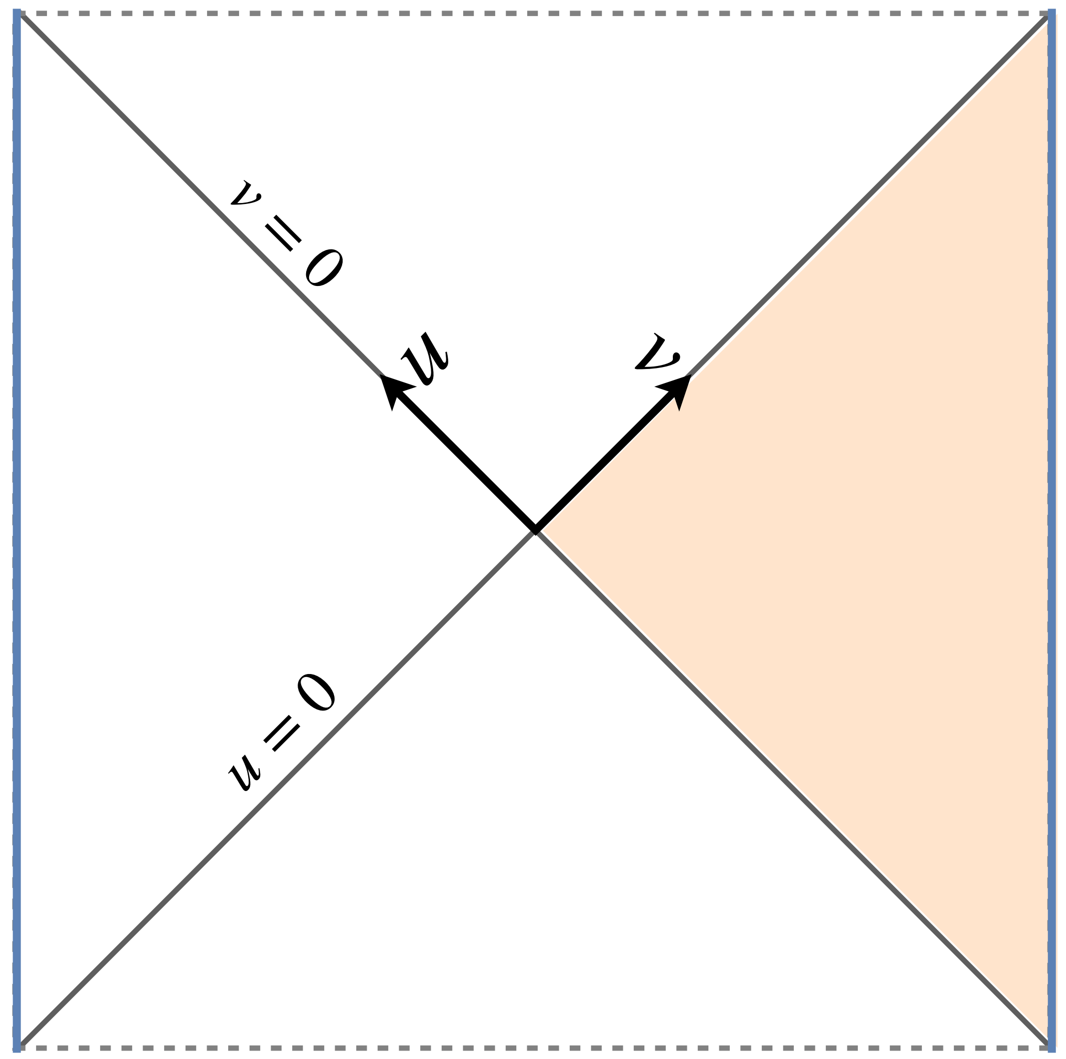
$$A \propto \int ds ds' \frac{(\dot{x}_i(s) - \dot{x}_i(s'))^2}{(s-s')^2}.$$

In terms of the Wilson line defect CFT, this implies:

$$\langle \mathbb{D}_i(x_1) \mathbb{D}_j(x_2) \rangle \propto \frac{\delta^2 A}{\delta x^i(x_1) x^i(x_2)} \propto \frac{1}{(x_1 - x_2)^4}.$$

Kruskal and Schwarzschild coordinates on AdS_2





$$ds^2 = -\frac{4dudv}{(1+uv)^2} = -(r^2-1)dt^2 + \frac{dr^2}{r^2-1}$$

$$u = -\sqrt{\frac{r-1}{r+1}}e^{-t}, \quad v = \sqrt{\frac{r-1}{r+1}}e^{t}$$