Instanton effects on extended strings from $\mathcal{N} = 4$ SYM

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Based on arXiv:2305.08297 with V. Rodriguez and Y. Wang and earlier work with D. Binder, S. Chester, M. Green, Y. Wang, C. Wen

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Introduction

- This talk is about QFT calculations in $\mathcal{N} = 4$ SYM theory which have string theory interpretations.
- In string theory, we study **scattering amplitudes**.
- $\mathcal{N} = 4$ SYM is a CFT $\rightarrow$ we study **correlation functions**.
- Last few years: lots of quantitative **precision tests**.
  - $\mathcal{N} = 4$ SYM correlators at large $N \Longrightarrow$ graviton scattering amplitude.
- So far: many impressive checks
- **This talk:** new string theory scattering amplitudes from CFT.

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What's new?

- Integrated correlator of 2 local ops and a Wilson (or 't Hooft or Wilson-'t Hooft) line $\mathbb{W}$

$$\int d^4x\, d^4y\, \mu(\vec{x},\vec{y}) \langle \mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\mathbb{W}\rangle$$

  in $\mathcal{N}=4$ SYM in "very strong coupling" limit.

- Implications for scattering of gravitons from long strings.

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Graviton scattering in 10d type IIB string theory

- $2 \to 2$ scattering of gravitons + superpartners

$$\textbf{SUSY} \quad \Longrightarrow \quad \boxed{\mathcal{A}_{2\to 2} = \underbrace{\delta^{16}(Q)}_{\text{polarizations}} \; f(\mathbf{s}, \mathbf{t})}$$

  (where $\mathbf{s}$ and $\mathbf{t}$ (and $\mathbf{u} = -\mathbf{s} - \mathbf{t}$) are Mandelstam invariants) is captured by the 10d *effective action*:

$$S_{\text{10d}} = \int d^{10}x \sqrt{g} \Big[R + \ell_s^6 R^4 + \ell_s^{10} D^4 R^4 + \ell_s^{12} D^6 R^4 + \cdots \Big] + \text{(SUSic completion)}$$

- String theory has parameters: $\alpha' = \ell_s^2$, string coupling $g_s = \langle e^{-\phi} \rangle$, and $\chi = \langle C_0 \rangle$. Denote $\tau = \chi + i/g_s$.
- $f(\mathbf{s}, \mathbf{t})$ is known in various limits:
  - 3 lowest orders in g_s^2 at all orders in $\ell_s \times$ momentum.
  - 4 lowest orders in $\ell_s \times$ momentum at all orders in g_s and χ.

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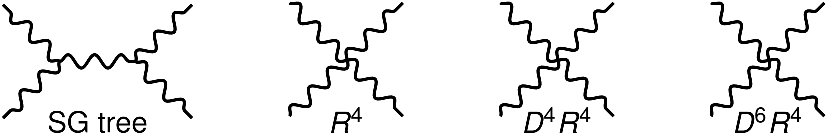
Graviton scattering at tree level

- At leading order in string coupling g_s, i.e. tree level (see Polchinski (12.4.30)):

$$f(\mathbf{s},\mathbf{t}) \sim \frac{\Gamma(-\frac{1}{4}\alpha'\mathbf{s})\Gamma(-\frac{1}{4}\alpha'\mathbf{t})\Gamma(-\frac{1}{4}\alpha'\mathbf{u})}{\Gamma(1+\frac{1}{4}\alpha'\mathbf{s})\Gamma(1+\frac{1}{4}\alpha'\mathbf{t})\Gamma(1+\frac{1}{4}\alpha'\mathbf{u})} + O(g_s^2)$$

- Further expansion at small momentum gives interpretation in terms of diagrams:

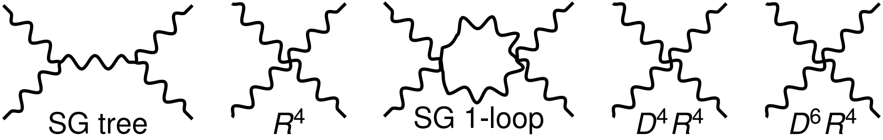
$$f(\mathbf{s},\mathbf{t}) = \underbrace{\frac{1}{\mathbf{stu}}}_{\text{SG}} + \underbrace{\ell_s^6 \frac{\zeta(3)}{32}}_{R^4} + \underbrace{\ell_s^{10}\frac{\zeta(5)}{2^{10}}(\mathbf{s}^2+\mathbf{t}^2+\mathbf{u}^2)}_{D^4R^4} + \underbrace{\ell_s^{12}\frac{\zeta(3)^2}{2^{11}}\mathbf{stu}}_{D^6R^4} + \cdots + O(g_s^2)$$



Small momentum expansion

- Small momentum at fixed $\tau = \chi + i/g_s$ [Green, Gutperle, Vanhove, Sethi; Wang, Yin '15] :

$$f(\mathbf{s},\mathbf{t}) = \underbrace{\frac{1}{\mathbf{stu}}}_{\text{SG tree}} + \underbrace{\ell_s^6 \frac{g_s^{\frac{3}{2}}}{64} E(\tfrac{3}{2},\tau,\bar\tau)}_{R^4} + \underbrace{\ell_s^8 g_s^2 f_{R|R}(\mathbf{s},\mathbf{t})}_{\text{SG one-loop}}$$
$$+ \underbrace{\ell_s^{10} \frac{\mathbf{s}^2+\mathbf{t}^2+\mathbf{u}^2}{2^{11}} g_s^{\frac{5}{2}} E(\tfrac{5}{2},\tau,\bar\tau)}_{D^4R^4} + \underbrace{\ell_s^{12} \frac{\mathbf{stu}}{2^{12}} g_s^3 \mathcal{E}(3,\tfrac{3}{2},\tfrac{3}{2},\tau,\bar\tau)}_{D^6R^4} + \cdots .$$



Eisenstein series

- The Eisenstein series are **modular invariants** defined by

$$E(s,\tau,\bar\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m+n\tau|^{2s}}\,, \qquad \tau \equiv \tau_1 + i\tau_2\,.$$

- For example,

$$E(\tfrac{3}{2},\tau,\bar\tau) = \frac{2\zeta(3)}{g_s^{\frac{3}{2}}} + \frac{2\pi^2}{3} g_s^{\frac{1}{2}} + \frac{8\pi}{\sqrt{g_s}} \sum_{k=1}^{\infty} \frac{\sigma_{-2}(k)}{k} K_1\left(\frac{2\pi k}{g_s}\right)\cos(2\pi k\chi)$$

where $\sigma_p(k) = \sum_{d|k} d^p$.

- The last term is non-perturbative in g_s:

$$g_s^{\frac{3}{2}} E(\tfrac{3}{2},\tau,\bar\tau) = 2\zeta(3) + \frac{2\pi^2}{3} g_s^2 + e^{-\frac{2\pi}{g_s}} \cos(2\pi\chi)\left[4\pi g_s^{\frac{3}{2}} + \frac{3}{4} g_s^{\frac{5}{2}} + \cdots\right] + e^{-\frac{4\pi}{g_s}} \cos(4\pi\chi)\left[\cdots\right] + \cdots\,.$$

$\mathcal{N} = 4$ SYM correlators

- $\mathcal{N} = 4$ SYM: vector field A_μ + scalars X_I ($I = 1, \ldots, 6$) + fermions in adjoint of $SU(N)$, w/ gauge coupling

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}\,, \qquad \lambda = g^2 N\,.$$

- Limits:
  - 't Hooft strong coupling: N large w/ λ fixed, then λ large (In this limit, $g \to 0$)
  - "very strong coupling": N large with τ fixed
  - Dual: IIB strings on $AdS_5 \times S^5$, with $\frac{L}{\ell_s} = \lambda^{1/4}$, $g_s = \frac{g^2}{4\pi}$, $\chi = \frac{\theta}{2\pi}$.
- Operator

$$\mathcal{O}_{IJ} = \operatorname{tr}\left(X_I X_J - \frac{\delta_{IJ}}{6} X_K X_K\right)$$

has $\Delta = 2$ and is $\frac{1}{2}$-BPS. Same multiplet as stress tensor.

AdS/CFT $\implies$ $\mathcal{O} \longleftrightarrow$ (metric + 4-form) fluctuations

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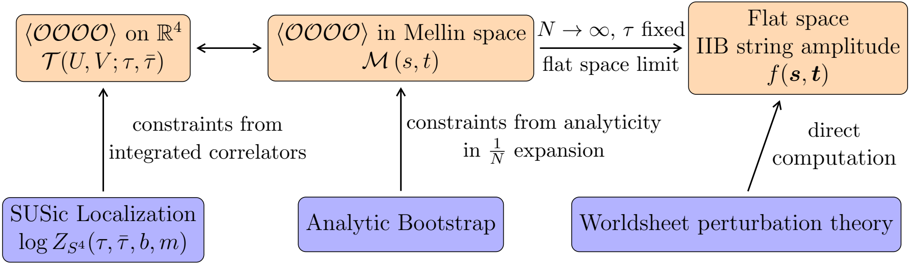
Relation b/w SYM correlator and graviton scattering



- **SUSY** $\Longrightarrow \langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle$ is expressed in terms of a single function $\mathcal{T}$:

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle = \text{(free part)} + \text{(prefactor)}\mathcal{T}(U, V; \tau, \bar{\tau})$$

where, with $\vec{x}_{ij} = \vec{x}_i - \vec{x}_j$,

$$U = \frac{\vec{x}_{12}^2 \vec{x}_{34}^2}{\vec{x}_{13}^2 \vec{x}_{24}^2}, \qquad V = \frac{\vec{x}_{14}^2 \vec{x}_{23}^2}{\vec{x}_{13}^2 \vec{x}_{24}^2}$$

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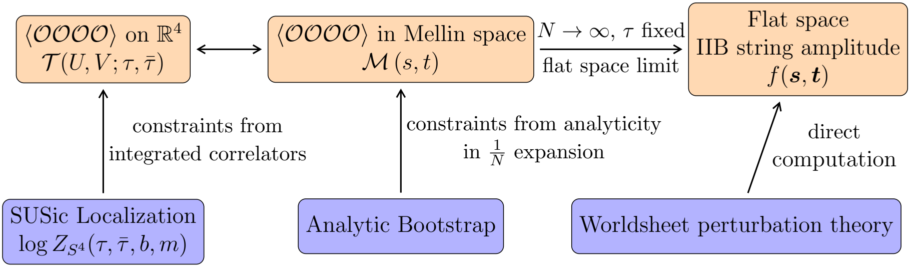

- Mellin space (with $u = 4 - s - t$):

$$\mathcal{T}(U, V) = \int_{c-i\infty}^{c+i\infty} \frac{ds\, dt}{(4\pi i)^2} \mathcal{M}(s, t)\, U^{\frac{s}{2}} V^{\frac{u}{2}-2} \Gamma\left(\frac{4-s}{2}\right)^2 \Gamma\left(\frac{4-t}{2}\right)^2 \Gamma\left(\frac{4-u}{2}\right)^2$$

- $\mathcal{M}(s, t)$ is the AdS analog of the scattering amplitude $f(\mathbf{s}, \mathbf{t})$.
- $N \to \infty$, τ fixed: $f(\mathbf{s}, \mathbf{t})$ extracted from large s, t limit of $\mathcal{M}(s, t)$
  [Giddings '99; Polchinski '99; Susskind '99; Penedones '10; Fitzpatrick, Kaplan '11]

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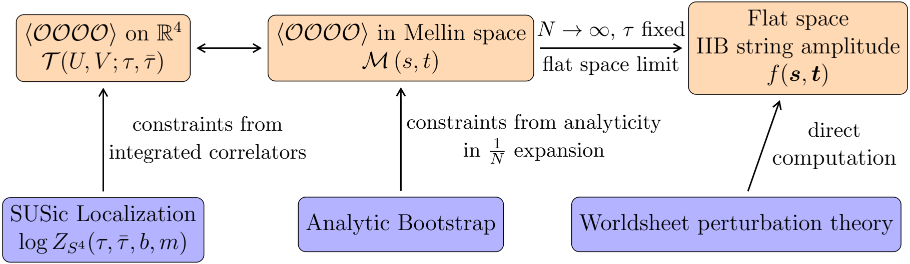

Integrated correlators (can be computed using SUSic localization) [Binder, Chester, SSP, Wang '19; Chester, SSP '20] :

$$(\text{Im }\tau)^2 \partial_\tau \partial_{\bar{\tau}} \partial_m^2 \log Z_{S^4} \Big|_{\substack{m=0 \\ b=1}} = \int dU\, dV\, \mu_1(U, V) \mathcal{T}(U, V)\,,$$

$$\partial_m^4 \log Z_{S^4} \Big|_{\substack{m=0 \\ b=1}} = (\text{free part}) + \int dU\, dV\, \mu_2(U, V) \mathcal{T}(U, V)\,.$$

where μ_1 and μ_2 are SUSY-preserving measures.

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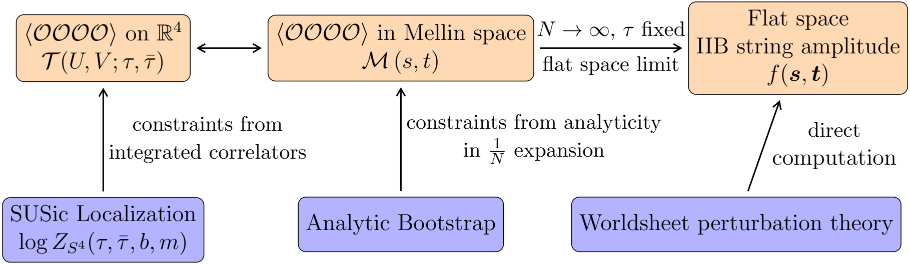

Analytic bootstrap:

- At each order in deriv expansion, $\mathcal{M}(s, t)$ is determined by analyticity + crossing + SUSY up to a few constants
- 2 constants at each order can be fixed using the derivs of Z_{S^4}.
- Can (re)derive low-energy expansion of $f(\mathbf{s}, \mathbf{t})$ from $\mathcal{N} = 4$ SYM!

Take-away: Same fns of $(\tau, \bar{\tau})$ appear in derivs of $\log Z_{S^4}$ and in $f(\mathbf{s}, \mathbf{t})$ at low orders in derivative expansion. (E.g. the Eisenstein series!!)

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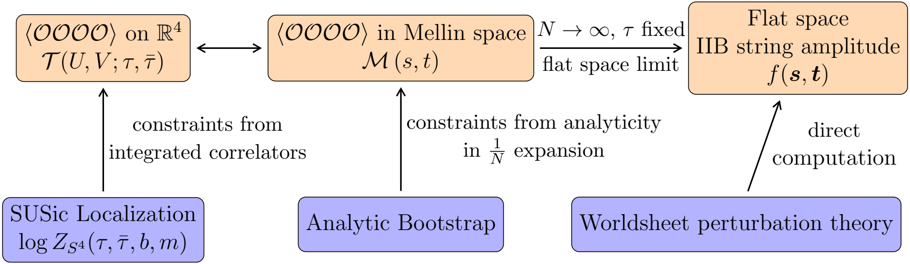

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How do we see the Eisenstein series from Z_{S^4}?

- [Pestun '07] : S^4 partition fn of $\mathcal{N} = 2^*$ theory (i.e. $\mathcal{N} = 4$ SYM deformed by an $\mathcal{N} = 2$-preserving mass m for the adj hyper)

$$Z_{S^4} = \int d^{N-1}a \prod_{i<j}(a_i - a_j)^2 e^{-\frac{8\pi^2}{g^2}\sum_i a_i^2} Z_{\text{pert}}(a_i, m)|Z_{\text{inst}}(a_i, \tau, m)|^2$$

- Z_{pert} comes from perturbative contirbution

$$Z_{\text{pert}}(a_i, m) = \frac{H^2(a_i - a_j)}{H(m)^{N-1}\prod_{i\neq j} H(a_i - a_j + m)}$$

where $H(x) = e^{-(1+\gamma)x^2} G(1 + ix)G(1 - ix)$.

- Z_{inst} comes from instantons at the poles of S^4 and is complicated: sum of N-tuples of Young diagrams that represent contour integrals [Nekrasov '02; Nekrasov, Okounkov '03]

- For $\partial_m^2 \log Z_{S^4}\big|_{m=0}$ only rectangular Young diagrams contribute!

How do we see the Eisenstein series from Z_{S^4}?

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$$\partial_m^2 \log Z_{S^4}\big|_{m=0} = \langle Z''_{\text{pert}}(a_i, 0)\rangle_{GMM} + \langle Z''_{\text{inst}}(a_i, \tau, 0)\rangle_{GMM} + \langle \bar{Z}''_{\text{inst}}(a_i, \bar{\tau}, 0)\rangle_{GMM}$$

  by writing each term as an expectation value in the Gaussian matrix model. Use resolvents & topological recursion & Mellin transforms to systematically evaluate each term [Chester, Green, SSP, Wang, Wen '19, '20; SSP, Rodriguez, Wang '23] .

- Expand in very strong coupling limit (large N, fixed τ) [Chester, Green, SSP, Wang, Wen '19, '20; Dorigoni, Green, Wen '21] :

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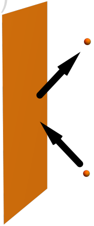
A defect CFT $\longleftrightarrow$ open-closed string system example

- So far: CFT 4-pt function $\longleftrightarrow$ $2 \to 2$ scattering of closed strings
- For the **rest of talk**, another setup: open-closed amplitudes in type IIB string theory.
- $1 \to 1$ scattering of a graviton from a $1/2$-BPS extended string (long fundamental string, D1-brane, or (p, q)-string)



$$\textbf{SUSY} \quad \Longrightarrow \quad \mathcal{A}_{1\to 1} = \underbrace{\delta^8(Q^{||})}_{\text{polarizations}} \; f_{\mathbb{L}}(\mathbf{s}, \mathbf{t})$$

$$\mathbf{s} = 2p_{1||}^2 = 2p_{2||}^2 \,, \qquad \mathbf{t} = p_1 \cdot p_2 \,.$$

- Setup is not $SL(2, \mathbb{Z})$ invariant. Under $\tau \to \frac{a\tau + b}{c\tau + d}$, we have $(p \;\; q) \to (p \;\; q) \begin{pmatrix} a & -c \\ -b & d \end{pmatrix}$.

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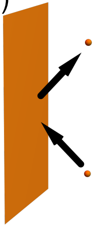

$$\textbf{SUSY} \quad \Longrightarrow \quad \mathcal{A}_{1\to 1} = \underbrace{\delta^8(Q^{||})}_{\text{polarizations}} \; f_{\mathbb{L}}(\mathbf{s}, \mathbf{t})$$

$$\mathbf{s} = 2p_{1||}^2 = 2p_{2||}^2 \,, \qquad \mathbf{t} = p_1 \cdot p_2 \,.$$

- Setup is not $SL(2,\mathbb{Z})$ invariant. Under $\tau \to \frac{a\tau+b}{c\tau+d}$, we have $\begin{pmatrix} p & q \end{pmatrix} \to \begin{pmatrix} p & q \end{pmatrix} \begin{pmatrix} a & -c \\ -b & d \end{pmatrix}$.

What's known

- Much less is known about this setup. Schematically:

$$S = S_{\text{10d}} + \int d^2x\sqrt{-g}\left[\text{DBI} + R^2 + D^2R^2 + D^4R^2 + \cdots\right] .$$

  $R =$ ambient space-time curvature, $D =$ tangential deriv.

- Amplitude known **only** at leading order in g_s (*disk diagram*) for D1-brane and $(n, 1)$ string. For **D1-brane** [Garousi-Myers '96; Hashimoto, Klebanov '96; Basu '08] :

$$f_{\mathbb{L}} \propto \frac{\ell_s^4\Gamma(\frac{\alpha'\mathbf{s}}{2})\Gamma(\frac{\alpha'\mathbf{t}}{2})}{\Gamma(1+\frac{\alpha'(\mathbf{s}+\mathbf{t})}{2})} + O(g_s) = \underbrace{\frac{4}{\mathbf{st}}}_{\text{SG+DBI}} \underbrace{-\frac{\pi^2}{6}\ell_s^4}_{R^2} + \underbrace{\frac{1}{2}(\mathbf{s}+\mathbf{t})\zeta(3)\ell_s^6}_{D^2R^2} + \cdots + O(g_s)$$

- $O(g_s)$ not fully understood—*annulus diagram* diverges!

- Small momentum, all orders in g_s: Coeffs of R^2, D^2R^2, D^4R^2, etc. should be fns of $(\tau, \bar{\tau})$. NOT modular invariant and not known.

Extended string $\longleftrightarrow$ CFT defect

- $\mathcal{N}=4$ SYM: two-pt function of $\mathcal{O}$ in presence of $1/2$-BPS line defect (Wilson line, 't Hooft line, (p,q) Wilson-'t Hooft dyonic line) [Barrat, Liendo, Plefka '20]

$$\textbf{SUSY} \quad \Longrightarrow \quad \langle \mathcal{O}\mathcal{O}\mathbb{L}\rangle = (\text{prefactor})\, \mathcal{T}_{\mathbb{L}}(U,V)$$

$$U = \frac{x_1^\perp \cdot x_2^\perp}{|x_1^\perp||x_2^\perp|}\,, \qquad V = \frac{(\vec{x}_1 - \vec{x}_2)^2}{|x_1^\perp||x_2^\perp|}$$

- Simplest case: Wilson loop $\mathbb{L} = \mathbb{W}$ in fundamental representation:

$$\mathbb{W} = \text{tr}_{\text{fund}}\, P \exp\left[i \oint ds\, \left(A_\mu(x(s))\dot{x}^\mu(s) + X_6(x(s))|\dot{x}(s)|\right)\right]\,.$$

- On S^4, can compute

$$\mathcal{I}_{\mathbb{W}} = \partial_m^2 \log\langle \mathbb{W}\rangle\big|_{m=0} = \text{integrated } \langle \mathcal{O}\mathcal{O}\mathbb{W}\rangle$$

using localization. (Insert $\sum_i e^{2\pi a_i}$ in matrix model.)

- $SL(2,\mathbb{Z})$ relates amplitudes $\langle \mathcal{O}\mathcal{O}\mathbb{W}\rangle$ to $\langle \mathcal{O}\mathcal{O}\mathbb{L}\rangle$ with coprime (p,q).

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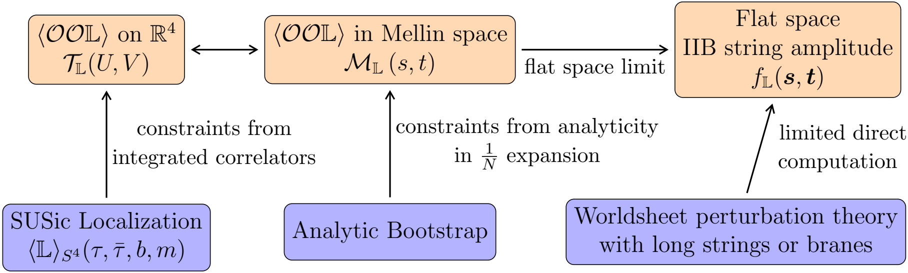
Scattering from extended string from CFT

Expect similar connections, but more conjectural in this case:



Expect:

Same fns of $(\tau, \bar{\tau})$ appear in derivs of $\partial_m^2 \log \langle \mathbb{L} \rangle \big|_{m=0}$ and in $f_{\mathbb{L}}(\mathbf{s}, \mathbf{t})$ at low orders in the derivative expansion.

Two-point function in presence of Wilson loop

- For line operator $\mathbb{L}$, expand

$$\partial_m^2 \log\langle\mathbb{L}\rangle\big|_{m=0} = \mathcal{I}_{\mathbb{L},-1/2}\sqrt{N} + \mathcal{I}_{\mathbb{L},0} + \frac{\mathcal{I}_{\mathbb{L},1/2}}{\sqrt{N}} + \frac{\mathcal{I}_{\mathbb{L},1}}{N} + \frac{\mathcal{I}_{\mathbb{L},3/2}}{N^{3/2}} + \cdots .$$

- For $\mathbb{L} = \mathbb{W}$ (Wilson loop), matrix model gives [SSP, Rodriguez, Wang '23] :

$$\begin{aligned}
\mathcal{I}_{\mathbb{W},-1/2} &= g\,,\\
\mathcal{I}_{\mathbb{W},0} &= \frac{1}{2} - \frac{\pi^2}{3}\,,\\
\mathcal{I}_{\mathbb{W},1/2} &= \frac{3}{8g} - \frac{g^3}{32}\,,\\
\mathcal{I}_{\mathbb{W},1} &= \frac{3(1+4\zeta(3))}{8g^2} + \frac{g^6}{11520} - \frac{3g^2}{2\pi^2}\sum_{k=1}^{\infty}\cos(k\theta)K_2(8\pi^2 k/g^2)\sigma_{-2}(k)\,,\\
\mathcal{I}_{\mathbb{W},3/2} &= \frac{3(21+64\zeta(3))}{128g^3} - \frac{g}{256} + \frac{7g^5}{10240} - \frac{g^9}{1935360} + \text{(non-pert)}
\end{aligned}$$

- We determined $\mathcal{I}_{\mathbb{W},k}(\tau,\bar{\tau})$ for $k \leq 3/2$. NOT modular invariant.

Two-point function in presence of 't Hooft loop

- $SL(2,\mathbb{Z})$ duality relates $\mathbb{W}$ to $\mathbb{L}$ for Wilson-'t Hooft loops $\longleftrightarrow$ (p,q)-strings.
- For example, S-duality when $\theta = 0$ gives for the 't Hooft loop $\mathbb{T}$:

$$
\begin{aligned}
\mathcal{I}_{\mathbb{T},-1/2} &= \frac{4\pi}{g}\,, \\
\mathcal{I}_{\mathbb{T},0} &= \frac{1}{2} - \frac{\pi^2}{3}\,, \\
\mathcal{I}_{\mathbb{T},1/2} &= \frac{-\frac{8\pi^4}{g^3} + \frac{3}{8}g}{4\pi}\,, \\
\mathcal{I}_{\mathbb{T},1} &= \frac{24\pi\zeta(3)}{g^4} - \frac{\pi^2}{g^2} + \frac{\pi}{6} + \frac{3g^2}{128\pi^2} + \frac{3g}{2\pi^{\frac{3}{2}}} e^{-\frac{8\pi^2}{g^2}} \left(1 - \frac{25g^2}{64\pi^2} + \mathcal{O}(g^4)\right) \\
&\quad + \mathcal{O}\left(e^{-\frac{16\pi^2}{g^2}}\right)\,.
\end{aligned}
$$

- Finite number of perturbative terms + infinite number of non-perturbative corrections for both $\mathbb{W}$ and $\mathbb{T}$!

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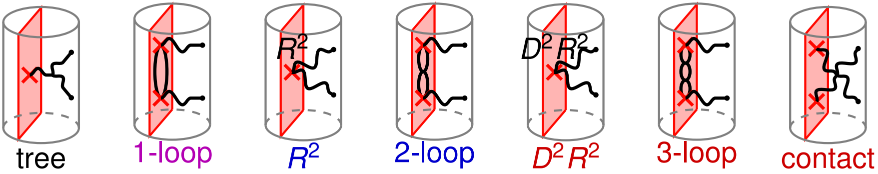
String theory interpretation

- At each order, **tree level diagrams** + **loop diagrams**:

$$\underbrace{\mathcal{I}_{\mathbb{L},-1/2}\sqrt{N}}_{\text{tree}} + \underbrace{\mathcal{I}_{\mathbb{L},0}}_{\text{1-loop}} + \underbrace{\frac{\mathcal{I}_{\mathbb{L},1/2}}{\sqrt{N}}}_{R^2,\ \text{2-loop}} + \underbrace{\frac{\mathcal{I}_{\mathbb{L},1}}{N}}_{D^2R^2,\ \text{3-loop, bulk contact}} + \cdots$$

- Some examples of diagrams are:



- At each order in $1/\sqrt{N}$, the contact and loop diagrams can be distinguished by their $(\tau, \bar{\tau})$ dependence!

Conjecture for graviton scattering amplitude

- Let me make a **conjecture** for $\mathbb{L} = \mathbb{W}$ (**Wilson loop**) $\longleftrightarrow$ **fundamental string**
- Bulk **effective** action is Nambu-Goto $+R^2 + D^2R^2 + D^4R^2 + \cdots$, but better to think about it in terms of scattering amplitude.
- Scattering amplitude $\boxed{\mathcal{A}_{1\to 1} = (\text{prefactor}) f_{\mathbb{W}}(\mathbf{s}, \mathbf{t})}$, where:

$$f_{\mathbb{W}}(\mathbf{s},\mathbf{t}) = \frac{1}{\mathbf{st}} + \ell_s^2 f_{\text{1-loop}}(\mathbf{s},\mathbf{t}) + \ell_s^4 \left(f_{\text{2-loop}}(\mathbf{s},\mathbf{t}) + \underbrace{c_1 g_s^2}_{R^2} \right)$$
$$+ \ell_s^6 \left(f_{\text{3-loop}}(\mathbf{s},\mathbf{t}) + g_s^2 f_{\text{bulk contact}}(\mathbf{s},\mathbf{t}) + \underbrace{c_2 g_s \mathcal{I}_{\mathbb{W},1}(\tau,\bar{\tau})(\mathbf{s}+\mathbf{t})}_{D^2R^2} \right) + \cdots$$

- Apply $SL(2,\mathbb{Z})$ to this formula $\Longrightarrow$ scattering from (p,q) strings.

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- Apply $SL(2,\mathbb{Z})$ to this formula $\Longrightarrow$ scattering from (p,q) strings.

Conjecture for graviton scattering amplitude

- D1-brane at $\chi = 0$: $\boxed{\mathcal{A}_{1\to 1} = (\text{prefactor}) f_{\mathbb{T}}(\mathbf{s}, \mathbf{t})}$, where:

$$f_{\mathbb{T}}(\mathbf{s}, \mathbf{t}) = \frac{1}{\mathbf{st}} + \ell_s^2 g_s f_{\text{1-loop}}(\mathbf{s}, \mathbf{t}) + \ell_s^4 \Bigg(g_s^2 f_{\text{2-loop}}(\mathbf{s}, \mathbf{t}) + \underbrace{c_1}_{R^2} \Bigg)$$

$$+ \ell_s^6 \Bigg(g_s^3 f_{\text{3-loop}}(\mathbf{s}, \mathbf{t}) + g_s f_{\text{bulk contact}}(\mathbf{s}, \mathbf{t}) + \underbrace{c_2 g_s^2 \mathcal{I}_{\mathbb{T},1}(\mathbf{s} + \mathbf{t})}_{D^2 R^2} \Bigg) + \cdots$$

where

$$g_s^2 \mathcal{I}_{\mathbb{T},1} = \frac{3\zeta(3)}{2\pi} - \frac{\pi g_s}{4} + \frac{\pi g_s^2}{6} + \frac{3 g_s^3}{32\pi} + \frac{3 g_s^3}{\pi} e^{-\frac{2\pi}{g_s}} \left(1 - \frac{25 g_s}{16\pi} + \mathcal{O}(g_s^2)\right)$$
$$+ \mathcal{O}\left(e^{-\frac{4\pi}{g_s}}\right)$$

- $\zeta(3)$ matches disk amplitude [Hashimoto, Klebanov '96; Basu '08] .

Conclusion

- We determined new functions of $(\tau, \bar{\tau})$ that appear in the effective actions at orders $D^2 R^2$ and $D^4 R^2$ on extended (p, q)-strings.

Didn't talk about:

- Actual computations using matrix model

For the future:

- Make connections precise between two-pt function in presence of line defect and graviton scattering from a long string.
- Generalization to Wilson loops in other irreps $\rightarrow$ effective action on D3- and D5-branes.

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