

# Twisted supergravity and exceptional super Lie algebras

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In recent work with Surya Raghavendran (PI/UoT/Yale) and Ingmar Saberi (LMU) we have uncovered rich algebraic structures underlying twists of supergravity.

In  $M$ -theory there are simple exceptional super Lie algebras associated of eleven-dimensional supergravity associated to twisted  $AdS$  geometries. Our goal is to use the corresponding exceptional symmetries to paint a clearer picture of the worldvolume theories associated to stacks of twisted (but not  $\Omega$ -deformed!!) M2/M5 branes.

Twisted supergravity

Costello and Li have introduced a way to study twists of supergravity.

**Idea:** work in background where one of the ghosts for supersymmetry takes a nonzero value. If supergravity couples to a gauge theory, then this returns the usual notion of twisting.

Inspired by topological string theory, Costello and Li give conjectural descriptions of twists of ten-dimensional supergravity in terms of Kodaira–Spencer theory.

BCOV identifies Kodaira–Spencer theory as the closed string field theory associated to the topological  $B$ -model on a CY3. Costello and Li showed how Kodaira–Spencer theory can be extended to Calabi–Yau manifolds in *any* dimension.

The space of fields is the cyclic cohomology of the category of coherent sheaves on a CY  $X$

$$\mathrm{Cyc}^\bullet(\mathrm{Coh}(X)).$$

This is the  $S^1$ -homotopy equivariant local observables of the  $B$ -model.

Via HKR there is a model for this as polyvector fields on  $X$ . Most importantly, we recover the Beltrami equation

$$\begin{aligned} \mu \in \Omega^{0,1}(X, \mathbb{T}_X), \quad \bar{\partial}\mu + \frac{1}{2}[\mu, \mu] = 0 \\ \partial_\Omega \mu = 0. \end{aligned}$$

as part of the equations of motion.

In even dimensions, a generic twist of a supersymmetric gauge theory can be placed in a background of a complex manifold. In general, the background geometry of a generic twist is that of a transverse holomorphic foliation (THF).

If  $M$  is a smooth manifold a THF is a codimension  $n$  foliation

$$\mathcal{F} \subset T_M^{\mathbf{C}} = T_M \otimes_{\mathbf{R}} \mathbf{C} \quad (1)$$

where the fiber over a point of  $T_M^{\mathbf{C}}/\mathcal{F}$  is spanned by  $\{\partial_{z_i}\}$ . Locally, a THF manifold admits a coordinate atlas consisting of local trivializations  $\mathbf{R}^m \times \mathbf{C}^n$  which are smooth along  $\mathbf{R}^m$  and holomorphic along  $\mathbf{C}^n$ .

There is a cohomology theory associated to a THF structure which we denote by  $\mathcal{A}^\bullet(M)$ . When the THF is  $T_M$  this is simply the de Rham complex. When the THF is  $T_M^{0,1}$  this is the Dolbeault complex. Generally, these two structures mix.

In eleven-dimensional supersymmetry there are essentially two classes of twists: minimal and non-minimal. The non-minimal twist is topological in seven directions and can be placed on geometry of the form

$$G_2 \times CY2 \tag{2}$$

The minimal twist exists on geometries of the form

$$M = \mathbf{R} \times Z \tag{3}$$

where  $Z$  is a CY5, or more generally on any eleven-manifold  $M$  equipped with a THF of complex codimension five which is equipped with a ‘holomorphic volume form’.

With Saberi, we have used the theory of pure spinors to deduce the underlying free theory of the minimal twist.

The underlying free theory has formal similarities with BCOV theory on a CY5. The linear BRST complex (including all BV fields) is

$$\begin{array}{ccc}
 & \text{odd} & \text{even} \\
 \mathcal{A}^\bullet(M, T_M)_\mu & \xrightarrow{\partial_\Omega} & \mathcal{A}^\bullet(M)_\nu \\
 \\
 \mathcal{A}^\bullet(M)_\beta & \xrightarrow{\partial} & \mathcal{A}^\bullet(M, \Omega^1)_\gamma.
 \end{array}$$

When split  $M = \mathbf{R} \times Z$ , have

$$\mathcal{A}^\bullet(M, \Omega^1) = \Omega^\bullet(\mathbf{R}) \otimes \Omega^{1, \bullet}(Z) \tag{4}$$

for example.



Assume  $M = \mathbf{R} \times Z$ . The field  $\mu \in C^\infty(\mathbf{R}) \otimes \text{PV}^{1,1}(Z)$  is a component of the eleven-dimensional metric  $g$  which is left over after twisting.

One of the key fields in 11-dimensional supergravity is a higher 3-form gauge field, otherwise known as the ‘supergravity  $C$ -field’

$$C \in \Omega^3(M^{11}).$$

It appears in the action through the ‘Chern–Simons’ term  $\int C \wedge G \wedge G$ . After twisting, only some components of the  $C$ -field remain. These are the 3-form components of

$$\beta \in \Omega^\bullet(\mathbf{R}) \otimes \Omega^{0,\bullet}(Z), \quad \gamma \in \Omega^\bullet(\mathbf{R}) \otimes \Omega^{1,\bullet}(Z).$$

Motivated by Costello and Li's topological string models of twists of superstrings, We proposed the following action functional for twisted 11-dimensional supergravity

$$\begin{aligned} & \int^{\Omega} \gamma d' \mu + \int^{\Omega} \beta d' \nu + \int^{\Omega} \beta \partial_{\Omega} \mu \\ & + \frac{1}{2} \int^{\Omega} (\mu \wedge \mu) \vee \partial \gamma + \dots \\ & + \frac{1}{6} \int \gamma \partial \gamma \partial \gamma. \end{aligned}$$

First line simply encodes the linear BRST complex described above. In the second line there are actually infinitely many terms, this must be understood as a formal series in the space of fields. (Analogous to the genus zero BCOV action.)

The last line is a twisted incarnation of the Chern–Simons action for the higher 3-form gauge field  $C$ .

Exceptional symmetry in twisted supergravity

On flat space  $M = \mathbf{R} \times \mathbf{C}$ , the linear BRST complex is quasi-isomorphic to

$$\begin{array}{ccc} \text{even} & & \text{odd} \\ \hline \text{Vect}(\mathbf{C}^5) & \xrightarrow{\partial_\Omega} & \mathcal{O}(\mathbf{C}^5) \\ & & \mathcal{O}(\mathbf{C}^5) \xrightarrow{\partial} \Omega^1(\mathbf{C}^5). \end{array}$$

Taking the remaining cohomology, this is

$$\begin{array}{ccc} \text{Vect}_0(\mathbf{C}^5) & & \\ & \mathbf{C} & \Omega_{cl}^2(\mathbf{C}^5) \end{array}$$

where we used  $\partial: \Omega^1(\mathbf{C}^5)/d\Omega^0(\mathbf{C}^5) \simeq \Omega_{cl}^2(\mathbf{C}^5)$ , given by  $[\gamma] \mapsto \partial\gamma$ . We will call a general closed two-form by  $\alpha$ .

The action functional induces the structure of a Lie algebra on the cohomology. For the moment, forget about the constants  $\mathbf{C}$ .

On divergence-free vector fields there is the obvious Lie bracket

$$[\mu, \mu'] = L_\mu(\mu') \in \text{Vect}_0(\mathbf{C}^5).$$

Also, divergence-free vector fields act on closed two-forms by Lie derivative

$$[\mu, \alpha] = L_\mu \alpha \in \Omega_{cl}^2(\mathbf{C}^5).$$

Most interestingly, the Chern–Simons term induces the following (anti) bracket

$$[\alpha, \alpha'] = \Omega_{\mathbf{C}^5}^{-1} \vee (\alpha \wedge \alpha') \in \text{Vect}_0(\mathbf{C}^5).$$

These brackets describe precisely the exceptional simple super Lie algebra

$$E(5|10) = \text{Vect}_0(\mathbf{C}^5) \oplus \Pi\Omega^{2,cl}(\mathbf{C}^5)$$

as part of the classification of  $\infty$ -dimensional linearly compact simple super Lie algebras obtained by Kac.

### Theorem (Saber-Raghavendran-W.)

*The symmetry algebra of the minimal twist of eleven-dimensional supergravity on flat space is a central extension of the exceptional simple super Lie algebra  $E(5, 10)$ .*

The  $L_\infty$  central extension can be computed by homotopy transfer:

$$(\mu, \mu', \alpha) \mapsto \langle \mu \wedge \mu', \alpha \rangle|_{z=0} \in \mathbf{C}.$$

This is an incarnation of the  $M2$ -brane extension arising in the brane scan.

Twisted  $AdS$  geometries

In the usual situation, the geometries

$$AdS_4 \times S^7, \quad AdS_7 \times S^4 \quad (5)$$

arise from backreacting some number of  $M2$ ,  $M5$  branes on flat space, respectively.

We will look at the restriction of the eleven-dimensional theory to the complement of the location of the brane; the coupling of  $M2$  and  $M5$  branes will dictate how this geometry is deformed.

We start with twisted  $M2$  branes wrapping

$$\mathbf{R} \times \mathbf{C} \times 0 \subset \mathbf{R} \times \mathbf{C} \times \mathbf{C}^4.$$

The backreaction will deform the geometry

$$\mathbf{R} \times \mathbf{C}^5 - \mathbf{R} \times \mathbf{C} \simeq \mathbf{R}_+ \times (\mathbf{R} \times \mathbf{C}) \times S^7,$$

as a manifold with a THF.\*



To first order, the coupling of  $M2$  branes to our bulk eleven-dimensional model is easy to describe.

Recall the field

$$\gamma \in \mathcal{A}^\bullet(M, \Omega^1) = \Omega^\bullet(\mathbf{R}) \otimes \Omega^{1,\bullet}(\mathbf{C}^5).$$

The coupling is

$$\int_{\mathbf{R} \times \mathbf{C}} \gamma.$$

Notice that this picks out a particular piece of the three-form component of  $\gamma$ .

The backreacted geometry is obtained as a solution to the equations of motion upon deforming the eleven-dimensional action by this coupling.

The solution is  $\mu = F_{M2}$  where

$$F_{M2} = \frac{6}{(2\pi i)^4} \frac{\sum_{a=1}^4 \bar{w}_a d\bar{w}_1 \cdots \widehat{d\bar{w}_a} \cdots d\bar{w}_4}{\|w\|^8} \partial_z$$

where  $z$  is a holomorphic coordinate along the brane and  $\{w_i\}$  is a holomorphic coordinate transverse to the brane.

Geometrically, the 7-form

$$d^4 w \wedge (F_{M2} \vee \partial_z) \in \Omega^{4,3}(\mathbf{C}^4 - 0).$$

represents the higher residue class of  $S^7 \subset \mathbf{C}^4 - 0$ .

There is a similar analysis for twisted  $M5$  branes and the twisted  $AdS_7$  geometry.

In the twisted  $AdS_4$ ,  $AdS_7$  geometries, we can consider the compactification of the theory along the corresponding sphere  $S^7, S^4$ .

Following the standard procedure, we define the space of twisted supergravity states  $\tilde{\mathcal{H}}_{AdS}$  to be the space of local observables of the compactified theory at  $\infty \in AdS$ . One may think of such boundary values as arising from modifications of a vacuum boundary condition at a point.

We can perform stringent tests of the feasibility of our mathematically twisted model by comparing these twisted state spaces  $\tilde{\mathcal{H}}_{AdS}$  to protected states in physical supergravity on  $AdS$ .

With Raghavendran we find an exact agreement of the superconformal index for both the twisted  $AdS_4$  and  $AdS_7$  geometries, for example.

Enhanced exceptional symmetry

Both the  $AdS_4, AdS_7$  supergravity backgrounds carry a global symmetry algebra  $\mathfrak{osp}(8|4)$  (the real forms differ).

After twisting, this symmetry is broken to  $\mathfrak{osp}(6|2)$ . It is not difficult to see that our twisted  $AdS$  backgrounds have this symmetry.

What is surprising is that after twisting there is an  $\infty$ -dimensional enhancement of this symmetry. This holds for both the  $AdS_4$  and  $AdS_7$  cases.

In the twisted  $AdS_7$  geometry there is an enhanced symmetry by the exceptional simple super Lie algebra

$$E(3|6)$$

(also defined by Kac) whose even and odd parts are:

- even:  $\text{Vect}^{hol}(\mathbf{C}^3) \ltimes \mathfrak{sl}(2) \otimes \mathcal{O}(\mathbf{C}^3)$ .
- odd:  $\Omega^{1,hol}(\mathbf{C}^3, K^{-1/2}) \otimes \mathbf{C}^2$ .

As expected, this is smaller than the symmetry algebra of twisted supergravity on flat space, which we identified with a central extension of  $E(5|10)$ .

There is, however, a systematic relationship between these two algebras.

There is a grading on  $E(5|10)$  of the form

$$E(5|10) = \bigoplus_{j \geq -1} V_j$$

induced from scaling of the transverse directions away from the brane. Years ago, Kac and Rudakov independently found this grading. The weight zero piece is

$$V_0 \simeq E(3|6).$$

and all  $V_j$ 's are irreducible  $E(3|6)$  representations which admit elegant descriptions as quotients of certain Verma modules.

There is an induced grading on the single particle twisted supergravity states in  $\tilde{\mathcal{H}}_{AdS_7}$ . In particular,  $\tilde{\mathcal{H}}_{AdS_7}$  is naturally a representation for  $E(3|6)$ .

Immediately, this gives the following.

### Theorem

*The supergravity superconformal index for  $AdS_7$  is a character for the exceptional simple super Lie algebra  $E(3|6)$ .*

We view this as an analog (or possibly a lift) of modularity properties about certain specializations of superconformal indices.

With Saberi we conjecture the following.

### Conjecture

The local operators in the holomorphic twist of any six-dimensional superconformal theory admits a symmetry by  $E(3|6)$ .

We show this explicitly for the minimal twist of the theory of the six-dimensional tensor multiplet (the theory on a single  $M5$  brane).



There is a similar result for twisted  $AdS_4$ . In this case, the enhanced symmetry is another exceptional simple super Lie algebra called  $E(1|6)$ .

It is deceptively similar to known superconformal algebras in two dimensions. (Kac has shown that this is not the case, however.)

With Raghavendran and Nik Garner (UW) we have seen how this exceptional super Lie algebra appears in holomorphic twists of three-dimensional ABJM type theories.

Further questions, future work

In our approach we actually find a rich algebraic structure present in the space of twisted supergravity states.

For a simpler example, recall the twisted holography example of Costello and Gaiotto where they backreact  $D1$  branes in the topological  $B$ -model on  $\mathbf{C}^3$ ; the deformed geometry is the deformed conifold  $SL(2, \mathbf{C})$ .

Furthermore, the space of states has the structure of a **vertex algebra**. As a test of twisted holography, they identify this vertex algebra with the large  $N$  limit of a simple gauged  $\beta\gamma$  system.

Costello performs a similar analysis for  $M5$  branes in twisted and  $\Omega$ -deformed  $M$ -theory.

In our situation, we find that the space of twisted states  $\tilde{\mathcal{H}}_{AdS_4}, \tilde{\mathcal{H}}_{AdS_7}$  has the structure of a *factorization algebra* on  $\mathbf{R} \times \mathbf{C}, \mathbf{C}^3$ , respectively.

## Goal

Characterize the complex three-dimensional vertex algebra  $\tilde{\mathcal{H}}_{AdS_7}$ .

There is a ‘punctured’ version of  $E(3|6)$  which amounts to replacing the  $\mathbf{C}^3$  boundary of twisted  $AdS_7$  with  $\mathbf{C}^3 - 0$ . What results is a three-dimensional (derived) version of Witt algebra  $\text{Vect}(\mathbf{C}^\times)$ .

The Witt algebra quantizes to the Virasoro algebra by turning on a central extension. An analogous central extension is present for  $E(3|6)|_{\mathbf{C}^3-0}$ .

Holographically, this central extension arises from a certain Witten diagram which is cubic in the backreaction which introduces a central term in the OPE like  $\sim N^3$ . Current work in progress is to characterize this central extension term explicitly. With this, we understand the zeroth piece  $\tilde{\mathcal{H}}_{AdS_7}^{(0)}$  as a vacuum module for this higher Virasoro algebra.

The twisted holographic proposal of Costello, Gaiotto, Li, and Paquette states (roughly) that there is a map of factorization algebras

$$\tilde{\mathcal{H}}_{AdS} \rightarrow \tilde{\text{Obs}}_{CFT}^N$$

where  $\tilde{\text{Obs}}_{CFT}^N$  is the algebra of observables on a stack of  $N$  twisted branes. Further, this becomes an equivalence in the large  $N$  limit.

For finite  $N$  this map is neither injective or surjective in cohomology. Nevertheless, we find:

- The character of the  $(-1)$  piece  $\tilde{\mathcal{H}}_{AdS_7}^{(-1)}$  agrees with the superconformal index of a single  $M5$  brane  $\text{Obs}_{CFT}^1$ .
- The character of the zeroth piece  $\tilde{\mathcal{H}}_{AdS_7}^{(0)}$  agrees with the conjectured superconformal index of a theory on a stack of two  $M5$  branes  $\text{Obs}_{CFT}^2$  to order  $q^4$ . The discrepancy takes the form of another irreducible module for  $E(3|6)$  which we would like to interpret in terms of black hole microstates.

From the perspective of the holomorphic twist, one can understand the  $\Omega$ -deformed supergravity in terms of a particular deformation of the theory by another odd symmetry  $S$ ; there is a compatible deformation along the brane that we denote by the same symbol.

Under this deformation our eleven-dimensional supergravity becomes Costello's five-dimensional non-commutative Chern–Simons theory. In particular we aim to show that

$$\tilde{\mathcal{H}}_{AdS_7} \xrightarrow{S} \text{State space of the } W_{1+\infty} \text{ vertex algebra,}$$

and that

$$\tilde{\mathcal{H}}_{AdS_7}^{(0)} \xrightarrow{S} \text{State space of the Virasoro vertex algebra.}$$

We have seen that the lowest piece, namely the holomorphic twist of the theory on a single  $M5$  brane does localize to the right thing.

Thank you!