General bounds on KK masses

Alessandro Tomasiello

Università di Milano-Bicocca

based on 2104.12773 with G.B. De Luca,
2109.11560 + 2212.02511 + WIP
with De Luca (physics-Stanford),
N. De Ponti (math-Bicocca), A. Mondino (math–Oxford)

CERN, 5 June, 2023

Introduction

KK spectrum: one of the most important pieces of data associated to a compactification

In holography:

- Checks of operator/state correspondence

[Kim, Romans, Van Nieuwenhuizen '85; Fabbri, Fré, Gualtieri, Termonia '99; Ceresole, Dall'Agata, D'Auria, Ferrara '99...]

- Relevant for scale separation

Exactly known vacua: $m_{\mathrm{KK}} \sim \frac{1}{\mathrm{diam}} \sim \frac{1}{r_{\mathrm{AdS}}} \sim \sqrt{|\Lambda|}$ 'no scale separation'

But in CFT: generically, no susy protection ⇨ many heavy operators?

[Polchinski, Silverstein '09]

This talk: several mathematical bounds
on the masses of spin-two fields...

- valid for many gravity compactifications (not just string/M-theory)
- often also in presence of singularities (D-branes, O-planes...)
- in terms of both ‘size’ and ‘shape’

We will not solve the problem of scale separation, but constrain it in several ways.

This talk: several mathematical bounds on the masses of spin-two fields…

- valid for many gravity compactifications (not just string/M-theory)
- often also in presence of singularities (D-branes, O-planes…)
- in terms of both 'size' and 'shape'

We will not solve the problem of scale separation, but constrain it in several ways.

For example:

- $m_k^2 < 600 k^2 \max\left\{ m_1^2,\ |\Lambda| + \sigma^2 \right\}$ $\qquad \sigma = \sqrt{D-2}\,\sup|\mathrm{d}A|$
- $m_k^2 \leqslant \left(|\Lambda| + (D-1)\sigma^2 \right) + \gamma \frac{k^2}{\mathrm{diam}^2}$ $\qquad$ max. distance among any two points
- $m_k^2 \geqslant \frac{C}{k^6} h_k^2$ $\qquad$ Cheeger constants: quantify 'small necks'

Plan

- Mathematical background

  Curvature, warping, and the weighted Raychaudhuri equation

- Overview of bounds

  in terms of Planck mass; Cheeger constant; diameter

- Examples and applications

  scale separation; gravity localization

Mathematical background

- Spin-two masses: eigenvalues of weighted Laplacian

$$\Delta_f \psi \equiv -\mathrm{e}^{-f}\nabla^m(\mathrm{e}^f\nabla_m\psi)$$

[Csaki, Erlich, Hollowood, Shirman'00; Bachas, Estes '11]

total dimension

$$f = (D-2)A$$

$$\mathrm{d}s^2_D = \mathrm{e}^{2A}(\mathrm{d}s^2_d + \mathrm{d}s^2_n)$$

warping

Mass operators for other fields:
not universal, only known in some cases.

e.g. Freund–Rubin: [Duff, Nilsson, Pope '86]

Mathematical background

- Spin-two masses: eigenvalues of weighted Laplacian

$$\Delta_f \psi \equiv -\mathrm{e}^{-f}\nabla^m(\mathrm{e}^f \nabla_m \psi)$$

[Csaki, Erlich, Hollowood, Shirman'00; Bachas, Estes '11]

total dimension

$$f = (D-2)A$$

$$\mathrm{d}s^2_D = \mathrm{e}^{2A}(\mathrm{d}s^2_d + \mathrm{d}s^2_n)$$

warping

Mass operators for other fields:
not universal, only known in some cases.

e.g. Freund–Rubin: [Duff, Nilsson, Pope '86]

- Presence of warping invalidates old theorems on 'usual' Laplace–Beltrami

long history: [Lichnerowicz '58, Cheeger '70, Cheng '75, Li, Yau '80, Buser '82...]

- Natural in the setup of metric measure spaces:

integration measure $=$ not just $\sqrt{g}$ but $e^f\sqrt{g}$

- Many ideas from recent progress in field of 'Optimal transport'

- Consider a distribution of particles moving geodesically $\rho(x)$ such that $\int_M \sqrt{g}\rho = 1$

Entropy: $S = -\int_M \sqrt{g}\rho \log \rho$

velocity field

using Raychaudhuri eq.: $\partial_t^2 S = -\int_M \sqrt{g}\rho(\nabla_m U_n \nabla^m U^n + R_{mn} U^m U^n)$

$$R_{mn} \geqslant 0 \;\Rightarrow\; \partial_t^2 S \leqslant 0$$

- Consider a distribution of particles moving geodesically $\rho(x)$ such that $\int_M \sqrt{g}\rho = 1$

Entropy: $S = -\int_M \sqrt{g}\rho \log \rho$

velocity field

using Raychaudhuri eq.: $\partial_t^2 S = -\int_M \sqrt{g}\rho(\nabla_m U_n \nabla^m U^n + R_{mn} U^m U^n)$

$$R_{mn} \geqslant 0 \;\Rightarrow\; \partial_t^2 S \leqslant 0$$

- Weighted 'Tsallis entropy': homogeneous (rather than extensive) [$\sim$ log Rényi entropy]

[Havrda, Charvat '67; Patil, Taillie '82; Tsallis '88]

$$S_{N,f} \equiv N\left(1 - \int_M \sqrt{g}\mathrm{e}^f \rho^{\frac{N-1}{N}}\right)$$

$$\partial_t^2 S_{N,f} \leqslant -\int_M \sqrt{g}\mathrm{e}^f \rho^{\frac{N-1}{N}} \underbrace{\left(R_{mn} - \nabla_m \nabla_n f + \frac{1}{n-N}\nabla_m f \nabla_n f\right)}_{\equiv\, R_{mn}^{N,f}} U^m U^n$$

[McCann '19; Mondino, Suhr '19; De Luca, De Ponti, Mondino, AT '22]

"Bakry–Émery curvature"

[Bakry, Émery '85]

N 'effective dimension': played $\sim$ role of rank of $\nabla_m U_n$

- Consider a higher-dimensional gravity $m_D^{D-2} \int \mathrm{d}^D x \sqrt{-g_D} R_D$ + matter

[De Luca, AT '20]

similar ideas in [Gautason, Schillo, Van Riet, Williams '15]

and a compactification $\mathrm{d}s_D^2 = \mathrm{e}^{2A}(\mathrm{d}s_d^2 + \mathrm{d}s_n^2)$

max. symmetric

- Consider a higher-dimensional gravity $m_D^{D-2} \int \mathrm{d}^D x \sqrt{-g_D} R_D$ + matter

[De Luca, AT '20]

similar ideas in [Gautason, Schillo, Van Riet, Williams '15]

and a compactification $\mathrm{d}s^2_D = \mathrm{e}^{2A}(\mathrm{d}s^2_d + \mathrm{d}s^2_n)$

max. symmetric

EoM: $R_{MN} = \frac{1}{2} m_D^{2-D} \left(T_{MN} - \frac{1}{D-2} g_{MN} T \right) \equiv \hat{T}_{MN}$

internal:

$$R_{mn} + (D-2)(-\nabla_m \nabla_n A + \partial_m A \partial_n A) = ((D-2)|\mathrm{d}A|^2 + \nabla^2 A) g_{mn} + \hat{T}_{mn}$$

[De Luca, AT '20]

similar ideas in [Gautason, Schillo, Van Riet, Williams '15]

- Consider a higher-dimensional gravity $m_D^{D-2}\int \mathrm{d}^D x\sqrt{-g_D}R_D$ + matter

and a compactification $\mathrm{d}s_D^2 = \mathrm{e}^{2A}(\mathrm{d}s_d^2 + \mathrm{d}s_n^2)$

max. symmetric

EoM: $R_{MN} = \frac{1}{2}m_D^{2-D}\left(T_{MN} - \frac{1}{D-2}g_{MN}T\right) \equiv \hat{T}_{MN}$

$\Lambda - \frac{1}{d}\hat{T}_{(d)}$ external

internal:

$$R_{mn} + (D-2)(-\nabla_m\nabla_n A + \partial_m A\partial_n A) = \underbrace{((D-2)|\mathrm{d}A|^2 + \nabla^2 A)}_{= \Lambda - \frac{1}{d}\hat{T}_{(d)}} g_{mn} + \hat{T}_{mn}$$

$$= \Lambda g_{mn} + (\hat{T}_{mn} - \frac{1}{d}g_{mn}\hat{T}_{(d)})$$

- Consider a higher-dimensional gravity $m_D^{D-2} \int \mathrm{d}^D x \sqrt{-g_D} R_D$ + matter

[De Luca, AT '20]

similar ideas in [Gautason, Schillo, Van Riet, Williams '15]

and a compactification $\mathrm{d}s_D^2 = \mathrm{e}^{2A}(\mathrm{d}s_d^2 + \mathrm{d}s_n^2)$

max. symmetric

EoM: $R_{MN} = \frac{1}{2} m_D^{2-D} \left(T_{MN} - \frac{1}{D-2} g_{MN} T \right) \equiv \hat{T}_{MN}$

internal:

$\Lambda - \frac{1}{d}\hat{T}_{(d)}$ external

$$R_{mn} + (D-2)(-\nabla_m \nabla_n A + \partial_m A \partial_n A) = (\underbrace{(D-2)|\mathrm{d}A|^2 + \nabla^2 A}_{= \Lambda - \frac{1}{d}\hat{T}_{(d)}}) g_{mn} + \hat{T}_{mn}$$

$$= \Lambda g_{mn} + \underbrace{(\hat{T}_{mn} - \tfrac{1}{d} g_{mn} \hat{T}_{(d)})}_{\text{non-negative}} \geqslant \Lambda g_{mn}$$

non-negative

- for all bulk fields in type II and $d = 11$ sugra
- potentials
- for brane sources

["Reduced Energy Condition"]

[De Luca, AT '20]

similar ideas in [Gautason, Schillo, Van Riet, Williams '15]

- Consider a higher-dimensional gravity $m_D^{D-2}\int \mathrm{d}^D x\sqrt{-g_D}R_D$ + matter

and a compactification $\mathrm{d}s_D^2 = \mathrm{e}^{2A}(\mathrm{d}s_d^2 + \mathrm{d}s_n^2)$

max. symmetric

EoM: $R_{MN} = \frac{1}{2}m_D^{2-D}\left(T_{MN} - \frac{1}{D-2}g_{MN}T\right) \equiv \hat{T}_{MN}$

external: $\Lambda - \frac{1}{d}\hat{T}_{(d)}$

internal:

$$R_{mn} + (D-2)(-\nabla_m\nabla_n A + \partial_m A\partial_n A) = \underbrace{((D-2)|\mathrm{d}A|^2 + \nabla^2 A)}_{=\,\Lambda - \frac{1}{d}\hat{T}_{(d)}}g_{mn} + \hat{T}_{mn}$$

$$R_{mn}^{N,f} = \Lambda g_{mn} + \underbrace{(\hat{T}_{mn} - \tfrac{1}{d}g_{mn}\hat{T}_{(d)})}_{\text{non-negative}} \geqslant \Lambda g_{mn}$$

BE curvature

$f = (D-2)A$

$N = 2 - d < 0$

but still OK

non-negative

- for all bulk fields in type II and $d = 11$ sugra
- potentials
- for brane sources

["Reduced Energy Condition"]

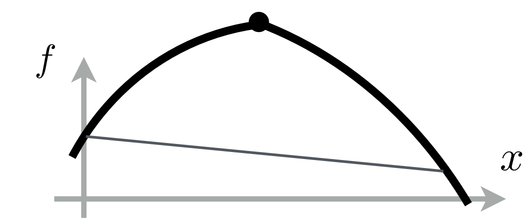
Sources create singularities. It would be best to avoid derivatives...

- Inspiration: functions of one variable

$$f'' \leqslant 0$$

generalize to non-smooth functions:

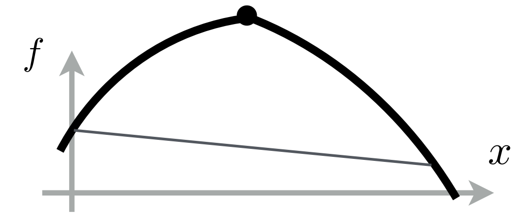
concavity



Sources create singularities. It would be best to avoid derivatives...

- Inspiration: functions of one variable

$$f'' \leqslant 0 \xrightarrow{\text{generalize to non-smooth functions:}} \text{concavity}$$



$$R^{N,f}_{mn} \geqslant 0 \Rightarrow \partial_t^2 S_{N,f} \leqslant 0 \xrightarrow{\text{generalize to non-smooth spaces:}} \text{concavity of } S_{N,f}$$

this leads to the 'Riemann-Curvature-Dimension' [RCD] condition [Sturm '06; Lott, Villani '07; Ambrosio, Gigli, Savaré 14]

[One can also reformulate the Einstein equations in this language] [McCann '19; Mondino, Suhr '19; De Luca, De Ponti, Mondino, AT '22]

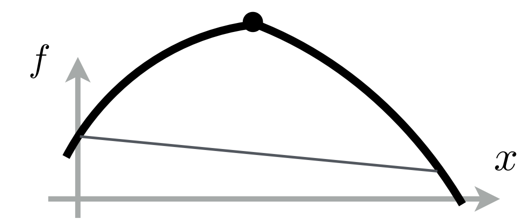
Sources create singularities. It would be best to avoid derivatives...

- Inspiration: functions of one variable

$f'' \leqslant 0$

generalize to non-smooth functions:

concavity



$R^{N,f}_{mn} \geqslant 0 \Rightarrow \partial_t^2 S_{N,f} \leqslant 0$

generalize to non-smooth spaces:

concavity of $S_{N,f}$

this leads to the ‘Riemann-Curvature-Dimension’ [RCD] condition

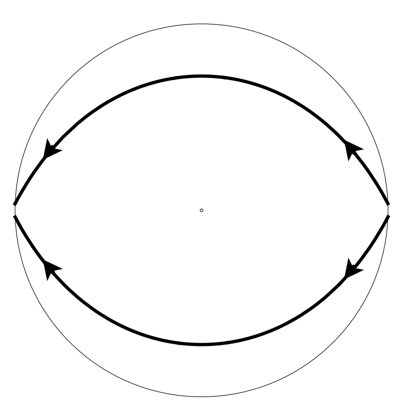
[Sturm ’06; Lott, Villani ’07; Ambrosio, Gigli, Savaré 14]

[One can also reformulate the Einstein equations in this language]

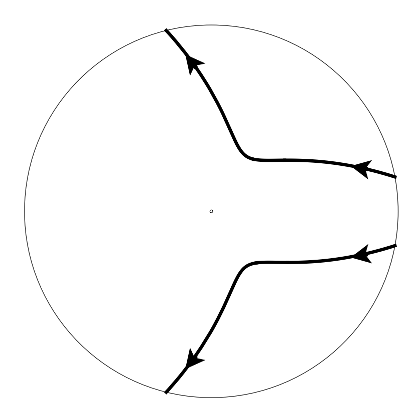
[McCann ’19; Mondino, Suhr ’19; De Luca, De Ponti, Mondino, AT ‘22]

- Theorem: spaces with D-branes are RCD.

[De Luca, De Ponti, Mondino, AT ‘22]



intuitively, they attract particles.



... while this fails for O-planes, which repel

Overview of bounds

[De Luca, AT '20; De Luca, De Ponti, Mondino, AT '21,'22]

Rather than showing all bounds in detail, a summary. Definitions:

- 4d Planck mass $M_4^2 \sim M_D^{D-2} \int_M \sqrt{g} e^{(D-2)A}$ [if unwarped: int. volume]

- diameter: max. distance between any two points in M

Overview of bounds

[De Luca, AT '20; De Luca, De Ponti, Mondino, AT '21,'22]

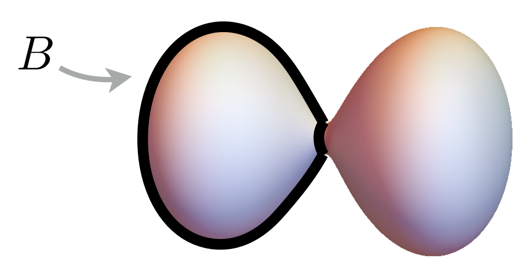
Rather than showing all bounds in detail, a summary. Definitions:

- 4d Planck mass $M_4^2 \sim M_D^{D-2} \int_M \sqrt{g} \mathrm{e}^{(D-2)A}$ [if unwarped: int. volume]

- diameter: max. distance between any two points in M

- Cheeger constant h_1: small when space has small 'neck'

$h_1 = \min_B \frac{\mathrm{vol}_A(\partial B)}{\mathrm{vol}_A(B)}$ 'min. of $\frac{\text{perimeter}}{\text{area}}$'



$\mathrm{vol}_A(B) \equiv \int_B \sqrt{g} \mathrm{e}^{(D-2)A}$

	4d Planck mass	Cheeger	diameter
upper bound	m_k {smooth; warp.}	m_1 {D-branes; warp.} $\frac{m_k}{m_1^2}$ {O-planes}	m_k {smooth; warp.}
lower bound		m_k {O-planes}	m_1 {D-branes}

- {smooth} $\subset$ {D-branes} $\subset$ {O-planes}

  spaces with D-brane sing. — spaces with also O-plane sing.

- {warp.}: bound contains $\sigma \equiv \sup_M \sqrt{D-2}|\mathrm{d}A|$

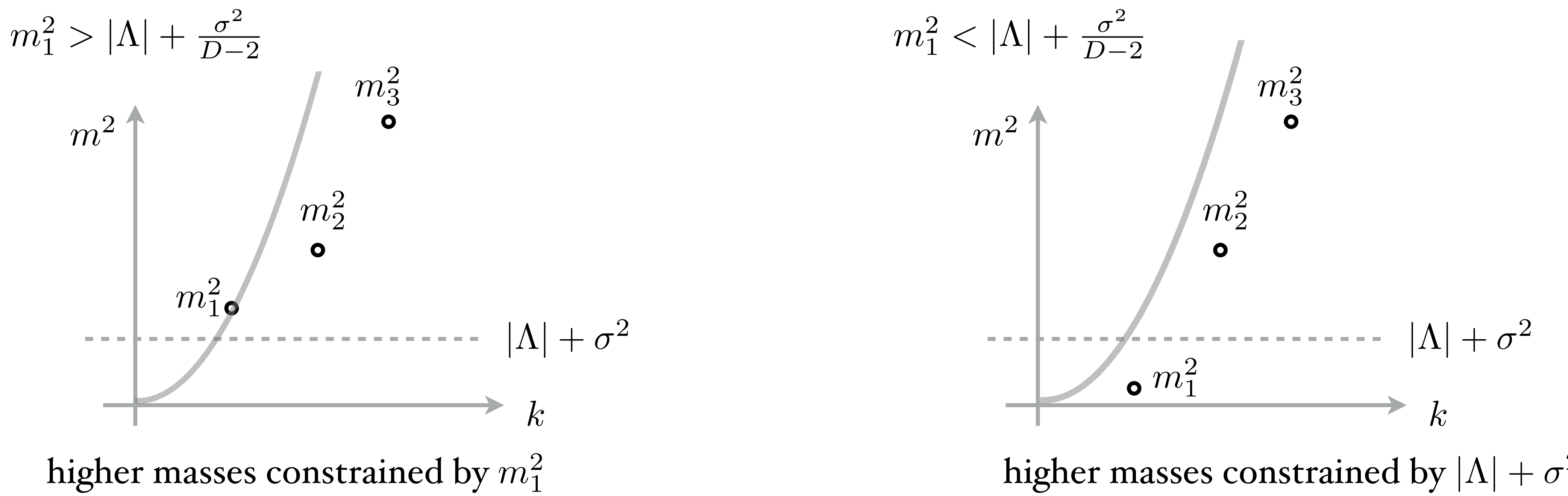
Scale separation

total dimension

$$D = d + n$$

$$\sigma = \sqrt{D-2}\,\sup|\mathrm{d}A|$$

- combining several bounds: $$m_k^2 < 600 k^2 \max\left\{m_1^2,\ |\Lambda| + \sigma^2\right\}$$



- a version of scale separation: second, third mass etc. can't be arbitrarily heavy

- first case in agreement with the Spin-2 conjecture;
  in second case, counterexamples

[Klaewer, Lüst, Palti '18]
[de Rham, Heisenberg, Tolley '18]
[Bachas '19]

- how about the first mass?

total dimension

$$D = d + n$$

[M_n smooth]

$\sigma = \sqrt{D-2}\,\sup|\mathrm{d}A|$

$$m_k^2 \leqslant \max\left\{(D-2)\sigma^2,\ \tfrac{1}{n-1}(|\Lambda| + \sigma^2)\right\} + \beta(k\, m_D^{D-2} m_d^{2-d})^{2/n}$$

[$\alpha, \beta, \gamma \sim 10^4$]

[Planck masses]

[De Luca, AT '21]
using [Hassannezhad '12]

but this doesn't exclude scale separation:

e.g. $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_p \to$ large second term

$p \to \infty$

[Gautason, Schillo, Van Riet, Williams '15]
[Cribiori, Junghans, Van Hemelryck, Van Riet, Wrase '21]

- how about the first mass?

$[M_n$ smooth$]$

total dimension

$$D = d + n$$

$$\sigma = \sqrt{D-2}\,\sup|\mathrm{d}A|$$

$$m_k^2 \leqslant \max\left\{(D-2)\sigma^2,\ \tfrac{1}{n-1}(|\Lambda|+\sigma^2)\right\} + \beta(k\, m_D^{D-2} m_d^{2-d})^{2/n}$$

[Planck masses]

$[\alpha, \beta, \gamma \sim 10^4]$

[De Luca, AT '21]
using [Hassannezhad '12]

but this doesn't exclude scale separation:

e.g. $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_p \to$ large second term

$p \to \infty$

[Gautason, Schillo, Van Riet, Williams '15]
[Cribiori, Junghans, Van Hemelryck, Van Riet, Wrase '21]

- this issue is eliminated working with the diameter:

[De Luca, AT '21] using [Setti '98]

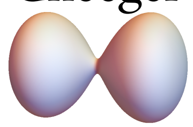
$$m_k^2 \leqslant \left(|\Lambda| + (D-1)\sigma^2\right) + \gamma\frac{k^2}{\mathrm{diam}^2}$$

but now problem is 'nonlocal': how large is diam?

for sphere quotients:
[Greenwald '00,
Gorodski, Lange, Lytchak, Mendes '19,
Collins, Jafferis, Vafa, Xu, Yau '22]

- lower bounds can be useful for establishing scale separation in a given solution.

Cheeger

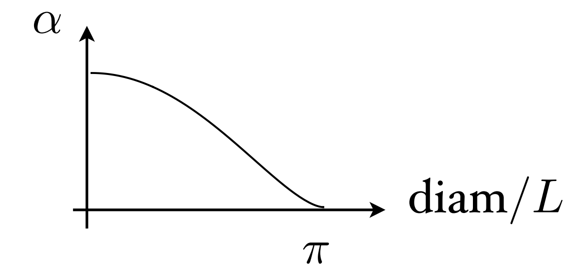


$$\frac{h_1^2}{4} \leqslant m_1^2$$

[even with O-planes]

$$\alpha\left(\frac{\mathrm{diam}}{L_{\mathrm{AdS}}}\right)\frac{1}{\mathrm{diam}^2} \leqslant m_1^2$$

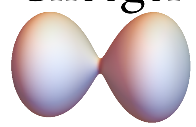
[even with D-branes]



[De Luca, De Ponti, Mondino, AT '21,'22];
diam. bound inspired by [Calderon '19]

- lower bounds can be useful for establishing scale separation in a given solution.

Cheeger

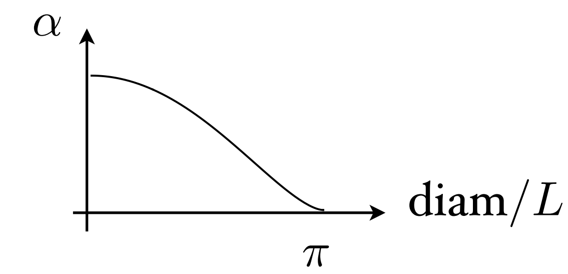


$$\frac{h_1^2}{4} \leqslant m_1^2$$

[even with O-planes]

$$\alpha\left(\frac{\text{diam}}{L_{\text{AdS}}}\right)\frac{1}{\text{diam}^2} \leqslant m_1^2$$

[even with D-branes]



[De Luca, De Ponti, Mondino, AT '21,'22];
diam. bound inspired by [Calderon '19]

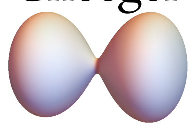
- $\text{AdS}_4\times$ CY: estimate of Cheeger on 10d solution confirms [$\int F_4 \sim N$]

$$m_1 \geqslant \frac{h_1}{2} \sim N^{-1/4} \gg \sqrt{|\Lambda|} \sim N^{-3/4}$$

[DeWolfe, Giryavets, Kachru, Taylor '05]
[Acharya, Benini, Valandro '06;
Junghans '20; Marchesano, Palti, Quirant, AT '20]

- lower bounds can be useful for establishing scale separation in a given solution.

Cheeger

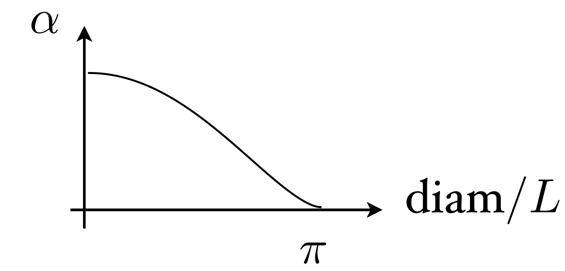


$$\frac{h_1^2}{4} \leqslant m_1^2$$

[even with O-planes]

$$\alpha\left(\frac{\text{diam}}{L_{\text{AdS}}}\right)\frac{1}{\text{diam}^2} \leqslant m_1^2$$

[even with D-branes]



[De Luca, De Ponti, Mondino, AT '21,'22]; diam. bound inspired by [Calderon '19]

- $\text{AdS}_4 \times$ CY: estimate of Cheeger on 10d solution confirms [$\int F_4 \sim N$]

$$m_1 \geqslant \frac{h_1}{2} \sim N^{-1/4} \gg \sqrt{|\Lambda|} \sim N^{-3/4}$$

[DeWolfe, Giryavets, Kachru, Taylor '05]
[Acharya, Benini, Valandro '06;
Junghans '20; Marchesano, Palti, Quirant, AT '20]

- A potentially simpler example: M-theory lift smooth $\text{AdS}_4 \times (\text{weak}G_2)_7$? $F_4 \sim N \text{vol}_{\text{AdS}_4}$

$$m_1 \geqslant \frac{c}{\text{diam}} \sim N^{-11/48} \gg \sqrt{|\Lambda|} \sim N^{-7/24}$$

[Cribiori, Junghans, Van Hemelryck, Van Riet, Wrase '19]

- Another way to obtain scale separation: Casimir energy

[De Luca, De Ponti, Mondino, AT '22]

inspired by
[De Luca, Silverstein, Torroba '21]
[Arkani-Hamed, Dubovsky, Nicolis, Villadoro '07]

$$\langle T^{\rm Cas}_{\mu\nu}\rangle = \frac{\ell_P}{R_7^{11}} g_{\mu\nu} \qquad \langle T^{\rm Cas}_{mn}\rangle = -\frac{4}{7}\frac{\ell_P}{R_7^{11}} g_{mn}$$

'Freund–Rubin' $\mathrm{AdS}_4 \times T^7$: $\quad F_4 \sim N \mathrm{vol}_{\mathrm{AdS}_4}$

$$m_1 \geqslant \frac{c}{\rm diam} \sim N^{-2/3} \gg \sqrt{|\Lambda|} \sim N^{-11/3}$$

$m_1 \sim |\Lambda|^{1/11}$ much slower than conjectural bound $|\Lambda|^{1/4}$ [Rudelius '21, Castellano, Herráez, Ibáñez '21]

Gravity localization

- Similar bounds also apply when $\int_M \sqrt{g} e^{(D-2)A} = \infty$

$\Rightarrow$ no massless graviton: $m_0 \neq 0$

(could be evaded if $\exists\, L^2$ harmonic function: but this never happens)

[De Luca, De Ponti, Mondino, AT *to appear*]

Gravity localization

- Similar bounds also apply when $\int_M \sqrt{g} e^{(D-2)A} = \infty$

$\Rightarrow$ no massless graviton: $m_0 \neq 0$

(could be evaded if $\exists\, L^2$ harmonic function: but this never happens) [De Luca, De Ponti, Mondino, AT *to appear*]

- maybe one can still obtain 4d gravitational potential in some regime?

5d model: [Karch, Randall '00]

$\mathrm{AdS}_5 \,|\, \mathrm{AdS}_5$

AdS_4 'brane'

$$V \sim G M_1 M_2 \left(\frac{1}{R} + \frac{L_5^2}{L_4 R^2} + \dots \right)$$

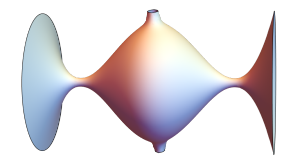
$\Rightarrow$ 4d behavior for $R \gg L_5^2 / L_4$

relies on:

- $m_1 \ll \sqrt{|\Lambda|} \ll m_2$
- wave-functions for $m_{k>1}$ small near brane

- string theory realizations?
  - hol. duals of defects in $\mathcal{N}=4$ super-YM

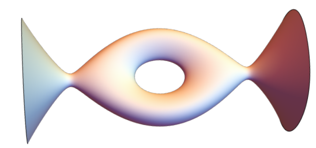
$S^2 \times S^2$ fibred on



[Bachas, Lavdas '17, '18] based on
[D'Hoker, Estes, Gutperle '07]
[Assel, Bachas, Estes, Gomis '11]

  - MN-type solutions over ∞-volume Riemann surfaces

top. spheres fibred on

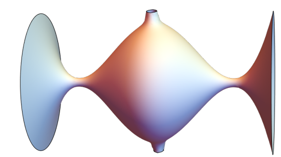


[Maldacena, Nuñez '00...]

- string theory realizations?
    - hol. duals of defects in $\mathcal{N} = 4$ super-YM

      $S^2 \times S^2$ fibred on

      D5

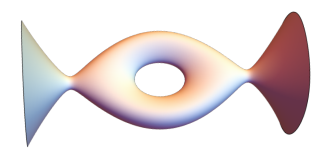
      

      NS5

      [Bachas, Lavdas '17, '18] based on [D'Hoker, Estes, Gutperle '07] [Assel, Bachas, Estes, Gomis '11]

    - MN-type solutions over ∞-volume Riemann surfaces

      top. spheres fibred on

      [Maldacena, Nuñez '00...]

- a version of the scale separation problem:

$$m_0 \ll \sqrt{|\Lambda|} \ll m_1$$

easy

problematic, because:

$$m_k^2 < 150k^2 \text{max}\left\{m_0^2,\ |\Lambda| + \sigma^2\right\}$$

⇨ 'localization' only up to cosmological scale?

- wave-function suppression might help, but hard to assess with our methods.

Conclusions

- No math evidence against scale separation; not even without sources

- Nontrivial relations among different masses; 'bootstrap' flavor

- We should improve by removing warping.

  requires advance in theory of optimal transport with 'negative' dimensions

- Bounds in terms of diameter, Cheeger constant:
  useful to estimate masses without computations

- Optimal transport approach to curvature in terms of entropy:
  might be of deeper significance