SC magnet design – EM part I

Stefania Farinon - INFN Genova
(stefania.farinon@ge.infn.it)

CERN Accelerator School, Normal and Superconducting Magnets – St. Pölten, Austria
November 2023

Lecture based on E. Todesco, "Masterclass -Design of superconducting magnets for particle accelerators", https://indico.cern.ch/category/12408/

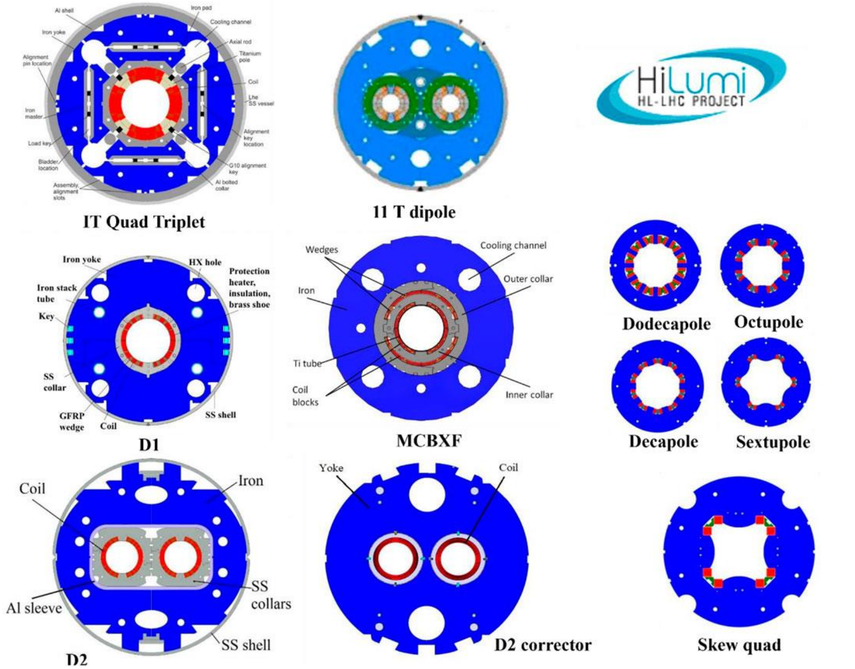
OUTLINE OF LECTURE I

- SC magnet design – EM part I

- Recap of field harmonics
- How to make multipoles with current lines
  - Perfect dipoles
  - Canted cosθ dipoles
  - Sector dipoles
  - Block-coils
  - Perfect quadrupoles
  - Sector quadrupoles

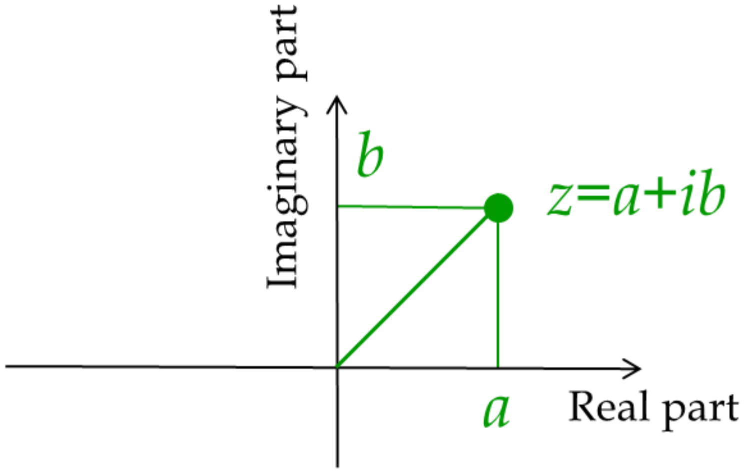
ACCELERATOR MAGNETS

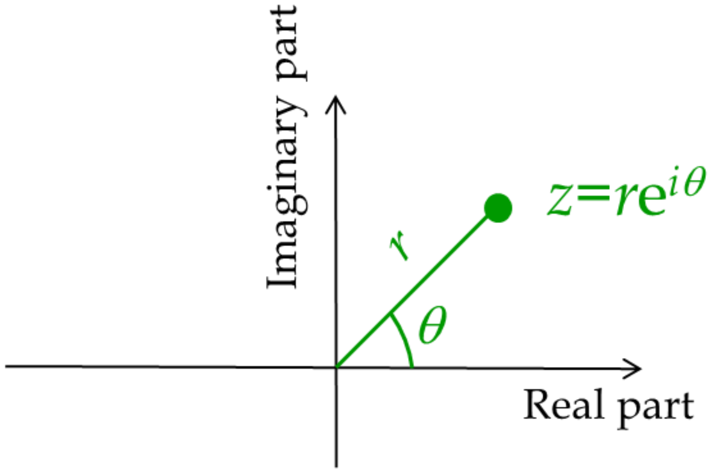
- Accelerator magnets exhibit a cross-section that extends over a length significantly greater than their cross-sectional dimensions:
  - electromagnetic design can effectively be treated as a 2D problem
  - coil heads can be considered as end effects
- The main accelerator magnet families are:
  - Dipoles
    to achieve uniform beam bending, dipoles must generate a constant magnetic field across the aperture
  - Quadrupoles
    Quadrupoles generate a linear variation (gradient) in the magnetic field across the aperture; beam that is radially focused is vertically defocused or vice-versa
  - Sextupoles
    Sextupoles generate a quadratic variation (gradient) in the magnetic field across the aperture and correct beam chromaticity



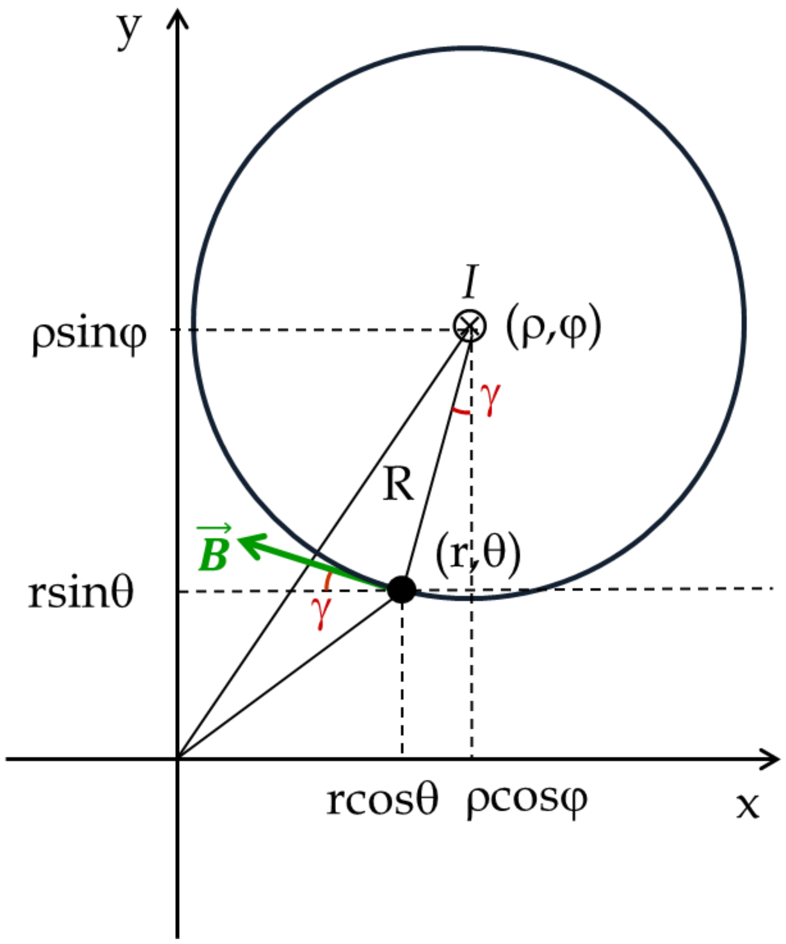
Magnets developed for High Luminosity LHC

- A **complex number** is an element of a number system that extends the real numbers with a specific element denoted i, called the imaginary unit and satisfying the equation $i^2=-1$.
- A complex number has two components and can be written:
  - In cartesian form as $z=a+ib$
  - In exponential form as $z=re^{i\theta}$
- Both the notations can be represented in the complex plane:
  - $r=\sqrt{(a^2+b^2)}$
  - $\theta=\text{atan}(b/a)$





FIELD FROM A CURRENT LINE



- In complex notation:

$$\boldsymbol{B}(\boldsymbol{z}) = \frac{\mu_0 I}{2\pi(\boldsymbol{z}-\boldsymbol{z_0})}, \text{ con } \boldsymbol{z} = re^{i\vartheta} \text{ e } \boldsymbol{z_0} = \rho e^{i\varphi}$$

- This can be easily checked:

$$\boldsymbol{B}(\boldsymbol{z}) = \frac{\mu_0 I}{2\pi(re^{i\vartheta} - \rho e^{i\varphi})}$$

$$= \frac{\mu_0 I}{2\pi[(r\cos\vartheta - \rho\cos\varphi) + i(r\sin\vartheta - \rho\sin\varphi)]}$$

$$= \frac{\mu_0 I[(r\cos\vartheta - \rho\cos\varphi) - i(r\sin\vartheta - \rho\sin\varphi)]}{2\pi[r^2 + \rho^2 - 2r\rho\cos(\varphi - \vartheta)]}$$

$$= \frac{\mu_0 I}{2\pi R}\frac{(r\cos\vartheta - \rho\cos\varphi) - i(r\sin\vartheta - \rho\sin\varphi)}{R}$$

$$= \frac{\mu_0 I}{2\pi R}(\sin\gamma + i\cos\gamma)$$

$$= B_y + iB_x$$

FIELD FROM A CURRENT LINE inside the filament (r<ρ)

- $\boldsymbol{B}(\boldsymbol{z}) = B_y + iB_x = \frac{\mu_0 I}{2\pi(\boldsymbol{z}-\boldsymbol{z_0})} = \frac{\mu_0 I}{2\pi(r\mathrm{e}^{i\vartheta}-\rho\mathrm{e}^{i\varphi})} = -\frac{\mu_0 I}{2\pi\rho\mathrm{e}^{i\varphi}} \frac{1}{1-\frac{r}{\rho}\mathrm{e}^{i(\vartheta-\varphi)}}$
- If $\epsilon < 1$: $\frac{1}{1-\epsilon} = \sum_{n=1}^{\infty} \epsilon^{n-1}$
- $B_y + iB_x = -\frac{\mu_0 I}{2\pi\rho}\mathrm{e}^{-i\varphi} \sum_{n=1}^{\infty} \left[\frac{r}{\rho}\mathrm{e}^{i(\vartheta-\varphi)}\right]^{n-1} = -\frac{\mu_0 I}{2\pi\rho} \sum_{n=1}^{\infty} \mathrm{e}^{-in\varphi}\mathrm{e}^{i(n-1)\vartheta}\left(\frac{r}{\rho}\right)^{n-1}$
- $B_y + iB_x = \sum_{n=1}^{\infty} \left[-\frac{\mu_0 I}{2\pi\rho}\mathrm{e}^{-in\varphi}\left(\frac{R_{ref}}{\rho}\right)^{n-1}\right]\left[\mathrm{e}^{i(n-1)\vartheta}\left(\frac{r}{R_{ref}}\right)^{n-1}\right]$

Dimensionless term that includes information about the location where the field is calculated $\boldsymbol{z} = re^{i\vartheta}$

Dimensioned term [T] that includes information about the location of the current line $\boldsymbol{z_0} = \rho e^{i\varphi}$

- $B_y + iB_x = \sum_{n=1}^{\infty}(B_n + iA_n)(\cos(n-1)\vartheta + i\sin(n-1)\vartheta)\left(\frac{r}{R_{ref}}\right)^{n-1}$

HARMONICS FROM A CURRENT LINE

- $B_y + iB_x = \sum_{n=1}^{\infty} \left[-\frac{\mu_0 I}{2\pi\rho} e^{-in\varphi} \left(\frac{R_{ref}}{\rho} \right)^{n-1} \right] \left[e^{i(n-1)\vartheta} \left(\frac{r}{R_{ref}} \right)^{n-1} \right]$
- $B_y + iB_x = \sum_{n=1}^{\infty} (B_n + iA_n)(\cos(n-1)\vartheta + i\sin(n-1)\vartheta) \left(\frac{r}{R_{ref}} \right)^{n-1}$
- With $B_n = -\frac{\mu_0 I}{2\pi\rho} \left(\frac{R_{ref}}{\rho} \right)^{n-1} \cos n\varphi$ and $A_n = \frac{\mu_0 I}{2\pi\rho} \left(\frac{R_{ref}}{\rho} \right)^{n-1} \sin n\varphi$

$$B_n = -\frac{\mu_0 I}{2\pi R_{ref}} \left(\frac{R_{ref}}{\rho} \right)^{n} \cos n\varphi$$

$$= \frac{\mu_0 I}{2\pi R_{ref}} \left(\frac{R_{ref}}{\rho} \right)^{n} \sin n\varphi$$

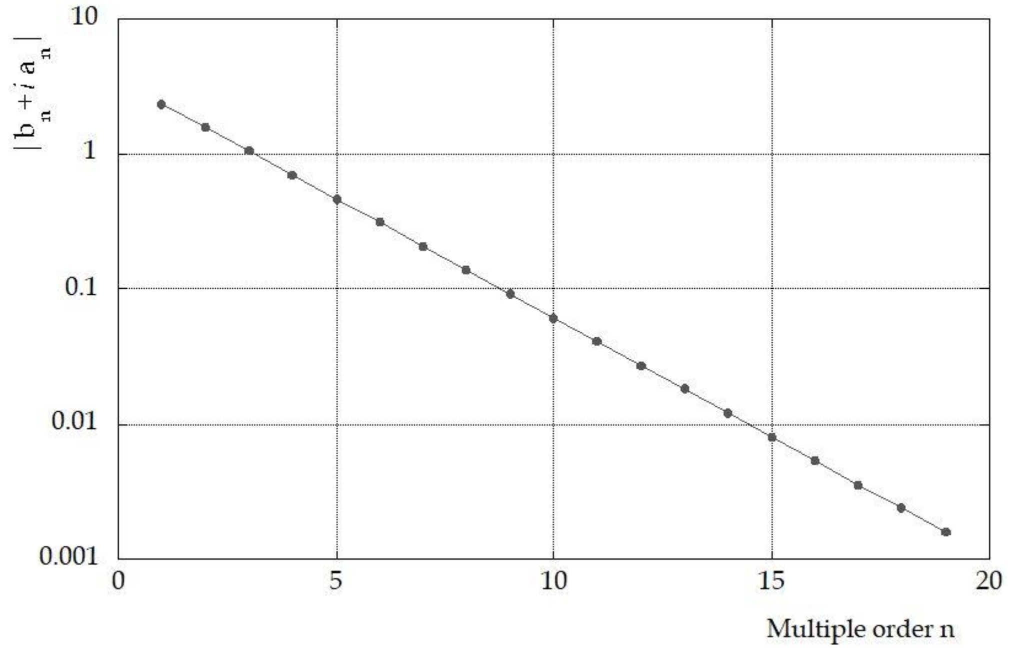
- The field harmonics B_n and A_n [T] can be rewritten in **normalized multipoles** b_n and a_n[dimensionless] as:

$$B_y + iB_x = 10^{-4} B_{R_{ref}} \sum_{n=1}^{\infty} (b_n + ia_n)(\cos(n-1)\vartheta + i\sin(n-1)\vartheta)\left(\frac{r}{R_{ref}}\right)^{n-1}$$

  - b_n are the normal components, a_n are the skew components (dimensionless)
  - The reference radius is introduced to separate, in the series, the term with information on the current line position to the term with information about the location where the field is calculated. It has no physical meaning and is usually chosen as 2/3 of the aperture radius.
  - We factorize 10^{-4} since the deviations from ideal field in superconducting magnets for particle accelerators should be of the order of 1‱ (per ten thousand)
  - $B_{R_{ref}}$ is the amplitude [T] of the fundamental harmonic at the reference radius. For example, in dipoles $B_{R_{ref}} = B_0$, in quadrupoles $B_{R_{ref}} = G \times R_{ref}$, etc.

- Multipoles given by a current line decay with the order:
  - $(B_n + iA_n) = -\frac{\mu_0 I}{2\pi\rho} e^{-in\varphi} \left(\frac{R_{ref}}{\rho}\right)^{n-1}$
  - $(b_n + ia_n) = -\frac{\mu_0 I}{2\pi\rho} \frac{10^4}{B_{R_{ref}}} e^{-in\varphi} \left(\frac{R_{ref}}{\rho}\right)^{n-1} = -\frac{\mu_0 I}{2\pi R_{ref}} \frac{10^4}{B_{R_{ref}}} \left(\frac{R_{ref}}{\rho e^{i\varphi}}\right)^{n} = -\frac{\mu_0 I}{2\pi R_{ref}} \frac{10^4}{B_{R_{ref}}} \left(\frac{R_{ref}}{\mathbf{z_0}}\right)^{n}$
  - $\ln(|b_n + ia_n|) = \ln\left(\frac{\mu_0 |I|}{2\pi R_{ref}} \frac{10^4}{B_{R_{ref}}}\right) + n\ln\left(\frac{R_{ref}}{\mathbf{z_0}}\right)$

$\mathbf{z_0} = \rho e^{i\varphi}$ is the location of the current line

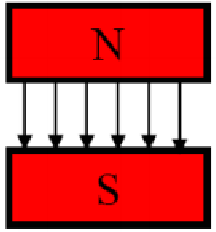


- In a semi-logarithmic scale, the slope of the linear decay is $\ln\left(\frac{R_{ref}}{\mathbf{z_0}}\right)$
  - This explains why only low order multipoles, in general, are relevant
  - It can help can detecting assembly errors in real magnets

EXAMPLES OF MAGNETS WITH $b_n \neq 0$, $a_n = 0$

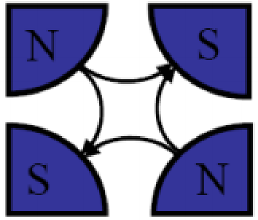
(skew harmonics are obtained by rotating the magnets by $\pi/2n$)

n=1: Dipole



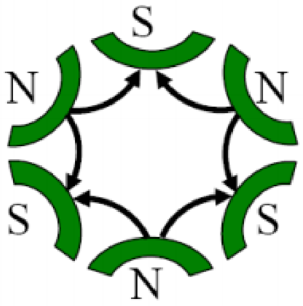
180° between poles

n=2: Quadrupole



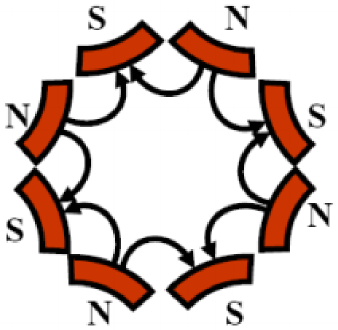
90° between poles

n=3: Sextupole

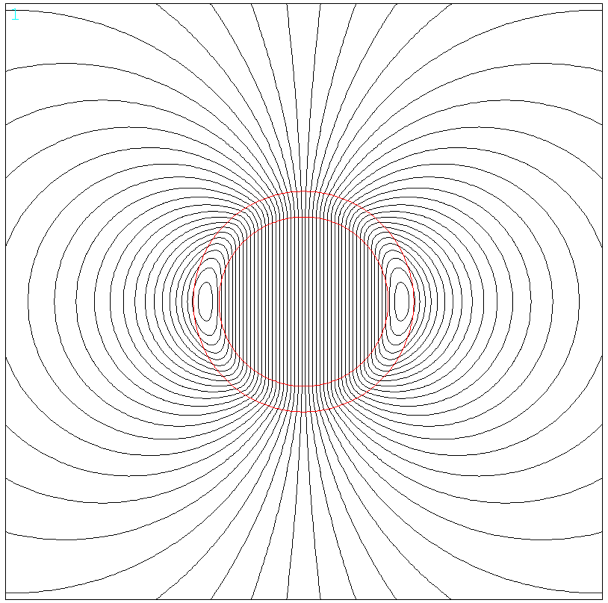


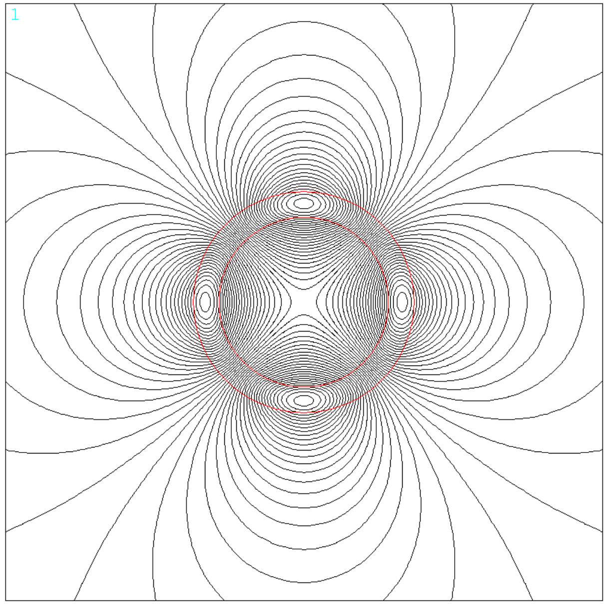
60° between poles

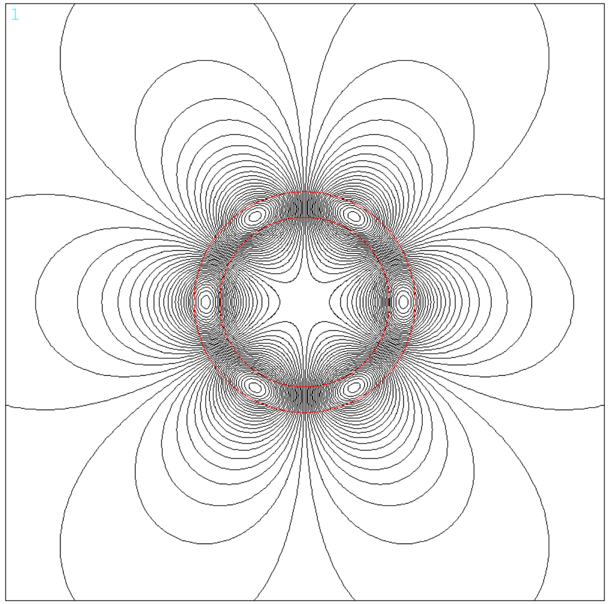
n=4: Octupole

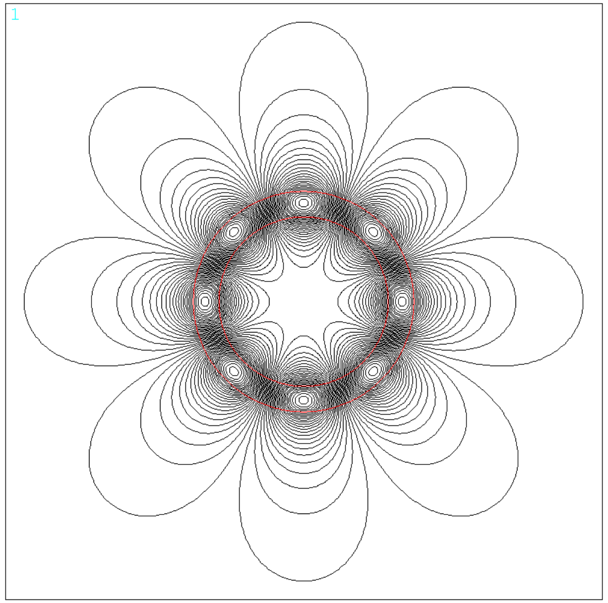


45° between poles









- The function $\boldsymbol{B}(\boldsymbol{z})$ is expressed through a Fourier series, enabling the utilization of corresponding inverse formulae to deduce the harmonic components from the field map:

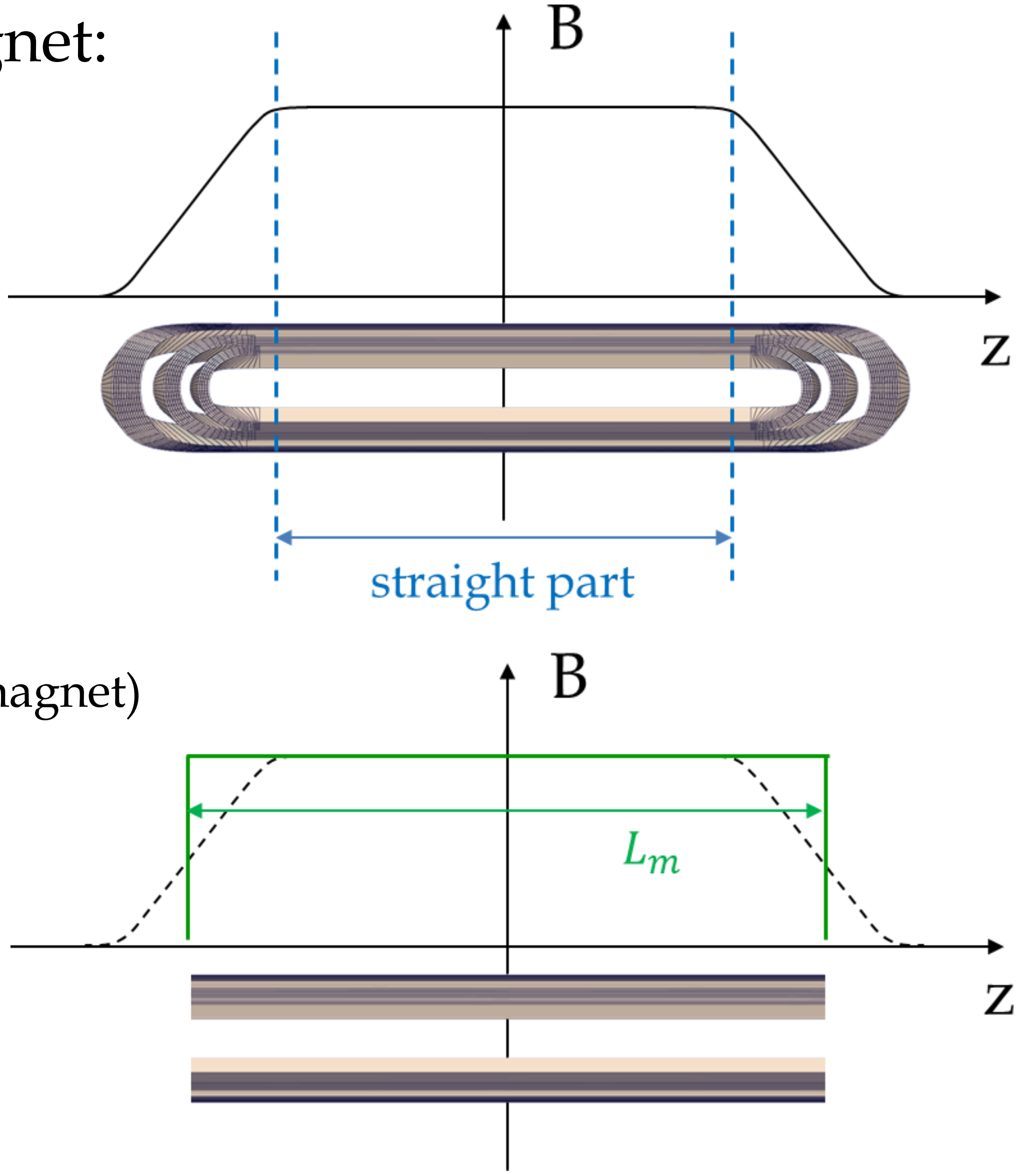
skew

$$a_n = \frac{10^4 n}{\pi R_{ref} B_{R_{ref}}} \int_0^{2\pi} A_z(R_{ref},\theta) \sin n\,\theta d\theta = \frac{10^4}{\pi B_{R_{ref}}} \int_0^{2\pi} B_x(R_{ref},\theta) \cos(n-1)\theta d\theta = -\frac{10^4}{\pi B_{R_{ref}}} \int_0^{2\pi} B_y(R_{ref},\theta) \sin(n-1)\theta d\theta$$

normal

$$b_n = -\frac{10^4 n}{\pi R_{ref} B_{R_{ref}}} \int_0^{2\pi} A_z(R_{ref},\theta) \cos n\,\theta d\theta = \frac{10^4}{\pi B_{R_{ref}}} \int_0^{2\pi} B_x(R_{ref},\theta) \sin(n-1)\theta d\theta = \frac{10^4}{\pi B_{R_{ref}}} \int_0^{2\pi} B_y(R_{ref},\theta) \cos(n-1)\theta d\theta$$

- The beam sees the field along the whole magnet:
  - Integrated strength [T · m]: $\int_{-\infty}^{+\infty} B_1(z)dz$
  - Main component: $\bar{B}_1 \equiv \frac{\int_{\text{straight part}} B_1(z)dz}{\int_{\text{straight part}} dz}$
    (average over the straight part)
  - Magnetic length: $L_m \equiv \frac{\int_{-\infty}^{+\infty} B_1(z)dz}{\bar{B}_1}$
    (length of the magnet as if there were no heads and the integrated force was the same as that of the actual magnet)
  - Average multipoles: $\bar{b}_n \equiv \frac{\int_{-\infty}^{+\infty} B_1(z)b_n(z)dz}{\int_{-\infty}^{+\infty} B_1(z)dz}$
    (weighted average with the main component)



DIPOLES

how to make dipoles with current lines

PERFECT DIPOLE 1: wall dipole

- Biot-Savart law for finite conductors:

$$\vec{B} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{j}\times\vec{r}}{|\vec{r}|^3}\,\mathrm{d}V \qquad \vec{j}\times\vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & j \\ x & y & z \end{vmatrix} = \begin{vmatrix} -yj \\ xj \\ 0 \end{vmatrix}$$

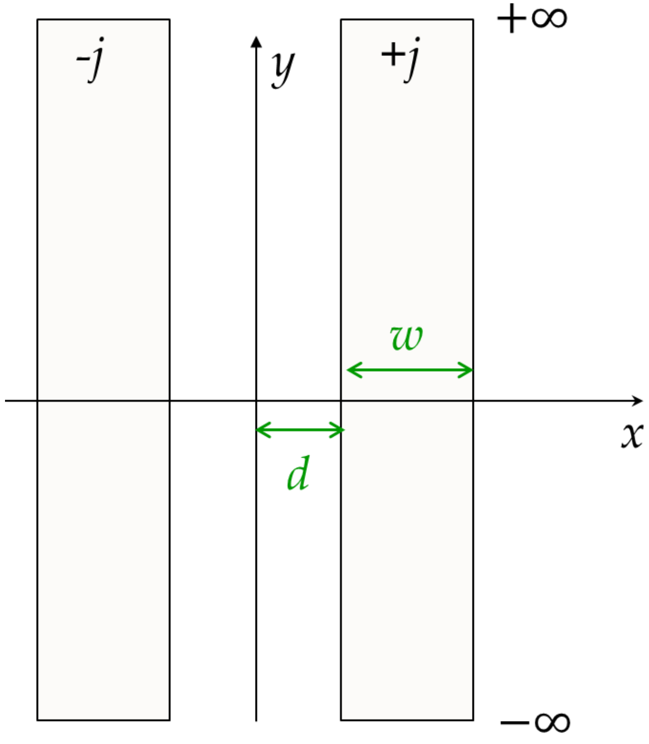
- Each wall contributes with:

$$B_y = \frac{\mu_0}{4\pi}\int_d^{d+w}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{xj}{(x^2+y^2+z^2)^{3/2}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \frac{\mu_0 j}{4\pi}\int_d^{d+w} x\mathrm{d}x \int_{-\infty}^{\infty}\mathrm{d}y \int_{-\infty}^{\infty}\frac{\mathrm{d}z}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{\mu_0 j}{4\pi}\int_d^{d+w} x\mathrm{d}x \int_{-\infty}^{\infty}\mathrm{d}y\,\frac{2}{x^2+y^2} = \frac{\mu_0 j}{2\pi}\int_d^{d+w} x\mathrm{d}x \int_{-\infty}^{\infty}\frac{\mathrm{d}y}{x^2+y^2}$$

$$= \frac{\mu_0 j}{2\pi}\int_d^{d+w} x\mathrm{d}x\,\frac{\pi}{x} = \frac{\mu_0 j}{2}\int_d^{d+w}\mathrm{d}x = \frac{\mu_0 j w}{2}$$



- The total magnetic field is then given by

$$B_y = \mu_0 j w \quad (B_x = 0)$$

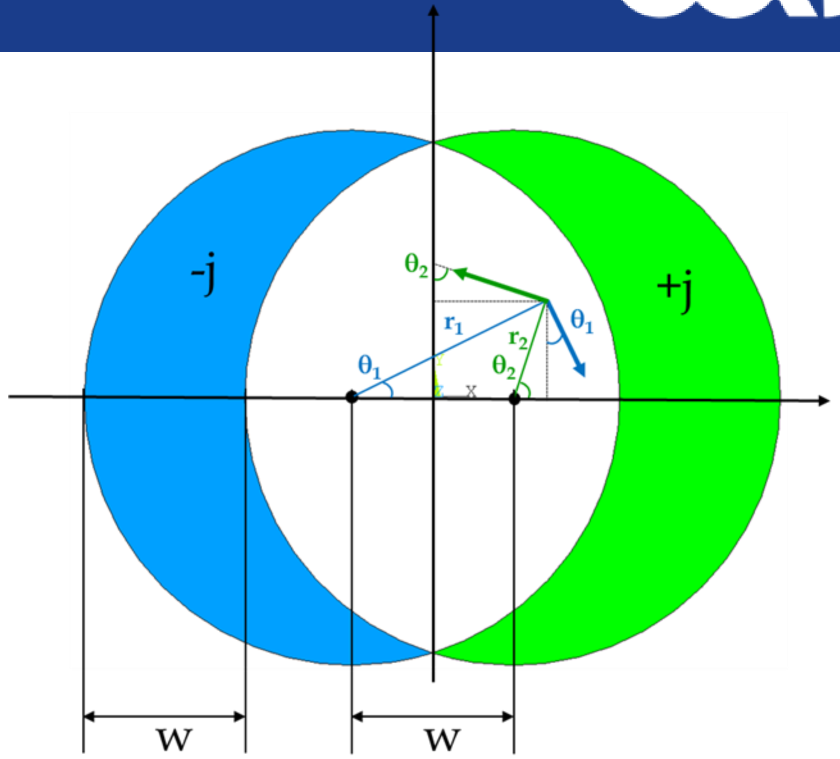
- mechanical structure and winding look easy
- the coil is infinite
- truncation gives reasonable field quality only for rather large height

PERFECT DIPOLE 2: intersecting circles (ellipses)

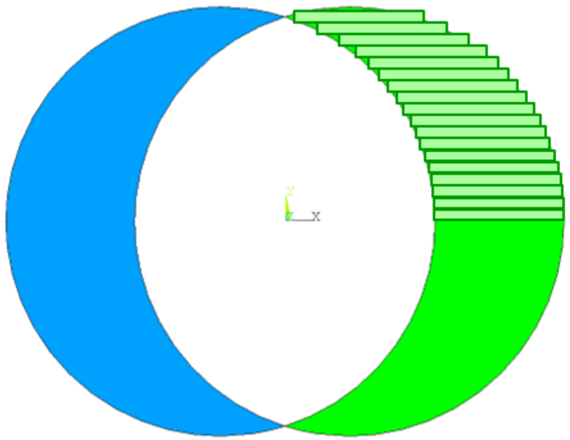
- A cylinder carrying a uniform current generates a magnetic field given by $B = \mu_0 jr/2$
  (Ampere's law at r gives $\oint \vec{B} d\ell = \mu_0 I \rightarrow B \times 2\pi r = \mu_0 j \pi r^2$)
- Combining the effect of the 2 cylinders:

$$B_y = \frac{\mu_0 j}{2}(-r_1 \cos\theta_1 + r_2 \cos\theta_2) = -\frac{\mu_0 jw}{2}$$

$$B_x = \frac{\mu_0 j}{2}(+r_1 \sin\theta_1 - r_2 \sin\theta_2) = 0$$



- the aperture is not circular
- the shape of the coil is not easy to wind with a flat cable (ends?)
- need of internal mechanical support that reduces available aperture



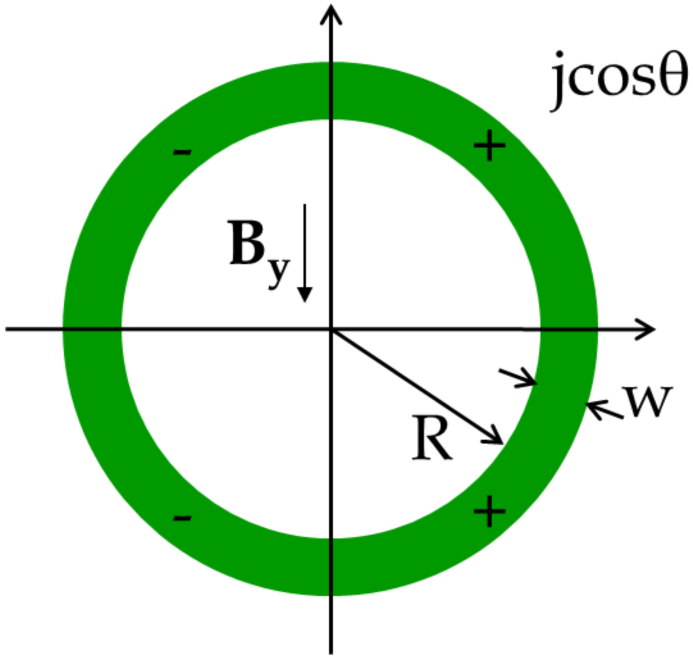
PERFECT DIPOLE 3: jcosθ current density distribution

- Let's consider a current density J=jcosθ distributed in a hollow cylinder of thickness w and inner radius R
- To calculate the resulting magnetic field, we can recall the field harmonics of a current line

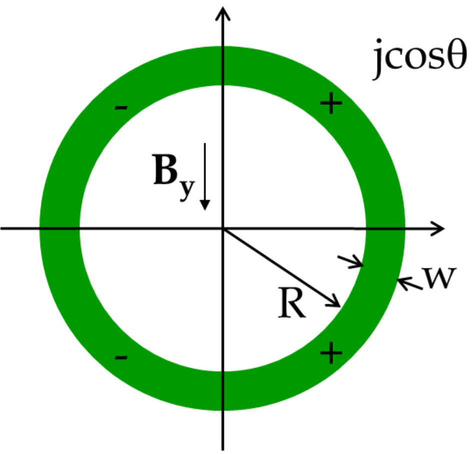
$$B_n(\rho,\theta) = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,\theta$$

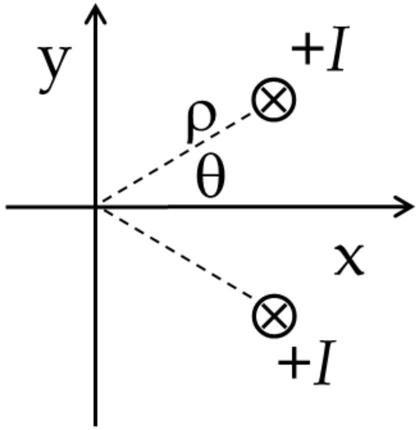
and integrate over the cross-section

$$\mathrm{I} \rightarrow JdS = j\cos\theta \cdot \rho d\rho d\theta$$



Up-down symmetry

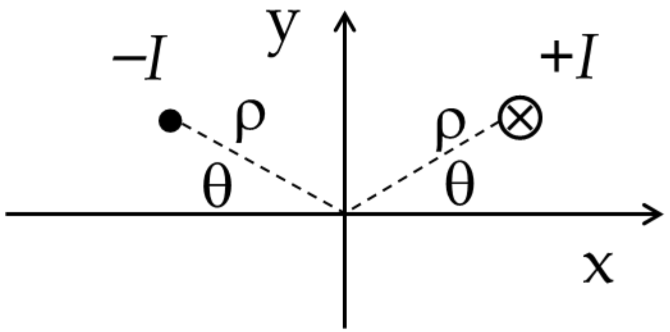




$$B_n = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,\theta - \frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,(-\theta)$$

$$= -2\,\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,\theta$$

Left-right anti-symmetry



$$B_n = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,\theta - \frac{\mu_0 (-I)}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,(\pi-\theta)$$

$$B_n = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n [\cos n\,\theta - \cos n\,(\pi-\theta)]$$

$$B_n = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,\theta[1-\cos n\,\pi]$$

$$= \begin{cases} -2\,\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,\theta & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

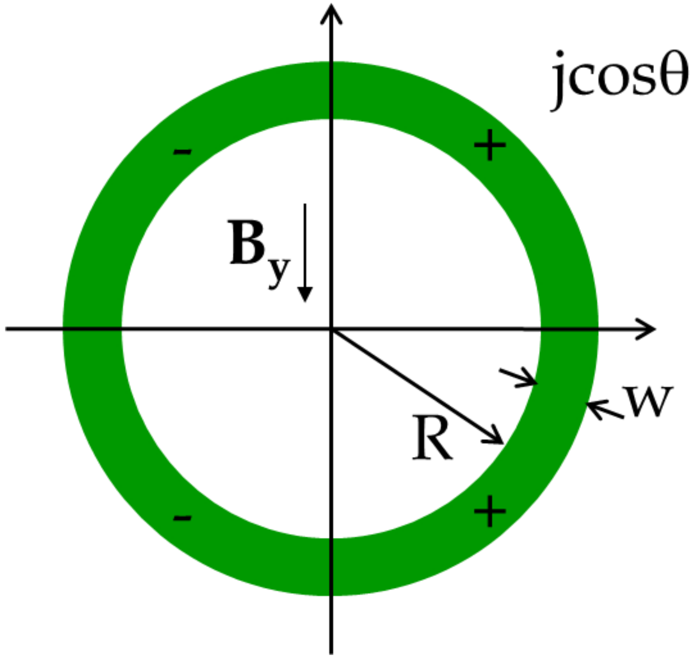
PERFECT DIPOLE 3: jcosθ current density distribution

- Let's consider a current density J=jcosθ distributed in a hollow cylinder of thickness w and inner radius R
- To calculate the resulting magnetic field, we can recall the field harmonics of a current line

$$B_n(\rho,\theta) = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,\theta$$

and integrate over the cross-section

$$\mathrm{I} \rightarrow JdS = j\cos\theta \cdot \rho d\rho d\theta$$



- $B_n = -4\frac{\mu_0 j}{2\pi R_{ref}}\int_R^{R+w}\left(\frac{R_{ref}}{\rho}\right)^n \rho d\rho \int_0^{\frac{\pi}{2}}\cos\theta\cos n\,\theta d\theta,\ \text{if}\ \ n\ \ \text{odd}$

since $\int_0^{\pi/2}\cos\theta\cos n\,\theta d\theta = \begin{cases}\pi/4 & \text{se}\ \ n=1\\ 0 & \text{se}\ \ n\neq 1\end{cases}$, the only surviving term is:

$$B_1 = -4\frac{\mu_0 j}{2\pi R_{ref}}\int_R^{R+w}\left(\frac{R_{ref}}{\rho}\right)\rho d\rho \cdot \frac{\pi}{4} = -\frac{\mu_0 j w}{2}$$

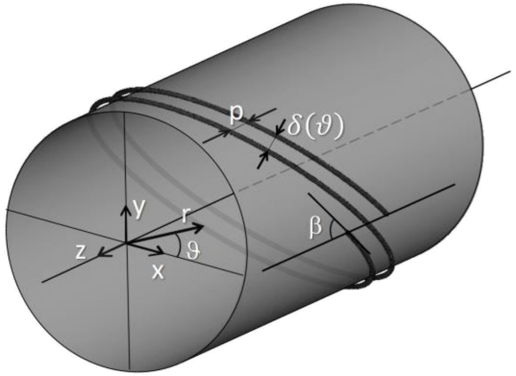
- self supporting structure (roman arch)
- the aperture is circular, the coil is compact
- winding is manageable

PERFECT DIPOLE 4: canted cosθ winding

- A current I flowing in an inclined solenoidal winding of equation

$$\boldsymbol{P}(\vartheta)=\begin{cases} r\cos\vartheta \\ r\sin\vartheta \\ \frac{p\vartheta}{2\pi}+A\sin\vartheta \end{cases}, \text{ corresponds to a current density } \begin{cases} j_r \\ j_\vartheta \\ j_z \end{cases} = \frac{I}{rp}\begin{cases} 0 \\ r \\ \frac{p}{2\pi}+A\cos\vartheta \end{cases}$$

(fully developed math in DOI:10.1109/TASC.2021.3053346)



- In a double winding with opposite inclination ($+A_1/-A_2$) and opposite current ($\pm I$), inner radius r_1 and outer radius r_2:
  - The solenoidal magnetic field cancels out (if the pitch is the same):

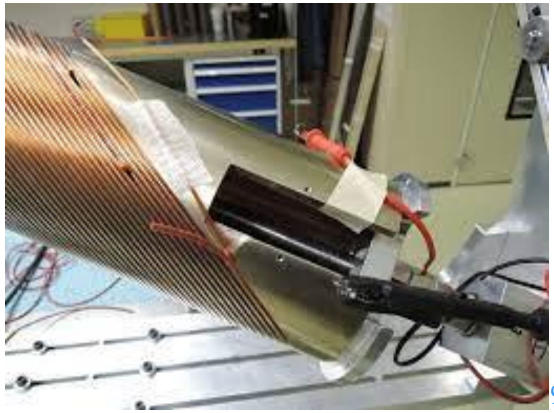
$$B_z=\mu_0\frac{I}{p}+\mu_0\frac{-I}{p}=0$$

  - The axial components adds up:

$$B_1=-\frac{\mu_0 I}{2}\frac{A_1}{r_1 p}-\frac{\mu_0(-I)}{2}\frac{-A_2}{r_2 p}=-\frac{\mu_0 I}{2p}\left(\frac{A_1}{r_1}+\frac{A_2}{r_2}\right)$$

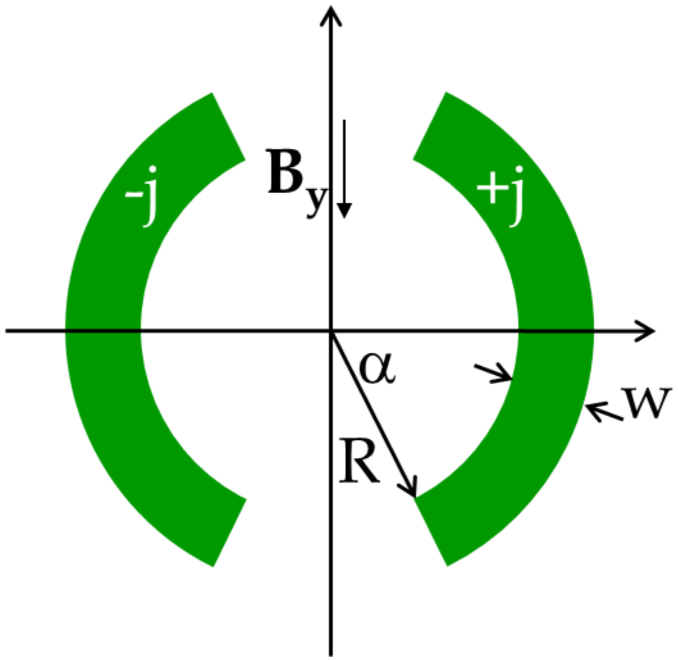
R.Meinke
https://accelconf.web.cern.ch/p03/papers/wpae025.pdf

- some conductor wasted to produce the solenoidal field
- easily generalized to quadrupoles and higher orders
- a former has grooves where the conductor (cable or wire) is wound
- no tooling, no collaring but no prestress



SECTOR DIPOLE – main field

- Sector coils are the practical solution to approximate the cos-theta layout by sectors with uniform current density (https://doi.org/10.15161/oar.it/143359)



- To calculate the resulting magnetic field, we can recall the field harmonics of a current line

$$B_n(\rho,\theta) = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,\theta$$

and integrate over the cross-section

$$\mathrm{I} \rightarrow j dS = j \cdot \rho d\rho d\theta$$

- $B_1 = -4\frac{\mu_0 j}{2\pi R_{ref}} \int_R^{R+w} \left(\frac{R_{ref}}{\rho}\right) \rho d\rho \int_0^{\alpha} \cos\theta\, d\theta = -\frac{2\mu_0 jw \sin\alpha}{\pi}$

    - $B_1 \propto$ current density (obvious)
    - $B_1 \propto$ coil width w (less obvious)
    - B_1 is independent on the aperture r (much less obvious)

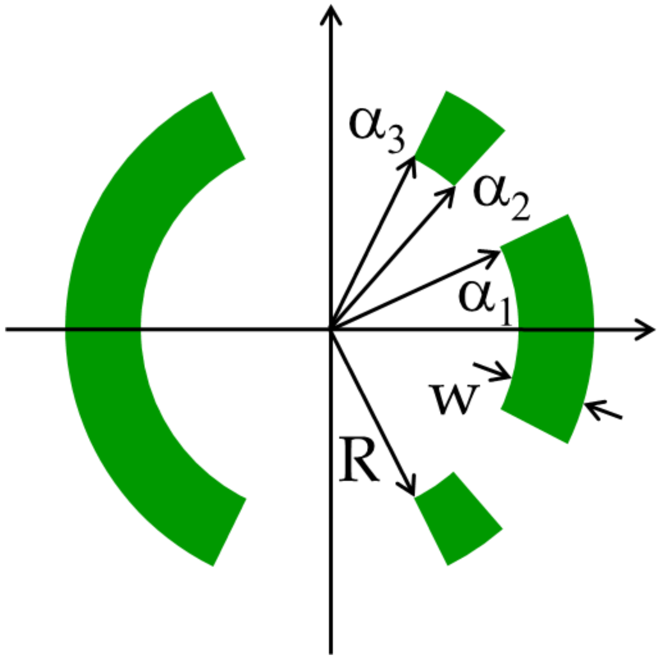
- $$B_n = -4\frac{\mu_0 j}{2\pi R_{ref}}\int_R^{R+w}\left(\frac{R_{ref}}{\rho}\right)^n \rho d\rho \int_0^\alpha \cos n\,\theta d\theta \quad \text{if } n \text{ odd}$$
  $$= -\frac{2}{n(n-2)}\frac{\mu_0 j R_{ref}^{n-1}}{\pi}\sin n\,\alpha\left(\frac{1}{R^{n-2}} - \frac{1}{(R+w)^{n-2}}\right)$$
- Normalizing to the dipole field:
  $$b_n = \frac{1}{n(n-2)}\frac{R_{ref}^{n-1}\,\mathbf{\sin n\,\alpha}}{w\sin\alpha}\left(\frac{1}{R^{n-2}} - \frac{1}{(R+w)^{n-2}}\right)\cdot 10^4 \text{if } n \text{ odd}$$
  - The only free term that can be made equal to zero is $\sin n\,\alpha$, leading to the solution $\alpha = \frac{\pi}{n} + k\frac{\pi}{n}$, $0 < \alpha < \frac{\pi}{2}$, k>0 integer → with one sector only one multiple can be made equal to zero
  - b_3=0 if α=60°
  - b_5=0 if α=36°, 72°
  - b_7=0 if α=25°.7, 51°.4, 77°.1

α	B_1 (T)	b_3 (units)	b_5 (units)	b_7 (units)	b_9 (units)
77.1	-5.9	-914	106	0	-8
60	-5.2	0	-239	61	0
51.4	-4.7	632	-298	0	22
36	-3.5	1844	0	-99	-17
25.7	-2.6	2560	431	0	-31

R=50 mm, w=15 mm, j=5 ·10^8 A/m^2

2-SECTOR DIPOLE

- To calculate the resulting magnetic field, we can recall the field harmonics of a current line $B_n(\rho,\theta) = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,\theta$ and integrate over the cross-section I $\rightarrow jdS = j \cdot \rho d\rho d\theta$
- $B_1 = -4\frac{\mu_0 j}{2\pi R_{ref}}\int_R^{R+w}\left(\frac{R_{ref}}{\rho}\right)\rho d\rho\left(\int_0^{\alpha_1}\cos\theta\, d\theta + \int_{\alpha_2}^{\alpha_3}\cos\theta\, d\theta\right) = -\frac{2\mu_0 jw\,(\sin\alpha_1 - \sin\alpha_2 + \sin\alpha_3)}{\pi}$



- Higher harmonics:

$$b_n = \frac{10^4}{n(n-2)}\frac{R_{ref}^{n-1}\,(\boldsymbol{\sin n\alpha_1 - \sin n\alpha_2 + \sin n\alpha_3})}{w\,(\sin\alpha_1 - \sin\alpha_2 + \sin\alpha_3)}\left(\frac{1}{R^{n-2}} - \frac{1}{(R+w)^{n-2}}\right)$$

- 3 components can be set to zero, as example:

$$\begin{cases}(\sin 3\,\alpha_1 - \sin 3\,\alpha_2 + \sin 3\,\alpha_3) = 0 & b_3 = 0\\ (\sin 5\,\alpha_1 - \sin 5\,\alpha_2 + \sin 5\,\alpha_3) = 0 & b_5 = 0\\ (\sin 7\,\alpha_1 - \sin 7\,\alpha_2 + \sin 7\,\alpha_3) = 0 & b_7 = 0\end{cases}$$

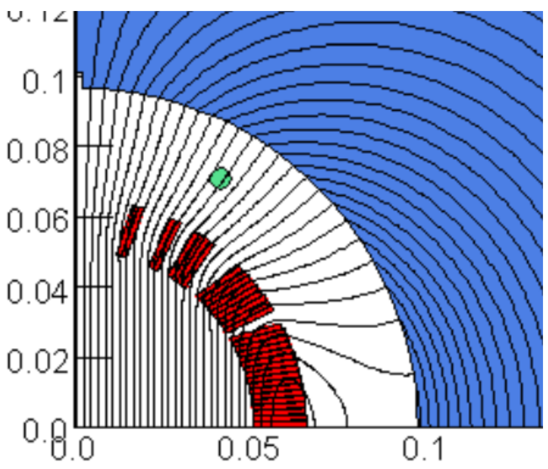
COMMENTS ON THE DIPOLE FIELD

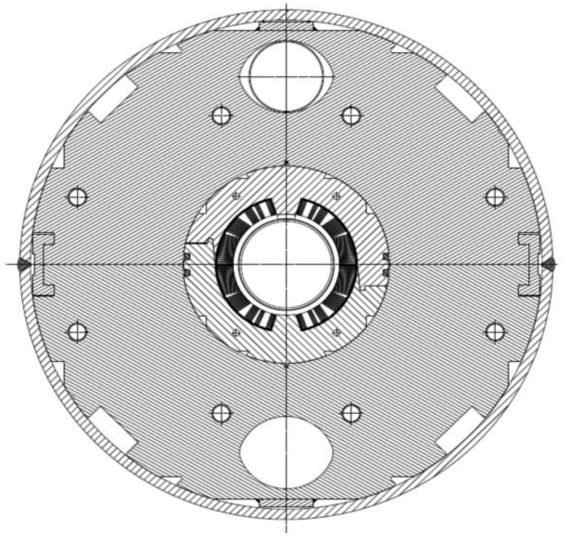
- Intercepting circles $B_1 = -\frac{\mu_0 jw}{2}$
- cosθ distribution $B_1 = -\frac{\mu_0 jw}{2}$
- 1-sector dipole $B_1 = -\frac{2\mu_0 jw \sin\alpha}{\pi}$
- 2-sector dipole $B_1 = -\frac{2\mu_0 jw\,(\sin\alpha_1 - \sin\alpha_2 + \sin\alpha_3)}{\pi}$

$$B_1 = -\gamma_c jw$$

$$\gamma_c = -\frac{B_1}{j\,w}$$

- The 60° sector dipole ($\gamma_c = \frac{2\mu_0 \sin 60}{\pi} = 6.9 \cdot 10^{-7}$ Tm/A) can be used to compare other layouts
  - *Example 1:* cosθ distribution and intercepting circle
  - $\gamma_c = 6.3 \cdot 10^{-7}$ Tm/A
  - *Example 2:* 2 sector dipole with $\alpha_1 = 43.2°$, $\alpha_2 = 52.2°$, $\alpha_3 = 67.3°$ ($b_3=b_5=b_7\sim 0$)
  - $\gamma_c = 6.5 \cdot 10^{-7}$ Tm/A
  - *Example 3:* the SIS300 dipole
    j=347 A/mm^2
    B_1=3.35 T (without iron, with iron B_1=4.5 T)
    w=15 mm
  - $\gamma_c = 6.4 \cdot 10^{-7}$ Tm/A





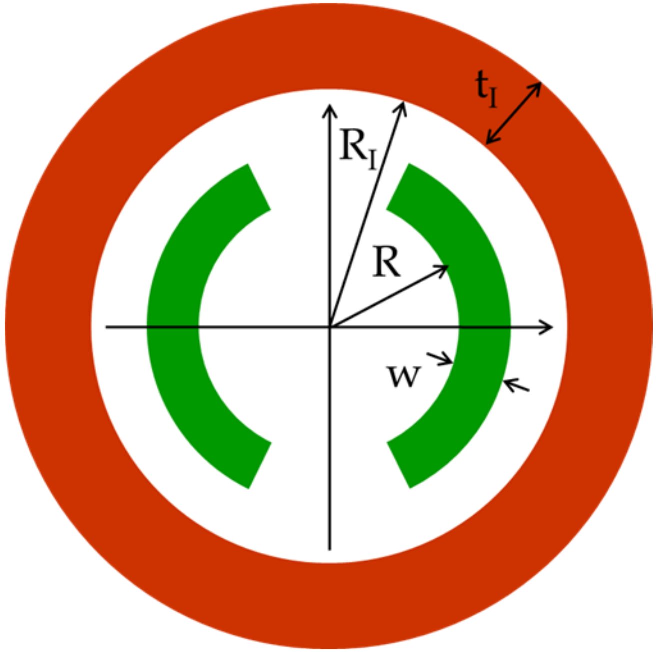
https://www.lnf.infn.it/sis/preprint/getfilepdf.php?filename=INFN-13-06-GE.pdf

EFFECT OF THE IRON YOKE

- It is possible to take into account the effect of an iron yoke of linear permeability μ_r, inner radius R_I and thickness t_I
- The correction to the field harmonics of a current line is given by:

$$B_n(\rho,\varphi) = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\varphi \left[1 + \boldsymbol{k}\left(\frac{\rho}{R_I}\right)^{2n}\right]$$

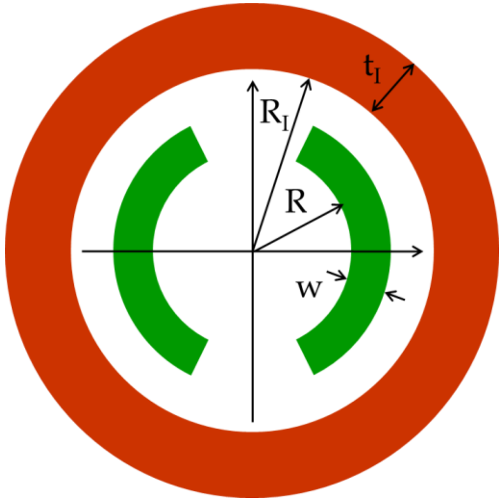
$$k = \frac{\mu_r - 1}{\mu_r + 1}\,\frac{1-\left(\frac{R_I}{R_I+t_I}\right)^{2n}}{1-\left(\frac{\mu_r-1}{\mu_r+1}\right)^2\left(\frac{R_I}{R_I+t_I}\right)^{2n}} \approx 1 \quad \text{if} \quad \mu_r >> 1$$



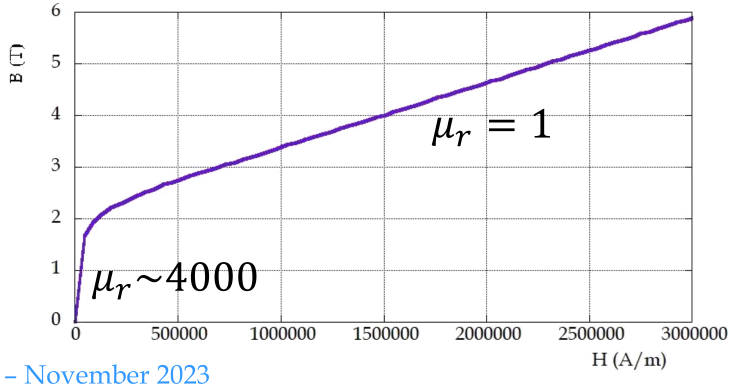
  - The derivation of the main physical quantities can be found at https://doi.org/10.15161/oar.it/143359
  - The iron contribution **has no additional angular dependence**, so the contribution is independent on the dipole layout
  - depending on $\boldsymbol{k(\rho/R_I)^{2n}}$ can be relevant only for:
    - small coil widths
    - low order multipoles (main component)
    - small collar widths

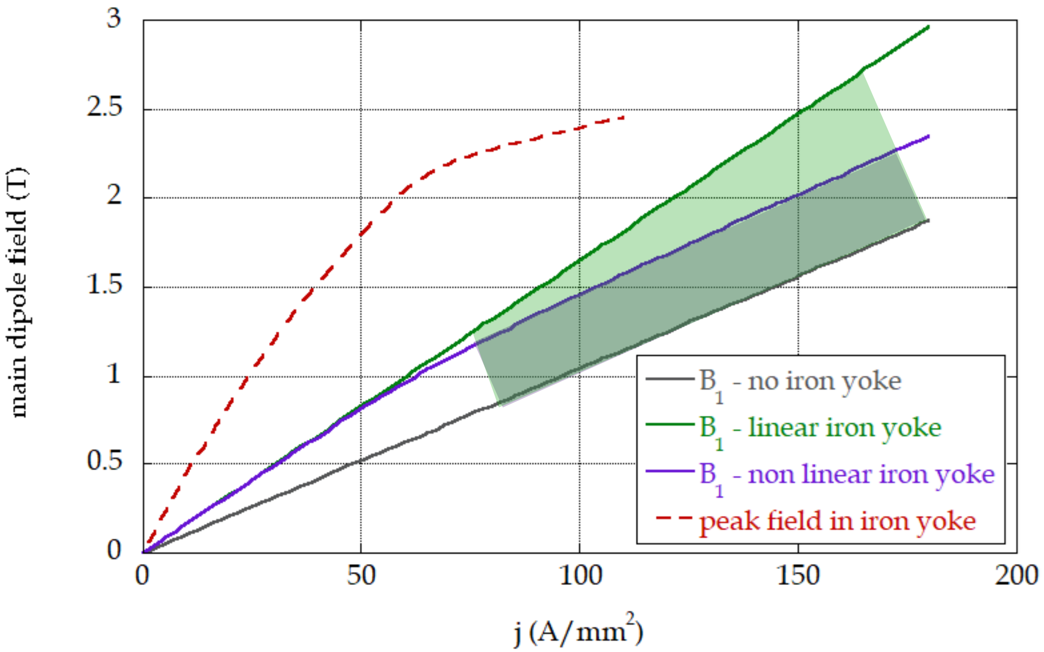
EFFECT OF THE IRON YOKE

- Main dipole field in presence of the iron yoke:
  - $$B_{1I} = -4\frac{\mu_0}{2\pi R_{ref}}\int_R^{R+w}\left(\frac{R_{ref}}{\rho}\right)\left[1+k\left(\frac{\rho}{R_I}\right)^2\right]\rho d\rho \int_{ang.ext.} j(\theta)\cos\theta\, d\theta$$
    $$= B_1\left(1+\frac{k}{R_I^2}\frac{(R+w)^3-R^3}{3w}\right) \sim B_1\left(1+\frac{R(R+w)}{R_I^2}\right)$$



- Field increase due to non saturated iron:
  - $B_{1I} = B_1(1+\Delta_I)$, $\Delta_I = \frac{R(R+w)}{R_I^2}$
- Limit of validity:
  - Iron yoke saturation ($B_{\text{sat}} \sim 2$ T)
  - Shielding condition: $t_I = \frac{RB_1}{B_{\text{sat}}}$

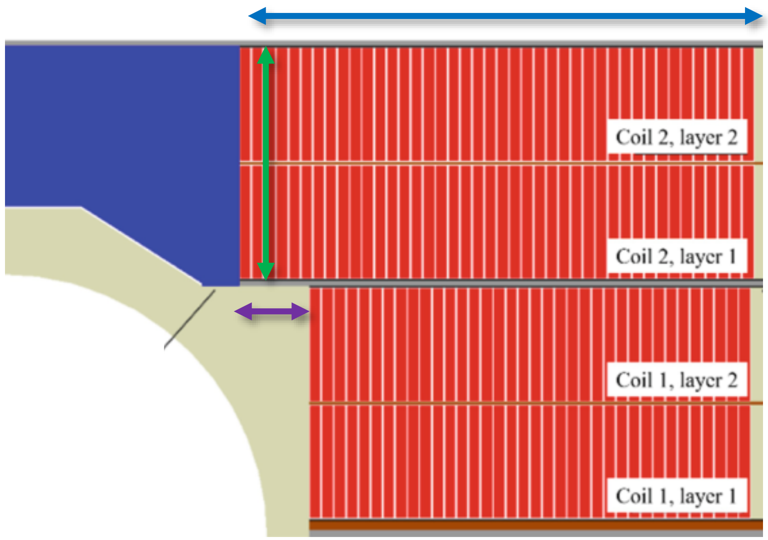




R=50 mm
w=15 mm
R_I=75 mm
t_I=25 mm

BLOCK COILS

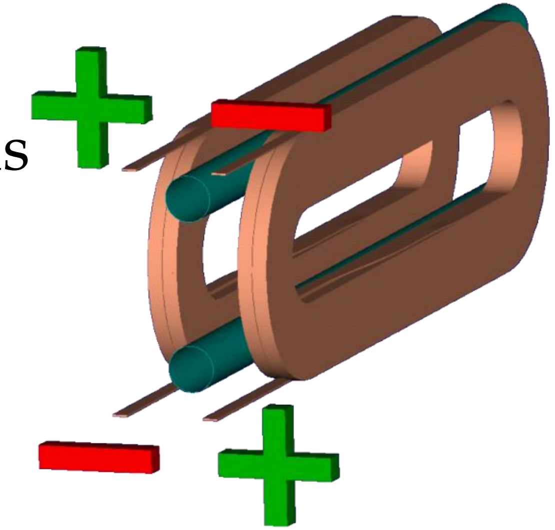
- A block layout has vertical cables
  - Need of internal support, reducing available aperture
  - Lack of roman arch gives a different distribution of forces
  - Saddle shape ends – no need of wedges, very simple coil
- Can field quality be optimized in a block layout?
  - without wedges there are 3 free parameters:
    - the total width of the coil
    - the height of the blocks (i.e. the cable width)
    - the indentation of the upper deck
  - one parameter can be used to increase the coil width, the other two to cancel b_3 and b_5



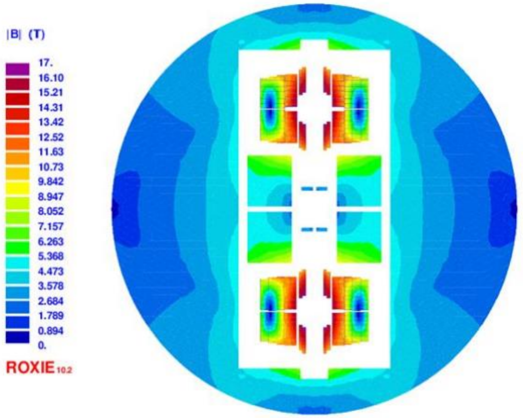
FRESCA II magnet cross section, E.Rochepault and P.Ferracin
https://link.springer.com/chapter/10.1007/978-3-030-16118-7_12

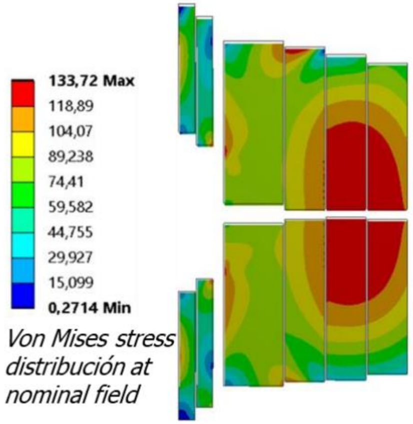
COMMON COILS

- The common coil design is based on the superposition of "racetrack coils" with simple ends that have a large bend radius
- The bend radius is determined by interbeam distance
- The dipole field is generated between the straight parts of the racetrack coils
- It is an intrinsically double aperture configuration



- Field quality ca be optimized piling up several racetracks with different dimensions
- Mechanics can be tricky





Design studies performed for 16 T common coil dipole for FCC by F.Toral ,CIEMAT

R.Gupta, https://wpw.bnl.gov/rgupta/wp-content/uploads/sites/9/2023/09/Snowmass-common-coil-white_paper-03-16-2022.pdf

FIELD FROM A CURRENT LINE outside the filament (r>ρ)

- To derive other quantities (Lorentz forces, stored energy) we need to determine the magnetic field generated by a current line at r>ρ

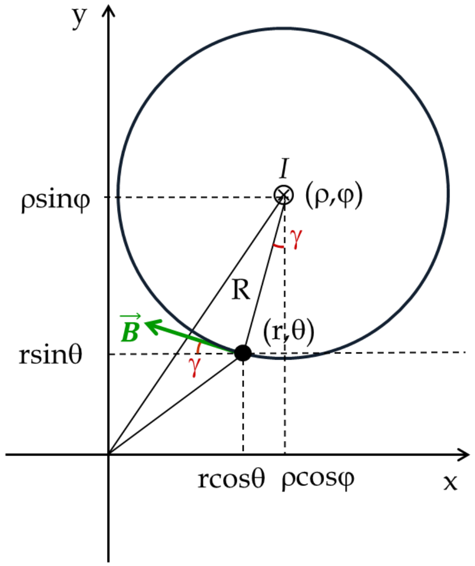
- $\boldsymbol{B}(\boldsymbol{z}) = \frac{\mu_0 I}{2\pi(\boldsymbol{z}-\boldsymbol{z_0})} = \frac{\mu_0 I}{2\pi(re^{i\vartheta}-\rho e^{i\varphi})} = \frac{\mu_0 I}{2\pi r e^{i\vartheta}} \frac{1}{1-\frac{\rho}{r}e^{i(\varphi-\vartheta)}}$

- If $\epsilon < 1$: $\frac{1}{1-\epsilon} = \sum_{n=1}^{\infty} \epsilon^{n-1} = 1 + \sum_{n=2}^{\infty} \epsilon^{n-1} = 1 + \sum_{m=1}^{\infty} \epsilon^{m}$

- $\boldsymbol{B}(\boldsymbol{z}) = \frac{\mu_0 I}{2\pi r e^{i\vartheta}} \left[1 + \sum_{n=1}^{\infty} \left(\frac{\rho}{r} e^{i(\varphi-\vartheta)}\right)^n\right] = \frac{\mu_0 I}{2\pi \boldsymbol{z}} \left[1 + \sum_{n=1}^{\infty} e^{in\varphi} \left(\frac{\rho}{\boldsymbol{z}}\right)^n\right]$

- To be noted that

  $\lim_{z\to\infty} \boldsymbol{B}(\boldsymbol{z}) = \frac{\mu_0 I}{2\pi \boldsymbol{z}}$ as expected from a current-carrying wire



OTHER QUANTITIES THAT CAN BE CALCULATED: Lorentz forces

- Complete derivation for cosθ and sector coils with and without iron yoke in https://doi.org/10.15161/oar.it/143359
- The Lorentz force density is given by $\vec{f}_L = \vec{j} \times \vec{B}$.
  - **$j_0$cosθ.** If the current density is $\vec{j} = (0,0,j_0\cos\theta)$ and the magnetic field is $\vec{B} = (B_r, B_\theta, 0)$
    - $f_r(r,\theta) = -j_0\cos\theta\, B_\theta = -\frac{\mu_0 j_0^2}{2}\cos^2\theta\left\{\frac{r^3-R^3}{3r^2} - (R+w-r)\right\}$
    - $f_\theta(r,\theta) = +j_0\cos\theta\, B_r = -\frac{\mu_0 j_0^2}{2}\cos\theta\sin\theta\left\{\frac{r^3-R^3}{3r^2} + (R+w-r)\right\}$
    - $f_z(r,\theta) = 0$
  - **Sector dipole.** If the current density is $\vec{j} = (0,0,j_0)$ when 0<θ<α_1 and the magnetic field is $\vec{B} = (B_r, B_\theta, 0)$
    - $f_r(r,\theta) = -j_0 B_\theta = -\sum_{n\ odd}\frac{2\mu_0 j_0^2}{n\pi}\cos n\theta\sin n\alpha_1\left\{\frac{r^{2+n}-R^{2+n}}{(2+n)r^{1+n}} - r^{n-1}\frac{(R+w)^{2-n}-r^{2-n}}{2-n}\right\}$
    - $f_\theta(r,\theta) = +j_0 B_r = -\sum_{n\ odd}\frac{2\mu_0 j_0^2}{n\pi}\sin n\theta\sin n\alpha_1\left\{\frac{r^{2+n}-R^{2+n}}{(2+n)r^{1+n}} + r^{n-1}\frac{(R+w)^{2-n}-r^{2-n}}{2-n}\right\}$
    - $f_z(r,\theta) = 0$

OTHER QUANTITIES THAT CAN BE CALCULATED: stored energy

- Complete derivation for cosθ and sector coils with and without iron yoke in https://doi.org/10.15161/oar.it/143359

- The easiest way to derive the stored energy is to calculate $\frac{E}{\ell} = \frac{1}{2}\int_{\text{conductors}} \vec{A}\cdot\vec{j}\,\mathrm{d}S$
  - **$j_0$cosθ.** If the current density is $\vec{j} = (0,0,j_0\cos\theta)$ and $\vec{A} = (0,0,A_z)$ inside the conductors
    - $A_z(r,\theta) = \frac{\mu_0 j_0}{2}\cos\theta\,\{r(R+w-r)+r^3-R^3\}$
    - $\frac{E}{\ell} = \frac{1}{2}\int_0^{2\pi}\mathrm{d}\theta\int_R^{R+w} A_z r\mathrm{d}r = \frac{\pi\mu_0 j_0^2}{24}\{(R+w)^4+3R^4-4R^3(R+w)\}$
  - **Sector dipole.** If the current density is $\vec{j} = (0,0,j_0)$ when 0<θ<α_1 and $\vec{A} = (0,0,A_z)$ inside the conductors
    - $A_z(r,\theta) = \sum_{n\ odd}\frac{2\mu_0 j_0}{n^2\pi}\cos n\theta\sin n\alpha_1\left\{\frac{r^{2+n}-R^{2+n}}{(2+n)r^n}+r^n\frac{(R+w)^{2-n}-r^{2-n}}{2-n}\right\}$
    - $\frac{E}{\ell} = \frac{1}{2}\int_0^{\alpha_1}\mathrm{d}\theta\int_R^{R+w} A_z r\mathrm{d}r = \sum_{n\ odd}\frac{4\mu_0 j_0^2}{n^3\pi}\sin^2 n\alpha_1\left\{\frac{(2-n)(R+w)^4+(2+n)R^4-4R^{2+n}(R+w)^{2-n}}{2(4-n^2)}\right\}$

$$\left.\frac{E}{\ell}\right|_{\text{first order}} = \frac{2}{3}\frac{\mu_0 j_0^2\sin^2 n\alpha_1}{\pi}\{(R+w)^4+3R^4-4R^3(R+w)\} = \frac{\pi B_1^2 R^2}{\mu_0}\left\{1+\frac{2}{3}\left(\frac{R+w}{R}-1\right)+\frac{1}{6}\left(\frac{R+w}{R}-1\right)^2\right\}$$

QUADRUPOLES

how to make quadrupoles with current lines

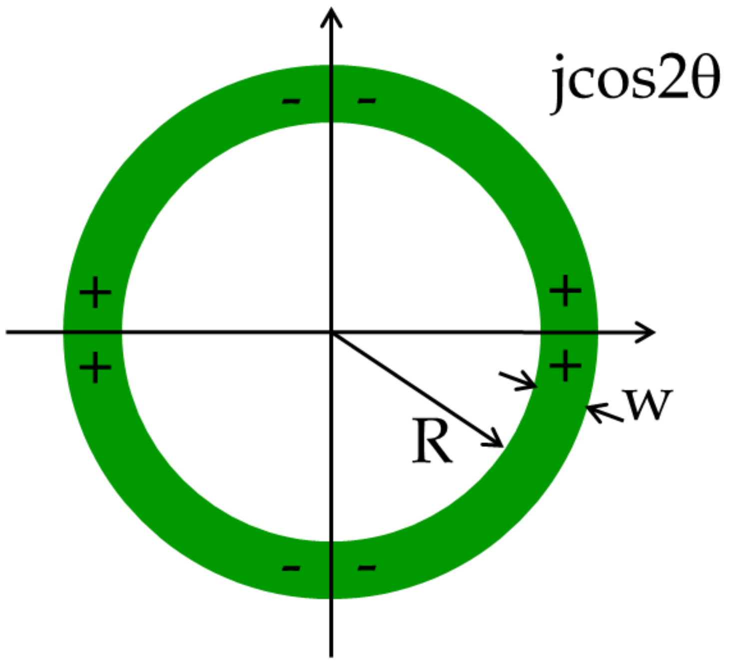
PERFECT QUADRUPOLE: jcos2θ current density distribution

- Let's consider a current density J=jcos2θ distributed in a hollow cylinder of thickness w and inner radius R
- To calculate the resulting magnetic field, we can recall the field harmonics of a current line

$$B_n(\rho,\theta) = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,\theta$$

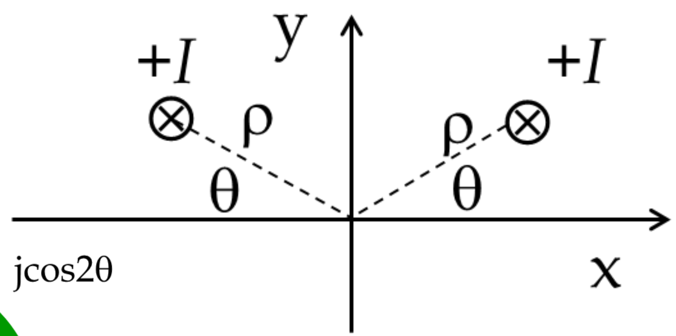
and integrate over the cross-section

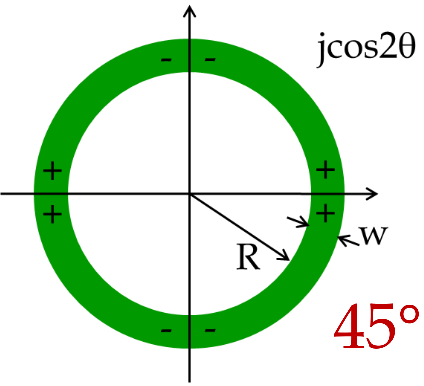
$$\mathrm{I} \rightarrow JdS = j\cos 2\theta \cdot \rho d\rho d\theta$$



Symmetry operation on current line

Left-right symmetry



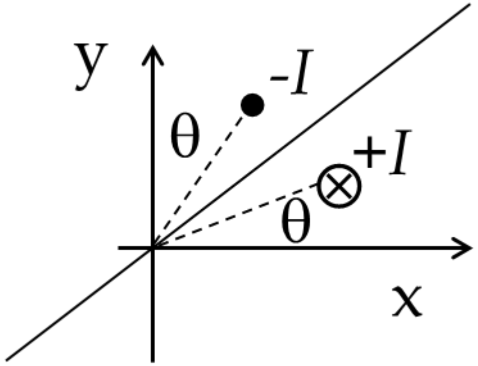


$$B_n = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,\theta - \frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,(\pi-\theta)$$

$$B_n = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,\theta[1+\cos n\,\pi]$$

$$= \begin{cases} -2\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,\theta & \text{if } \boldsymbol{n} \textbf{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$$

45° anti-symmetry



$$B_n = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,\theta - \frac{\mu_0(-I)}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,(\frac{\pi}{2}-\theta)$$

$$B_n = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \left[\cos n\,\theta - \cos n\,(\frac{\pi}{2}-\theta)\right]$$

n even

$$B_n = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,\theta\left[1-\cos\frac{n\pi}{2}\right]$$

$$= \begin{cases} -2\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,\theta & \text{if } \boldsymbol{\frac{n}{2}} \textbf{ odd} \\ 0 & \text{if } \frac{n}{2} \text{ even} \end{cases}$$

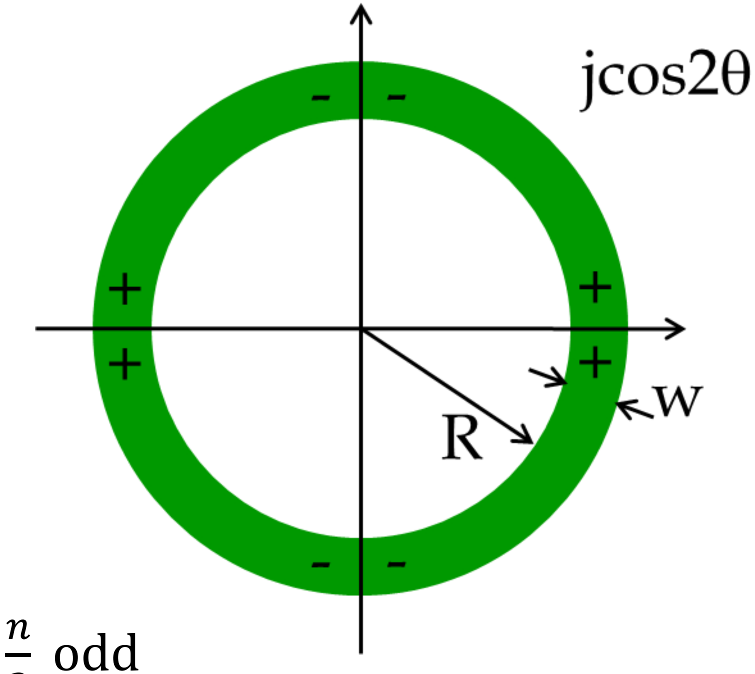
PERFECT QUADRUPOLE: jcos2θ current density distribution

- Let's consider a current density J=jcos2θ distributed in a hollow cylinder of thickness w and inner radius R
- To calculate the resulting magnetic field, we can recall the field harmonics of a current line

$$B_n(\rho,\theta) = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,\theta$$

and integrate over the cross-section

$$\mathrm{I} \rightarrow JdS = j\cos 2\theta \cdot \rho d\rho d\theta$$

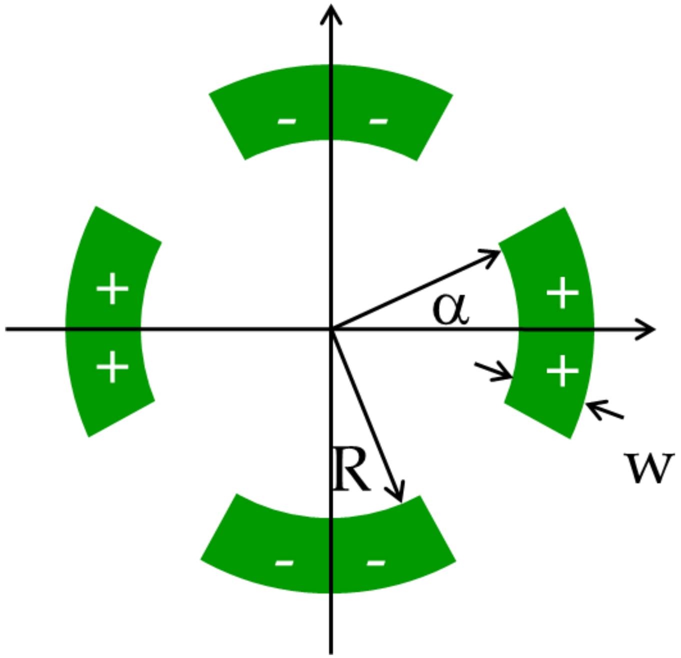


- $B_n = -8\frac{\mu_0 j}{2\pi R_{ref}}\int_R^{R+w}\left(\frac{R_{ref}}{\rho}\right)^n \rho d\rho \int_0^{\frac{\pi}{4}}\cos 2\theta\cos n\,\theta d\theta,$ if n even and $\frac{n}{2}$ odd

since $\int_0^{\pi/4}\cos 2\theta\cos n\,\theta d\theta = \begin{cases}\pi/8 & \text{se } n=2\\ 0 & \text{se } n\neq 2\end{cases}$, the only surviving term is:

$$B_2 = -8\frac{\mu_0 j}{2\pi R_{ref}}\int_R^{R+w}\left(\frac{R_{ref}}{\rho}\right)^2 \rho d\rho \cdot \frac{\pi}{4} = -\frac{\mu_0 j R_{ref}}{2}\ln\left(1+\frac{w}{R}\right) \Rightarrow G = \frac{B_2}{R_{ref}} = \frac{\mu_0 j}{2}\ln\left(1+\frac{w}{R}\right)$$

SECTOR QUADRUPOLES – dipole field



- To calculate the resulting magnetic field, we can recall the field harmonics of a current line

$$B_n(\rho,\theta) = -\frac{\mu_0 I}{2\pi R_{ref}}\left(\frac{R_{ref}}{\rho}\right)^n \cos n\,\theta$$

and integrate over the cross-section

$$\mathrm{I} \rightarrow j dS = j \cdot \rho d\rho d\theta$$

- $B_n = -8\frac{\mu_0 j}{2\pi R_{ref}}\int_R^{R+w}\left(\frac{R_{ref}}{\rho}\right)^n \rho d\rho \int_0^{\alpha}\cos n\theta\, d\theta$

$$B_n = \begin{cases} -\frac{2\mu_0 j R_{ref}}{\pi}\sin 2\,\alpha \ln\left(1+\frac{w}{R}\right) & n=2 \\ -\frac{4}{n(n-2)}\frac{\mu_0 j R_{ref}^{n-1}}{\pi}\sin n\,\alpha\left(\frac{1}{R^{n-2}}-\frac{1}{(R+w)^{n-2}}\right) & n=6,10,14,\ldots \end{cases}$$

$$G = \frac{B_2}{R_{ref}} = -\frac{2\mu_0 j}{\pi}\sin 2\,\alpha \ln\left(1+\frac{w}{R}\right)$$

SECTOR QUADRUPOLES – higher harmonics

- Normalizing to the quadrupole field B_2:

$$b_n = \frac{2}{n(n-2)} \frac{R_{ref}^{n-2} \sin n\alpha}{\sin 2\alpha \ln\left(1+\frac{w}{R}\right)} \left(\frac{1}{R^{n-2}} - \frac{1}{(R+w)^{n-2}}\right) \cdot 10^4 \quad \text{if } n \text{ even and } \frac{n}{2} \text{ odd } (n = 6,10,14,..)$$

  - The only free term that can be made equal to zero is $\sin n\,\alpha$, leading to the solution $\alpha = \frac{\pi}{n} + k\frac{\pi}{n}$, $0 < \alpha < \frac{\pi}{4}$, k>0 integer
    $\rightarrow$ with one sector only one multiple can be made equal to zero
  - b_6=0 if α=30°
  - b_{10}=0 if α=18°, 36°

a	G (T/m)	b_6 (units)	b_{10} (units)	b_{14} (units)
30	-91	0	-32	3
18	-62	660	0	-5
36	-100	-252	0	2

R=50 mm, w=15 mm, j=5 $\cdot 10^8$ A/m^2

THANKS FOR THE ATTENTION