Lecture HDG-1:
Magnetic Field Simulation by Finite Element Methods

Prof. Dr.-Ing. Herbert De Gersem

TECHNISCHE UNIVERSITÄT DARMSTADT

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Herbert De Gersem: my single slide

professor (100%, electrical engineering, TU Darmstadt)

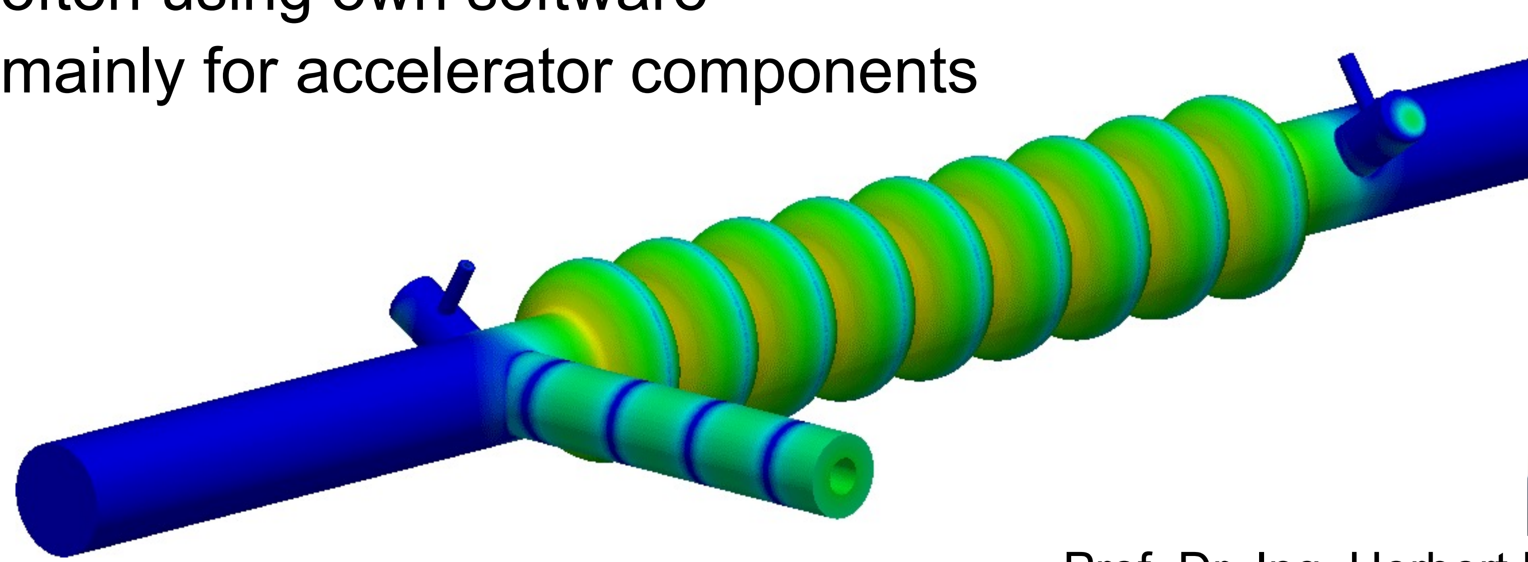
professor (10%, accelerator physics, KU Leuven)

electromagnetic (and other) field simulation

mainly by finite-element methods

often using own software

mainly for accelerator components



TESLA cavity
fundamental mode
W. Ackermann

Prof. Dr.-Ing. Herbert De Gersem

Herbert De Gersem: my 2nd single slide

Belgian (dutch speaking)
consultancy on Belgian beer (not the fruity ones)
consultancy on French wine (only the excellent ones)

fond of teaching (glad to be here!)
also hands-on sessions and projects

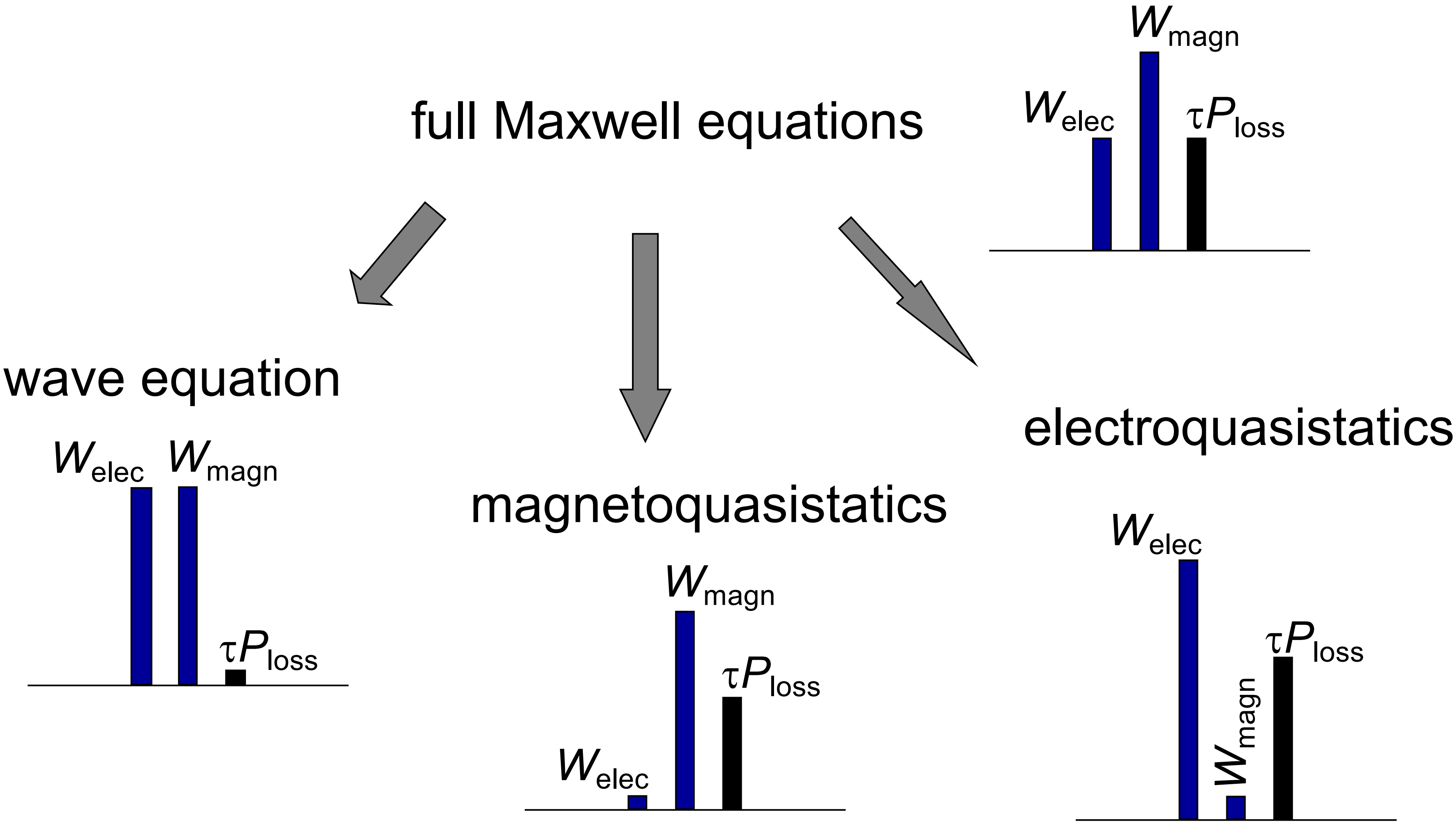
Herbert De Gersem

when not teaching,
when not doing research
practising French horn
playing classical music
in symphonic orchestras

Overview

1. magnetoquasistatic formulation
2. discretisation in space
3. finite-element shape functions
4. boundary and symmetry conditions
5. reduction to 2D models
6. modelling of coils and permanent magnets

Electromagnetic Field Simulation



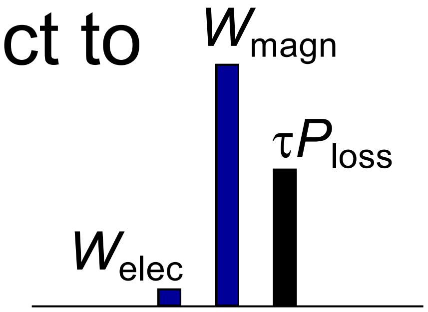
full Maxwell equations

wave equation

magnetoquasistatics

electroquasistatics

Magnetoquasistatics (1)



- neglect displacement currents with respect to conducting currents
  - Ampère-Maxwell

$$\nabla\times\vec{H} = \vec{J} + \cancel{\frac{\partial \vec{D}}{\partial t}}$$

- magnetic vector potential $\vec{A}$
  - conservation of magnetic flux

$$\nabla\cdot\vec{B} = 0 \quad \Rightarrow \quad \vec{B} = 0 + \nabla\times\vec{A}$$

- electric scalar potential φ
  - Faraday-Lenz

$$\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\nabla\times\frac{\partial \vec{A}}{\partial t} \quad \Rightarrow \quad \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla\varphi$$

Magnetoquasistatics (2)

Ampère

$$\nabla\times\vec{H} = \vec{J}$$

$$\Downarrow$$

$$\nabla\times\left(\nu\vec{B}\right) = \sigma\vec{E}$$

$$\Downarrow$$

$$\nabla\times\left(\nu\nabla\times\vec{A}\right) + \sigma\frac{\partial\vec{A}}{\partial t} = \underbrace{-\sigma\nabla\varphi}_{\vec{J}_s}$$

source current density

permeability

$$\vec{B} = \mu\vec{H} = \frac{1}{\nu}\vec{H}$$

reluctivity

$$\vec{J} = \sigma\vec{E}$$

conductivity

parabolic partial differential equation

↔ elliptic PDEs (e.g. electrostatics, magnetostatics)

↔ hyperbolic PDEs (e.g. wave equation)

Physical Meaning (1)

flux

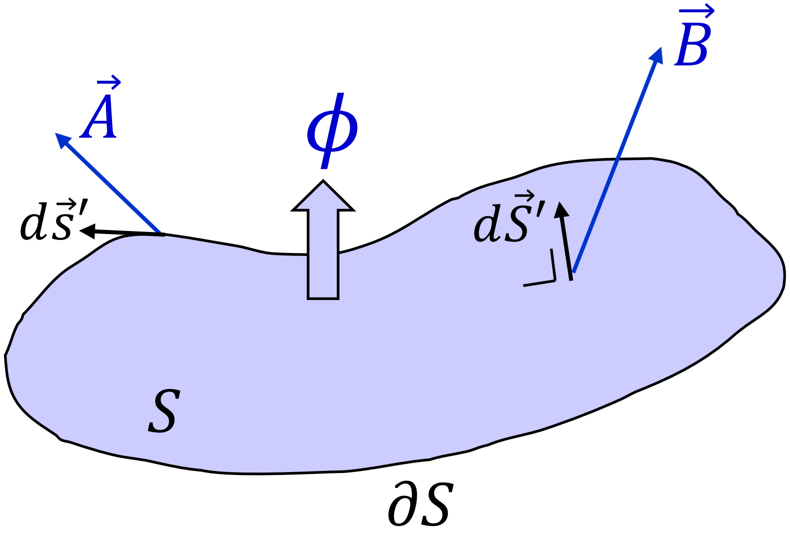
$$\phi = \int_S \vec{B} \cdot d\vec{S}'$$

definition magnetic vector potential

$$\phi = \int_S \nabla \times \vec{A} \cdot d\vec{S}'$$

Stokes

$$\phi = \oint_{\partial S} \vec{A} \cdot d\vec{s}'$$



induced voltage

$$u_{\mathrm{ind}} = -\frac{d}{dt} \oint_{\partial S} \vec{A} \cdot d\vec{s}'$$

Physical Meaning (2)

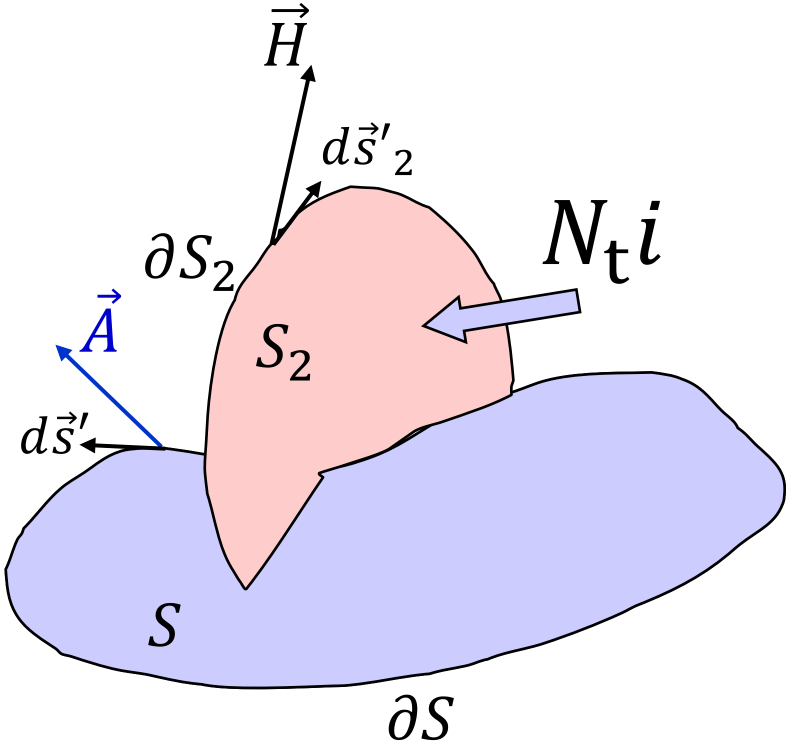
flux $\phi = \oint_{\partial S} \vec{A} \cdot d\vec{s}'$

induced voltage

$$u_{\mathrm{ind}} = -\frac{d}{dt} \oint_{\partial S} \vec{A} \cdot d\vec{s}'$$

Ampère

$$N_{\mathrm{t}} i = \oint_{\partial S_2} \vec{H} \cdot d\vec{s}'_2$$



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Spatial Discretisation (1)

- weighted residual approach

$$\nabla\times\left(\nu\nabla\times\vec{A}\right)+\sigma\frac{\partial\vec{A}}{\partial t}=\vec{J}_s \qquad \text{in} \quad V$$

$$\int_V\left(\nabla\times\left(\nu\nabla\times\vec{A}\right)+\sigma\frac{\partial\vec{A}}{\partial t}\right)\cdot\vec{w}_i\,\mathrm{d}V'=\int_V\vec{J}_s\cdot\vec{w}_i\,\mathrm{d}V' \qquad \forall\,\vec{w}_i(\vec{r})$$

$\vec{w}_i(\vec{r})$ vectorial „weighting functions“
vectorial „test functions“

- scalar product : $(\vec{\alpha},\vec{\beta})=\int_V\vec{\alpha}\cdot\vec{\beta}\,\mathrm{d}V'$

Spatial Discretisation (2)

→ weak formulation

$$\int_V \left(\nabla\times\left(\nu\nabla\times\vec{A}\right) + \sigma\frac{\partial\vec{A}}{\partial t}\right)\cdot\vec{w}_i\,\mathrm{d}V' = \int_V \vec{J}_s\cdot\vec{w}_i\,\mathrm{d}V' \qquad \forall\vec{w}_i(\vec{r})$$

$$\Big\Downarrow \qquad (\nabla\times\vec{v})\cdot\vec{w} = \nabla\cdot(\vec{v}\times\vec{w}) + \vec{v}\cdot\nabla\times\vec{w}$$

$$\int_V \left(\nabla\cdot\left(\left(\nu\nabla\times\vec{A}\right)\times\vec{w}_i\right) + \nu\nabla\times\vec{A}\cdot\nabla\times\vec{w}_i + \sigma\frac{\partial\vec{A}}{\partial t}\cdot\vec{w}_i\right)\mathrm{d}V' = \int_V \vec{J}_s\cdot\vec{w}_i\,\mathrm{d}V'$$

$$\Big\Downarrow \qquad \text{Gauss}$$

$$\oint_{\partial V} \underbrace{\left(\nu\nabla\times\vec{A}\right)}_{\vec{H}}\times\vec{w}_i\cdot\mathrm{d}\vec{S}' + \int_V \Big(\underbrace{\nu\nabla\times\vec{A}\cdot\nabla\times}\vec{w}_i + \sigma\frac{\partial\vec{A}}{\partial t}\cdot\vec{w}_i\Big)\mathrm{d}V' = \int_V \vec{J}_s\cdot\vec{w}_i\,\mathrm{d}V'$$

only first derivative required ⟹ „weak“ formulation

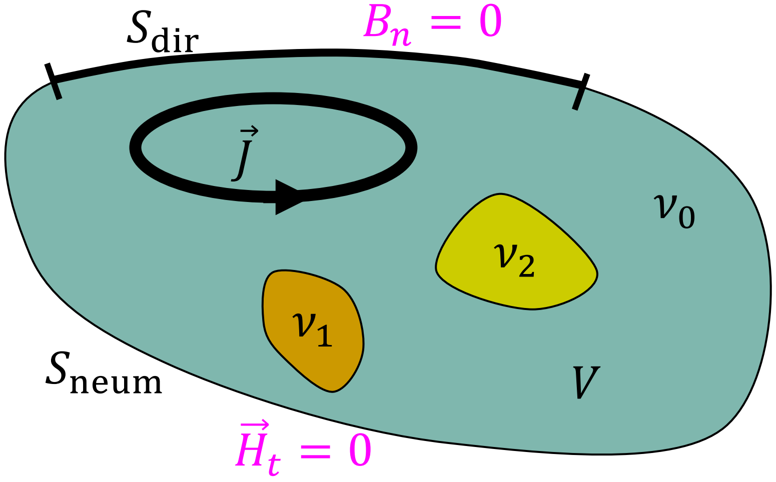
Spatial Discretisation (3)

Dirichlet BC at S_{dir}

$$\vec{A}\times\vec{n} = \vec{A}_{\mathrm{dir}}\times\vec{n} \quad \Leftrightarrow \quad \vec{B}\cdot\vec{n} = B_n$$

homogeneous Neumann BC at S_{neum}

$$\vec{H}_t = \left(\nu\nabla\times\vec{A}\right)\times\vec{n} = 0$$



$$\underbrace{\int_{S_{\mathrm{neum}}} \overbrace{\left(\nu\nabla\times\vec{A}\right)}^{\vec{H}}\times\vec{w}_i\cdot \mathrm{d}\vec{S}'}_{=0} + \underbrace{\int_{S_{\mathrm{dir}}} \left(\nu\nabla\times\vec{A}\right)\times\vec{w}_i\cdot \mathrm{d}\vec{S}'}_{=0} \qquad \forall\, \vec{w}_i(\vec{r})$$

$$\forall\, \vec{w}_i : \vec{w}_i\times\vec{n} = 0 \text{ at } S_{\mathrm{dir}}$$

„natural“ boundary condition

„essential“ boundary condition

Spatial Discretisation (4)

- discretisation

$$\vec{A}(\vec{r},t) = \sum_j a_j(t)\,\vec{v}_j(\vec{r}) \qquad \vec{v}_j(\vec{r})\times\vec{n} = 0 \qquad \text{at } S_{\mathrm{dir}}$$

$$\begin{cases} \vec{v}_j(\vec{r}) & \text{„shape/form functions“, „trial functions“} \\ a_j(t) & \text{unknowns, degrees of freedom} \end{cases}$$

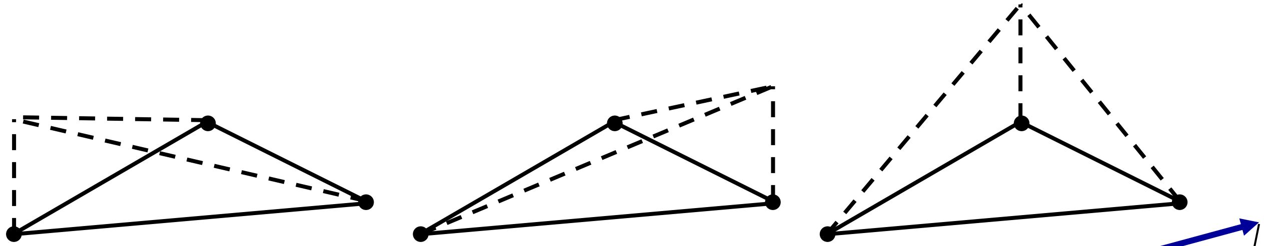
- Ritz-Galerkin method $\vec{v}_j(\vec{r}) = \vec{w}_j(\vec{r})$

- Petrov-Galerkin method $\vec{v}_j(\vec{r}) \neq \vec{w}_j(\vec{r})$

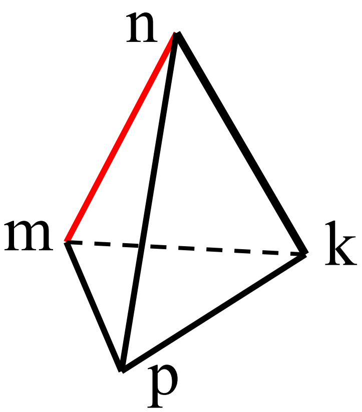
Spatial Discretisation (5)

- nodal shape functions

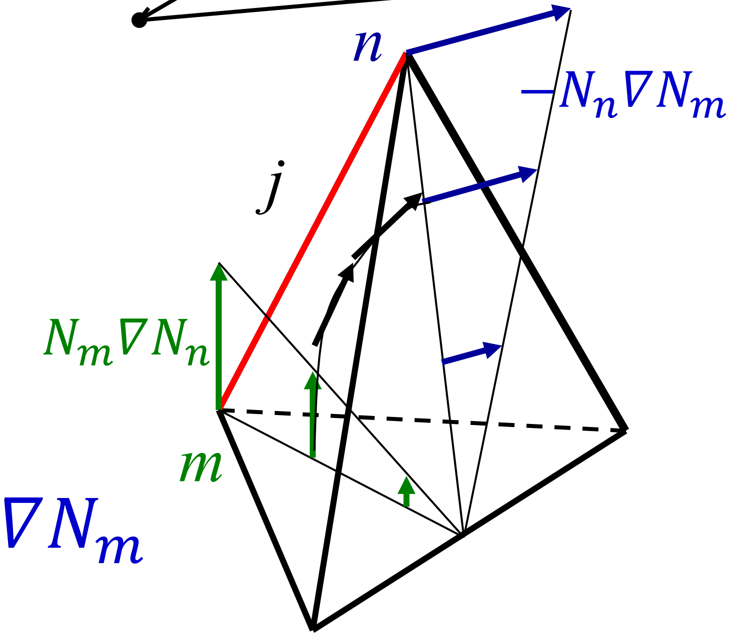
$$\varphi(x,y) = \quad u_1 N_1(x,y) \quad + u_2 N_2(x,y) \quad + u_3 N_3(x,y)$$



- edge shape functions



$$\vec{v}_j(\vec{r}) = N_m \nabla N_n - N_n \nabla N_m$$



Spatial Discretisation (6)

- discretization

$$\int_V \left(\nu \nabla\times\vec{A}\cdot\nabla\times\vec{w}_i + \sigma\frac{\partial\vec{A}}{\partial t}\cdot\vec{w}_i\right)\mathrm{d}V' = \int_V \vec{J}_s\cdot\vec{w}_i\,\mathrm{d}V' \qquad \forall\,\vec{w}_i(\vec{r})$$

$$\vec{A}(\vec{r},t)=\sum_j a_j(t)\,\vec{w}_j(\vec{r})$$

`curlcurl_ll.m`

`edgemass_ll.m`

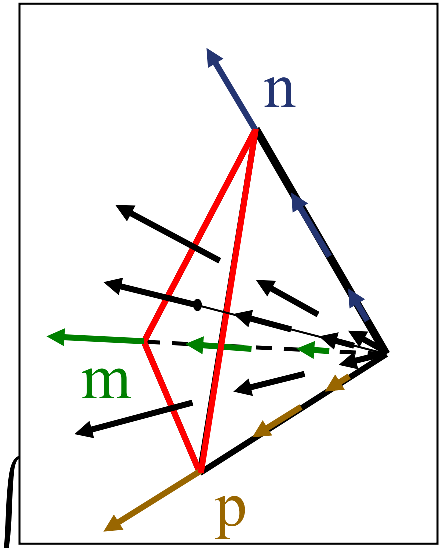
$$\sum_j \left(a_j \underbrace{\int_V \nu\nabla\times\vec{w}_j\cdot\nabla\times\vec{w}_i\,\mathrm{d}V'}_{=k_{ij}} + \frac{\mathrm{d}a_j}{\mathrm{d}t}\underbrace{\int_V \sigma\vec{w}_j\cdot\vec{w}_i\,\mathrm{d}V'}_{=m_{ij}}\right) = \underbrace{\int_V \vec{J}_s\cdot\vec{w}_i\,\mathrm{d}V'}_{=f_i}$$

$$[k_{ij}][a_j]+[m_{ij}]\left[\frac{\mathrm{d}a_j}{\mathrm{d}t}\right]=[f_i]$$

K and *M* symmetric,
semi-positive-definite

Spatial Discretisation (7)

$$\sum_j \left(a_j \int_V \nu \nabla\times\vec{w}_j \cdot \nabla\times\vec{w}_i \,\mathrm{d}V' + \frac{\mathrm{d}a_j}{\mathrm{d}t} \int_V \sigma \vec{w}_j \cdot \vec{w}_i \,\mathrm{d}V' \right)$$



$$\nabla\times\vec{w}_j(\vec{r}) = \sum_q c_{jq}\vec{z}_q(\vec{r})$$

$$\sum_j \left(\underbrace{a_j}_{\boxed{\hat{a}_j}} \sum_p \sum_q c_{ip} c_{jq} \underbrace{\int_V \nu\, \vec{z}_q \cdot \vec{z}_p \,\mathrm{d}V'}_{\boxed{M^{\mathrm{fe}}_{\nu,pq}}} + \frac{\mathrm{d}a_j}{\mathrm{d}t} \underbrace{\int_V \sigma \vec{w}_j \cdot \vec{w}_i \,\mathrm{d}V'}_{\boxed{M^{\mathrm{fe}}_{\sigma,ij}}} \right) = \underbrace{\int_V}_{\boxed{\hat{\hat{j}}_{\mathrm{s},i}}}$$

$$\tilde{C} M^{\mathrm{fe}}_{\nu} C \hat{a} + M^{\mathrm{fe}}_{\sigma} \frac{\mathrm{d}\hat{a}}{\mathrm{d}t} = \hat{\hat{j}}_{\mathrm{s}} \quad \Longrightarrow \quad K_\nu \hat{a} + M_\sigma \frac{\mathrm{d}\hat{a}}{\mathrm{d}t} = \hat{\hat{j}}_{\mathrm{s}}$$

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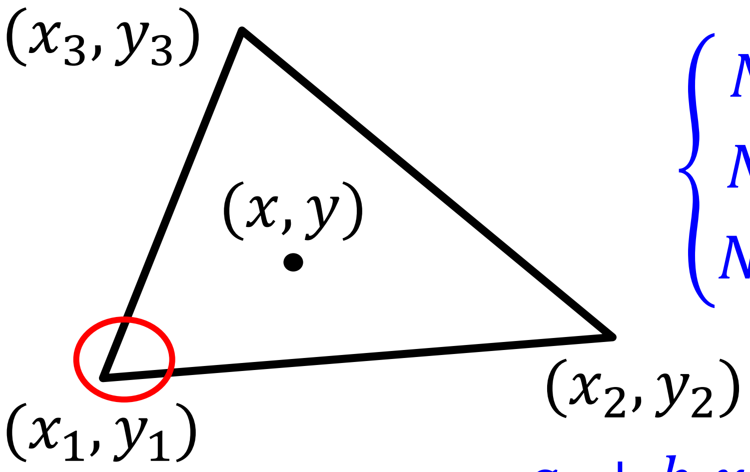
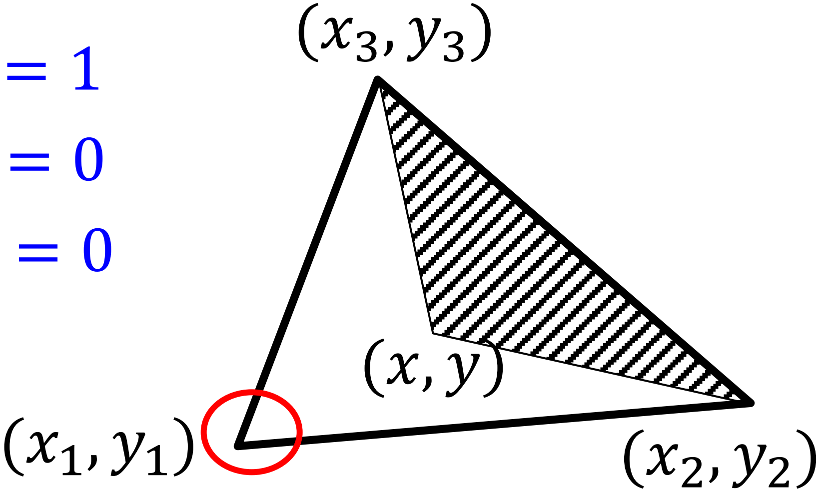
2D Nodal Shape Functions



$$\begin{cases} N_i(x_i, y_i) = 1 \\ N_i(x_j, y_j) = 0 \\ N_i(x_k, y_k) = 0 \end{cases}$$



$$N_i(x, y) = \frac{a_i + b_i x + c_i y}{2S_{ijk}}$$

$$\begin{cases} a_i = x_j y_k - x_k y_j \\ b_i = y_j - y_k \\ c_i = x_k - x_j \\ S_{ijk} = \text{element area} \end{cases}$$

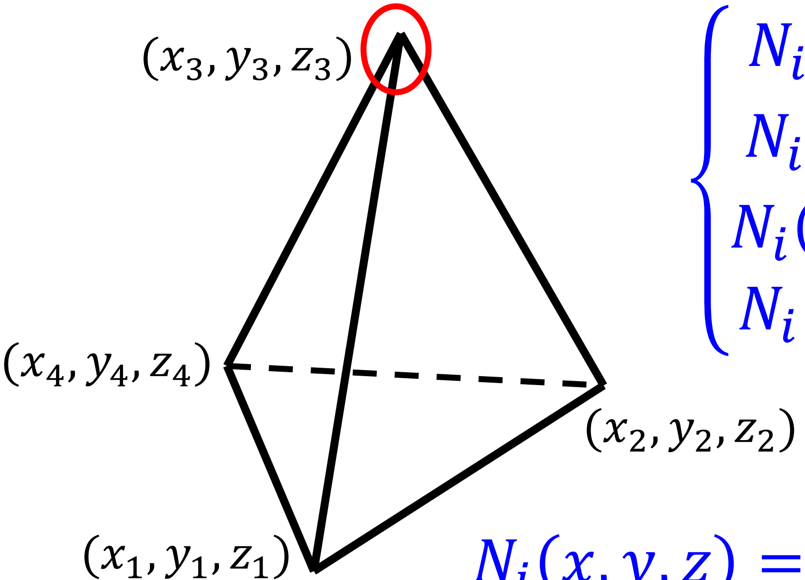
surface coordinates

$$N_i(x, y) = \frac{S_{pjk}}{S_{ijk}}$$

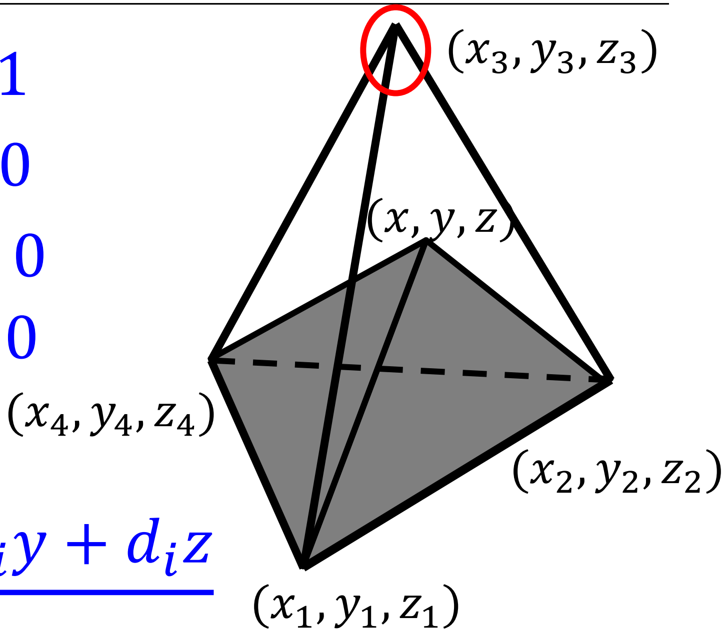
partition of unity

$$\sum_{i=1}^{3} N_i(x, y) = 1, \qquad \forall (x, y)$$

3D Nodal Shape Functions



$$\begin{cases} N_i(x_i, y_i, z_i) = 1 \\ N_i(x_j, y_j, z_j) = 0 \\ N_i(x_k, y_k, z_k) = 0 \\ N_i(x_\ell, y_\ell, z_\ell) = 0 \end{cases}$$



$$N_i(x, y, z) = \frac{a_i + b_i x + c_i y + d_i z}{6V_{ijk\ell}}$$

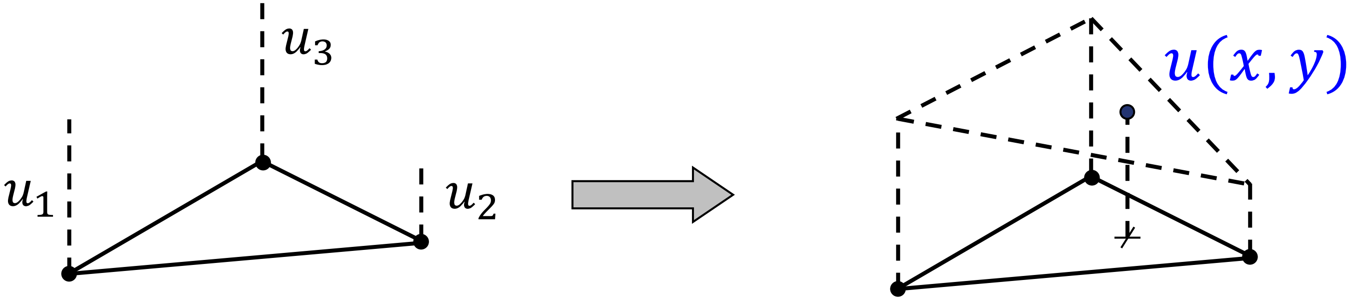
$$\begin{cases} a_i = \\ b_i = \\ c_i = \\ d_i = \\ V_{ijk\ell} = \text{element volume} \end{cases}$$

volume coordinates $N_i(x, y, z) = \frac{V_{pjk\ell}}{V_{ijk\ell}}$

partition of unity $\sum_{i=1}^{4} N_i(x, y, z) = 1, \forall (x, y, z)$

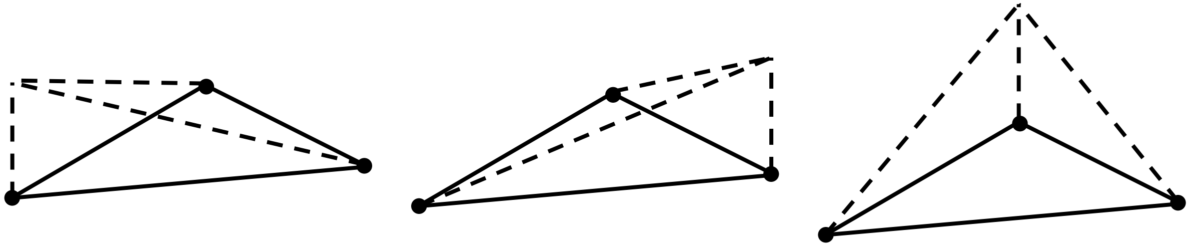
Dofs or Nodal Values ?

- solution $\varphi(x,y) = \varphi_{\mathrm{dir}}(x,y) + \sum_j u_j N_j(x,y)$
  - interpolation :
    nodal values & interpolating functions



  - series development :
    degrees of freedom & shape functions

$$u_1, u_2, u_3 \quad \Rightarrow \quad u_1 N_1(x,y) \quad + u_2 N_2(x,y) \quad + u_3 N_3(x,y)$$



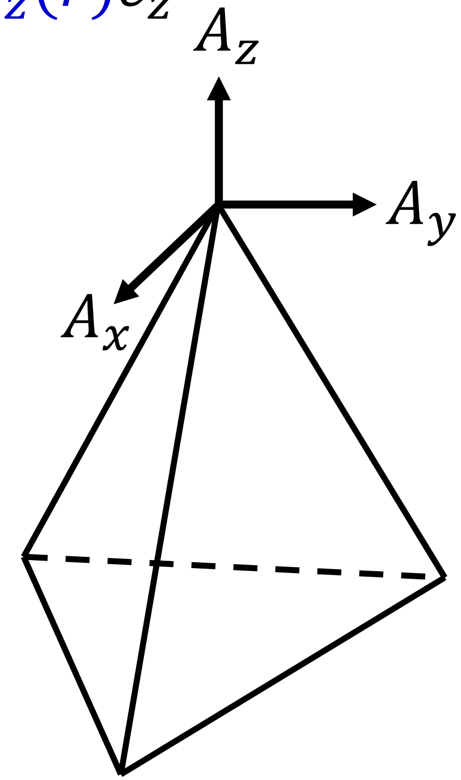
Vectorial Shape Functions

- vector field $\vec{A}(\vec{r}) = A_x(\vec{r})\vec{e}_x + A_y(\vec{r})\vec{e}_y + A_z(\vec{r})\vec{e}_z$
  - –interpolate each component separately by *scalar* shape functions

$$\vec{A}(\vec{r}) = \left(\sum_j u_{x,j}N_j(\vec{r})\right)\vec{e}_x$$

$$+\left(\sum_j u_{y,j}N_j(\vec{r})\right)\vec{e}_y + \left(\sum_j u_{z,j}N_j(\vec{r})\right)\vec{e}_z$$

  - –BUT
    - – too much continuity (both normal and tangential components)
    - – spurious modes

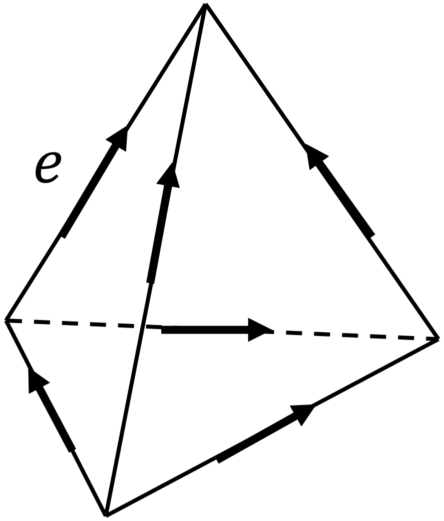


Edge Elements

- shape functions defined by

$$\begin{cases} \int_e \vec{w}_e \cdot d\vec{s}' = 1 \\ \int_{e'} \vec{w}_e \cdot d\vec{s}' = 0, e \neq e' \end{cases}$$

–linear combination $\vec{E}(\vec{r}) = \sum_{j=1}^{E} u_j \vec{w}_{e_j}(\vec{r})$



  - features tangential continuity (1-form)

–physical meaning of the degrees of freedom

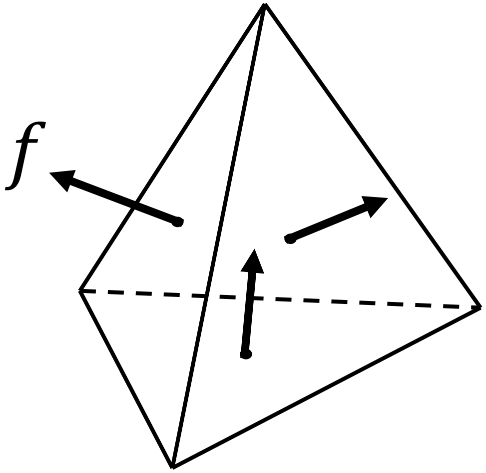
$$\int_e \vec{E} \cdot d\vec{s}' = \sum_{j=1}^{E} u_j \int_e \vec{w}_{e_j} \cdot d\vec{s}' = u_e \int_e \vec{w}_{e_j} \cdot d\vec{s}' = u_e$$

  - voltage drop along the edge

Face(t) Elements

- shape functions defined by

$$\begin{cases} \int_f \vec{w}_f \cdot d\vec{S}' = 1 \\ \int_{f'} \vec{w}_f \cdot d\vec{S}' = 0, f' \neq f \end{cases}$$



–linear combination $\vec{B}(\vec{r}) = \sum_{j=1}^{F} u_j \, \vec{w}_{f_j}(\vec{r})$

  - features normal continuity (2-form)

–physical meaning of the degrees of freedom

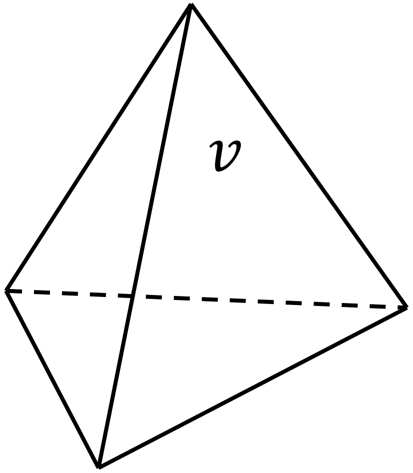
$$\int_f \vec{B} \cdot d\vec{S}' = \sum_{j=1}^{F} u_j \int_f \vec{w}_{f_j} \cdot d\vec{S}' = u_f \int_f \vec{w}_{f_j} \cdot d\vec{S}' = u_f$$

  - flux through the facet

Volume Elements

- shape functions defined by

$$\begin{cases} \int_{v} w_v dV' = 1 \\ \int_{v'} w_v dV' = 0, v' \neq v \end{cases}$$



–linear combination $\rho(\vec{r}) = \sum_{j=1}^{V} u_j \, w_{v_j}(\vec{r})$

  - commonly discontinuous (3-form)

–physical meaning of the degrees of freedom

$$\int_{v} \rho dV' = \sum_{j=1}^{V} u_j \int_{v} w_{v_j} dV' = u_v \int_{v} w_{v_j} dV' = u_v$$

  - charge within the element

Canonical Construction

- construction
  - –nodal shape functions

$$N_m(\vec{r}), N_n(\vec{r}), N_p(\vec{r}), N_q(\vec{r})$$

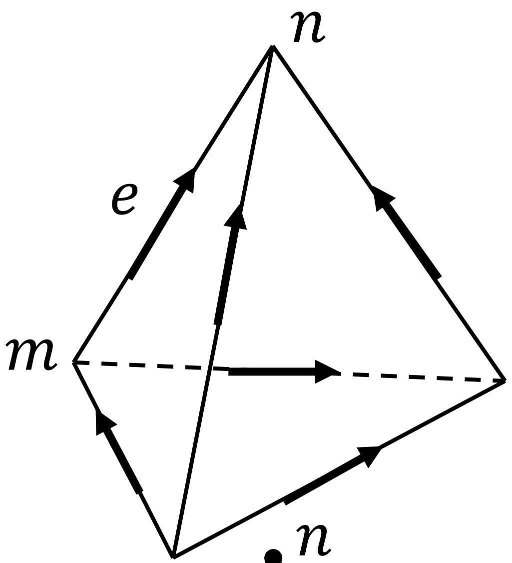
  - –edge elements

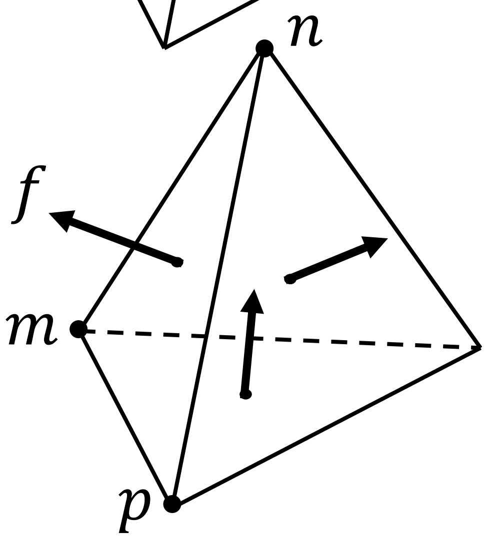
$$\vec{w}_e(\vec{r}) = N_m \nabla N_n - N_n \nabla N_m$$

  - –facet elements

$$\vec{w}_f(\vec{r}) = 2(N_m \nabla N_n \times \nabla N_p + N_n \nabla N_p \times \nabla N_m + N_p \nabla N_m \times \nabla N_n)$$

  - –volume elements $w_v(\vec{r}) = \frac{1}{V}$





Whitney Complex

- Whitney elements (for a given mesh)
  - W^0: nodal elements
  - W^1: edge elements
  - W^2: facet elements
  - W^3: volume elements
- Whitney complex

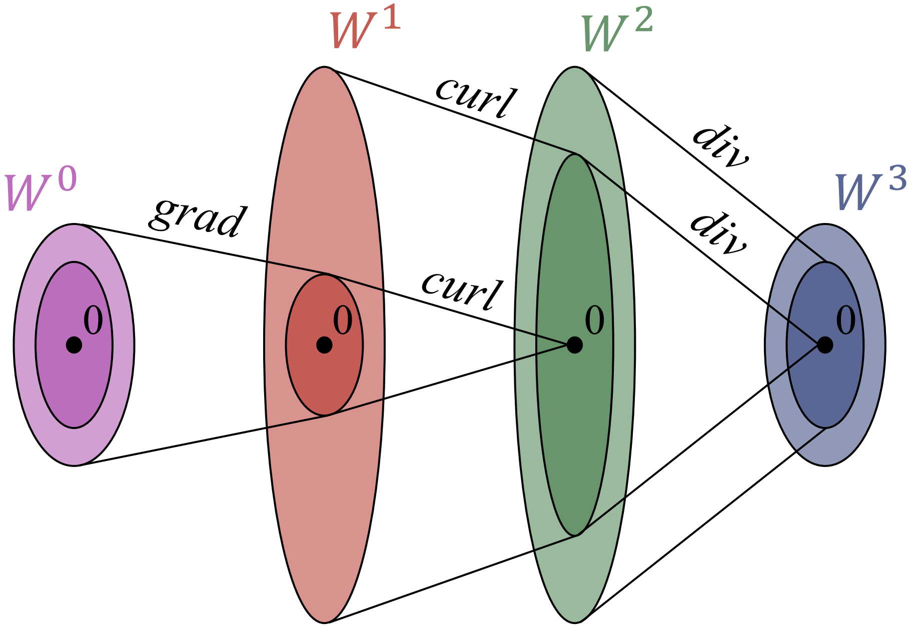
$$\operatorname{grad} W^0 \subset W^1$$

$$\operatorname{curl} W^1 \subset W^2$$

$$\operatorname{div} W^2 \subset W^3$$

$$\operatorname{curl} \operatorname{grad} W^0 = 0$$

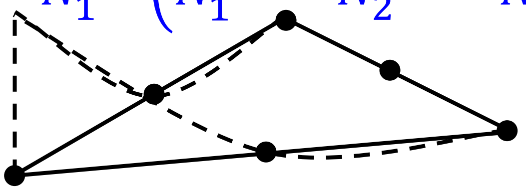
$$\operatorname{div} \operatorname{curl} W^1 = 0$$



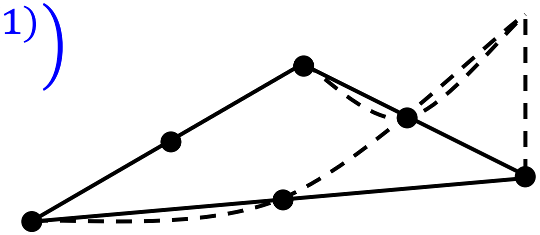
2D Higher-Order Elements

- nodal elements

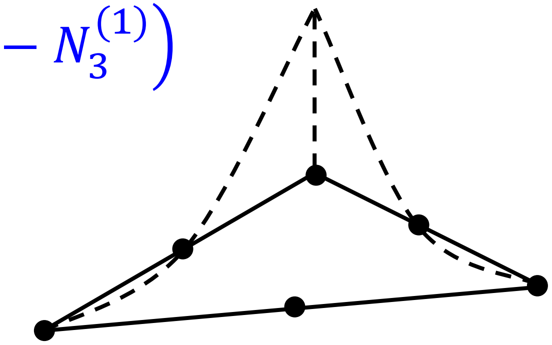
$$N_1^{(2)} = N_1^{(1)}\left(N_1^{(1)} - N_2^{(1)} - N_3^{(1)}\right)$$



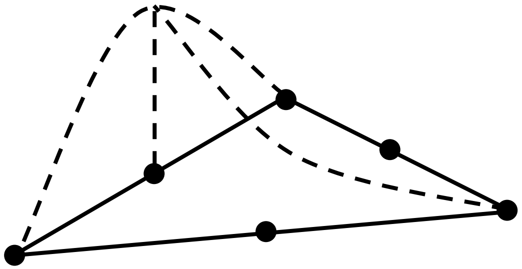
$$N_2^{(2)} = N_2^{(1)}\left(-N_1^{(1)} + N_2^{(1)} - N_3^{(1)}\right)$$



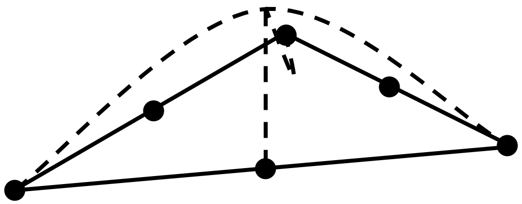
$$N_3^{(2)} = N_3^{(1)}\left(-N_1^{(1)} - N_2^{(1)} + N_3^{(1)}\right)$$



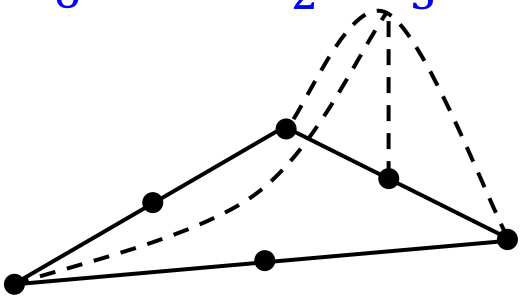
$$N_4^{(2)} = 4N_3^{(1)}N_1^{(1)}$$



$$N_5^{(2)} = 4N_1^{(1)}N_2^{(1)}$$



$$N_6^{(2)} = 4N_2^{(1)}N_3^{(1)}$$

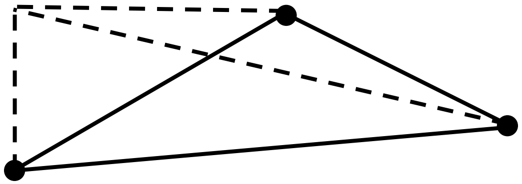


degrees of freedom == nodal values

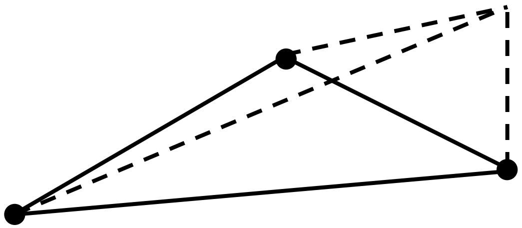
Hierarchical Finite Elements

- nodal elements

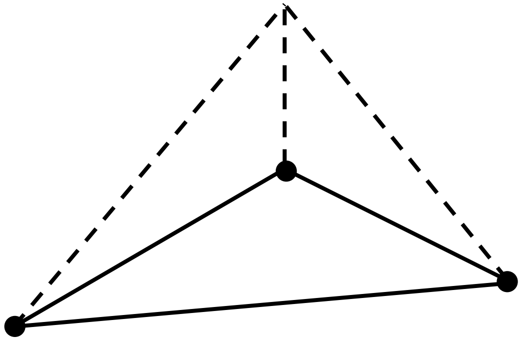
$N_1^{(2)} = N_1^{(1)}$



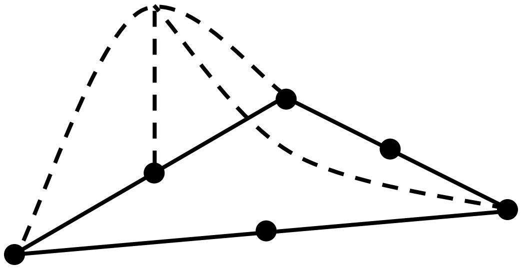
$N_2^{(2)} = N_2^{(1)}$



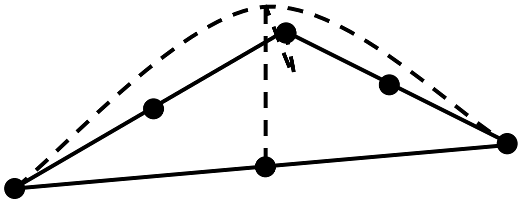
$N_3^{(2)} = N_3^{(1)}$



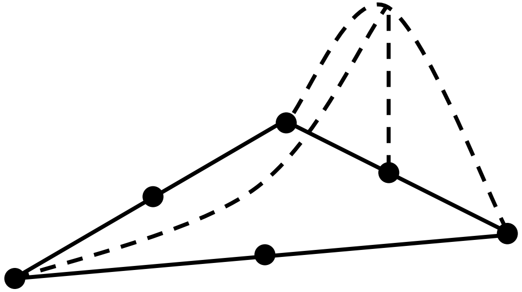
$N_4^{(2)} = 4N_3^{(1)}N_1^{(1)}$



$N_5^{(2)} = 4N_1^{(1)}N_2^{(1)}$



$N_6^{(2)} = 4N_2^{(1)}N_3^{(1)}$



degrees of freedom ≠ nodal values

Hierarchical Finite Elements

- first-order system of equations

$$\left[K^{(11)}\right]\left[u^{(1)}\right]=\left[f^{(1)}\right]$$

- second-order system of equations

$$\begin{bmatrix} K^{(11)} & K^{(12)} \\ K^{(21)} & K^{(22)} \end{bmatrix}\begin{bmatrix} u^{(1)} \\ u^{(2)} \end{bmatrix}=\begin{bmatrix} f^{(1)} \\ f^{(2)} \end{bmatrix}$$

first-order system embedded in second-order system

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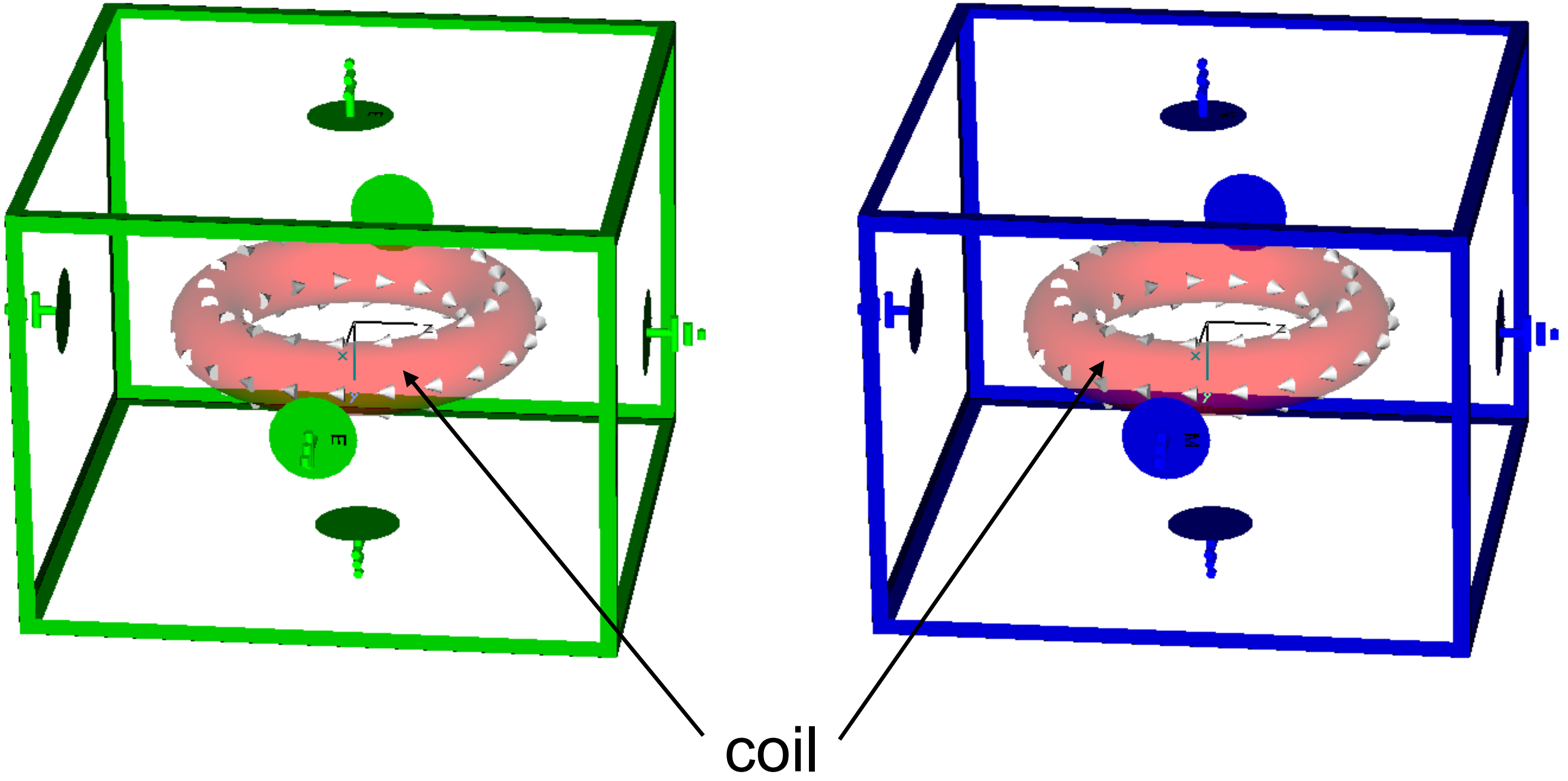
Boundary Conditions (1)

	electric BC „flux wall“ „current gate“ 🟩	magnetic BC „flux gate“ „current wall“ 🟦
definition	$\vec{E}_t = 0$	$\vec{H}_t = 0$
electric current	$\vec{J}_n \neq 0$	$\vec{J}_n = 0$
magnetic flux	$\vec{B}_n = 0$	$\vec{B}_n \neq 0$
magnetic vector potential formulation	Dirichlet BC	Neumann BC
magnetic scalar potential formulation	Neumann BC	Dirichlet BC

Boundary Conditions (2)

electric boundary conditions magnetic boundary conditions

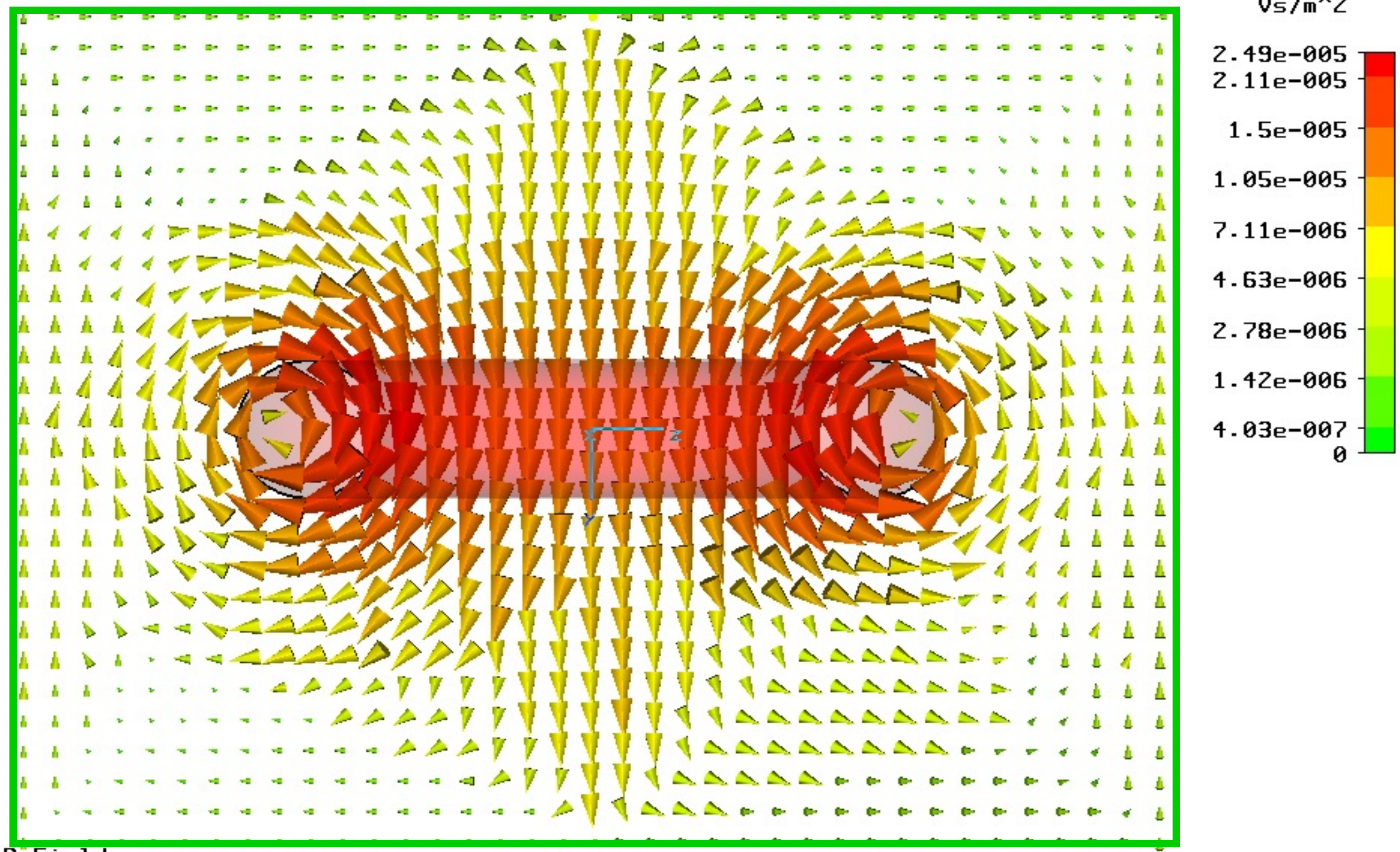


coil

Boundary Conditions (3)

electric boundary conditions

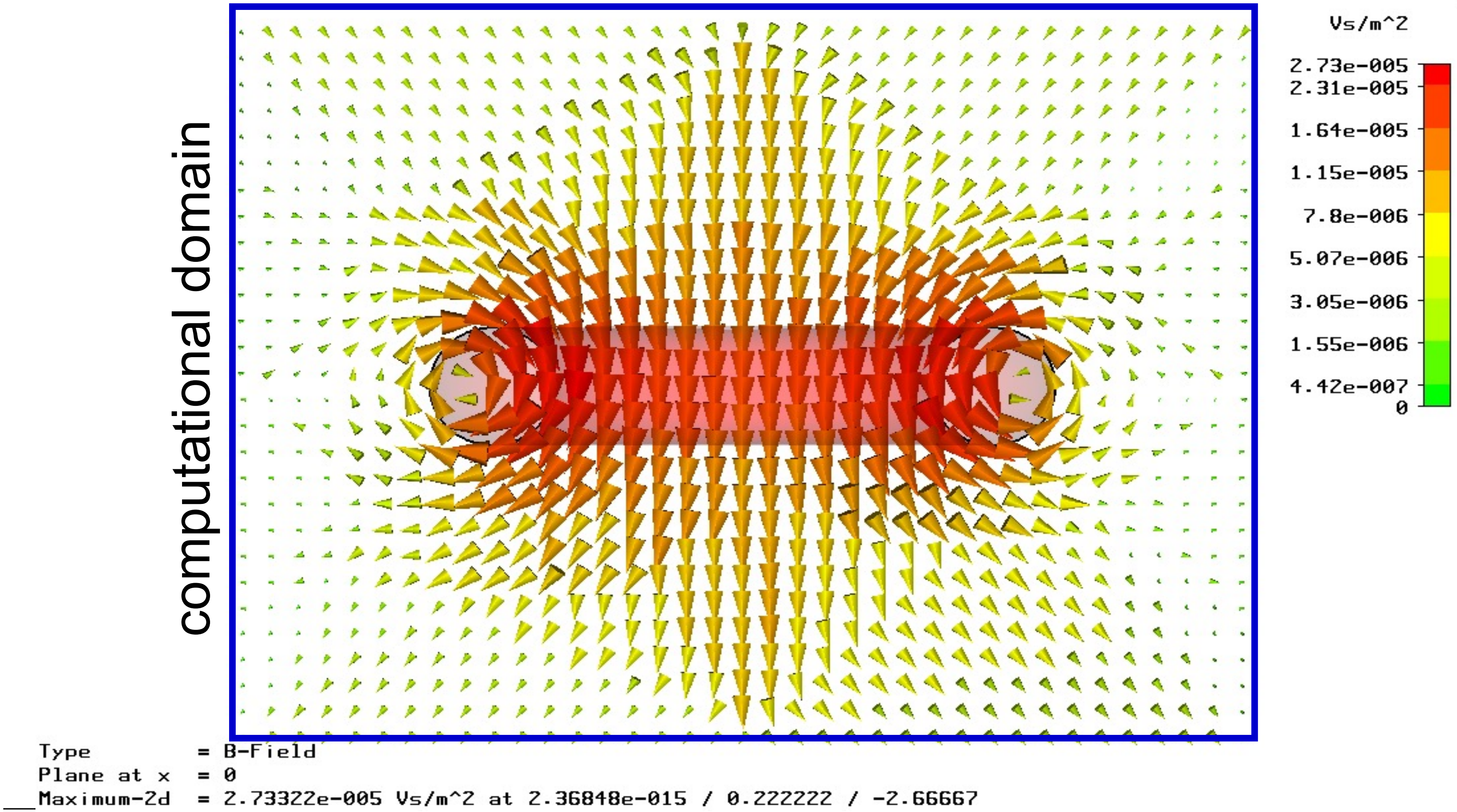
computational domain



```
Type          = B-Field
Plane at x    = 0
Maximum-2d    = 2.49377e-005 Vs/m^2 at 2.36848e-015 / 0.222222 / -2.66667
```

Boundary Conditions (4)

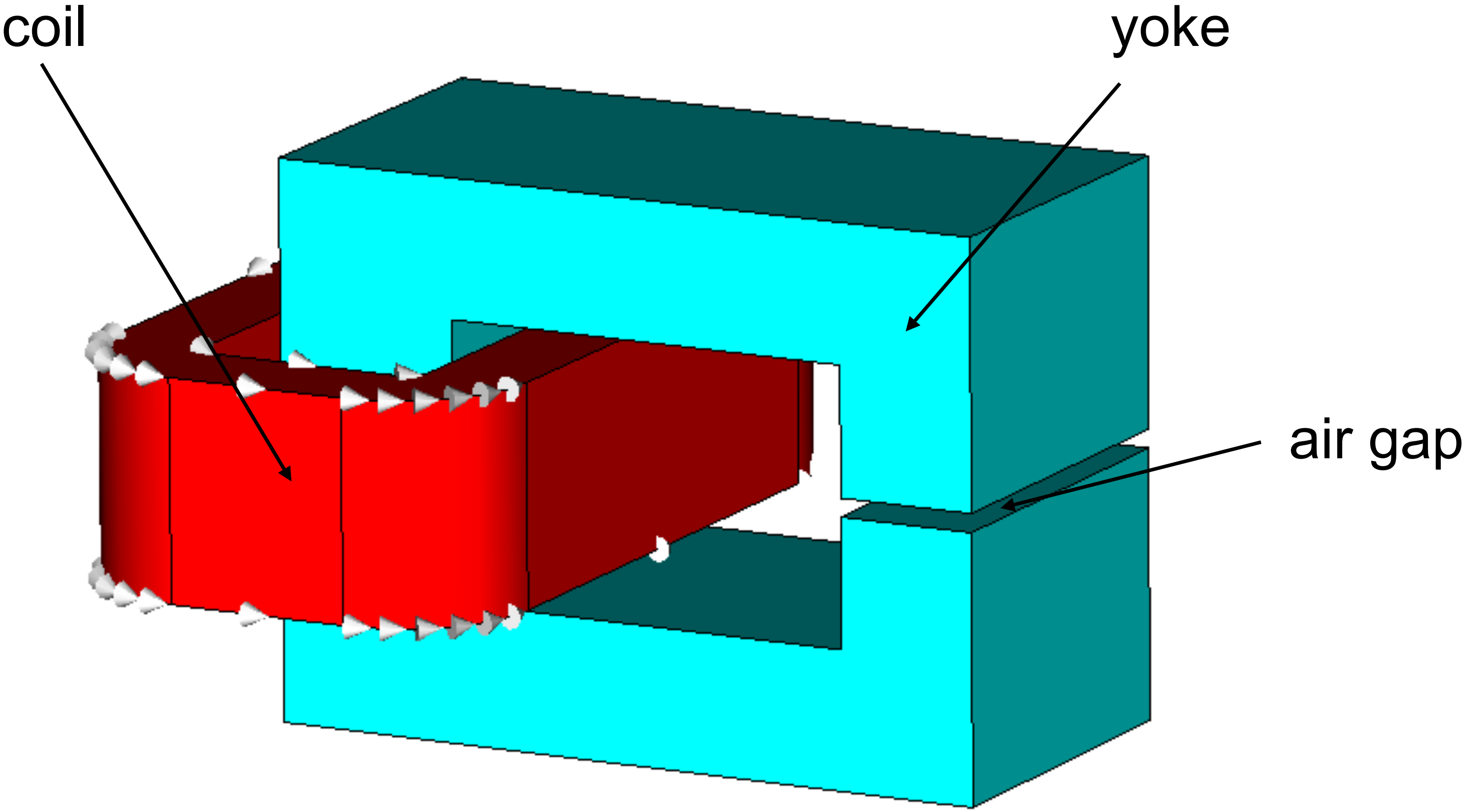
magnetic boundary conditions



computational domain

```
Type          = B-Field
Plane at x    = 0
Maximum-2d    = 2.73322e-005 Vs/m^2 at 2.36848e-015 / 0.222222 / -2.66667
```

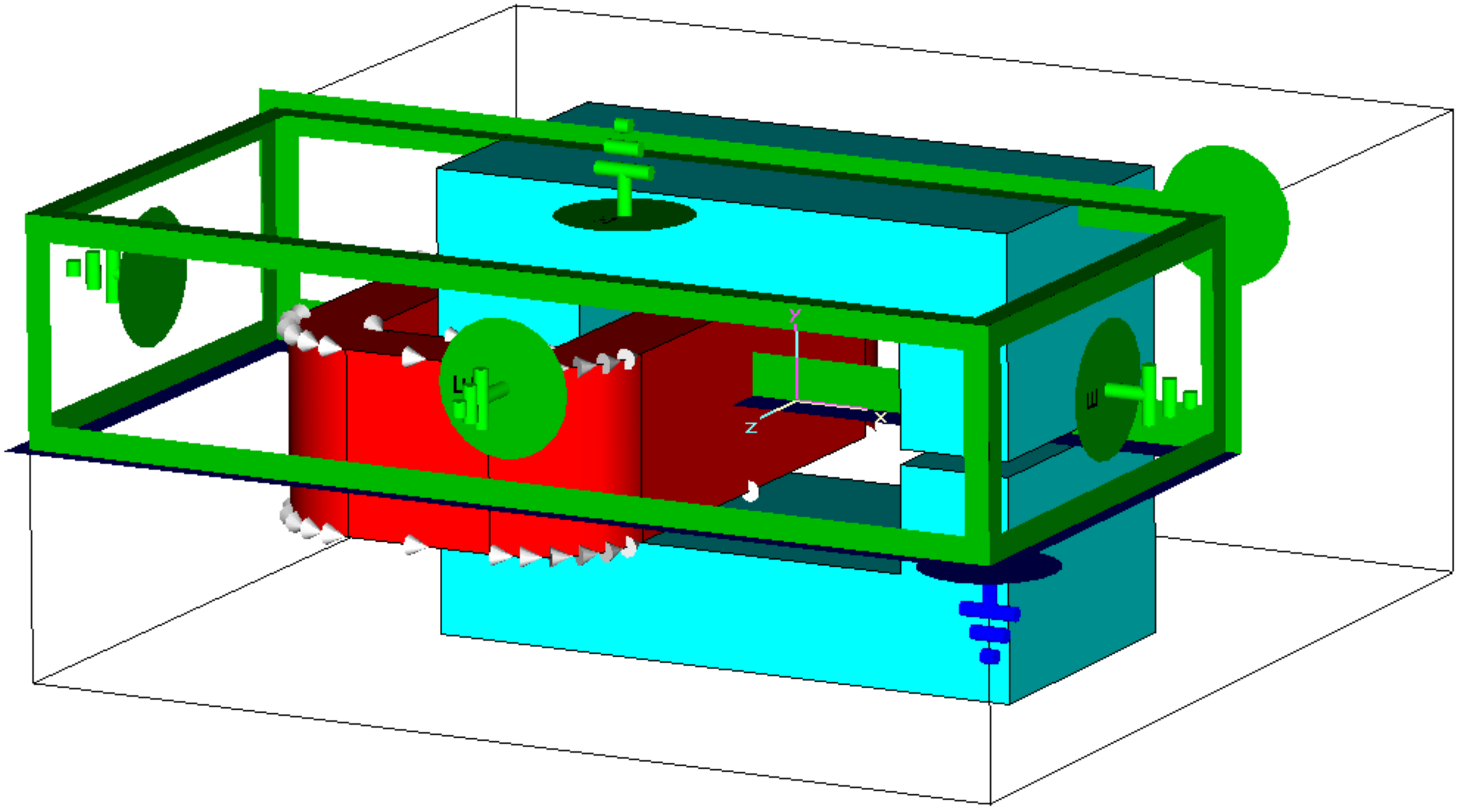
Symmetries (1)



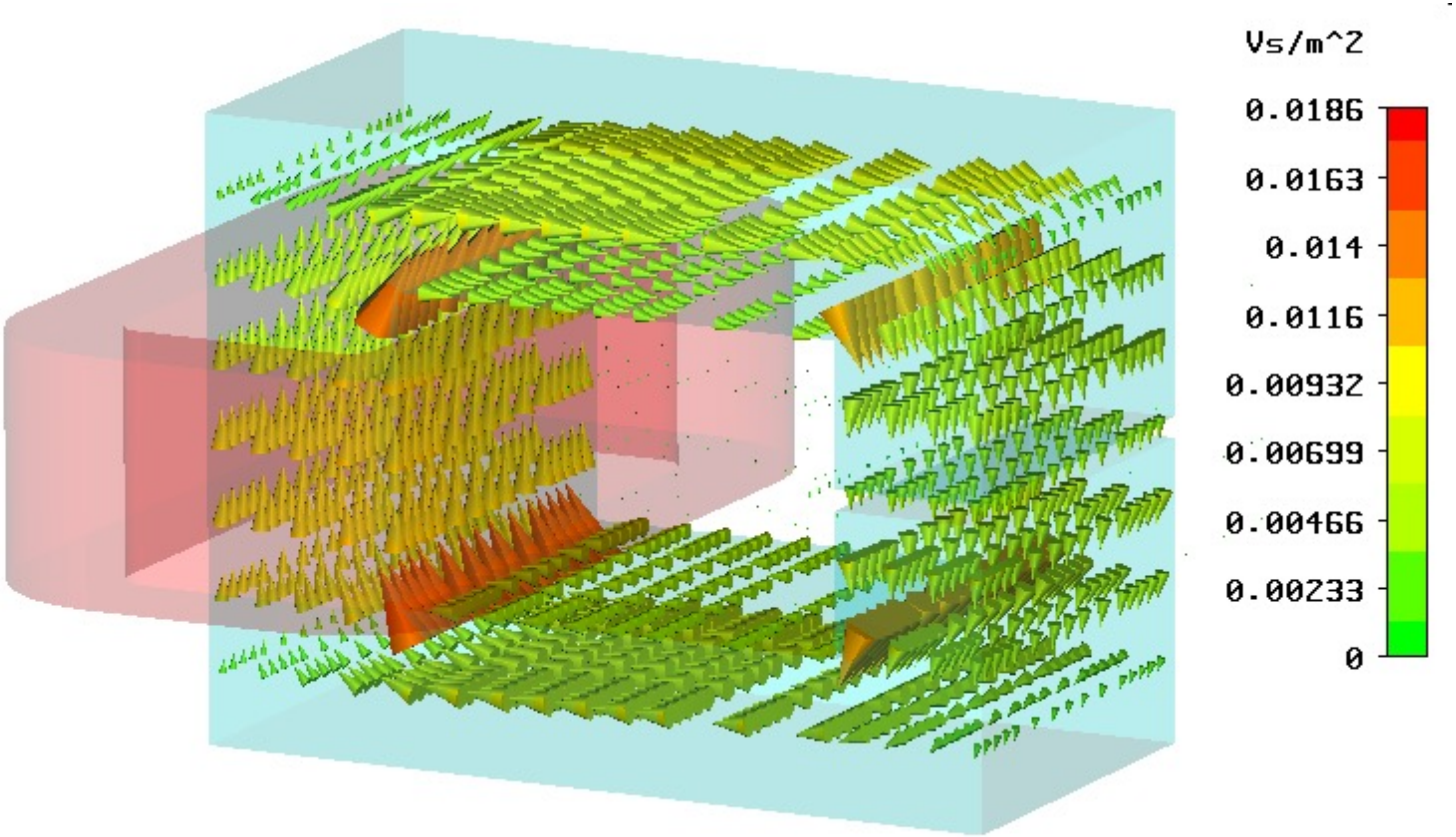
Symmetries (2)

🟩 electric boundary conditions

🟦 magnetic boundary conditions



Symmetries (3)



Inserting Boundary Conditions

$$\begin{bmatrix} K_{bb} & K_{bc} \\ K_{cb} & K_{cc} \end{bmatrix} \begin{bmatrix} u_b \\ \boxed{u_c} \end{bmatrix} + \begin{bmatrix} 0 \\ \boxed{g_c} \end{bmatrix} = \begin{bmatrix} f_b \\ f_c \end{bmatrix}$$

→ unknown boundary-integral term

→ potentials at Dirichlet boundaries

insert essential boundary conditions ⇨

$$\left[\begin{array}{cc:c} K_{bb} & K_{bc} & 0 \\ K_{cb} & K_{cc} & B_{cq} \\ \hdashline 0 & B_{qc} & 0 \end{array}\right] \left[\begin{array}{c} u_b \\ u_c \\ \hdashline y_q \end{array}\right] = \left[\begin{array}{c} f_b \\ f_c \\ \hdashline 0 \end{array}\right]$$

eliminate known potentials ⇨

$$K_{bb}u_b = f_b - K_{bc}u_c$$

K f → **`bdrycond_shrink.m`** → K_{bb} $f_b - K_{bc}u_c$

`backslash`

u ← **`bdrycond_inflate.m`** ← u_b

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Reduction to 2D Models (1)

- 2D cartesian models

$$\vec{J} = \left(0{,}0, J_z(x,y)\right)$$

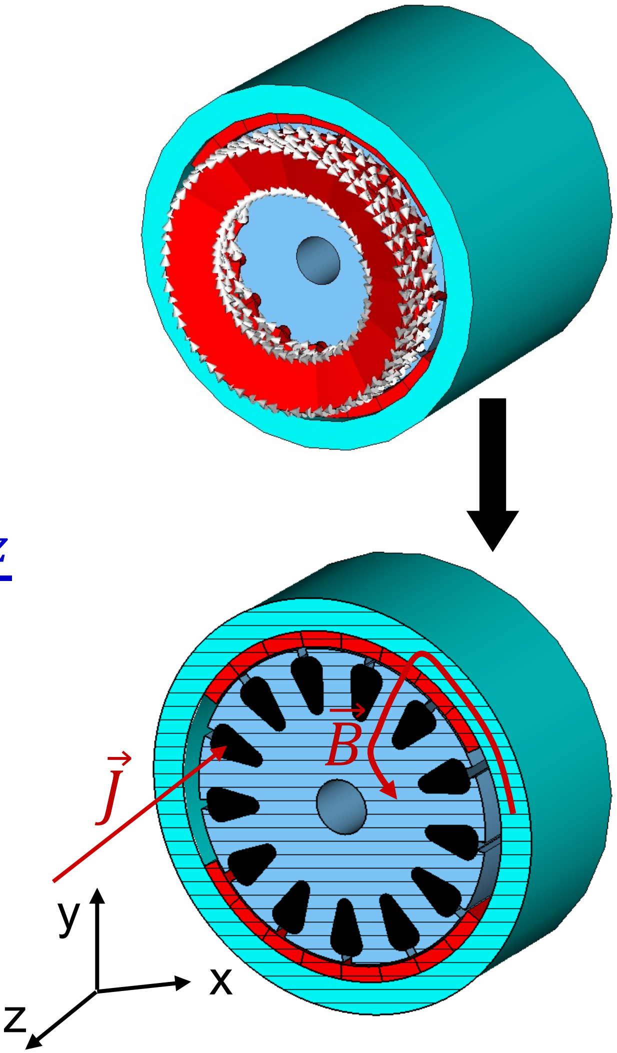
$$\vec{B} = \left(B_x(x,y), B_y(x,y), 0\right)$$

$$\vec{A} = \left(0{,}0, A_z(x,y)\right)$$

$$\vec{B} = \nabla\times\vec{A} \iff B_x = \frac{\partial A_z}{\partial y};\ B_y = -\frac{\partial A_z}{\partial x}$$

$$\nabla\cdot\vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_z}{\partial x \partial y} = 0$$

$$\frac{\partial A_z}{\partial z}, \frac{\partial B_x}{\partial z}, \frac{\partial B_y}{\partial z}, \frac{\partial J_z}{\partial z} = 0 \quad \text{but} \quad \frac{\partial \varphi}{\partial z} \neq 0$$



Reduction to 2D Models (2)

- 2D cartesian models

$$B_x = \frac{\partial A_z}{\partial y};\;\; B_y = -\frac{\partial A_z}{\partial x}$$

anisotropic material

$$\begin{bmatrix} H_x \\ H_y \end{bmatrix} = \begin{bmatrix} \nu_x & 0 \\ 0 & \nu_y \end{bmatrix} \begin{bmatrix} B_x \\ B_y \end{bmatrix}$$

$$H_x = \nu_x \frac{\partial A_z}{\partial y};\;\; H_y = -\nu_y \frac{\partial A_z}{\partial x}$$

$$J_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_{s,z} - \sigma \frac{\partial A_z}{\partial t}$$

Ampère
+ Faraday-Lenz

$$-\frac{\partial}{\partial x}\left(\nu_y \frac{\partial A_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\nu_x \frac{\partial A_z}{\partial y}\right) + \sigma \frac{\partial A_z}{\partial t} = J_{s,z}$$

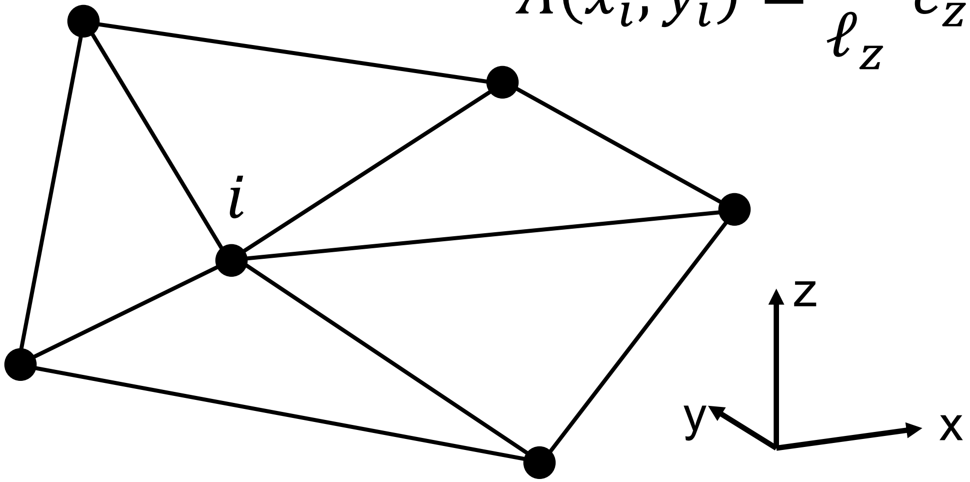
$$\Longleftrightarrow \qquad \nabla \times \left(\nu \nabla \times \vec{A}\right) + \sigma \frac{\partial \vec{A}}{\partial t} = \vec{J}_s$$

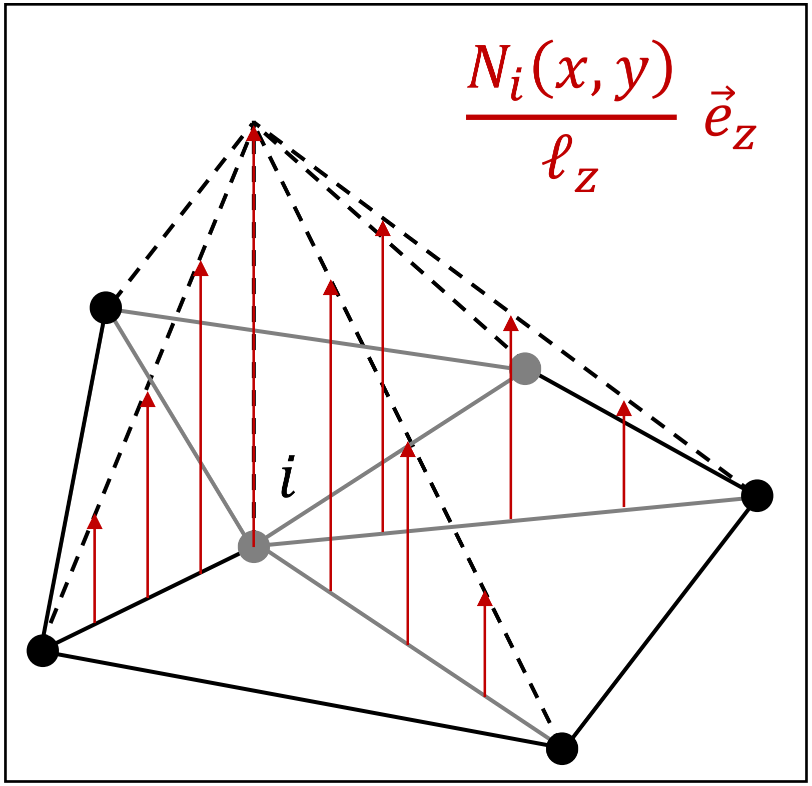
2D Discretisation (1)

$$\sum_j \left(a_j \int_V \nu \nabla\times\vec{w}_j \cdot \nabla\times\vec{w}_i \,\mathrm{d}V + \frac{\mathrm{d}a_j}{\mathrm{d}t} \int_V \sigma\vec{w}_j \cdot \vec{w}_i \,\mathrm{d}V \right) = \int_V \vec{J}_\mathrm{s} \cdot \vec{w}_i \,\mathrm{d}V$$

$$\vec{A} = \sum_j a_j \vec{w}_j = \sum_j a_j \frac{N_j(x,y)}{\ell_z} \vec{e}_z$$

$$\vec{A}(x_i,y_i) = \frac{a_i}{\ell_z} \vec{e}_z$$



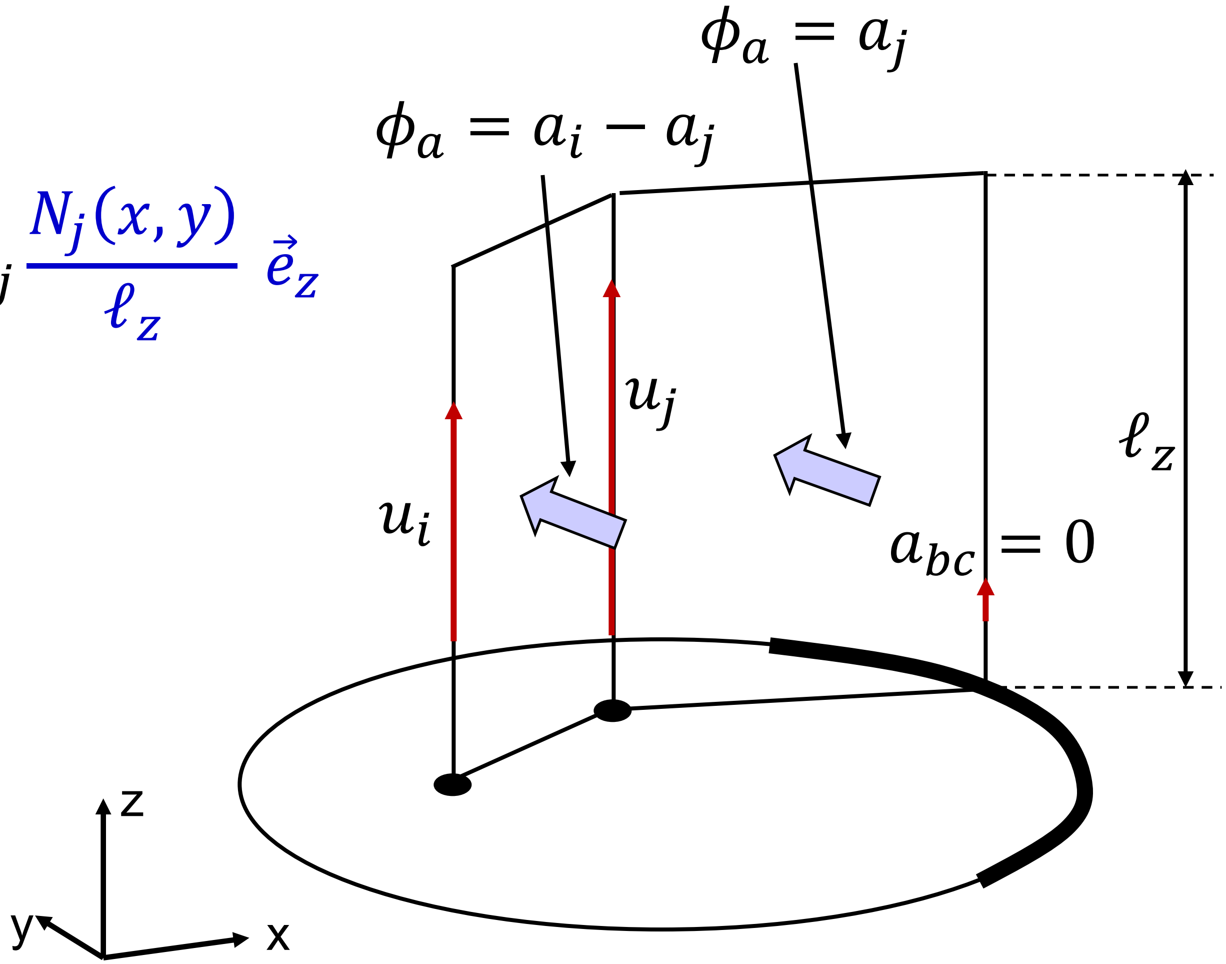


2D Discretisation (2)

$$\vec{A} = \sum_j a_j \vec{v}_j = \sum_j a_j \frac{N_j(x,y)}{\ell_z}\,\vec{e}_z$$

$$\vec{A}(x_i, y_i) = \frac{a_i}{\ell_z}\,\vec{e}_z$$

$$a_i = \ell_z \vec{A}(x_i, y_i) \cdot \vec{e}_z$$

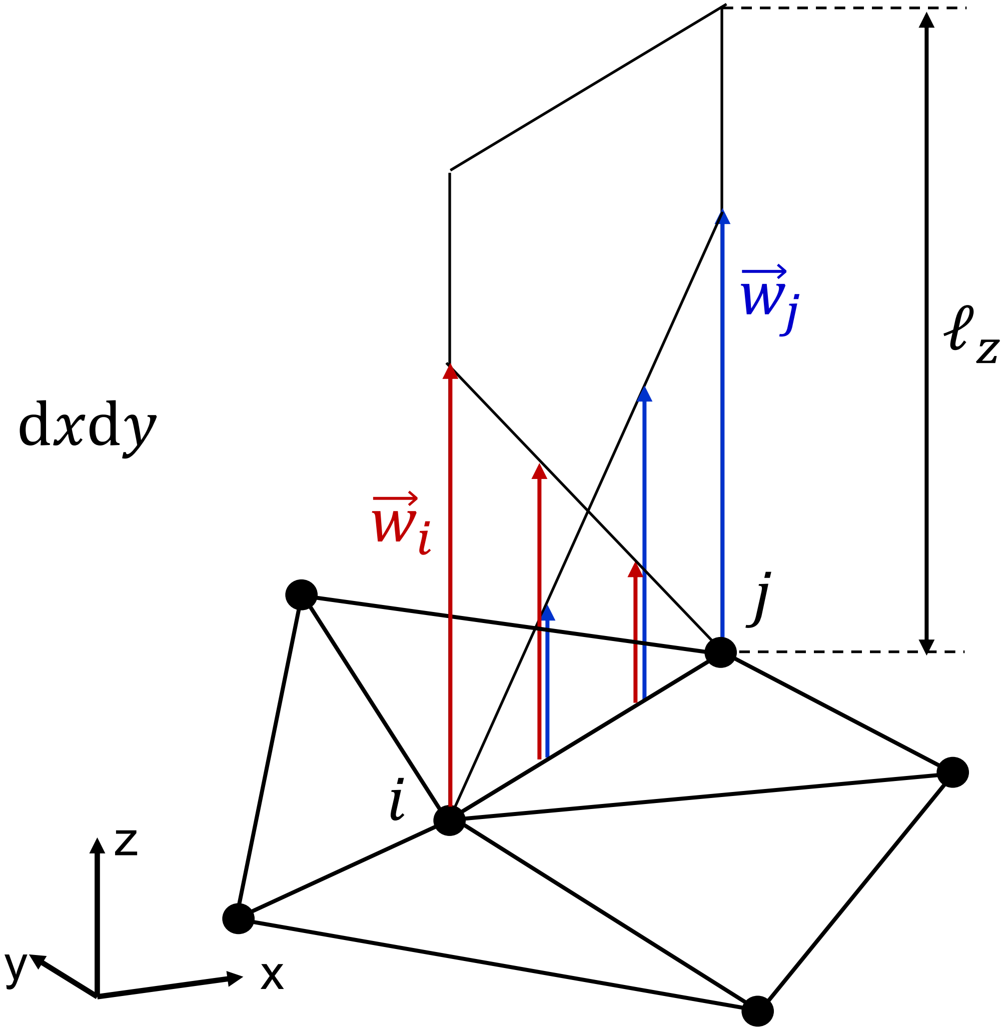


2D Discretisation (3)

$$M^{\mathrm{FE}}_{\sigma,i,j} = \int_V \sigma \vec{w}_j \cdot \vec{w}_i \, \mathrm{d}V'$$

$$M^{\mathrm{FE}}_{\sigma,i,j} = \int_{S_{2\mathrm{D}}} \sigma \frac{N_j}{\ell_z} \vec{e}_z \cdot \frac{N_i}{\ell_z} \vec{e}_z \, \ell_z \, \mathrm{d}x\mathrm{d}y$$

$$M^{\mathrm{FE}}_{\sigma,i,j} = \int_{S_{2\mathrm{D}}} \frac{\sigma}{\ell_z} N_j N_i \, \mathrm{d}x\mathrm{d}y$$



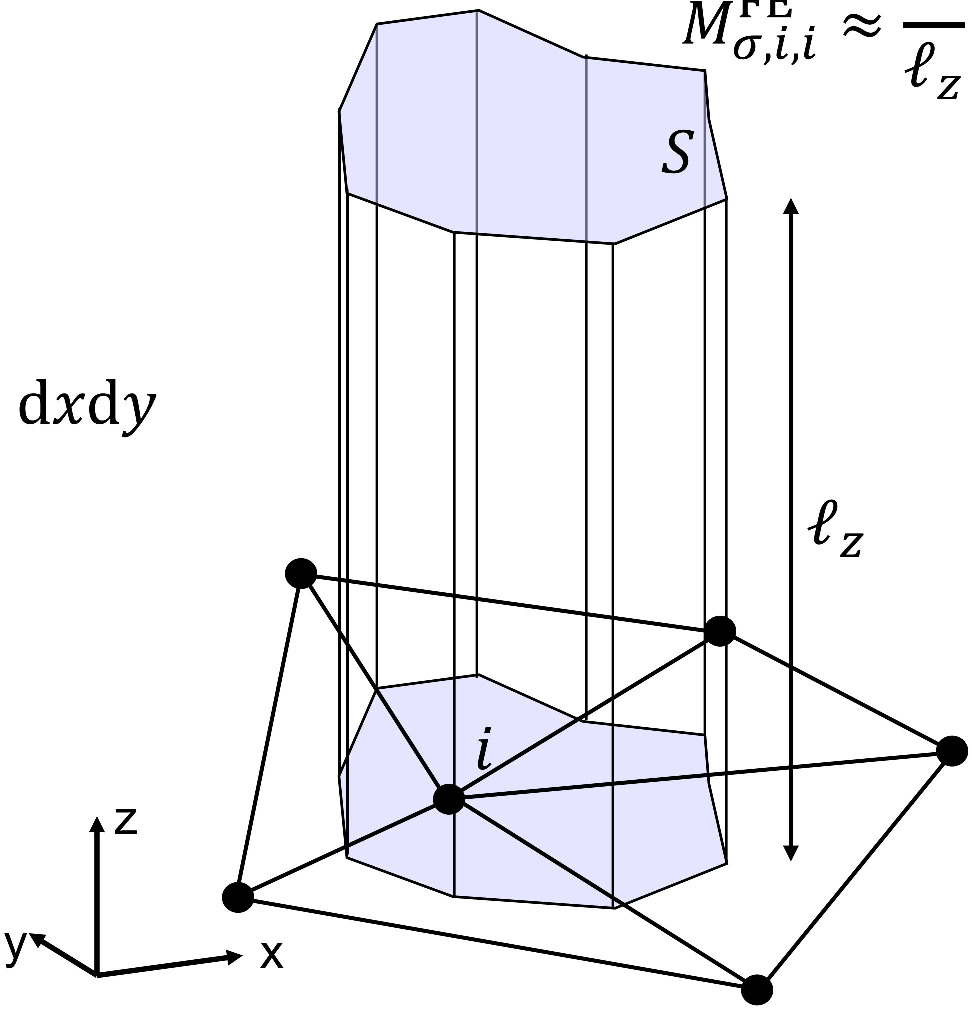
2D Discretisation (4)

$$M_{\sigma,i,j}^{\mathrm{FE}} = \int_V \sigma \vec{w}_j \cdot \vec{w}_i \, \mathrm{d}V'$$

$$M_{\sigma,i,j}^{\mathrm{FE}} = \int_{S_{2\mathrm{D}}} \sigma \frac{N_j}{\ell_z} \vec{e}_z \cdot \frac{N_i}{\ell_z} \vec{e}_z \, \ell_z \, \mathrm{d}x\mathrm{d}y$$

$$M_{\sigma,i,j}^{\mathrm{FE}} = \int_{S_{2\mathrm{D}}} \frac{\sigma}{\ell_z} N_j N_i \, \mathrm{d}x\mathrm{d}y$$

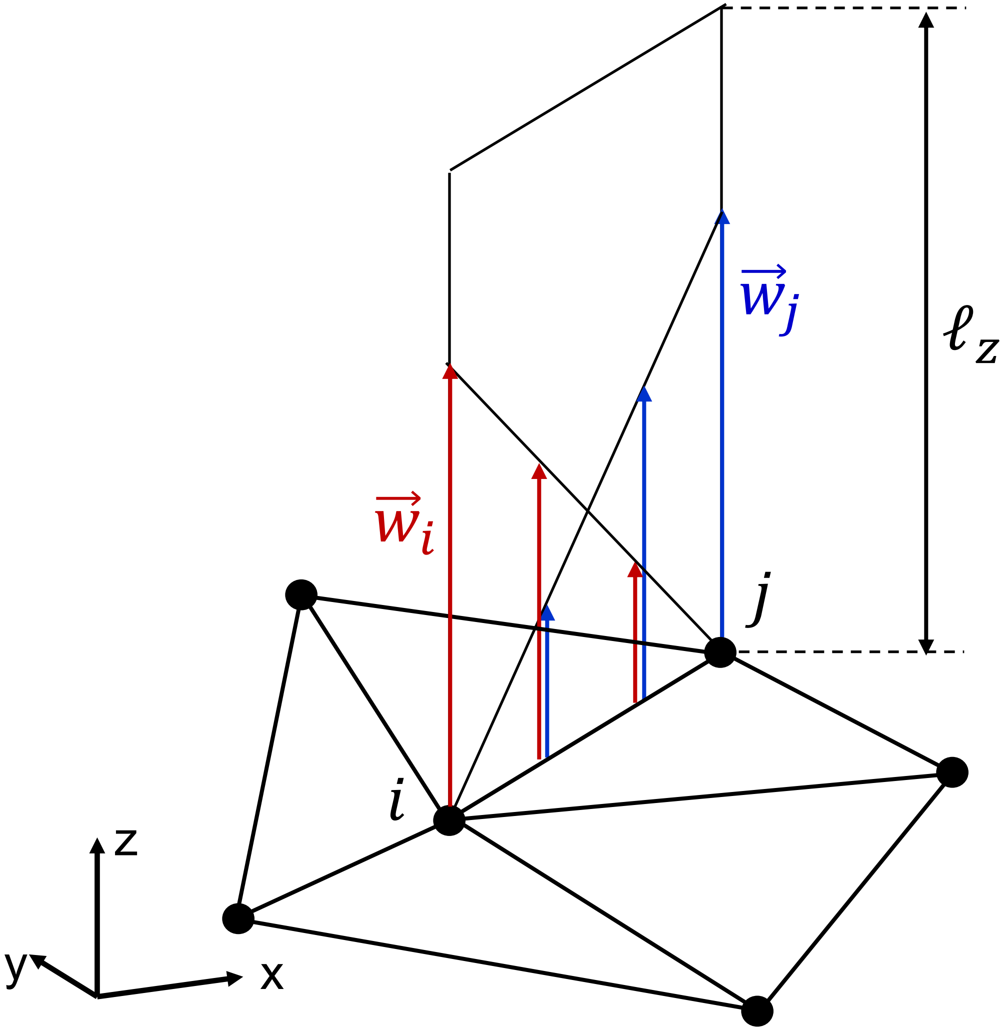
$$M_{\sigma,i,i}^{\mathrm{FE}} \approx \frac{\sigma}{\ell_z} S$$



2D Discretisation (5)

face functions

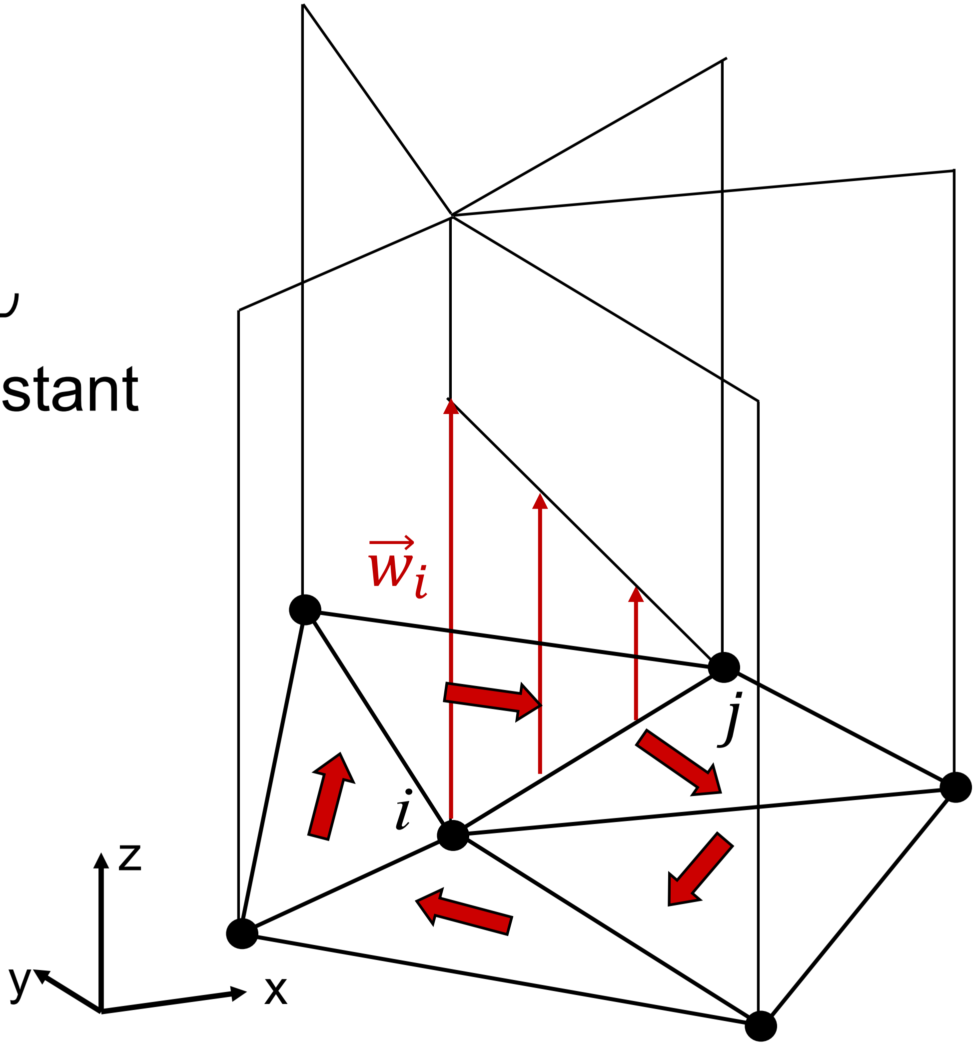
$$\vec{z}_{ij} = N_i \nabla\times\vec{w}_j - N_j \nabla\times\vec{w}_i$$



2D Discretisation (6)

face functions

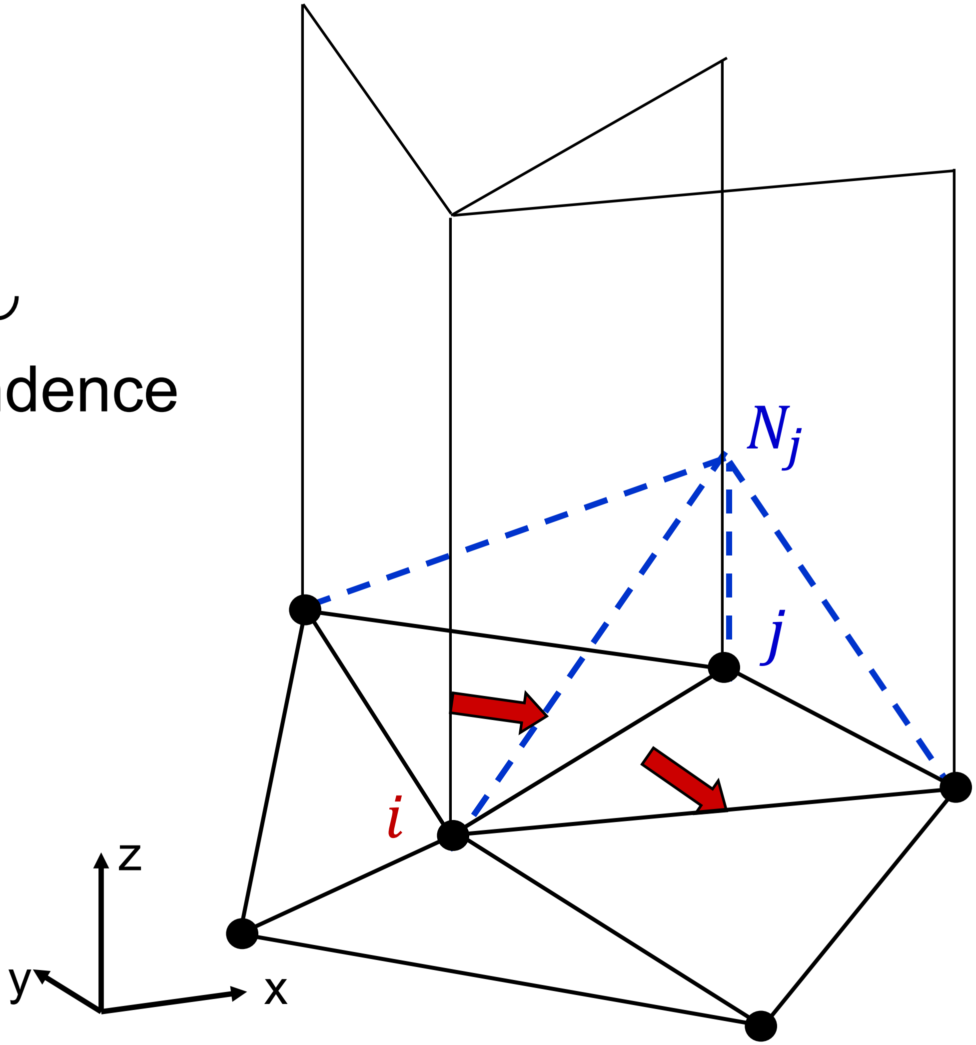
$$\vec{z}_{ij} = N_i \nabla\times\vec{w}_j - N_j \underbrace{\nabla\times\vec{w}_i}_{\text{piecewise constant}}$$



2D Discretisation (7)

face functions

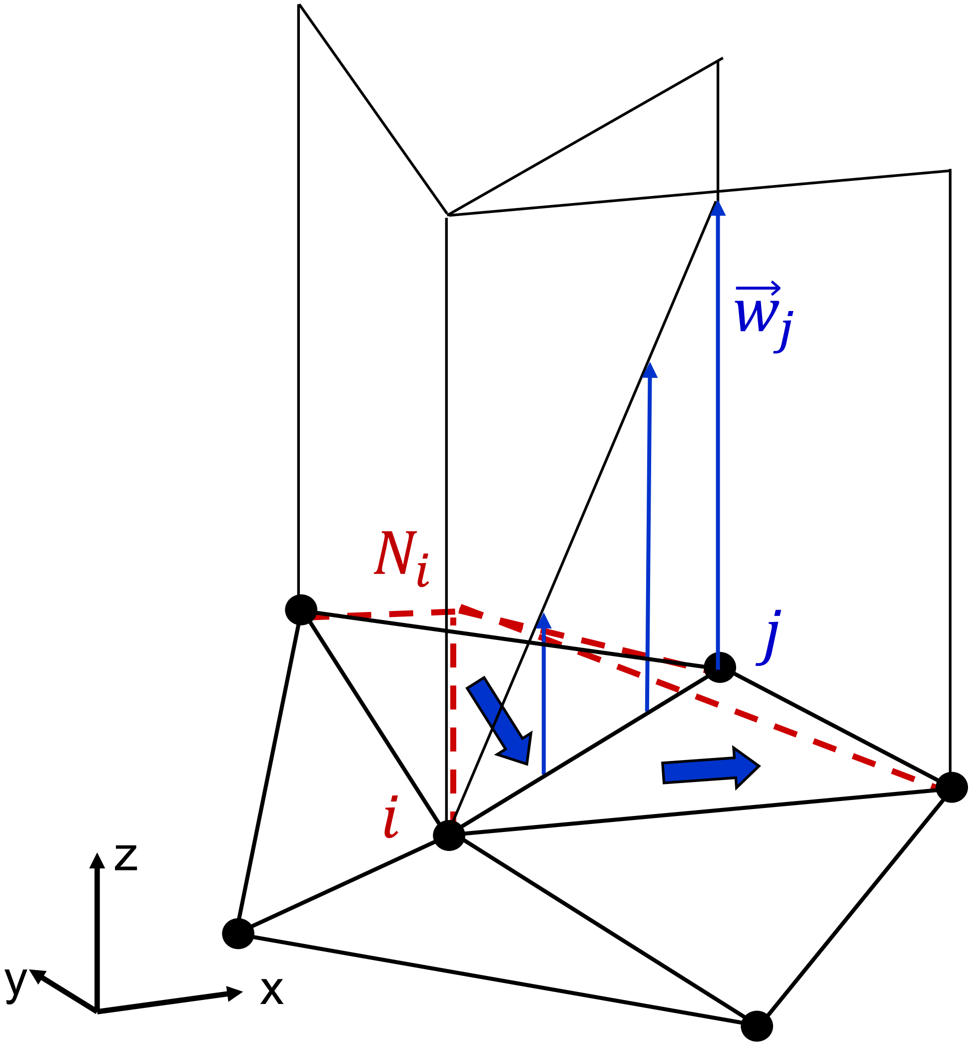
$$\vec{z}_{ij} = N_i \nabla \times \vec{w}_j - \underbrace{N_j \nabla \times \vec{w}_i}_{\text{linear dependence}}$$



2D Discretisation (8)

face functions

$$\vec{z}_{ij} = \underbrace{N_i \nabla\times\vec{w}_j}_{\text{linear dependence}} - N_j \nabla\times\vec{w}_i$$



2D Discretisation (9)

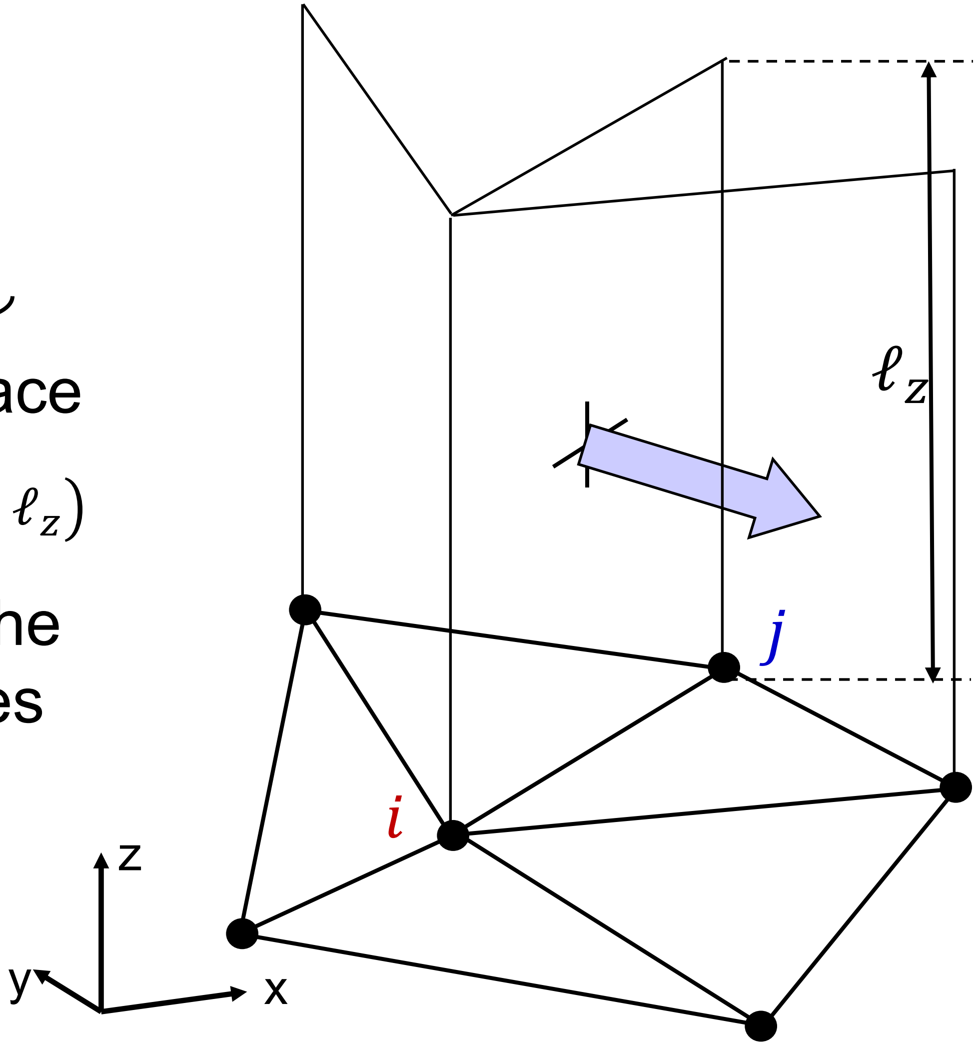
face functions

$$\vec{z}_{ij} = \underbrace{N_i \nabla \times \vec{w}_j - N_j \nabla \times \vec{w}_i}$$

flux through the face

$$(x_i \rightarrow x_j, x_i \rightarrow x_j, 0 \rightarrow \ell_z)$$

decaying to 0 at the neighbouring faces

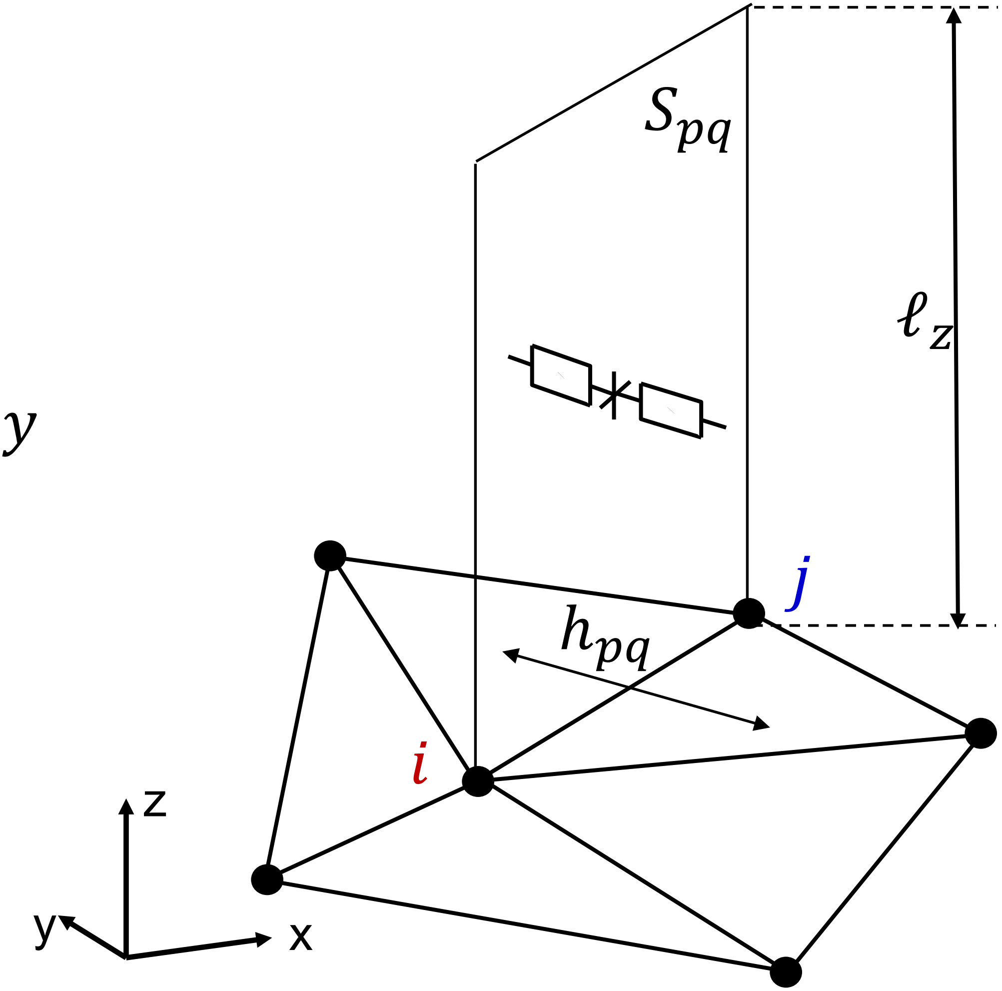


2D Discretisation (10)

$$\boldsymbol{M}^{\mathrm{FE}}_{\nu,p,q} = \int_{V} \nu \vec{z}_q \cdot \vec{z}_p \, \mathrm{d}V'$$

$$\boldsymbol{M}^{\mathrm{FE}}_{\nu,p,q} = \int_{S_{2\mathrm{D}}} \nu \vec{z}_q \cdot \vec{z}_p \, \ell_z \, \mathrm{d}x\mathrm{d}y$$

$$\boldsymbol{M}^{\mathrm{FE}}_{\nu,p,q} \approx \frac{\nu h_{pq}}{S_{pq}}$$



2D Discretisation (11)

Ampère‘s law

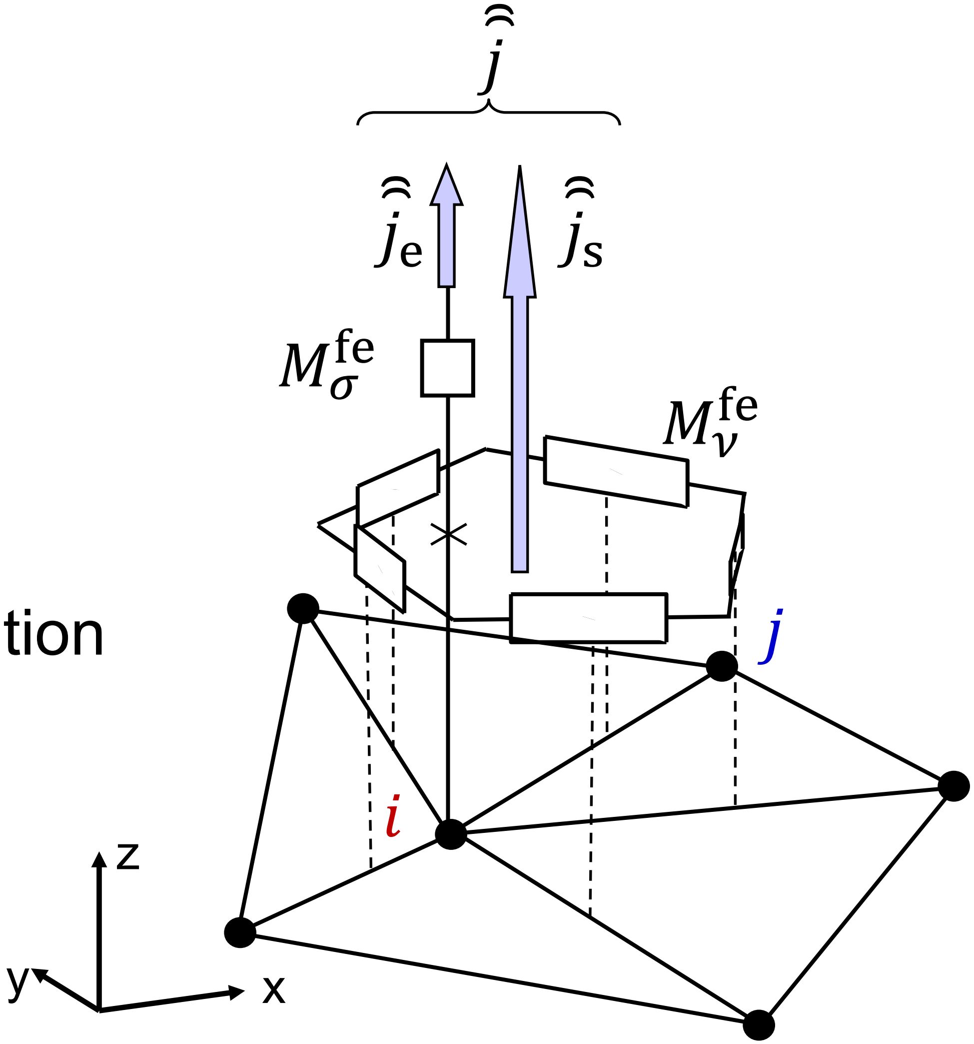
$$\tilde{C} M_{\nu}^{\mathrm{FE}} C \widehat{a} = \widehat{\widehat{j}}$$

Ohm + Faraday-Lenz

$$\widehat{\widehat{j}}_{\mathrm{e}} = -M_{\sigma}^{\mathrm{FE}} \frac{d\widehat{a}}{dt}$$

magnetoquasistatic formulation

$$\tilde{C} M_{\nu}^{\mathrm{fe}} C \widehat{a} + M_{\sigma}^{\mathrm{fe}} \frac{d\widehat{a}}{dt} = \widehat{\widehat{j}}_{\mathrm{s}}$$



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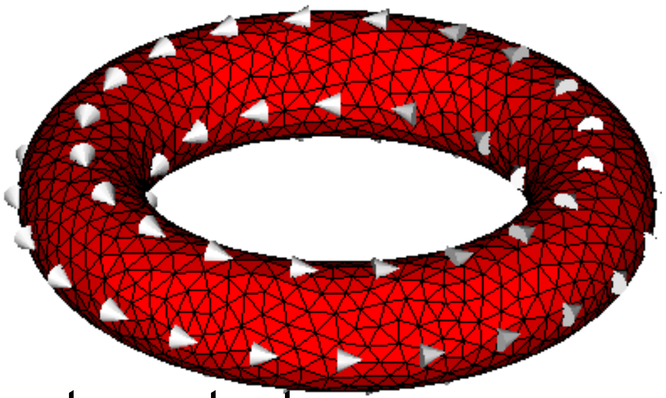
Coil Model (1)

assumptions

- homogeneous current distribution
- no eddy currents

notice (*model*)

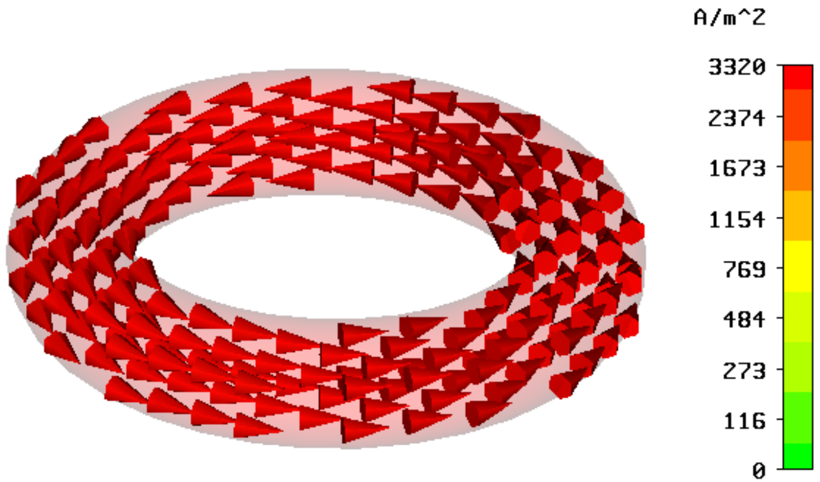
- there will be an induced voltage !!
- current density not ct when cross-section not constant



winding function $\vec{\chi}_{\mathrm{coil},q}(\vec{r})$ [1/m^2]

- computed geometrically
- by field solution

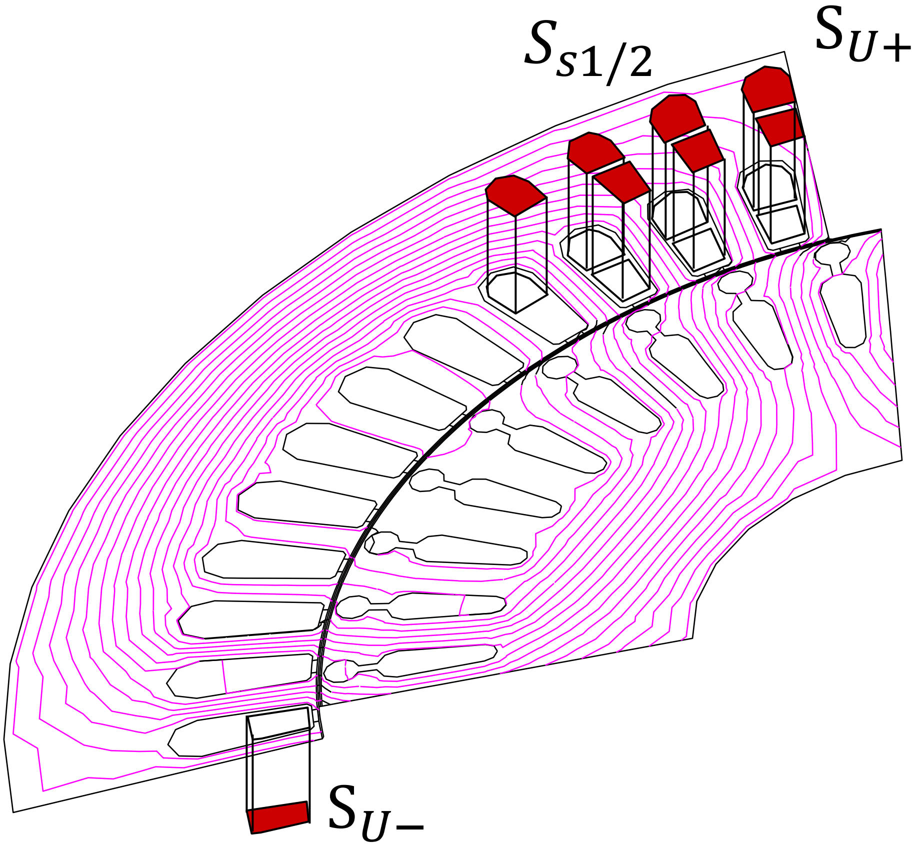
$$\vec{J}_{\mathrm{s}}(\vec{r},t) = \sum_{q=1}^{n_{\mathrm{coil}}} \underbrace{\vec{\chi}_{\mathrm{coil},q}(\vec{r})\, i_q(t)}_{\vec{J}_{\mathrm{coil},q}\,(\vec{r},t)}$$



Coil Model (2)

- in 2D: $\vec{J}(x,y,t) = \big(0,0,J_z(x,y,t)\big)$

$$\begin{cases} \vec{\chi}_{\mathrm{coil},U}(x,y) = +\dfrac{N_{\mathrm{t}}}{S_{s1/2}}\vec{e}_z & \text{in } S_{U+} \\ \vec{\chi}_{\mathrm{coil},U}(x,y) = -\dfrac{N_{\mathrm{t}}}{S_{s1/2}}\vec{e}_z & \text{in } S_{U-} \\ \vec{\chi}_{\mathrm{coil},U}(x,y) = 0 & \text{in } S_{\mathrm{2D}}\backslash S_{U+}\backslash S_{U-} \end{cases}$$



Coil Model (3)

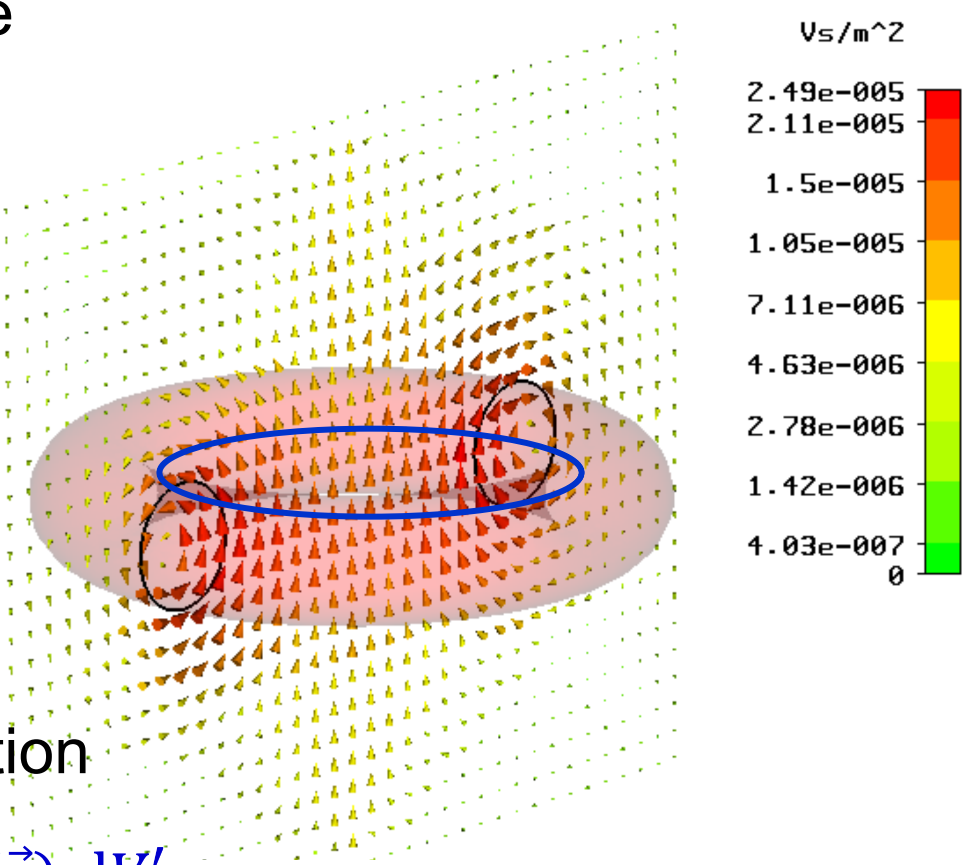
- induced voltage ~ flux linkage
- which flux is linked?

- for a single path

$$\phi = \oint_{\partial S} \vec{A} \cdot \mathrm{d}\vec{s}'$$

- for a coil
- integrating along the coil
- average at the coil cross-section

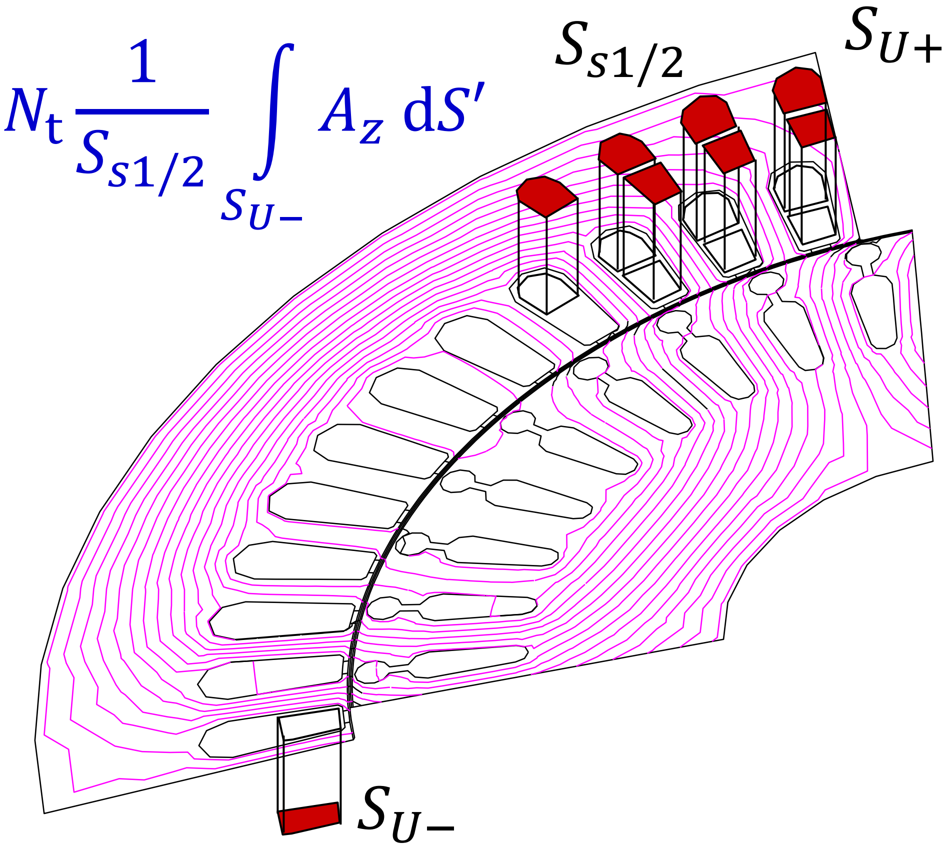
$$\psi_{\mathrm{coil},q}(t) = \int_V \vec{A}(\vec{r},t) \cdot \vec{\chi}_{\mathrm{coil},q}(\vec{r})\, \mathrm{d}V'$$



Coil Model (4)

- in 2D: $\psi_{\mathrm{coil},q} = \int\limits_V \vec{A} \cdot \vec{\chi}_{\mathrm{coil},q}\, dV'$

$$\psi_{\mathrm{coil},U} = N_\mathrm{t} \frac{1}{S_{s1/2}} \int\limits_{S_{U+}} A_z\, \mathrm{d}S' - N_\mathrm{t} \frac{1}{S_{s1/2}} \int\limits_{S_{U-}} A_z\, \mathrm{d}S'$$



Stranded Conductor Model (5)

$$\hat{\hat{j}}_i = \int_V \vec{J} \cdot {\color{red}\vec{v}_i}\, \mathrm{d}V'$$

$$\Big\Downarrow \qquad \vec{J}(\vec{r},t) = \sum_{q=1}^{n_{\mathrm{str}}} \vec{\chi}_{\mathrm{coil},q}(\vec{r})\, i_q(t)$$

$$\hat{\hat{j}}_i = \sum_{q=1}^{n_{\mathrm{coil}}} i_q(t) \underbrace{\int_V \vec{\chi}_{\mathrm{coil},q}(\vec{r}) \cdot {\color{red}\vec{v}_i(\vec{r})}\, \mathrm{d}V'}_{X_{\mathrm{coil},iq}}$$

$$\boxed{\hat{\hat{j}} = X_{\mathrm{coil}}\, i_{\mathrm{coil}}}$$

```
Xcoil = current_Pstr(prb);
```

Stranded Conductor Model (6)

$$u_{\mathrm{coil},q} = R_{\mathrm{coil},q}\, i_q + \frac{\mathrm{d}\psi_{\mathrm{coil},q}}{\mathrm{d}t}$$

$$\psi_{\mathrm{coil},q}(t) = \int_V \vec{\chi}_{\mathrm{coil},q}(\vec{r}) \cdot \vec{A}(\vec{r},t)\,\mathrm{d}V'$$

$$u_{\mathrm{coil},q} = R_{\mathrm{coil},q}\, i_q + \sum_j \frac{\mathrm{d}\hat{a}_{\mathbf{j}}}{\mathrm{d}t} \underbrace{\int_V \vec{\chi}_{\mathrm{coil},q}(\vec{r}) \cdot \vec{v}_j(\vec{r})\,\mathrm{d}V'}_{X_{\mathrm{coil},jq}}$$

$$\boxed{u_{\mathrm{coil}} = R_{\mathrm{coil}}\, i_{\mathrm{coil}} + X_{\mathrm{coil}}^T \frac{\mathrm{d}\hat{a}}{\mathrm{d}t}}$$

```
ucoil = Rcoil*icoil +1i*omega*Xcoil'*a;  % [V] : voltage
```

Stranded Conductor Model (7)

field model
+ stranded conductors
+ voltage sources

$$\begin{bmatrix} K_\nu & -X_{\text{coil}} \\ j\omega X_{\text{coil}}^T & R_{\text{coil}} \end{bmatrix} \begin{bmatrix} \widehat{a} \\ i_{\text{coil}} \end{bmatrix} = \begin{bmatrix} 0 \\ u_{\text{coil}} \end{bmatrix}$$

symmetrisation: multiply the circuit equations by $-\frac{1}{j\omega}$

no eddy-current term !

```
Acpl=[
  Knu             -Xcoil ;
  1i*omega*Xcoil'  Rcoil
  ];                        % coupled system matrix
bcpl=[ 0 ; ucoil ];         % coupled righthandside
```

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4. boundary and symmetry conditions
5. reduction to 2D models
6. modelling of coils and permanent magnets

Permanent Magnet (1)

scalar and linear permanent-magnet model

$$B = B_\mathrm{r} + \mu H$$

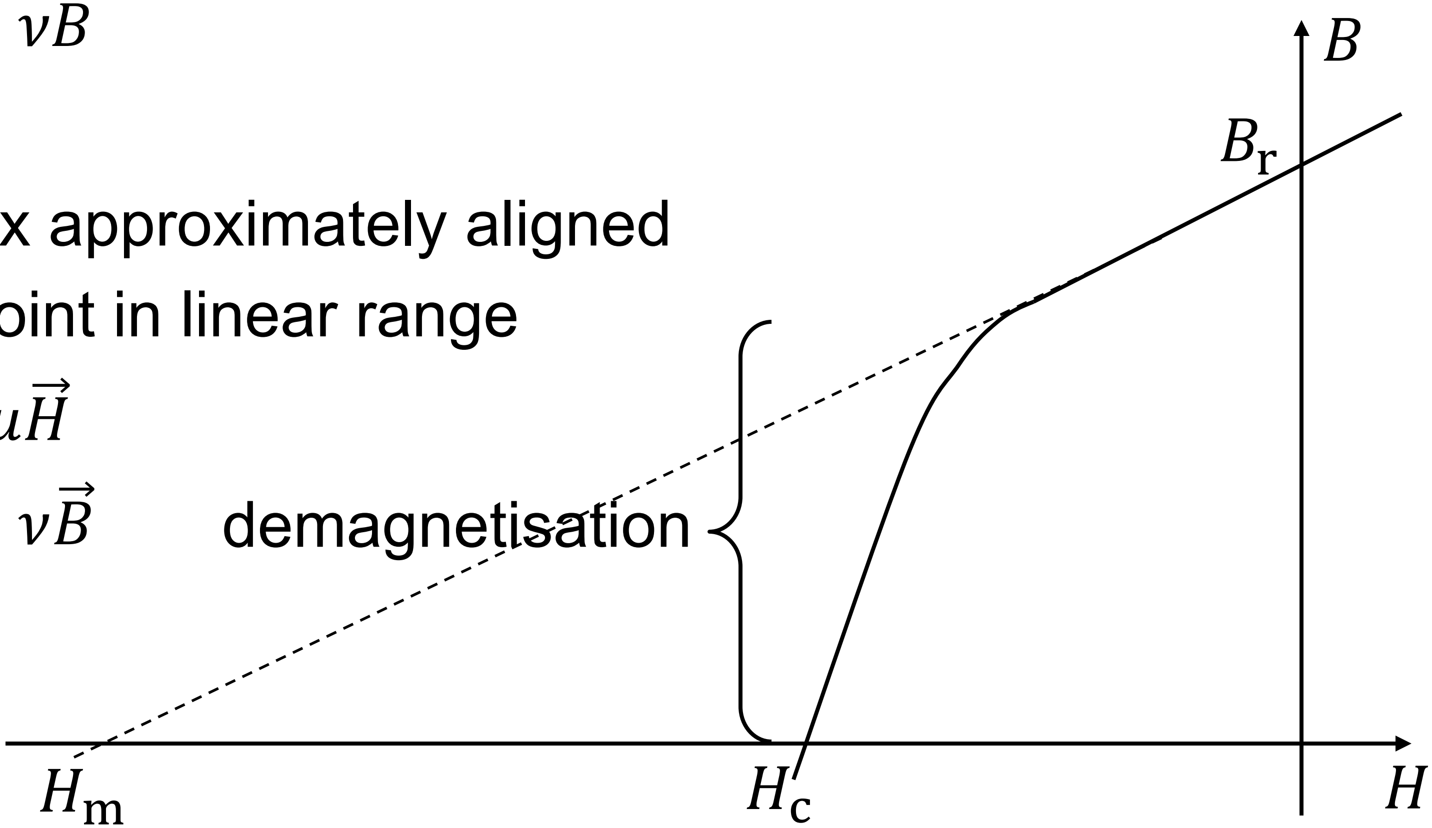
$$H = H_\mathrm{m} + \nu B$$

assumptions

- resulting flux approximately aligned
- operation point in linear range

$$\vec{B} = \vec{B}_\mathrm{r} + \mu \vec{H}$$

$$\vec{H} = \vec{H}_\mathrm{m} + \nu \vec{B}$$



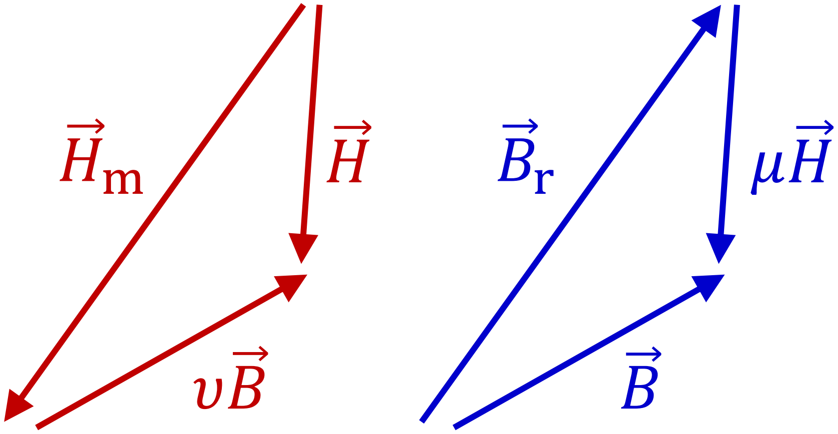
Permanent Magnet (2)

magnetoquasistatic formulation:

$$\vec{H} = \vec{H}_\mathrm{m} + \nu\vec{B}$$

$$\vec{H} = \vec{H}_\mathrm{m} + \nu\nabla\times\vec{A}$$

$$\nabla\times\left(\nu\nabla\times\vec{A}\right) = \vec{J} - \underbrace{\nabla\times\vec{H}_\mathrm{m}}_{\vec{J}_\mathrm{m}}$$



$\vec{J}_\mathrm{m}$ magnetisation current

in 2D:

$$-\frac{\partial}{\partial x}\left(\nu_y\frac{\partial A_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\nu_x\frac{\partial A_z}{\partial y}\right) = J_z - \frac{\partial H_{\mathrm{m},y}}{\partial x} + \frac{\partial H_{\mathrm{m},x}}{\partial y}$$

Permanent Magnet (3)

discretisation:

$$\mathrm{RHS} = \int_V \vec{J}_\mathrm{m} \cdot \vec{w}_i \,\mathrm{d}V'$$

$$\mathrm{RHS} = -\int_V \nabla\times\vec{H}_\mathrm{m} \cdot \vec{w}_i \,\mathrm{d}V'$$

$$\Big\Downarrow \; (\nabla\times\vec{v})\cdot\vec{w} = \nabla\cdot(\vec{v}\times\vec{w}) + \vec{v}\cdot\nabla\times\vec{w}$$

$$\mathrm{RHS} = \int_V \nabla\cdot\left(\vec{w}_i\times\vec{H}_\mathrm{m}\right)\mathrm{d}V' - \int_V \vec{H}_\mathrm{m}\cdot\nabla\times\vec{w}_i \,\mathrm{d}V'$$

$$\Big\Downarrow \quad \text{Gauss}$$

$$\mathrm{RHS} = \underbrace{\oint_{\partial V} \vec{w}_i\times\vec{H}_\mathrm{m}\cdot\mathrm{d}\vec{S}'}_{} - \int_V \vec{H}_\mathrm{m}\cdot\nabla\times\vec{w}_i \,\mathrm{d}V'$$

$= 0$ when no permanent magnets at the model boundary

Permanent Magnet (4)

discretisation (in 2D):

$$\mathrm{RHS} = -\int_V \vec{H}_\mathrm{m} \cdot \nabla \times \vec{w}_i \, \mathrm{d}V'$$

$$\Downarrow \quad \vec{w}_i = \frac{N_i(x,y)}{\ell_z} \vec{e}_z$$

$$\mathrm{RHS} = \int_V \frac{1}{\ell_z} \left(\frac{\partial N_i}{\partial y} H_{\mathrm{m}x} - \frac{\partial N_i}{\partial x} H_{\mathrm{m}y} \right) \mathrm{d}V'$$

$$\mathrm{RHS} = \int_{\emptyset V} \left(\frac{\partial N_i}{\partial y} H_{\mathrm{m}x} - \frac{\partial N_i}{\partial x} H_{\mathrm{m}y} \right) \mathrm{d}S'$$

Overview

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