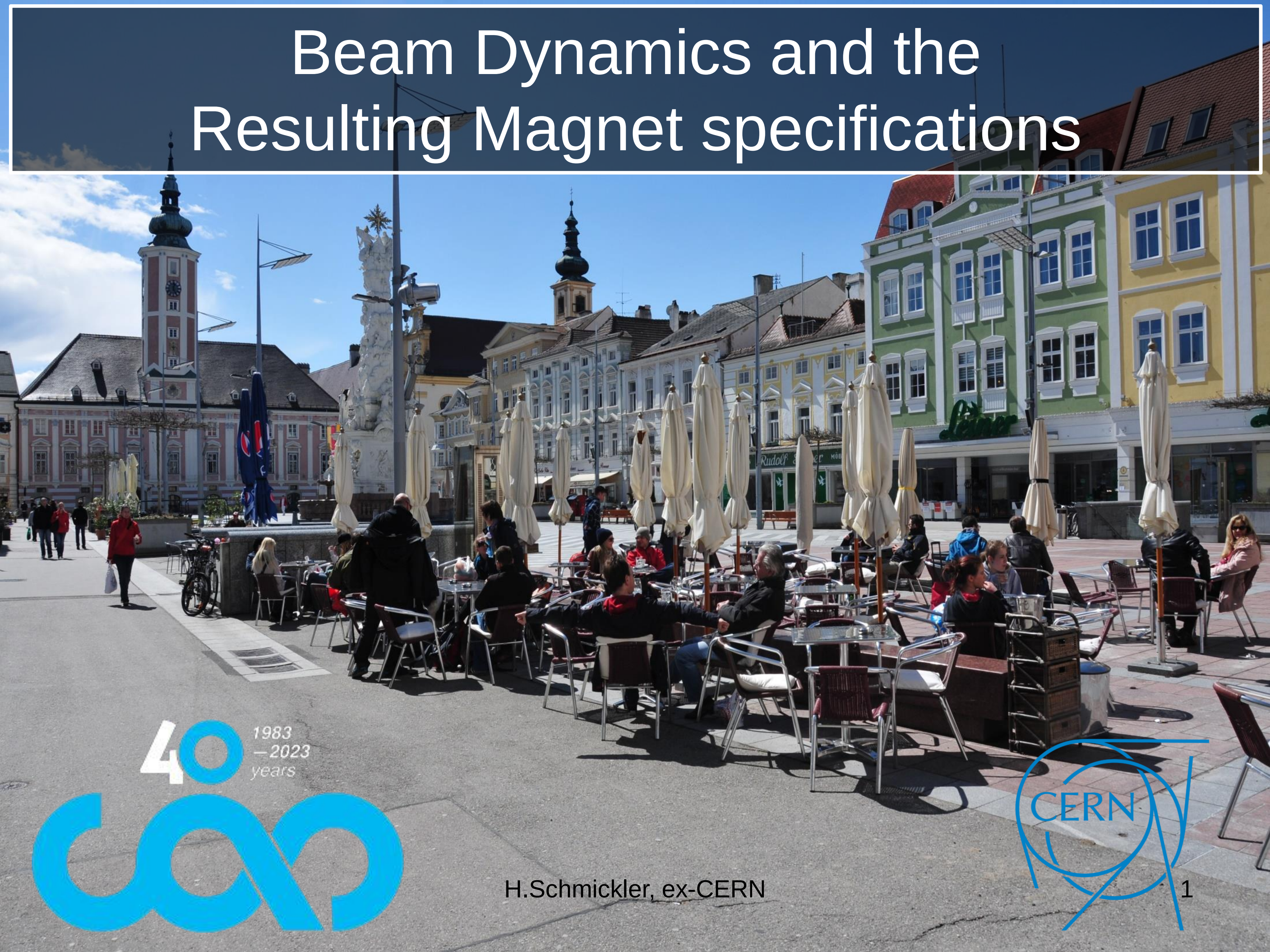
Beam Dynamics and the Resulting Magnet specifications



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Content

Basics (only 5 minutes):

- Phenomenology of Special relativity
- Lorentz Force

The classic pendulum described in three different formalisms:

- Differential equations
- Matrix formalism
- Hamiltonian formalism

1st hour

Main Dish: Primer on linear beam optics

Main application of magnets in accelerators:

- **dipoles**: - bending, orbit/trajectory corrections
  - spectrometer, separation of positive and negatively charged particles
- **quadrupoles:**
  - transverse focusing (FODO)
  - high luminosity insertions in colliders
- **sextupoles**:
  - correction of momentum dependence of quadrupole magnet strength
- **solenoids:**
  - magnetic field in HE physics detectors
- **wigglers, undulators**: light generation

2nd hour

1: Relativistic particles

Conservation of transverse momentum
→ A moving object in its frame S' has a mass m' = $^{m}/_{\gamma}$

Or $m = \gamma m_0 = \frac{m_0}{\sqrt{1-(\frac{v}{c})^2}} \cong m_0 + \frac{1}{2} m_0 v^2 (\frac{1}{c^2})$ (approximation for small v)

Multiplied by c^2:

$$mc^2 \cong m_0 c^2 + \frac{1}{2} m_0 v^2 = m_0 c^2 + T$$

Interpretation:
→ Total energy $E\ is \quad E = m \cdot c^2$
→ For small velocities the total energy is the sum of the kinetic energy plus the rest energy
→ Particle at rest has rest energy $E_0 = m_0 \cdot c^2$

→ **Always true (Einstein):** $\boldsymbol{E = m \cdot c^2 = \gamma m_0 \cdot c^2}$

Relativistic momentum $p = mv = \gamma m_0 v = \gamma m_0 \beta c$

From page before (squared):

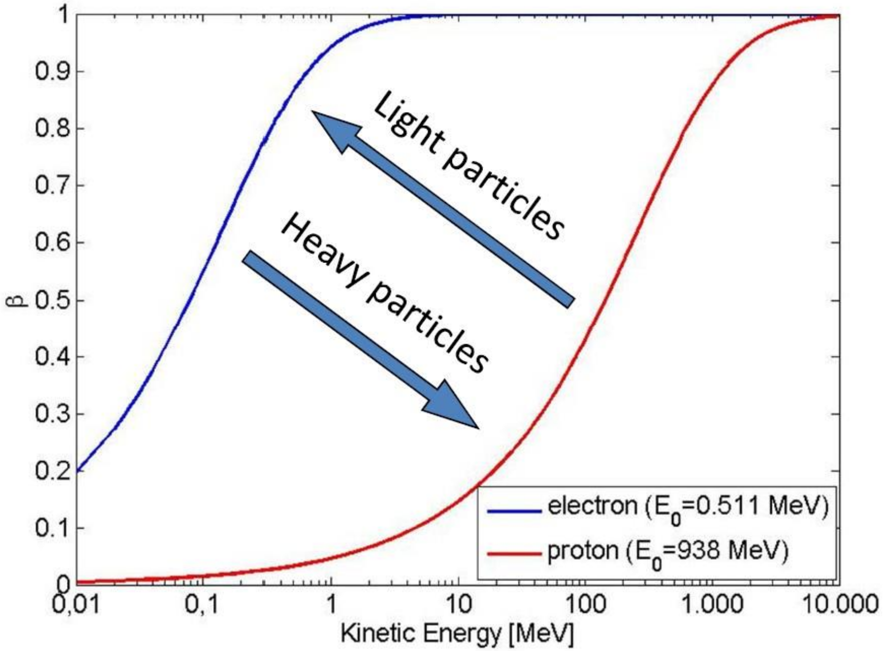
$$E^2 = m^2 c^4 = \gamma^2 {m_0}^2 c^4 = \left(\frac{1}{1-\beta^2}\right) {m_0}^2 c^4 = \left(\frac{1-\beta^2+\beta^2}{1-\beta^2}\right) {m_0}^2 c^4 = (1+\gamma^2\beta^2) {m_0}^2 c^4$$

$$E^2 = (m_0 c^2)^2 + (pc)^2 \Rightarrow \frac{E}{c} = \sqrt{(m_0 c)^2 + p^2}$$

Or by introducing new units [E] = eV ; [p] =eV/c ; [m] = eV/c^2

$$E^2 = {m_0}^2 + p^2$$

Due to the small rest mass electrons reach already almost the speed of light with relatively low kinetic energy, but protons only



Electromagnetic Fields and forces onto charged particles

- Described by Maxwell's equations and by the Lorentz-force
- Lots of mathematics, we will only "look" at the equations
- Only electric fields can transfer momentum to charged particles
  → EM cavities for acceleration
- Magnetic fields are used to bend or focus the trajectory of charged particles
  → construction of different types of accelerator magnets

- Also electrostatic forces can bend and focus beams; but since the forces are small we neglect this part in most cases

Integral form	Differential form
$\int_S \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$	$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
$\int_S \vec{B} \cdot d\vec{A} = 0$	$\nabla \cdot \vec{B} = 0$
$\oint_\Gamma \vec{E} \cdot d\vec{l} = -\frac{d\Phi(\vec{B})}{dt}$	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
$\oint_\Gamma \vec{B} \cdot d\vec{l} = \mu_0 \left(I + \epsilon_0 \frac{\partial}{\partial t} \int_S \vec{E} \cdot d\vec{A} \right)$	$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{j} + \epsilon_0 \frac{\partial}{\partial t} \vec{E} \right)$

Lorentz force $\vec{F} = q * (\vec{E} + \vec{v} \times \vec{B})$ (with $\vec{E}$ crossed out)

typical velocity in high energy machines: $v \approx c \approx 3*10^8 \, m/s$

Example:

$B = 1T \quad \rightarrow \quad F = q*3*10^8 \frac{m}{s} * 1 \frac{Vs}{m^2}$

$F = q*300 \frac{MV}{m}$

equivalent el. field E

technical limit for el. field

$E \leq 1 \frac{MV}{m}$

But: for specific cases we also use electrostatic elements

quadrupole

Separators for electron and positron beams in the same vacuum chamber

Different Mathematical descriptions for Particle motion in an accelerator…a real pain?

We use differential equations, matrix – formalism, Hamiltonians, perturbation theory…

- Is there a right or wrong?

- Is it personal likings?

→ Depending on the problem to solve (or the phenomenon to describe) one mathematical tool is more adequate than the other.

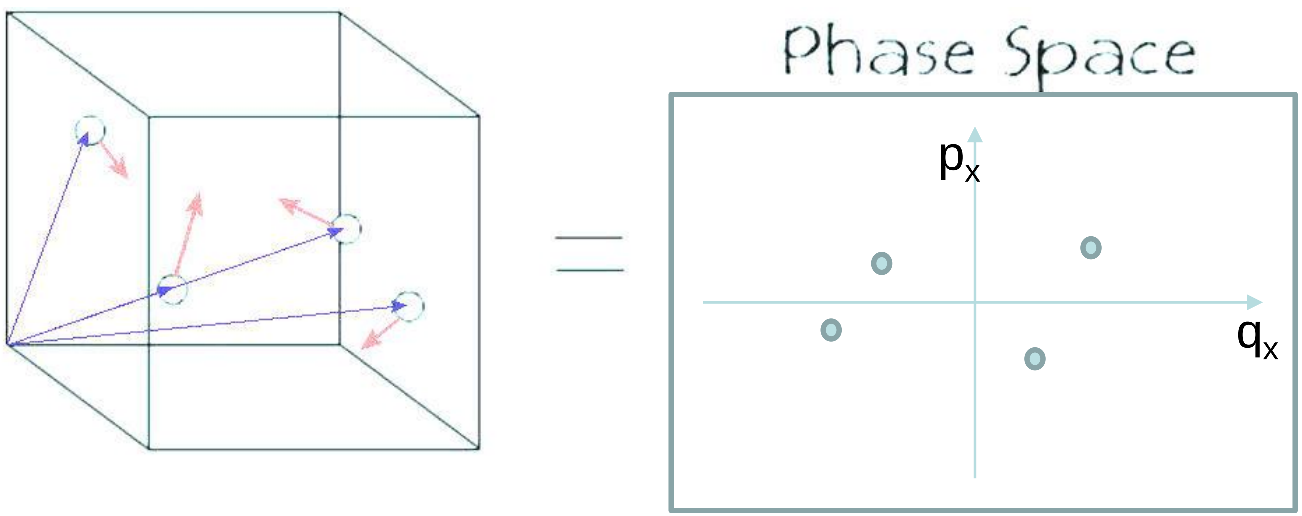
→ One should be aware of many of them in order to be able to choose the most adequate one.

In the following slides we will look at the very simple example of the classical spring-oscillator and describe it with a differential equation, with a matrix formalism and by using the Hamiltonian equations of motion.

But first another important concept: **Definition of phase space**

Phase Space

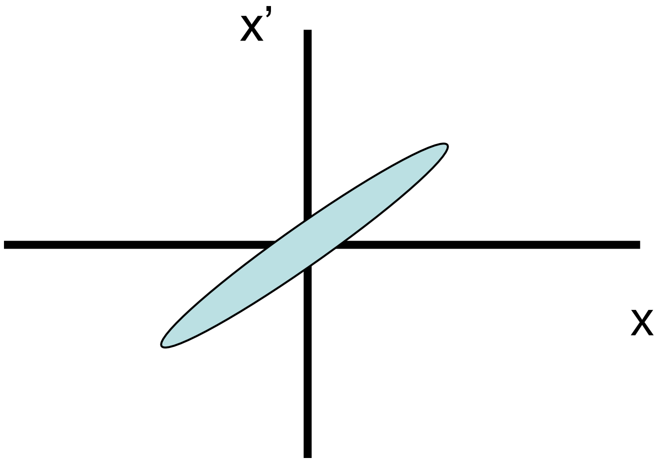
- We are used to describe a particle by its 3D position (x,y,z in carthesian Coordinates) (blue arrows below)
- In order to get the dynamics of the system, we need to know the momentum (px, py, pz); red arrows below
- In accelerators we describe a particle state as a 6D phase space point.
  Below the projection into a 2 D phase space plot.
  The points correspond to the x-position (q_x) and the x component of the p-vector (p_x).



This shows only one of the three possible phase space projections

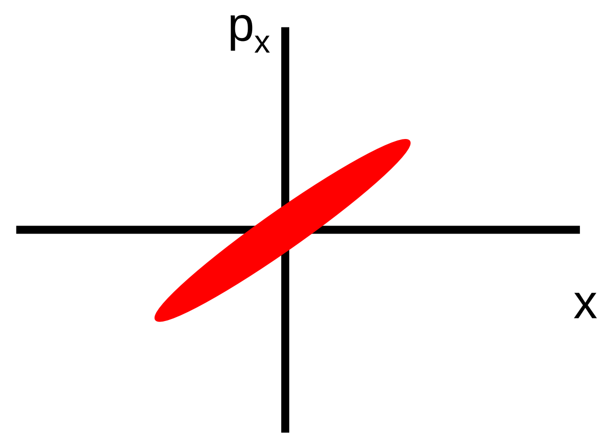
Warning: We often use the term phase space for the 6N dimensional space defined by x, x’ (space, angle), but this the “trace space” of the particles.
At constant energy phase space and trace space have similar physical interpretation

Trace space



$$x' = \frac{dx}{ds} = \frac{dx}{dt} \cdot \frac{dt}{ds} = \frac{\beta_x}{\beta_s}$$

Phase space



$$p_x = m_o \, c \, \gamma_{rel} \, \beta_x$$

An important argument to use the trace space is that in praxis we can measure angles of particle trajectories, but it is very difficult to measure the momentum of a particle.

Action functional S

Define action as S:= $\int_{t_1}^{t_2} p\, dq$

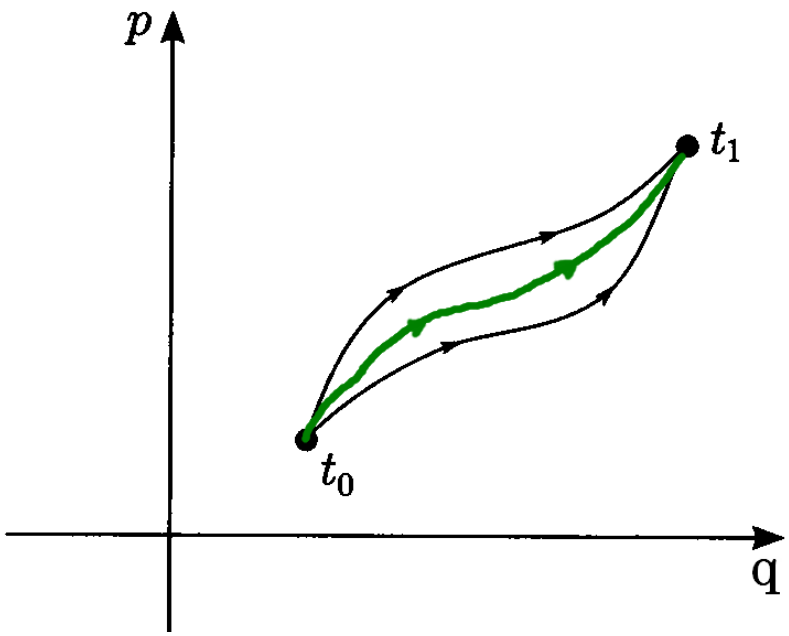
p: Generalized momentum; q: generalized space coordinate

No immediate physical interpretation of S

Much more important:

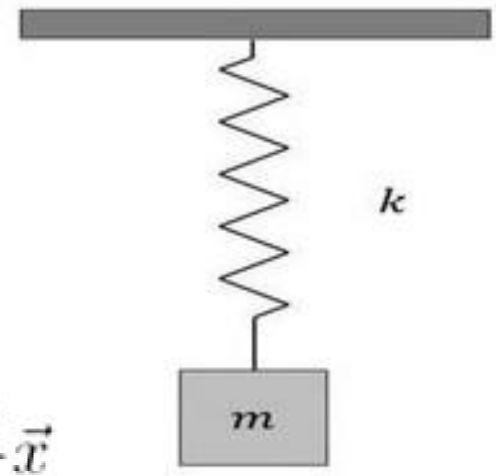
"Stationary" action principle:=
Nature chooses path from t_1 to t_2 such that the action integral is a minimum and stationary

→ we have a new invariant, which we can use to study the dynamics of the system



Harmonic oscillator (1/4)

Solved by using a **Differential equation**



Starting from:
Newton's Kraftansatz (F = m * a) and Hook's law (F = - k * x)

$$\vec{F} = m \cdot \vec{a} = -k \cdot \vec{x} \quad \text{or} \quad \ddot{\vec{x}} = \frac{k}{m}\vec{x}$$

As at school we "guess" the solution:

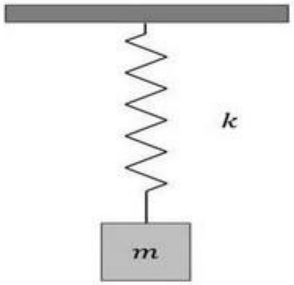
$$x(t) = A_0 \cdot \cos \omega t$$

And we find that with the angular frequency ω
we have found a description of the motion of our system.

$$\omega = \sqrt{\frac{k}{m}}$$

Harmonic oscillator (2/4)

Solved by using a **matrix formalism**



The general solution to the previous differential equation is a linear combination of a cosinus- and a sinus-term.
So after an additional differentiation we get:

$$x(t) = A_c \cdot \cos\omega t + A_s \cdot \sin\omega t$$
$$\dot{x}(t) = -\omega A_c \cdot \sin\omega t + \omega A_s \cdot \cos\omega t$$

Furthermore we have to introduce initial conditions x(0) = x_0 and $\dot{x}(0) = \dot{x}_0$ and the classical momentum $p = m \cdot \dot{x}$; ($p_0 = m \cdot \dot{x}_0$) which then yields:

$$x(t) = A_c \cdot \cos\omega t + A_s \cdot \sin\omega t$$
$$p(t) = -m\omega A_c \cdot \sin\omega t + p_0 \cdot \cos\omega t$$

By comparing coefficients we get $A_c = x_0$ and $A_s = p_0/m\omega$, which finally produces:

$$x(t) = x_0 \cdot \cos\omega t + \frac{p_0}{m\omega} \cdot \sin\omega t$$
$$p(t) = -m\omega x_0 \cdot \sin\omega t + p_0 \cdot \cos\omega t$$

or in matrix annotation:

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} \cos\omega t & \frac{1}{m\omega}\sin\omega t \\ -m\omega\sin\omega t & \cos\omega t \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ p_0 \end{pmatrix}$$

So we can stepwise develop our solution from a starting point x_0, p_0

Harmonic oscillator (3/4)

A little reminder of classical mechanics:

- Take a set of "canonical conjugate variables"
  (generalized coordinate q, momentum p in a single one dimensional case)
- Construct a function H, which satisfies the dynamical equations of the system:

$$\frac{\partial q}{\partial t} = \dot{q} = \frac{\partial H}{\partial p} \quad \text{and} \quad \frac{\partial p}{\partial t} = \dot{p} = -\frac{\partial H}{\partial q}$$

- H "= the Hamiltonian " of the system is a constant of motion
  (= H does not explicitly depend on t) .
- The Hamiltonian of a system is the total energy of the system: H = T +V
  (sum of potential and kinetic energy)

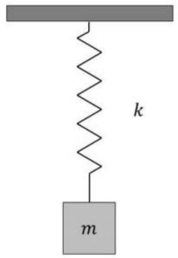
Proof:

$$\begin{aligned} \dot{H} &= \sum_{i=1}^{n} \frac{\partial H}{\partial x_i}\dot{x}_i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\dot{p}_i \\ &= \sum_{i=1}^{n} \frac{\partial H}{\partial x_i}\frac{\partial H}{\partial p_i} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\left(-\frac{\partial H}{\partial x_i}\right) = 0. \end{aligned}$$

Used x instead of q just to test your attention

Harmonic oscillator (4/4)

Back to our Example: Mass-spring system



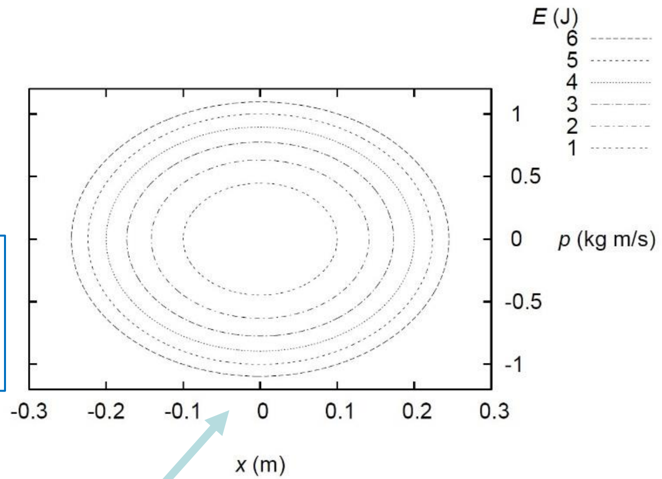
$Start\ with: H = T + V = \frac{1}{2}$ k $x^2 + \frac{p^2}{2m}$ = E

→ Hamiltonian formalism to obtain the equations of motion:

$\frac{\delta x}{\delta t} = \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$ or p=m$\dot{x}$ = mv

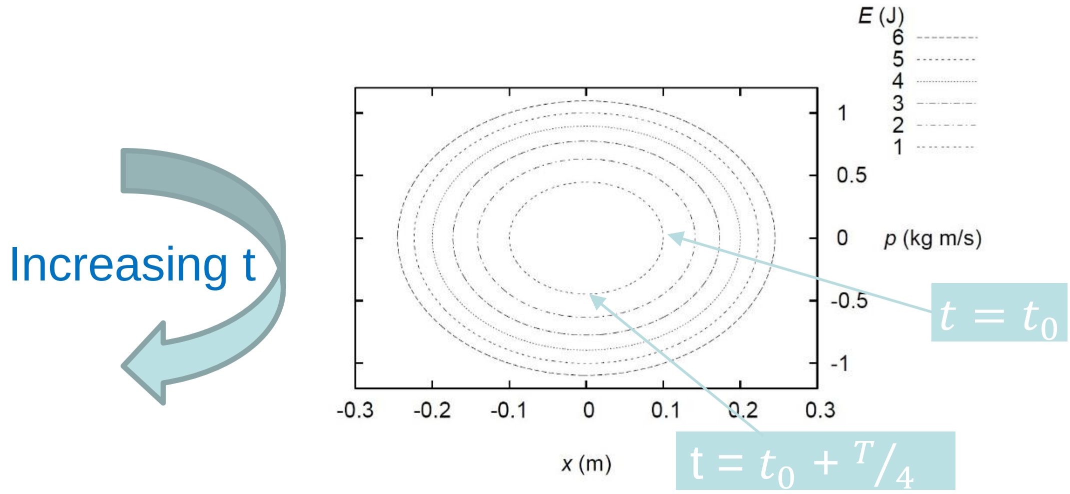
$\frac{\delta p}{\delta t} = \dot{p} = -\frac{\partial H}{\partial x}$ = -kx

This brings us back to the differential equation of solution 1:
$F = ma = m\ddot{x}$ = - kx
With the well known "guessed" sinusoidal solution for x(t).



Instead of guessing a solution for x(t) we look at the trajectory of the system in phase space.
In this simple case the Hamiltonian itself is the equation of an ellipse in phase space.

Outlook on Hamiltonian treatments



- In the example, the free parameter along the trajectory is time (we are used to express the space-coordinate and momentum as a function of time)
- This is fine for a linear one-dimensional pendulum, but it is not an adequate description for transverse particle motion in an accelerator.
  → we will choose soon "s", the path length along the particle trajectory as free parameter
- Any linear motion of the particle between two points in phase space can be written as a matrix transformation: $\binom{x}{x'}(s) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \binom{x}{x'}(s_0)$
- In matrix annotation we define an action "J" as product J:= $\frac{1}{2}\binom{x}{x'}(s)\ \binom{x}{x'}(s_0)$.
- J is a motion invariant and describes also an ellipse in phase space. The area of the ellipse is $2\pi J$

We get already a deep understanding of the motion by looking at phase space diagrams!

Hamiltonians of some machine elements (3D)

In general for multipole n:

$$H_n = \frac{1}{1+n} \mathcal{R}e\left[(k_n + \mathrm{i}k_n^{(s)})(x+\mathrm{i}y)^{n+1}\right] + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

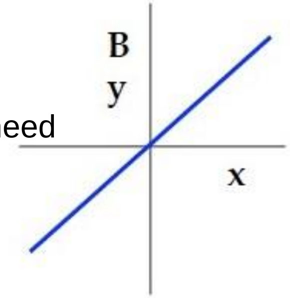
We get for some important types (normal components k_n only):

dipole: $$H = -\frac{-x\delta}{\rho} + \frac{x^2}{2\rho^2} + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

quadrupole: $$H = \frac{1}{2}k_1(x^2 - y^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

sextupole: $$H = \frac{1}{3}k_2(x^3 - 3xy^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

⇨ Such a field (force) we need for focusing



Weak focusing from dipoles

dipole: $$H = -\frac{-x\delta}{\rho} + \frac{x^2}{2\rho^2} + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

quadrupole: $$H = \frac{1}{2}k_1(x^2 - y^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

This means that we can construct a focusing circular accelerator based only on dipoles…in particular when ρ is small.

This has been done in the 1950's and it was called " a weak focusing synchrotron" (vacuum chamber: about 2m wide)

How about the vertical plane? There is no vertical dipole field.

Why do the particles not fall down? Discuss over a beer ☺

Now: how to describe particle motion in an accelerator?

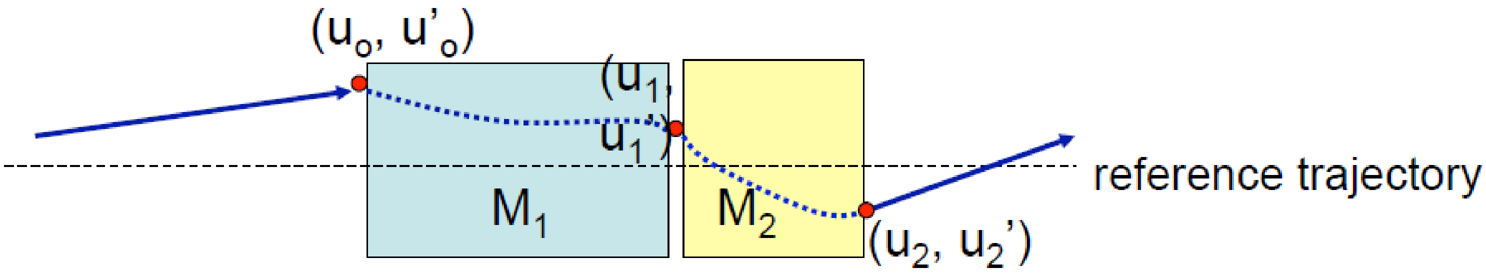
1. <u>Differential equations</u>? - yes, but:
   longitudinal plane: acceleration AND focusing done by a sinusoidal field in an RF cavity → need linearization for small amplitudes around working point

   transverse plane: some people do (“Hill’s equation”). There is only a solution for fully symmetric accelerator designs…but no real accelerator is fully symmetric.
2. <u>Hamiltonian approach</u>: Yes, but too involved for this course.
3. <u>Matrix approach</u>: That is what we will do!
   In short: “Find” a matrix describing the motion of a particle in each element of the accelerator → find the transport matrix through the whole chain by the Multiplication of all matrix elements → ideal for computer simulations! And the result we display as phase(trace)-space trajectory.

Of course the matrix approach works only for linear forces!
A more general approach is to use Maps instead of Matrices
(in our case needed for sextupoles)

Piecewise Constant Transport: Two Elements

The matrix representation is very convenient. For instance, what if we had two consecutive elements, with strengths K_1 and K_2? What is the final equation of transport for a particle through both elements?



The solution for the first element becomes the initial condition for the second element…

First, $\begin{pmatrix} u_1 \\ u_1' \end{pmatrix} = M_1 \begin{pmatrix} u_o \\ u_0' \end{pmatrix}$ then, $\begin{pmatrix} u_2 \\ u_2' \end{pmatrix} = M_2 \begin{pmatrix} u_1 \\ u_1' \end{pmatrix} = M_2 \left(M_1 \begin{pmatrix} u_o \\ u_0' \end{pmatrix} \right) = M_2 M_1 \begin{pmatrix} u_o \\ u_0' \end{pmatrix}$

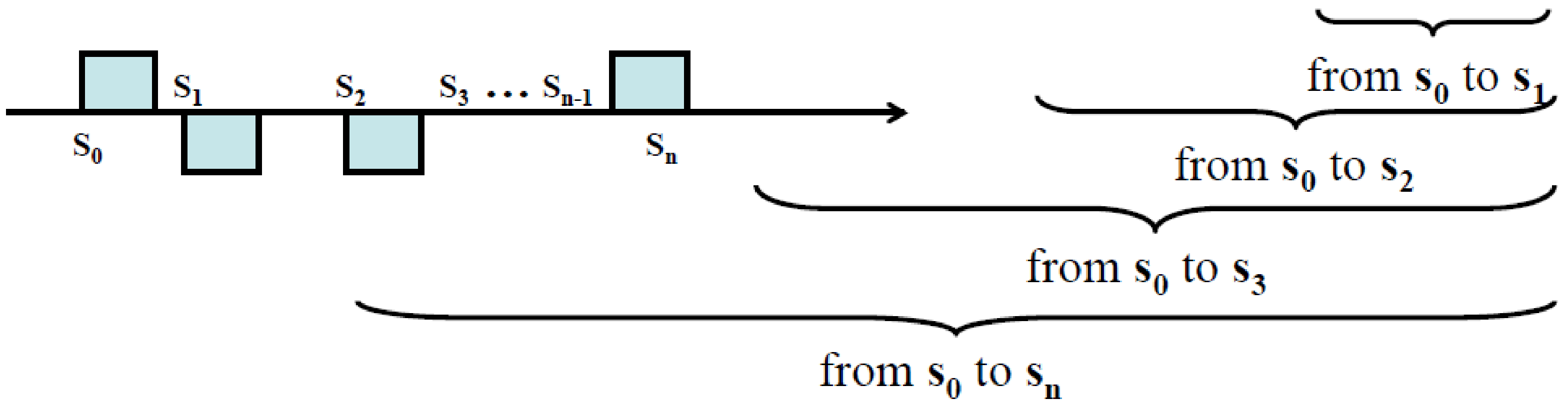
And finally, we have

$$\begin{pmatrix} u_2 \\ u_2' \end{pmatrix} = M(s_2|s_0) \begin{pmatrix} u_0 \\ u_0' \end{pmatrix}, \text{ where } M(s_2|s_1) = M_2 M_1$$

Piecewise Constant Transport: n Elements

For an arbitrary number of transport elements, each with a constant, but different, K_n, we have:

$$\mathcal{M}(s_n|s_0) = \mathcal{M}(s_n|s_{n-1})\ldots\mathcal{M}(s_3|s_2)\cdot\mathcal{M}(s_2|s_1)\cdot\mathcal{M}(s_1|s0)$$



$$\Rightarrow \begin{pmatrix} u_n \\ u_n' \end{pmatrix} = M(s_n|s_o)\begin{pmatrix} u_o \\ u_o' \end{pmatrix}$$

Thus by breaking up the parameter K(s) into piecewise constant chunks, $K(s)=\{K_1, K_2, \ldots K_n\}$, we have found a useful method for finding the particle transport equation through a long section of beamline with many elements.

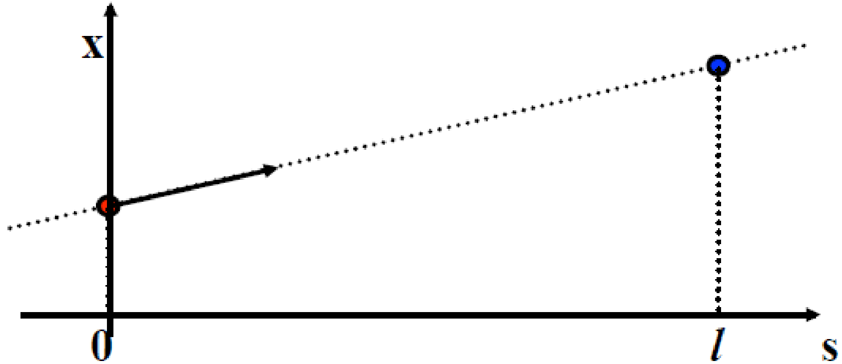
Transport Through a Drift

In a drift space, there is no change in the momentum of the particle. We take the limit of M as K-> 0.

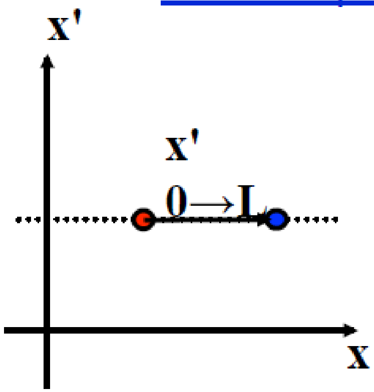
$$M_{drift} = \begin{pmatrix} \cos(\sqrt{K}l) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}l) \\ -\sqrt{K}\sin(\sqrt{K}l) & \cos(\sqrt{K}l) \end{pmatrix} \xrightarrow{K=0} M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_o \\ u_0' \end{pmatrix} \qquad \begin{aligned} u &= u_o + l u_0' \\ u' &= u_0' \end{aligned}$$

Real space (s,x)



Phase space (x, x')

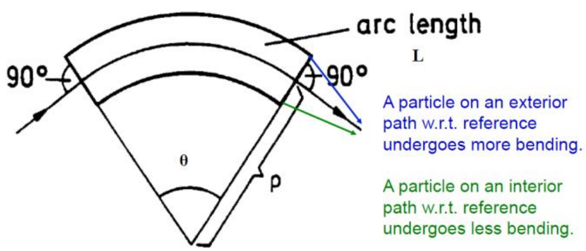


Transport in Pure Dipole Sector Magnet

In a pure sector dipole, we take the quad strength k, to be zero, k=0. In the deflecting plane, i.e, the plane of the bend (usually horizontal), we have:

$$M_{\mathrm{x,sector}} = \begin{pmatrix} \cos(\theta) & \rho_o \sin(\theta) \\ -\kappa_o \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$\theta = \kappa_o l, \quad \kappa_o = \frac{1}{\rho_o}$$

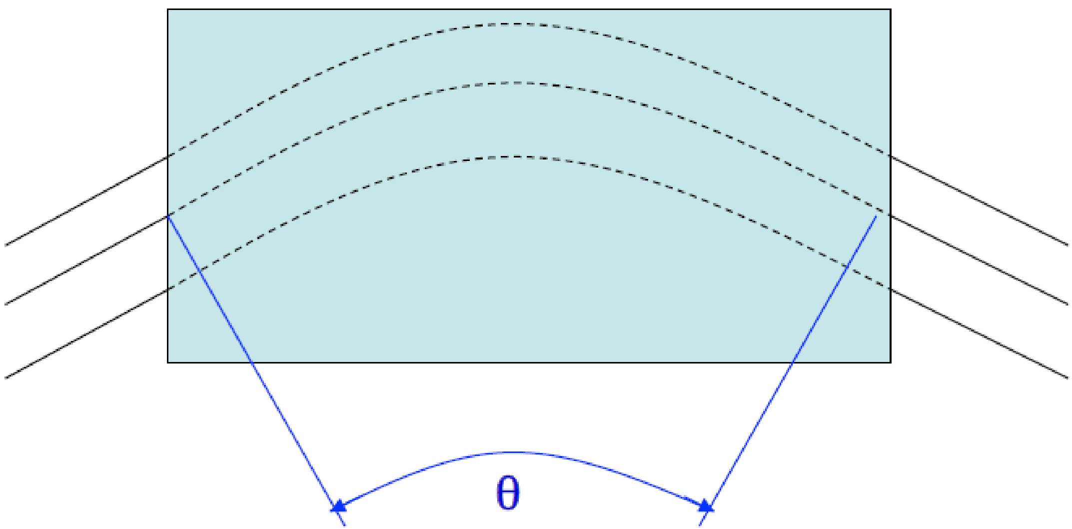


And in the non-deflecting plane, $\rho \rightarrow 0$, and we are left with a drift:

$$M_{\mathrm{y,sector}} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

Transport in Rectangular Dipoles

In a rectangular dipole, the particle path in the horizontal direction is the same for all trajectories, so there is no focusing in the horizontal direction.



$$M_{x,rect} = \begin{pmatrix} 1 & \rho \sin\theta \\ 0 & 1 \end{pmatrix}$$

In the horizontal direction the magnet transforms like a drift with length equal to the path length $\rho \sin\theta$.

Transport Through a Quadrupole

In the case of a quadrupole, there is no bending, so the only remaining term is the quad strength term.

$$K = K_n + \frac{1}{\rho^2} \xrightarrow{\rho=\infty} K_n$$

Focusing:

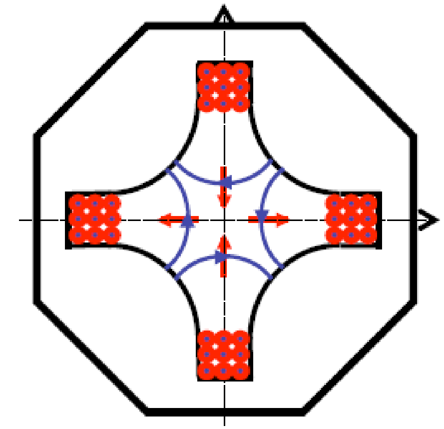
$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K_n}\, l) & \frac{1}{\sqrt{K_n}} \sin(\sqrt{K_n}\, l) \\ -\sqrt{K_n} \sin(\sqrt{K_n}\, l) & \cos(\sqrt{K_n}\, l) \end{pmatrix}$$

Defocusing:

$$M_{QD} = \begin{pmatrix} \cosh(\sqrt{|K_n|}\, l) & \frac{1}{\sqrt{|K_n|}} \sinh(\sqrt{|K_n|}\, l) \\ \sqrt{|K_n|} \sinh(\sqrt{|K_n|}\, l) & \cosh(\sqrt{|K_n|}\, l) \end{pmatrix}$$

Finite Length Quad Transport.

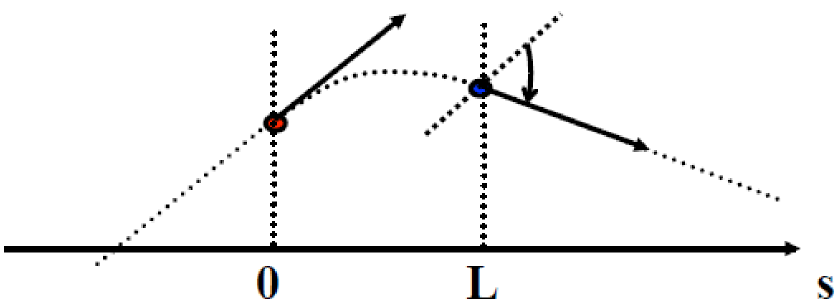
Now consider again the quadrupole with finite length, L. The angle is changed through the length, and the position as well. For instance, for K>0:



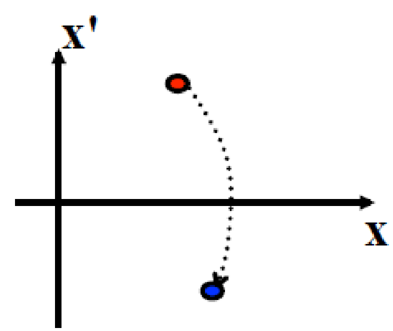
$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{K_n}\, l) & \frac{1}{\sqrt{K_n}} \sin(\sqrt{K_n}\, l) \\ -\sqrt{K_n} \sin(\sqrt{K_n}\, l) & \cos(\sqrt{K_n}\, l) \end{pmatrix} \begin{pmatrix} x_o \\ x_0' \end{pmatrix}$$

(**Examples**)

Real space (s, x):



Phase space (x,x'):



Thin Lens Approximation for a Quadrupole

In the "thin lens approximation", we let the length of the quadrupole approach zero while holding the focal length constant: $L \rightarrow 0$ as $1/f = KL = \text{constant}$.

(**Derivation/Example**)

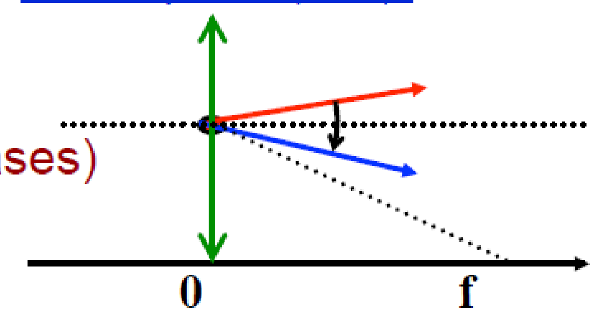
$$M_{Quad} = \begin{pmatrix} 1 & 0 \\ \mp \frac{1}{f} & 1 \end{pmatrix}$$

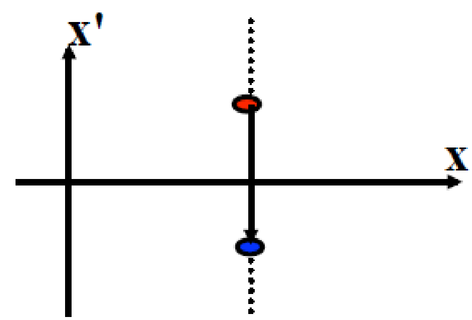
In this approximation, the position remains fixed, but the momentum changes:

Real space (s, x):

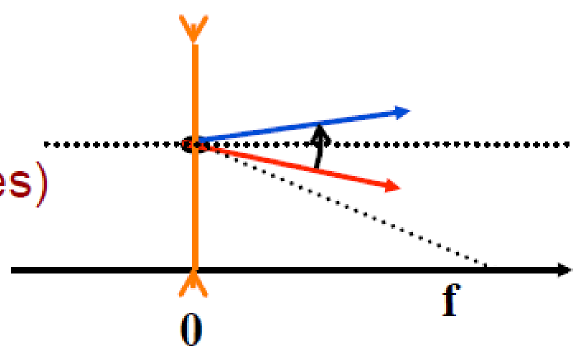
Phase space (x, x'):

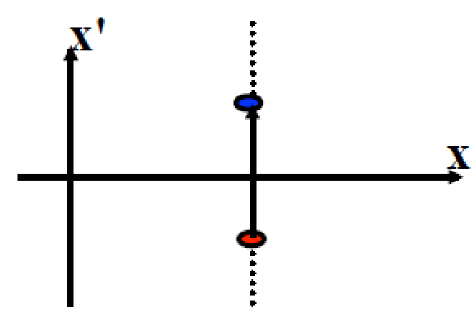
Focusing:
(slope decreases)



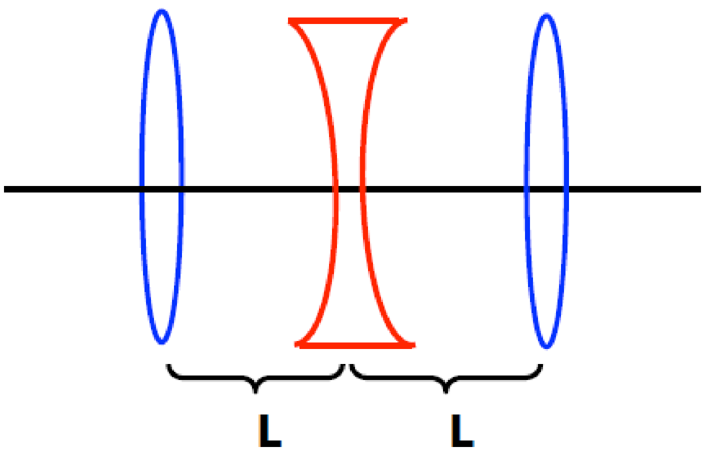


Defocusing:
(slope increases)





Example: FODO Channel



- Consider a defocusing quadrupole "sandwiched" by two focusing quadrupoles with focal lengths **f**.
- Note: Wiedemann's f is for half quad. I use f for full quad: $f=f_W/2$.
- The symmetric transfer matrix is taken from center to center of focusing quads (thus one full focusing quad and one full defocusing quad)

This arrangement is very common in beam transport lines.

$$M_{\text{FODO}} = M_{\text{Half QF}} M_{\text{Drift}} M_{\text{QD}} M_{\text{Drift}} M_{\text{Half QF}}$$

$$M_{\text{FODO}} = \begin{pmatrix} 1 & 0 \\ \frac{-1}{2f_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-1}{2f_1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-2\frac{L}{f^*} & 2L(1-\frac{L}{2f_2}) \\ -\frac{2}{f^*}(1-\frac{L}{2f_1}) & 1-2\frac{L}{f^*} \end{pmatrix}$$

$$\frac{1}{f^*} = \frac{1}{2f_1} + \frac{1}{2f_2} - \frac{L}{4f_1 f_2}$$

Stability criterion: $0 < L/2f^* < 1$ …focuses in BOTH planes

Sarah Cousineau, Jeff Holmes, Yan Zhang USPAS 2011

Transfer-Matrix/Map of

- Solenoids
- combined function magnets

are out of scope of this lecture

- Sextupoles
(quadratic dependence of force from centre)
first non-linear element in our lecture → transfer map M

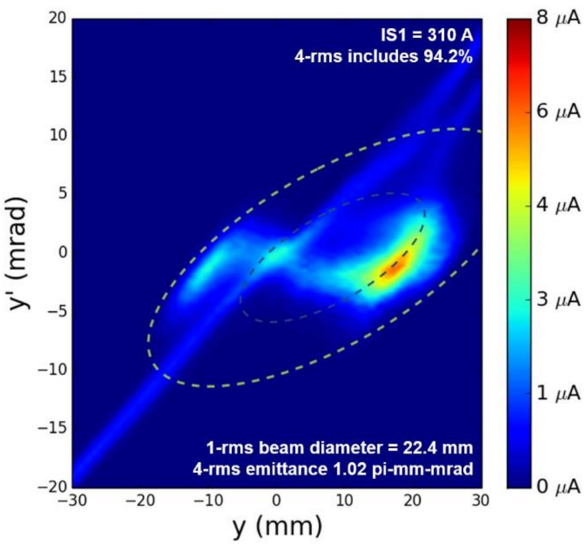
$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

$$\begin{aligned} x &\mapsto x, \\ p_x &\mapsto p_x - \frac{1}{2} k_2 L x^2 \end{aligned}$$

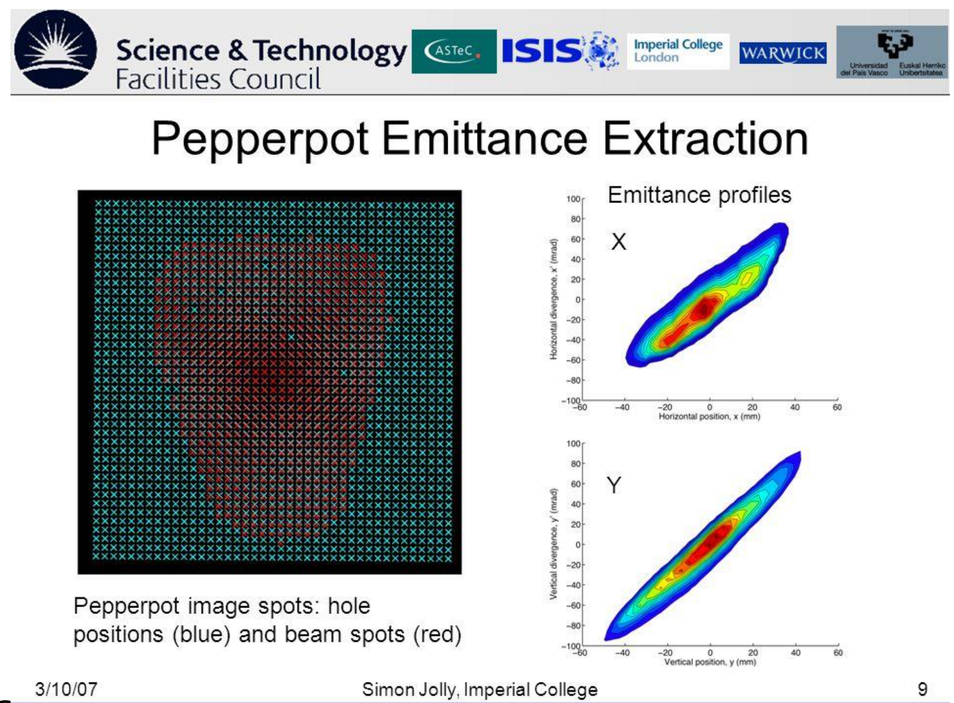
the impact on the phase space trajectories we shall see later.

So far: Motion of ONE particle → Now a whole BEAM

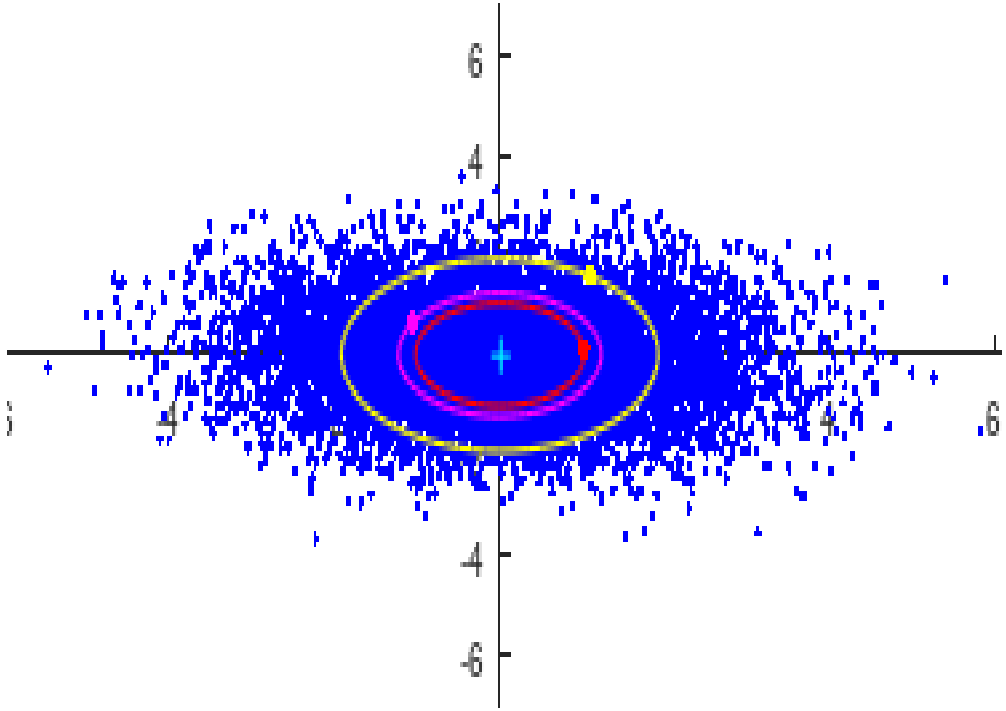
We focus on "bunched" beams, i.e. many (10^{11}) particles bunched together longitudinally.
From the generation of the beams the particles have transversally a spread in their original position and momentum.
Nevertheless for some studies the beam can be treated as one Macro-Particle!



Source: ISODAR (Isotope at rest experiment)

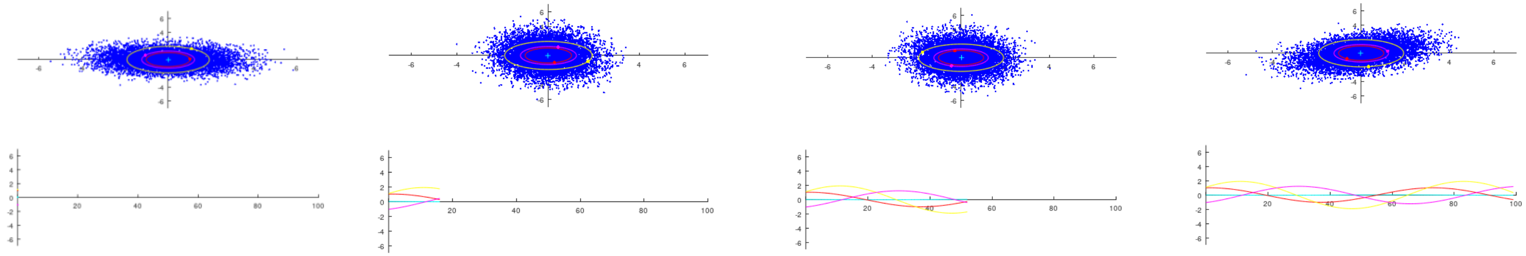


A beam (bunch): Motion of individual particles (1/4)

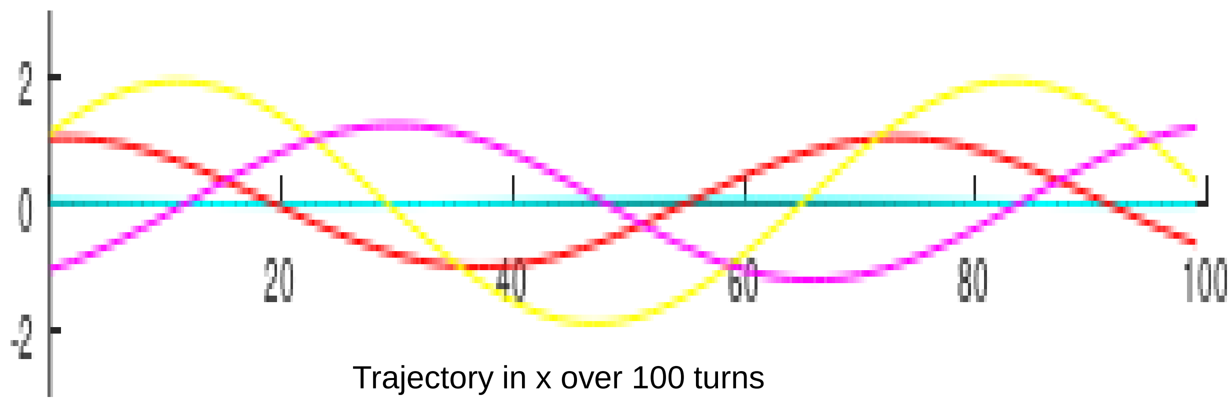


- Generate 10000 particle as a Gaussian distribution in x and p_x
- For illustration mark 3 particle in colours red, magenta and yellow
- The average (centre of charge) is indicated as cyan cross
- Make some turns (100 turns with 3 degrees phase advance par turn)

A beam (bunch): Motion of individual particles (2/4)



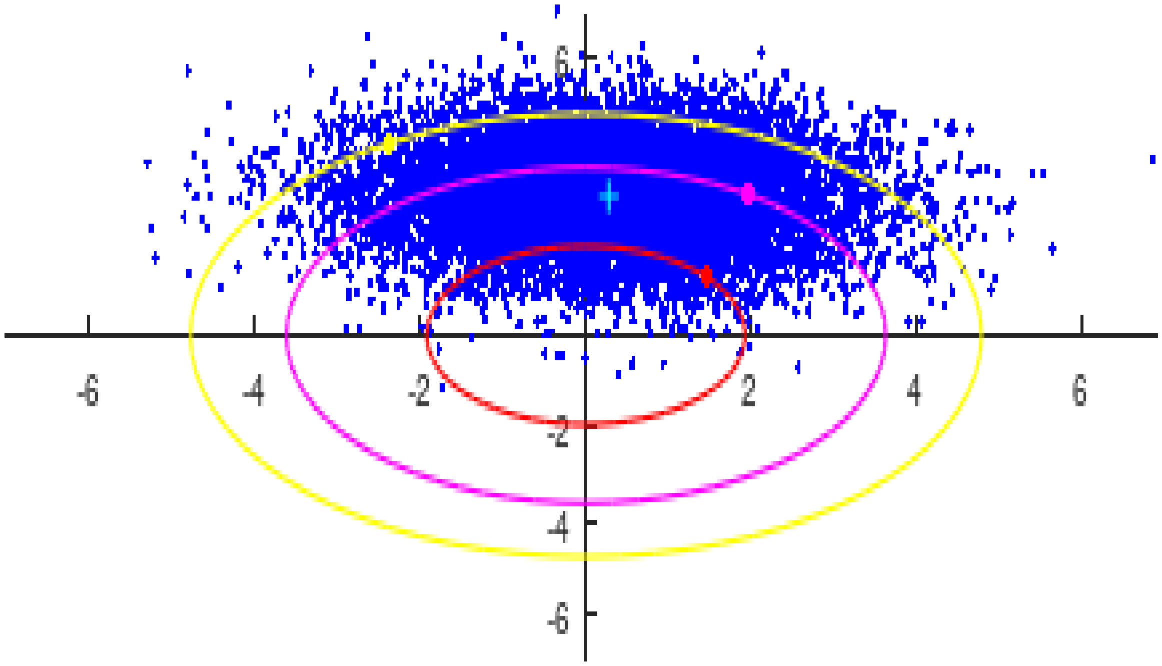
turn 0 turn 10 turn 53 turn 100



Trajectory in x over 100 turns

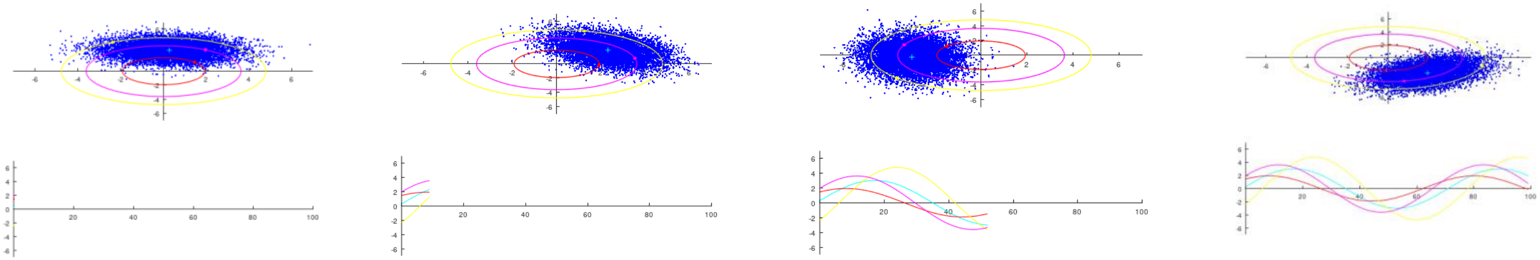
Individual particles perform betatron oscillations (incoherently!), the whole beam is “quiet”, it propagates without a coherent transverse motion.

A beam (bunch): Motion of individual particles (3/4)

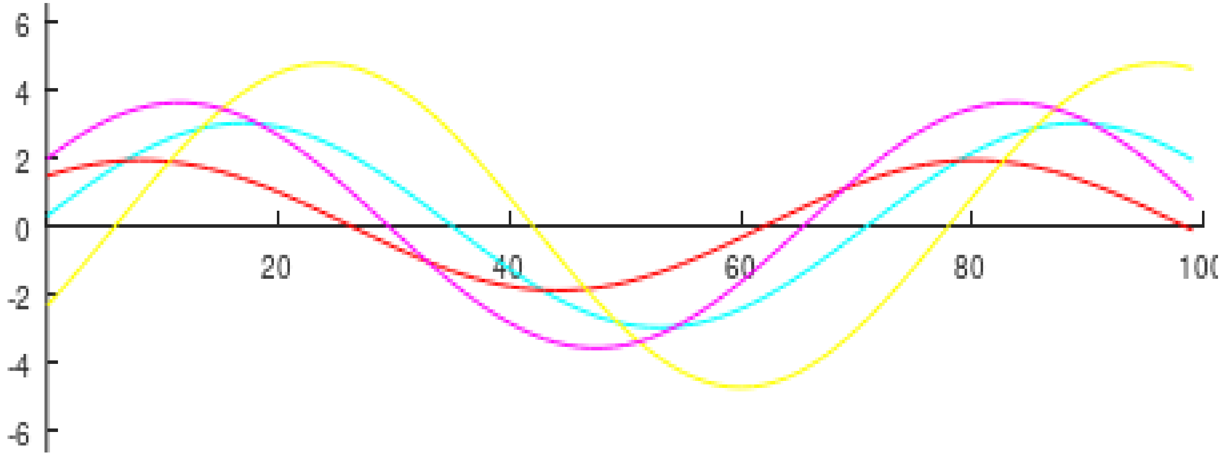


- The whole bunch receives (for example at injection) a transverse kick (additional momentum q) of 2 units
- Tracing over 100 turns as before

A beam (bunch): Motion of individual particles (4/4)



Turn 0 Turn 10 Turn 53 Turn 100



The incoherent motion of the particles remains the same, but this time the center of charge also moves (cyan curve). **The beam beforms a betatron oscillation.**

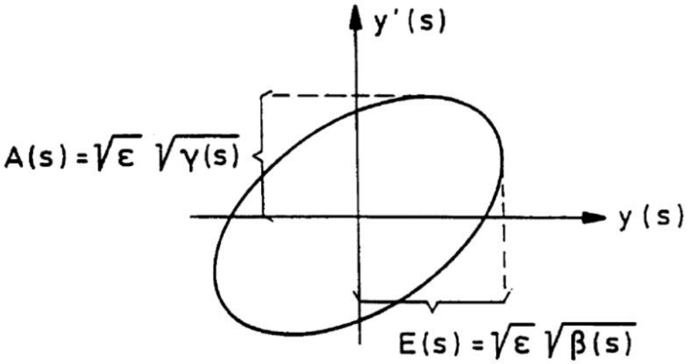
Definition of beam emittance ε

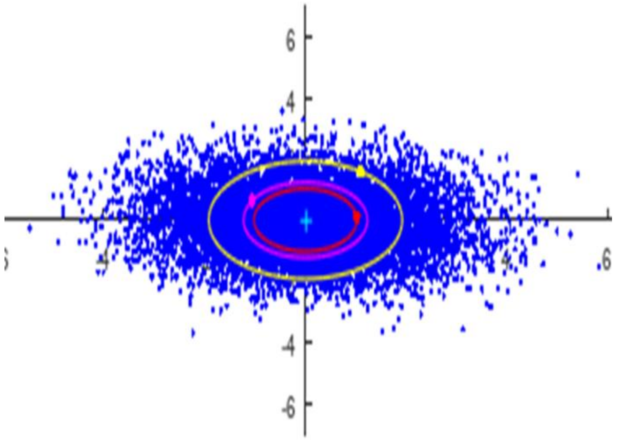
- From last slides: Individual particles continuously perform oscillations in phase space with constant action.

- Independent of the actual phase space distribution of the particles the average action is a very useful quantity to describe the volume in phase space occupied by the whole beam.

- We call this quantity **emittance ε.**
  $\varepsilon = \langle J \rangle$

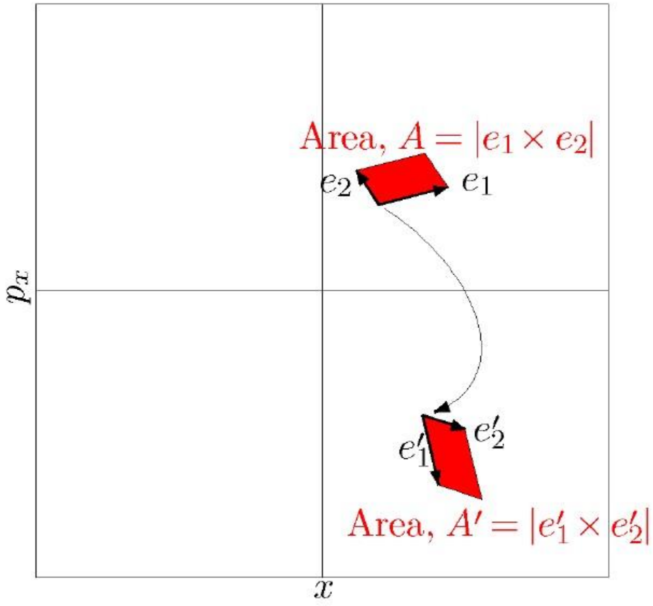
- The shape of the emittance rotates in phase space exactly as the phase space ellipses of single particles.
- Just now the ellipse is "full of particles" and not only a trajectory of a single particle!





Liouville's Theorem (1/2)

1. All particle rotate in phase space with the same angular velocity (in the linear case)
2. All particle advance on **their** ellipse of constant action



Physically, a symplectic transfer map conserves phase space volumes when the map is applied.

This is Liouville's theorem, and is a property of charged particles moving in electromagnetic fields, in the absence of radiation.

→ Since volumes in phase space are preserved, (1)+(2) means that the whole beam phase space density distribution transforms the same way as the individual constant action ellipses of individual particles.

Liouville's Theorem (2/2)

1. We have already identified the action as a preserved quantity in a conservative system, therefore as average action…
   …the emittance of a particle beam is preserved in a conservative beam line/accelerator.
   Attention: As soon as synchrotron light emission plays a role, the system is no longer conservative!
   (one of the next slides: radiation damping)
2. Let us be picky: The sentence above is often quoted as Liouville's theorem, but this is incorrect. Liouville's theorem describes the preservation of phase space volumes, the preservation of the phase space of a beam is then just results from the Hamiltonian description.

* There are several different definitions of the emittance ε, also different normalization factors. This depends on the accelerator type, but the definition as average action describes best the physics. The RMS emittance below is useful in the world of real measurements.

Another often used definition is called RMS emittance

$$\varepsilon = const * \langle x^2\rangle\langle p^2\rangle - \langle xp\rangle^2 \quad \text{or} \quad \varepsilon = const * \langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2$$

More on beam emittance

The reference momentum increases during acceleration

$P_0 = \beta_0\gamma_0 mc \rightarrow P_1 = \beta_1\gamma_1 mc$ $(\beta, \gamma \text{ relativistic parameters})$

we can show: $\beta_0\gamma_0\epsilon_0 = \beta_1\ \gamma_1\epsilon_1$

So the transverse emittance scales with the product $\beta\gamma$

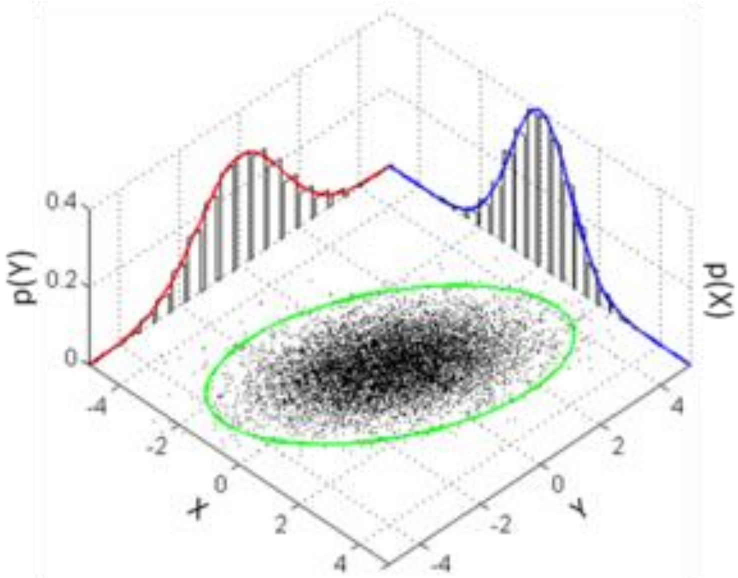
For this reason we define:

normalized emittance $\varepsilon_N := \beta\gamma\varepsilon$ while we call ε the geometric emittance
The "shrinking" of the transverse emittance during acceleration is called "adiabatic damping

Other ways to influence the emittance (advanced subjects):
- make it bigger by error (injection errors, resonances….)
- make it smaller by cooling (stochastic cooling; electron-cooling….)

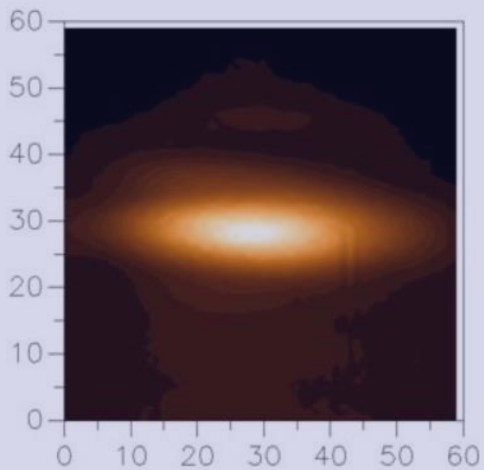
Not to be confused with:
Radiation damping = Reduction in emittance due to the emission of photons as synchrotron radiation

What do we normally measure from the phase-space ellipse?



- At a given location in the accelerator we can measure the position of the particles, normally it is difficult to measure the angle…so we measure the projection of the phase space ellipse onto the space dimension:
  →called a profile monitor

Attention! The standard 2 D image of a synchrotron light based beam image is NOT a phase space measurement



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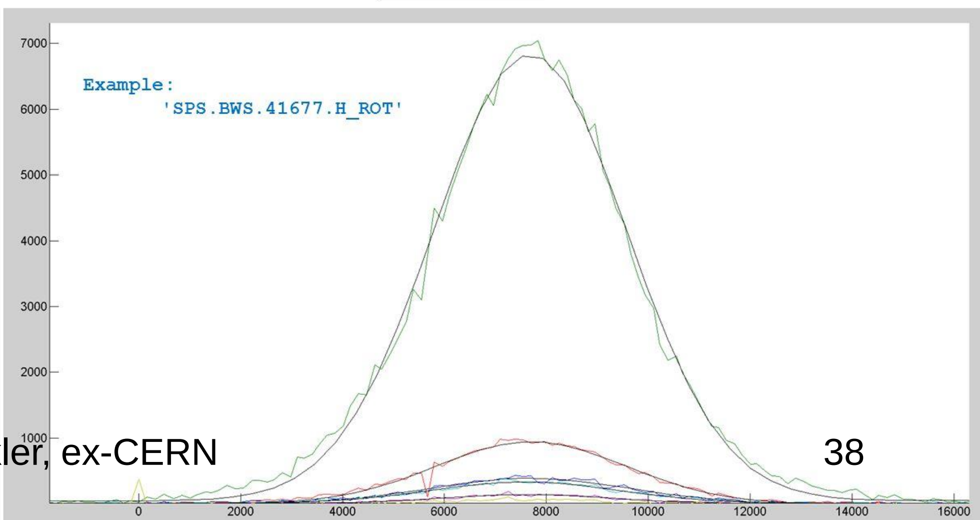
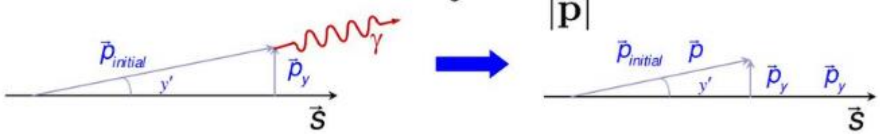


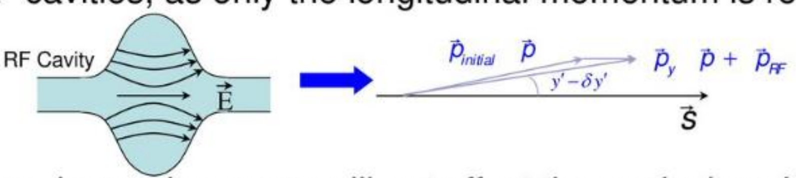
Radiation damping

Synchrotron radiation predominantly in electron storage rings leads to a shrinking of transverse momenta
→emittance not preserved

- Synchrotron radiation emitted in the direction of motion of electron, whose momentum is reduced
- This reduces the vertical component of the momentum but the angle remains the same $y' = \frac{\delta p_{\perp}}{|\mathbf{p}|}$



- The key for betatron damping is the energy recovery by the RF cavities, as only the longitudinal momentum is restored



- The change in energy will not affect the vertical position but the angle changes proportionally $\delta y' = y' \frac{\delta E}{E}$

$$P_s = \frac{e^2 c}{6\pi\varepsilon_0 (m_0 c^2)^4} \frac{E^4}{\rho^2}$$

Power inversely proportional to 4th power of rest mass (proton **2000 times** heavier than electron)
On the other hand, for multi TeV hadron colliders (LHC, FCCpp) synchrotron radiation is an important issue (protection with beam screens)

$$\Delta E = \frac{e^2}{3\varepsilon_0 (m_0 c^2)^4} \frac{E^4}{\rho}$$

By integrating around one revolution, the **energy loss per turn** is obtained**.** For the ILC DR, it is around **4.5 MeV/turn**. On the other hand, for LEPII (**120 GeV**) it was **6 GeV/turn**, or for FCCee (ttbar flavor at **175 GeV**), it will be **7.5GeV/turn** i.e. circular electron/positron machines of hundreds of GeV become quite demanding with respect to RF power (and extremely long)

Figures "stolen" from Y.Papaphillipou

Keywords concerning beam motion in circular accelerators at a glance

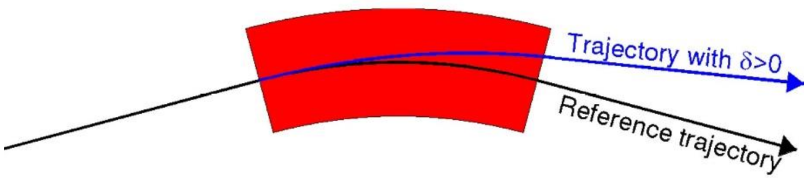
1. Dispersion (wherever we have dipoles)

2. Twiss parameters:
   Phase advance μ(s)
   Beta function β(s)

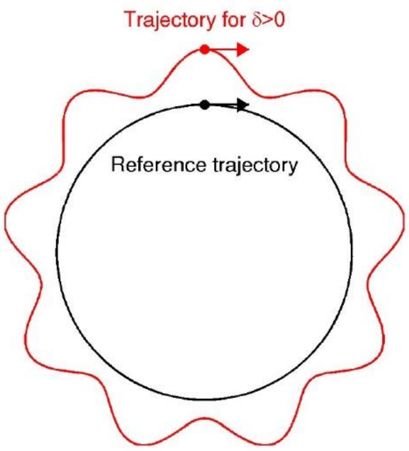
3. Betatron tunes (determined by quadrupoles), working diagram

4. Chromaticity
   (when we talk about sextupoles, 2nd hour)

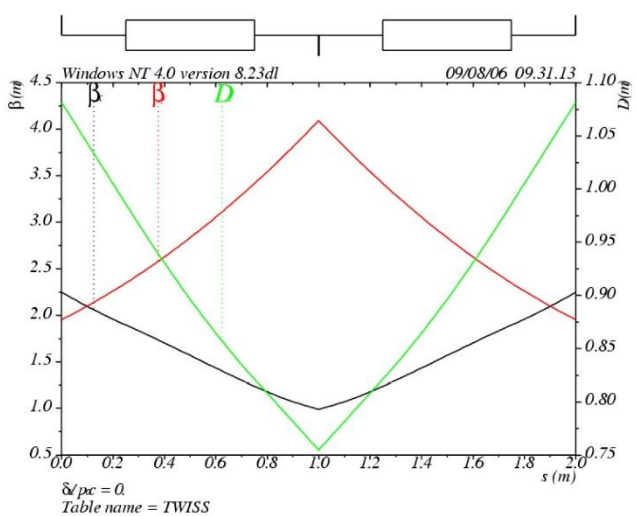
“off-momentum” particles in a synchrotron



What happens: A particle with a momentum deviation $\delta = \frac{\delta p}{p} > 0$ gets bent less in a dipole.



- In a weakly focusing synchrotron it would just settle to another circular orbit with a bigger diameter
- In an alternate gradient synchrotron it is more complicated: The focusing/defocusing is also dependent on the momentum, so the resulting orbit follows the optics of the

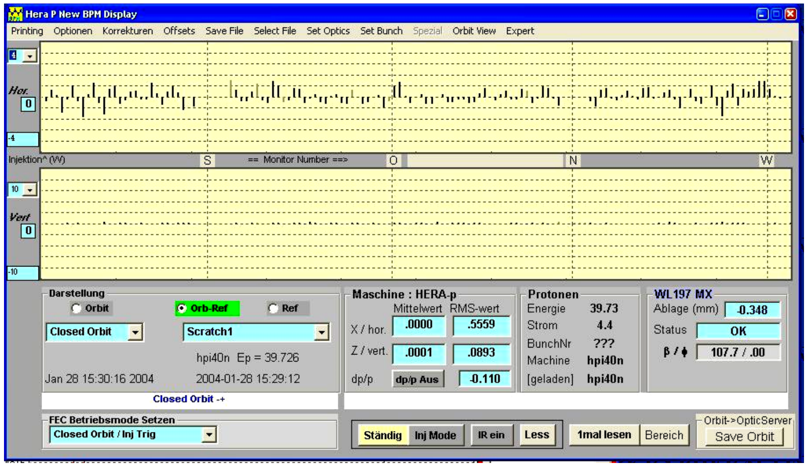


We describe the dispersion as a function of s as $D(s)$; the resulting position of a particle is thus simply:

$$x_{\delta p} = x_0 + D(s)\,\frac{\delta p}{p}$$

Typical values of D(s) are some meters, with $\frac{\delta p}{p} = 10^{-3}$ the orbit deviation becomes millimeters

Dispersion Measurement example



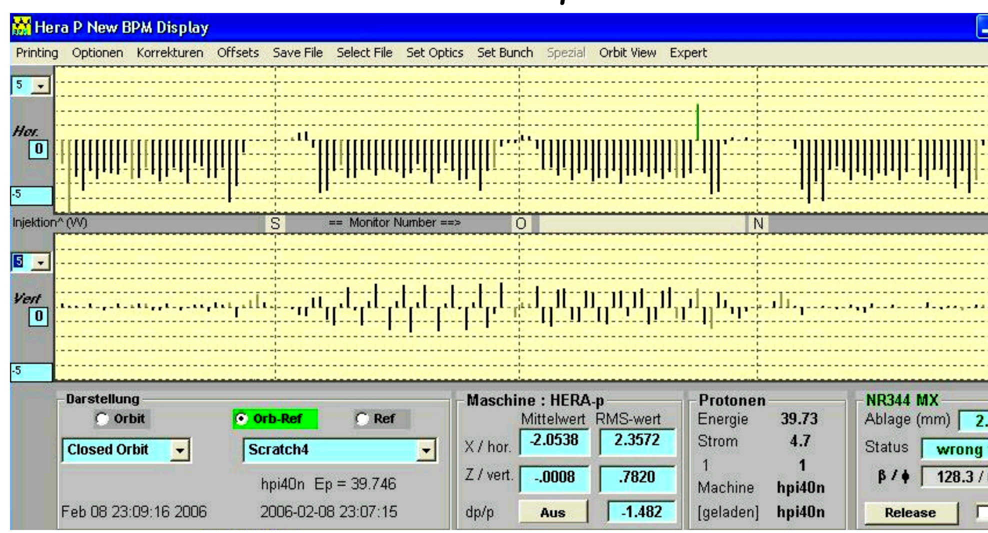
HERA typical orbit measurement

1) Measure orbit
2) Change momentum
3) Measure orbit again
4) Calculate Dispersion from difference orbit

dedicated momentum change of the stored beam
→ closed orbit is moved to a dispersion orbit

$$x_D = D(s) * \frac{\partial p}{p}$$

HERA Dispersion Orbit



Twiss parameters (1/2)

- Introduced in the late 50's by Corant/Snyder
- The classical way to parametrize the evolution of the phase space ellipse along the accelerator

Basic concept of this formalism:

1) Write the transfer matrix in this form (2 dimensional case):

$$M = I\ cos\mu + S\ \cdot A\ sin\mu$$

I $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; S $= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; A $= \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}$

2) M must be symplectic → $\beta\gamma - \alpha^2$ = 1

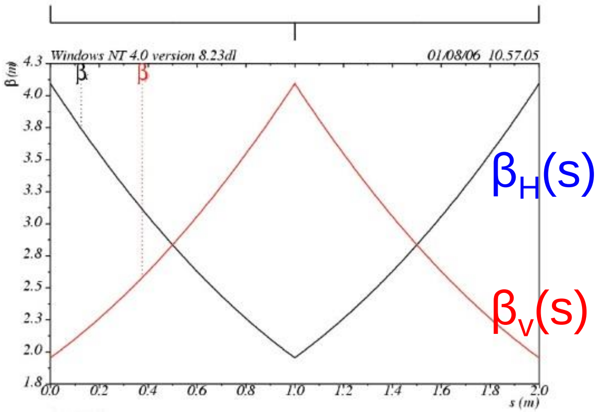
3) Four parameters: $\alpha(s);\ \beta(s);\ \gamma(s) and\ \mu(s)$, with one interrelation (2)
→ Three independent variables

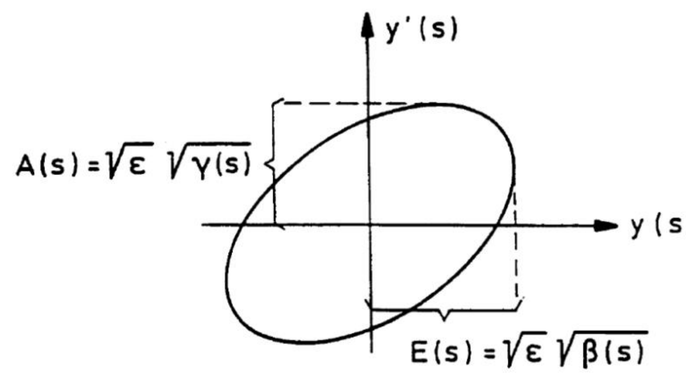
4) $$x(s) = \sqrt{\varepsilon} \cdot \sqrt{\beta(s)} \cdot cos\{\mu(s) + \varphi\}$$

Twiss parameters (2/2)

What is the power of this approach?

- Instead of computing step by step with consecutive Matrix-operations the phase space at a given point, the values of the Twiss parameters β(s), α(s) and μ(s) describe the beam at any point in the accelerator.
- It looks like as if we had found a closed solution of the differential equation (for ex. Hill's equation) for the accelerator.
- The twiss parameters are the output of any accelerator simulation tool.



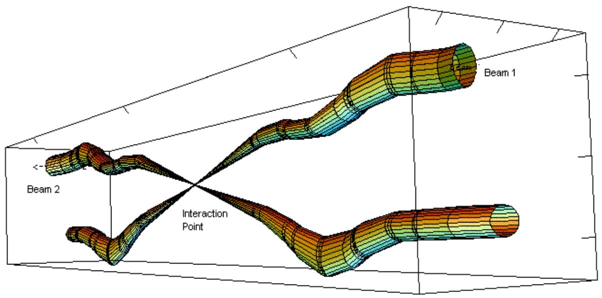


Beam size $\sigma(s) = \sqrt{\varepsilon\,\beta(s)}$

Importance of the Twiss parameters (1/2)

- Focusing quadrupole → low beta values
- The transverse beam envelope follows like a "squashed sausage" the square root of the beat function

J. Jowett

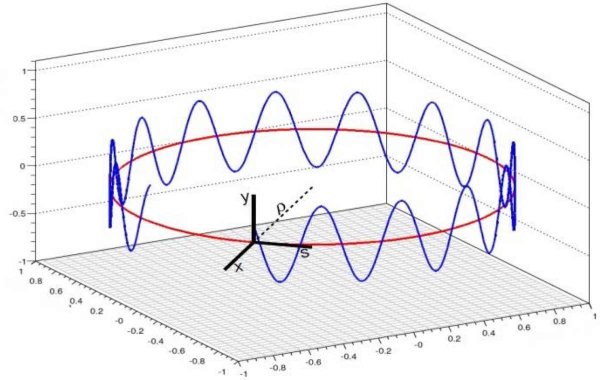


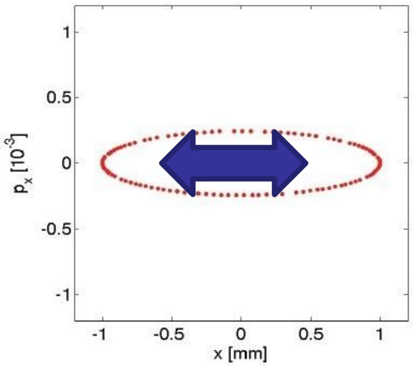
Relative beam sizes around IP1 (Atlas) in collision

- The shape of phase space changes along s.
  The projection of the phase space onto the space co-ordinate (=beam size) can perform a quasi harmonic oscillation with variable amplitude (again modulated by $\sqrt{\beta(s)}$)
  called **BETATRON-Oscillation**

$$\mu = \int_{s1}^{s2} \frac{1}{\beta} \mathrm{ds}$$

$$x(s) = \sqrt{\varepsilon\beta(s)} \cdot cos\{\mu(s) + \varphi\}$$





Importance of the Twiss parameters (2/2)

2.) $\alpha = -\frac{1}{2}\frac{d\beta}{ds}$

α indicates the rate of change of β along s

α zero at the extremes of beta (waist)

3.) $\mu = \int_{s1}^{s2} \frac{1}{\beta} \mathrm{ds}$

Phase Advance: Indication how much a particle rotates in phase space when advancing in s

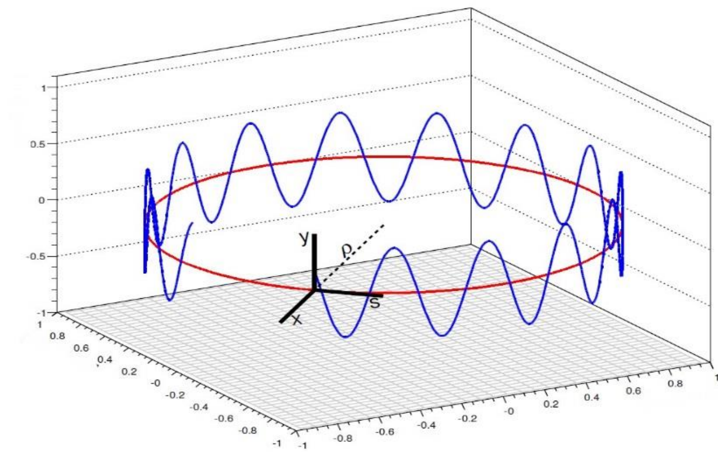
Of particular importance: Phase advance around a complete turn of a circular accelerator, called the betatron tune Q (H,V) of this accelerator

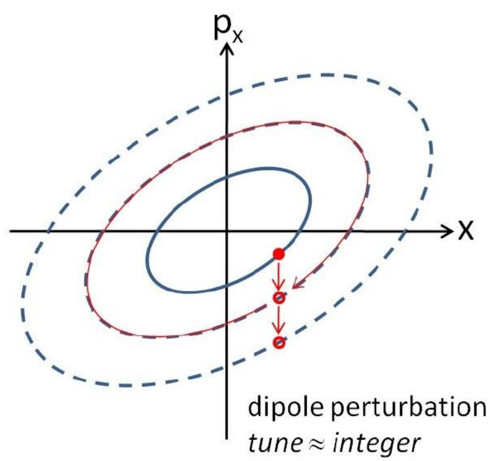
$$Q_{H,V} = \frac{1}{2\pi}\int_{0}^{C} \frac{1}{\beta_{H,V}} \, ds$$

The betatron tunes $Q_{H,V}$

- One of the most important parameters of a circular accelerator
- For a circular accelerator it is the phase advance over one turn in each respective plane.
- The equivalent in a linac is called “phase advance per cell”

- In large accelerators the betatron tunes are large numbers (LHC ˜ 65), i.e. the phase space ellipse turns about 65 times in one machine turn.
- We measure the tune by exciting transverse oscillations and by spectral analysis of the motion observed with one pickup.
  But this way we measure the **fractional part of the tune; often called $q_{H,V}$**





- Integer tunes (fractional part= 0) lead to resonant infinite growth of particle motion even in case of only small disturbances.

Importance of betatron tunes

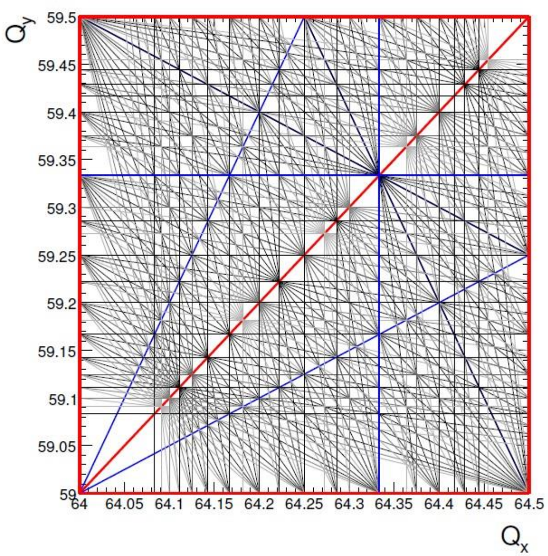
If we include vertical as well as horizontal motion, then we find that resonances occur when the tunes satisfy:

$$m_x \nu_x + m_y \nu_y = \ell,$$

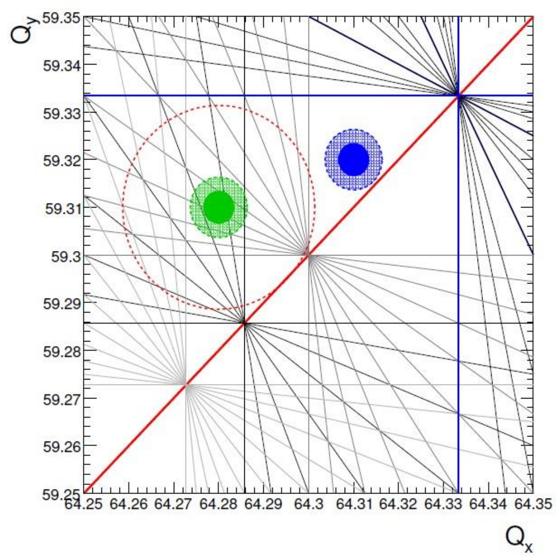
where m_x, m_y and ℓ are integers.

The order of the resonance is $|m_x| + |m_y|$.

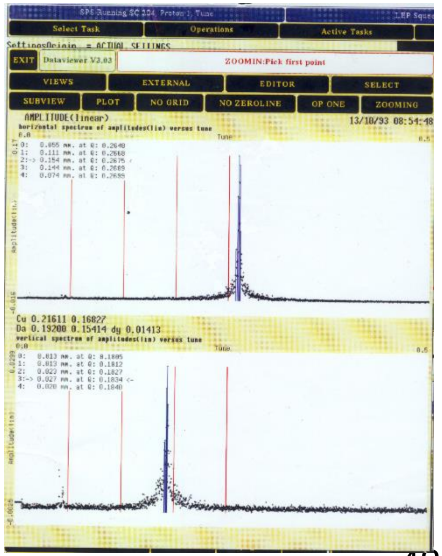
The couple (Q_H ,Q_V) is called the working point of the accelerator.
Below: tune measurement example from LEP



(a) Full tune diagram



(b) Zoom around LHC Q working points



Coffee break

Let’s talk about magnets!

- What is important for an accelerator physicist?

  - basic function: following slides
  - imperfections → direct impact on accelerator performance and operation

  - operational parameters:
    excitation curve, hysteresis, reproducibility, quench-limits, powering (single, serial), reference magnets

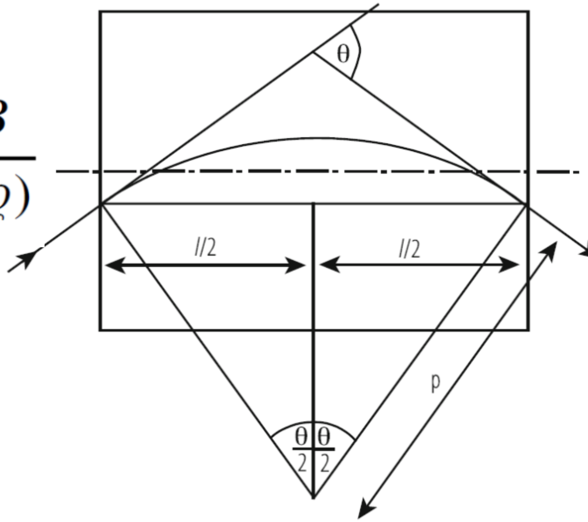
  - production tolerances → evtl. Measurement setups
    multipole components
  - theory, design, design-tools…

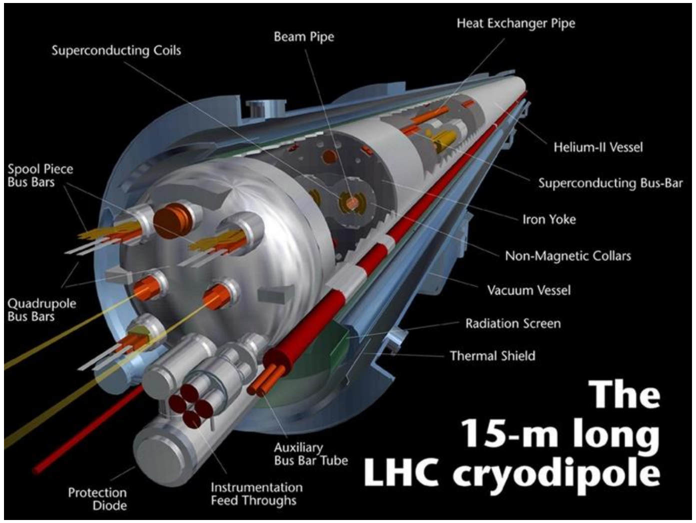
} Later This Course

Dipoles

- Main purpose: Bending of particle beams

$$\sin(\theta/2) = \frac{\ell}{2\varrho} = \frac{\ell B}{2(B\varrho)} \longrightarrow \theta \approx \frac{\ell B}{(B\varrho)}$$





Beam rigidity: $B\rho \; [\mathrm{T \cdot m}] = 3.3356 \; pc \; [\mathrm{GeV}]$

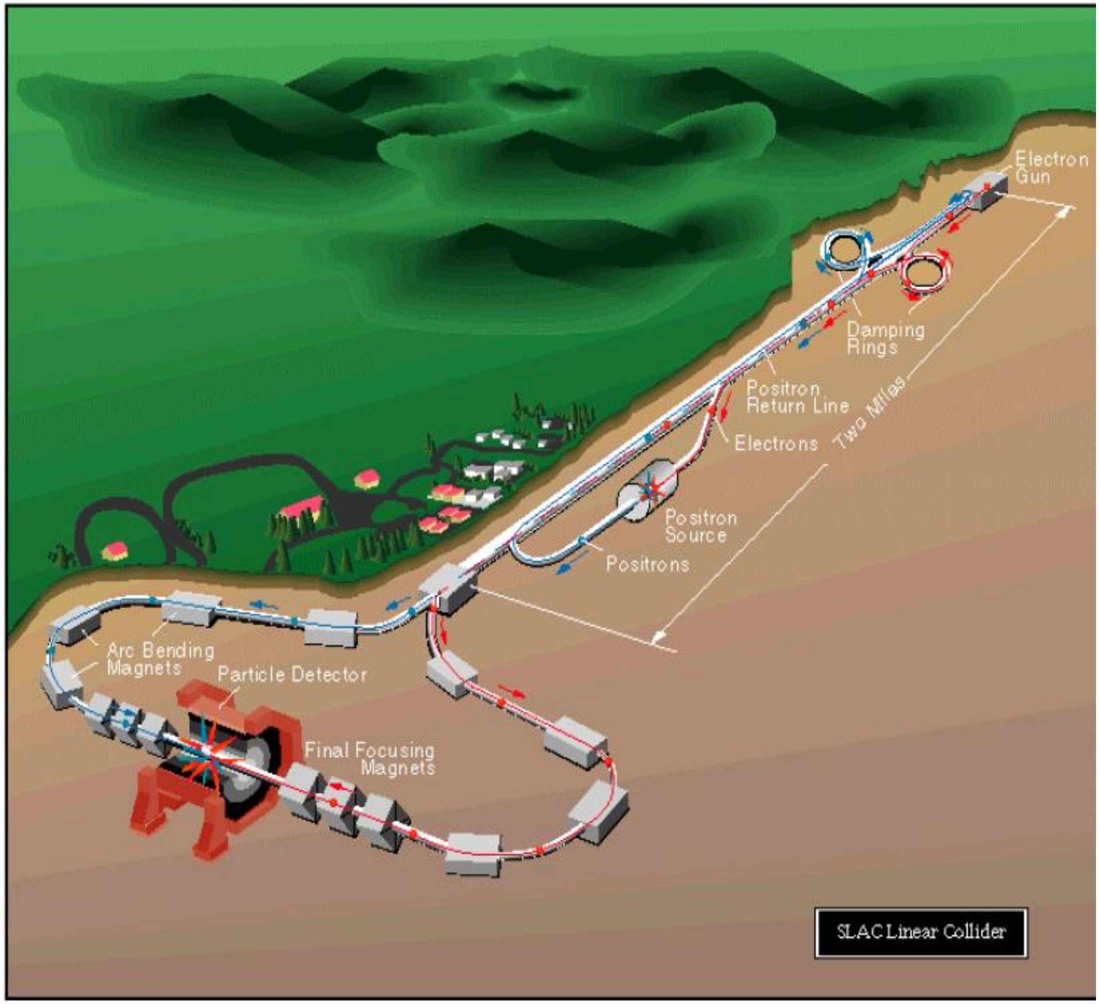
Example: LHC main dipole:
B ≈ 8T ; pc ≈ 13000 GeV → ρ ≈ 5420 m

Dipole as particle/antiparticle separator

Stanford Linear Collider (SLC):
So far the one and only e^+e^- linear collider using one linac!

e^+ and e^- accelerated on negative and positive crest of RF-wave.

Separated into collision arcs by a dipole



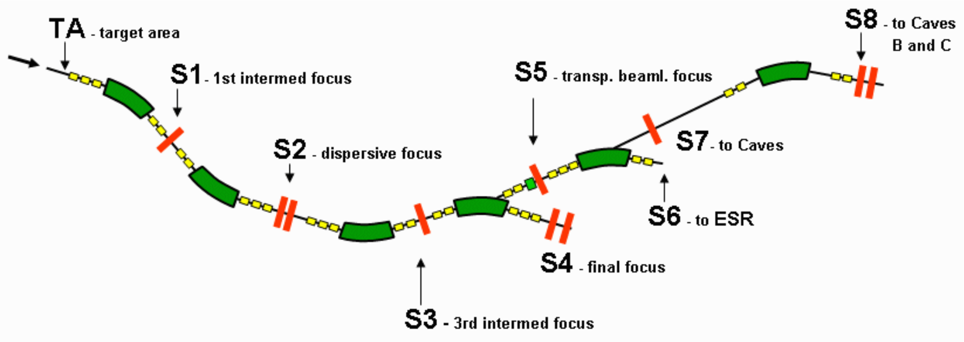
Dipoles as Separators

- Consider particles with fixed p but different M and Q in a dipole field B
  - assume B·L same for all M, Q
  - deflection angle given by

$$\theta = Q/M \cdot B{\cdot}L/p$$

- Charge separation
  - assume same M, but different Q
  - let only one angle θ pass and vary B, then Q proportional to 1/B
- Mass separation
  - assume same Q, but different M
  - let only one angle θ pass and vary B, then M proportional to B

Some separator magnets are rather large…

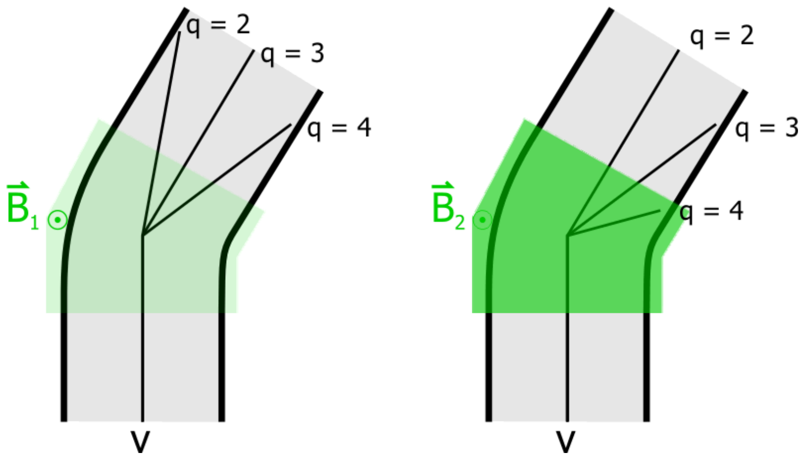


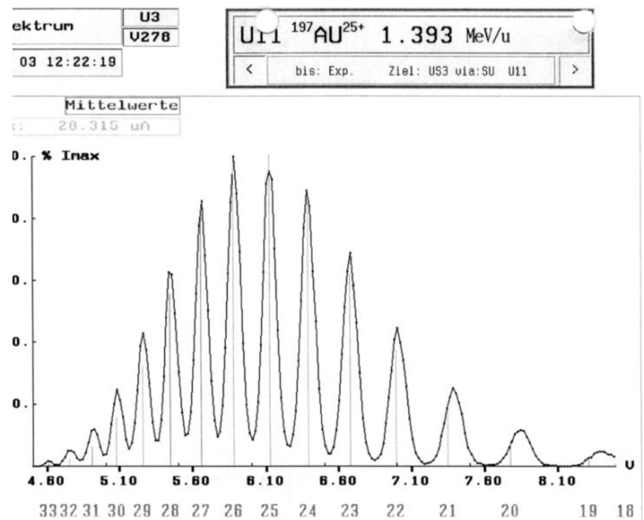
…and one can build separators with more dipoles.

Taken from D.Oneka (GSI)

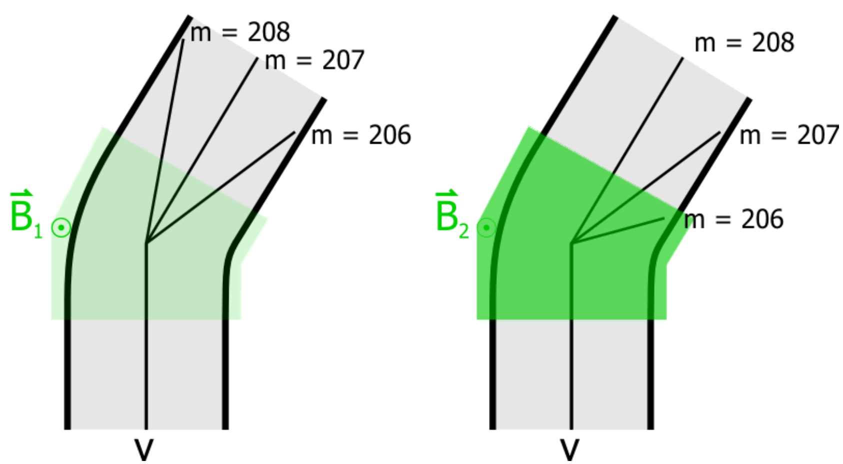
Example: Spectrometer

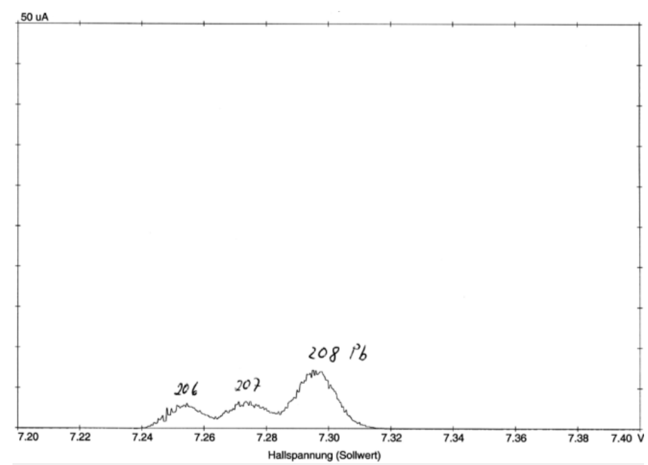
Charge separation behind gas stripper





Mass (isotope) separation

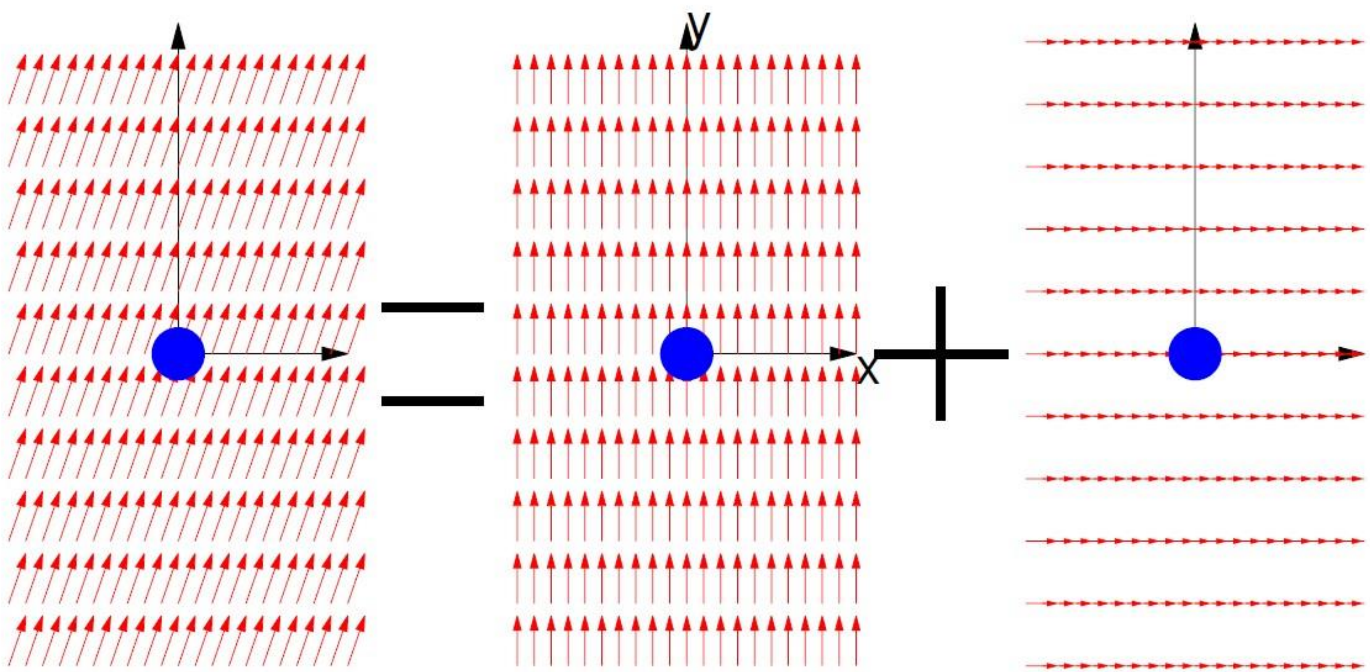




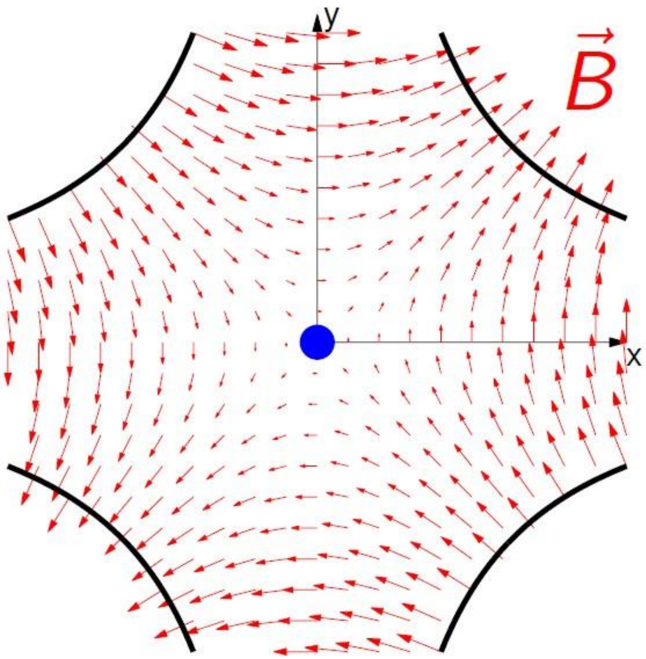
Taken from D.Oneka (GSI)

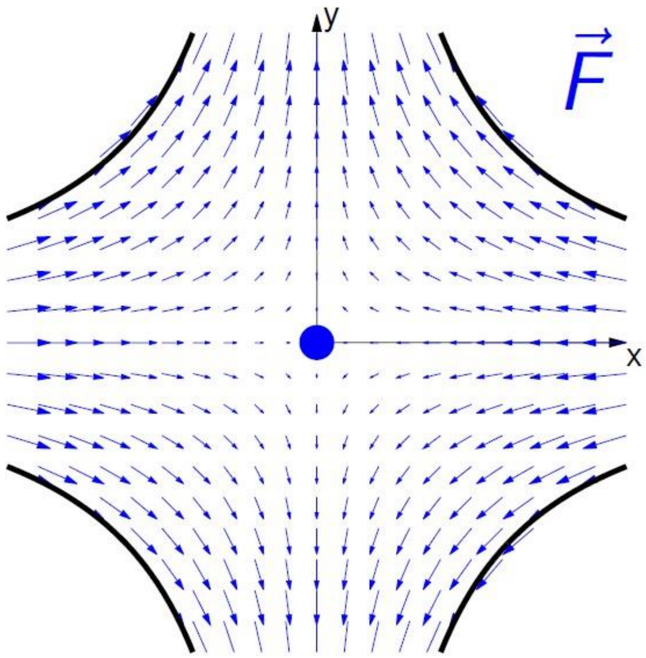
Dipole Errors

error	effect	correction
strength (k)	change in deflection	change excitation current, replace magnet
lateral shift	none	
tilt	additional vertical deflection	corrector dipole magnet



Quadrupole Errors (1/3)

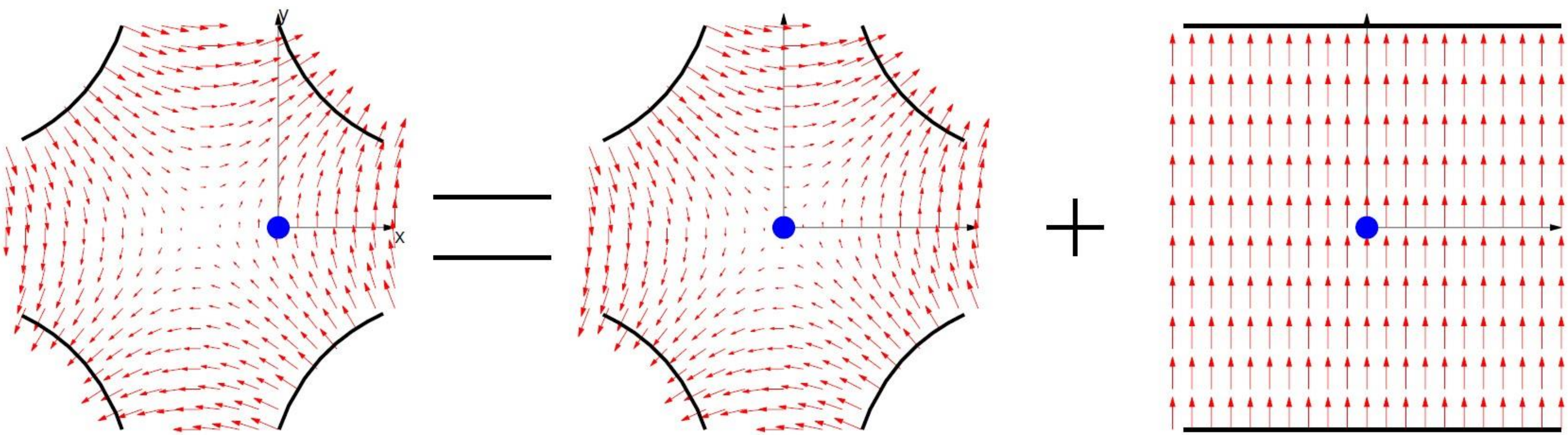




Note that $F_x = -kx$ and $F_y = ky$ making horizontal dynamics totally decoupled from vertical.

Quadrupole Errors 2/3

Error type	effect on beam	correction(s)
strength	Change in focusing, “beta-beating”	Change excitation current, Repair/Replace magnet
Lateral shift	Extra dipole kick	Excitation of a corrector dipole magnet
tilt	Coupling of the beam motion in the two planes	Excitation of a additional “skewed quadrupoles (45^0)

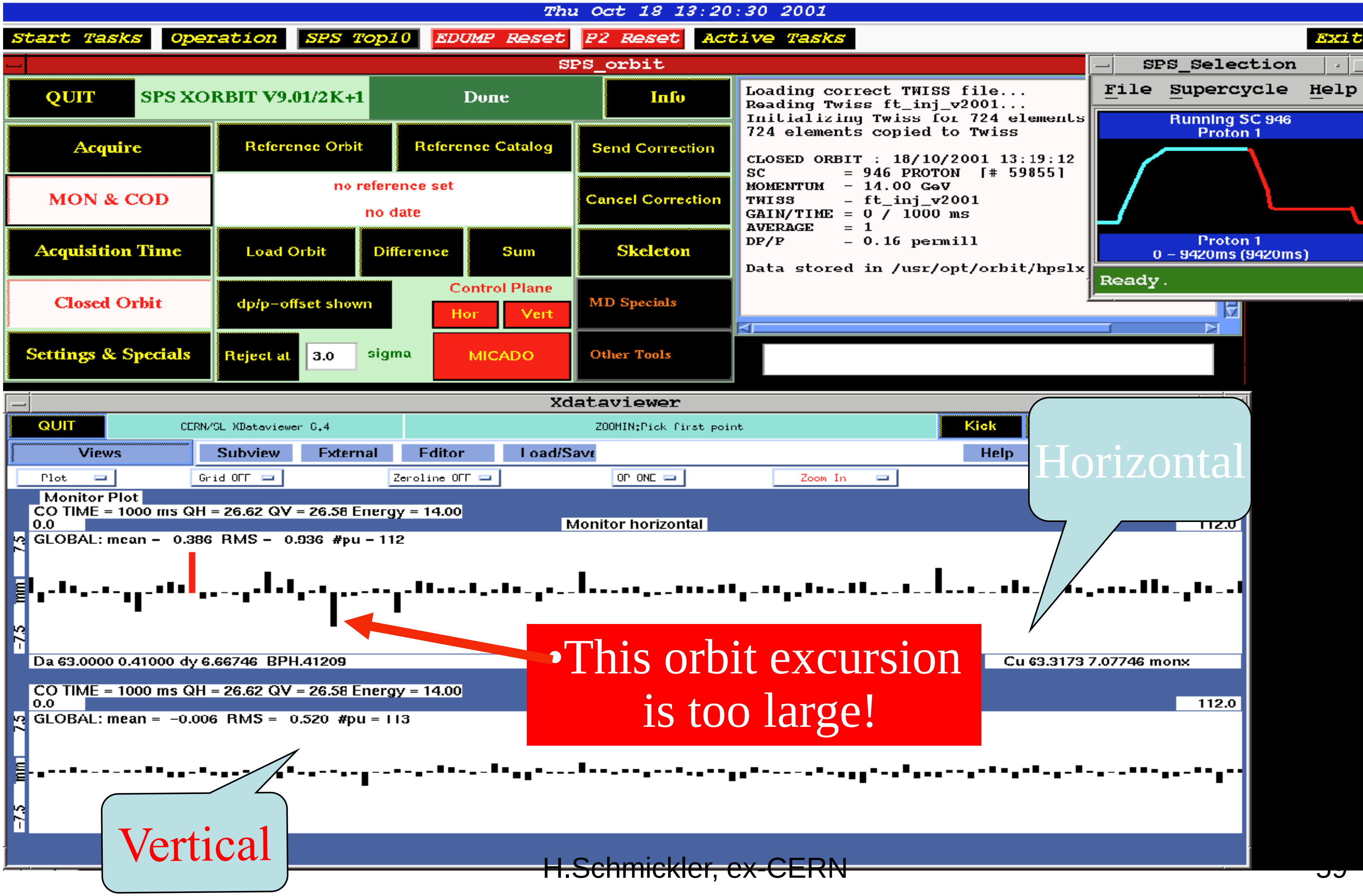


An offset quadrupole is seen as a centered quadrupole plus a dipole.

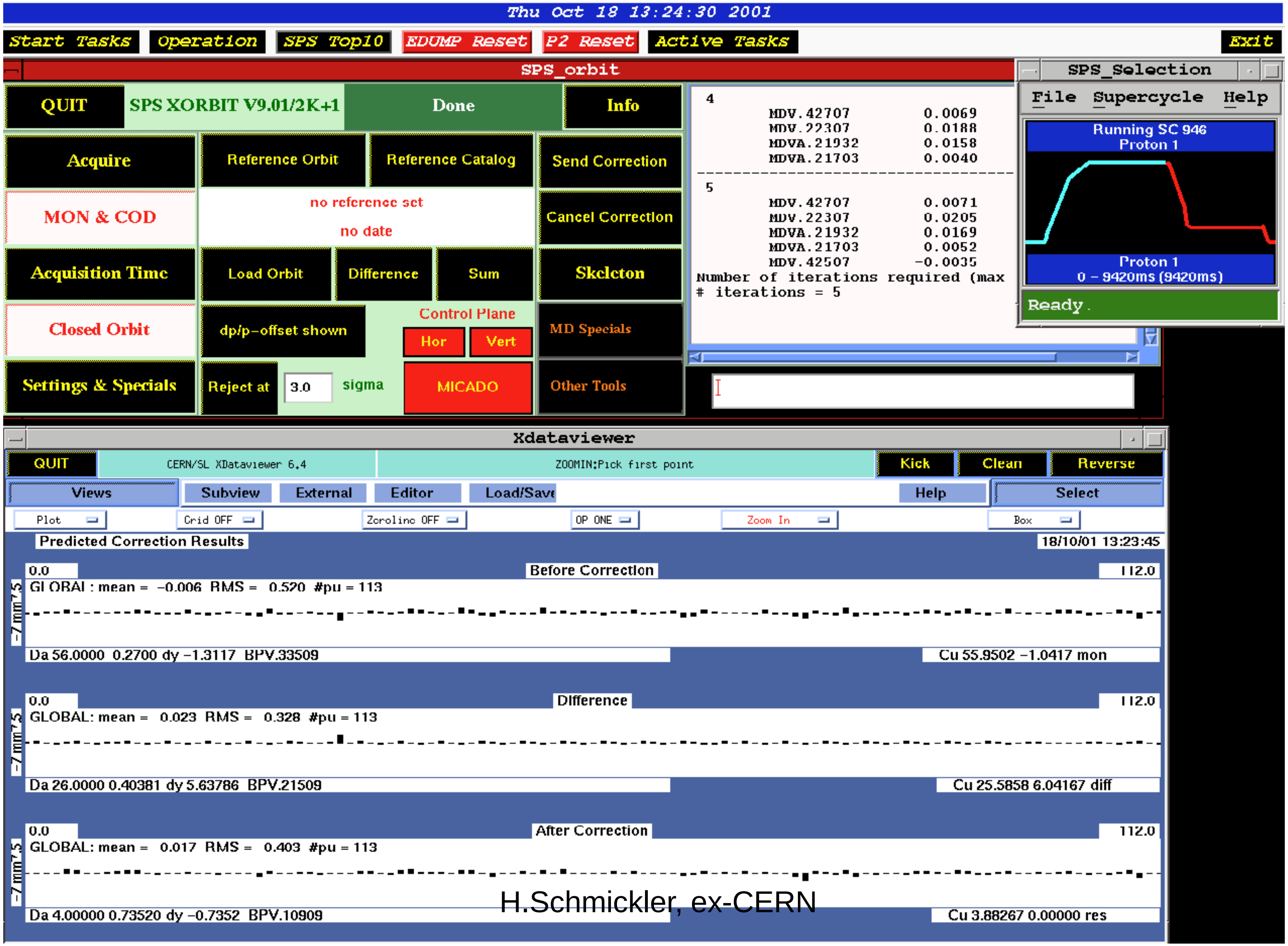
Need for steering dipoles

- Tilted dipoles give unwanted kicks in the vertical plane
- Shifted quadrupoles behave like an additional dipole ("downfeed")
  ← (Shifted sextupoles behave like an additional quadrupole)
- For best accelerator performance need extra small corrector dipoles with individual power-converter
- Placement:
  horizontal Dipoles on each F quadrupole
  vertical Dipoles on each D quadrupole
- Correction: Either by operator intervention (see example) or by automatic orbit-feedback software (in circular lightsources up to 1 kHz repetition rate)
- **Needs best possible knowledge of optical functions!**

Orbit Acquisition

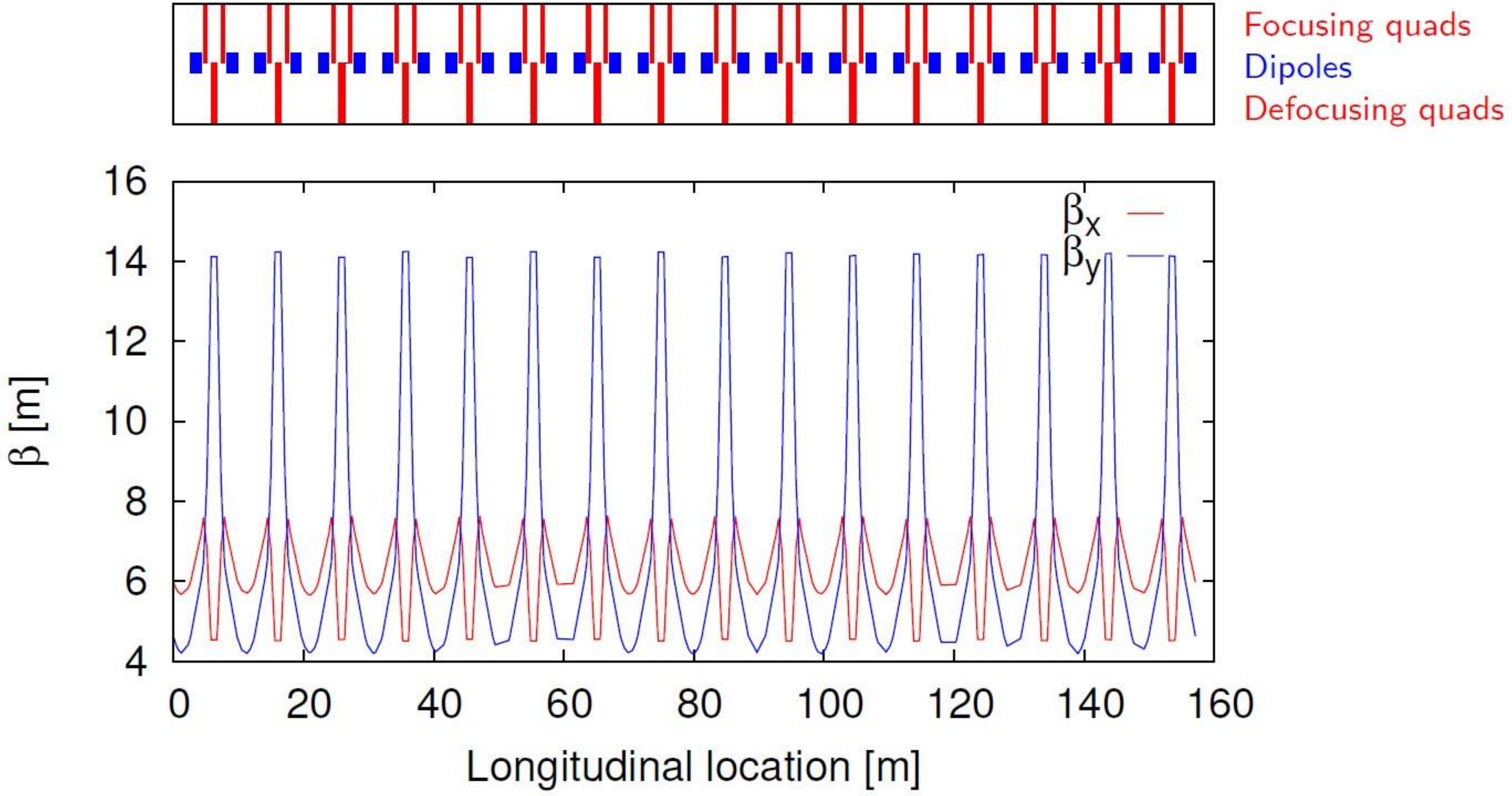


Orbit Correction (Operator Panel)

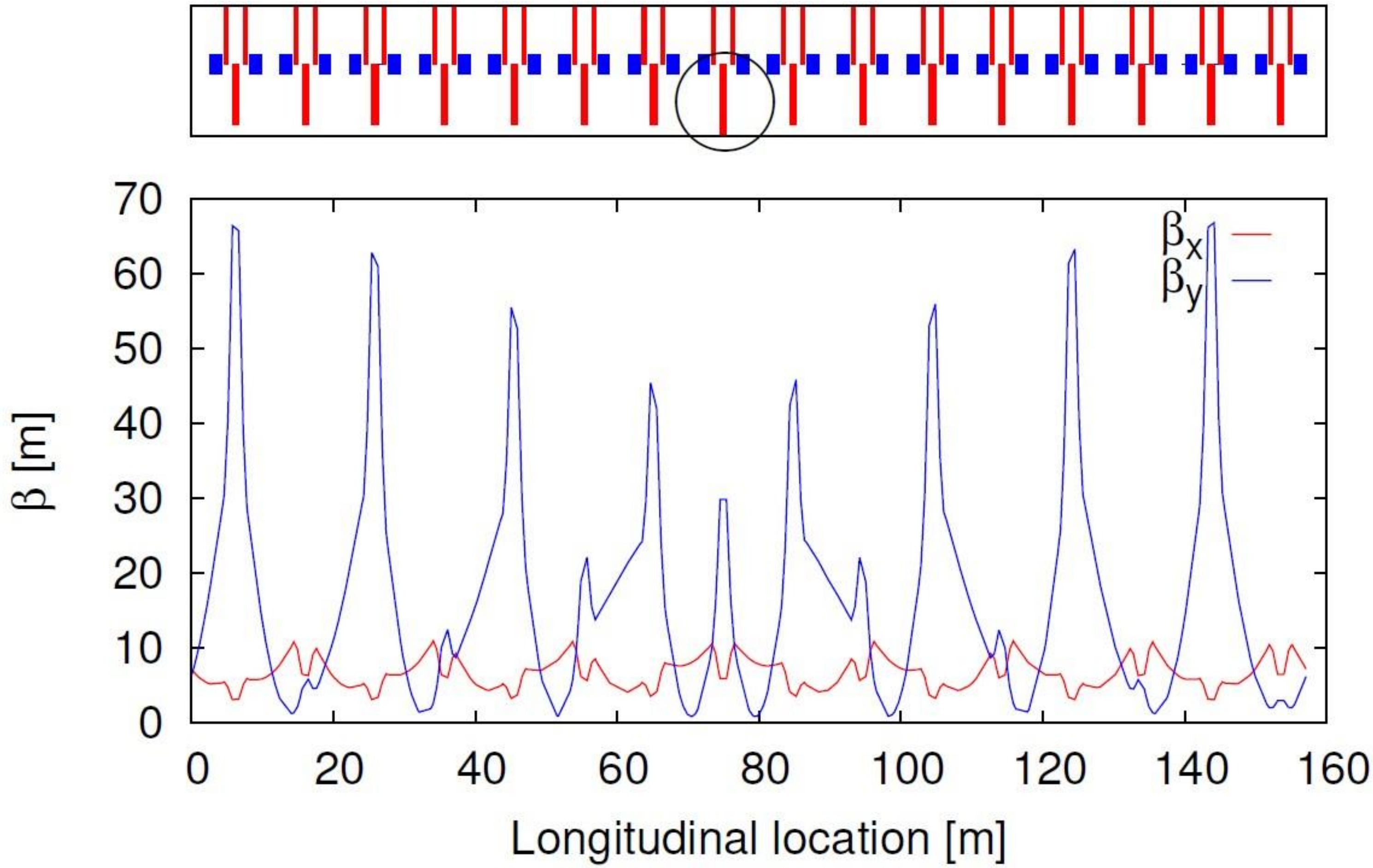


Orbit Correction (Detail)

Quadrupole strength error: Beta-beating (1/2)

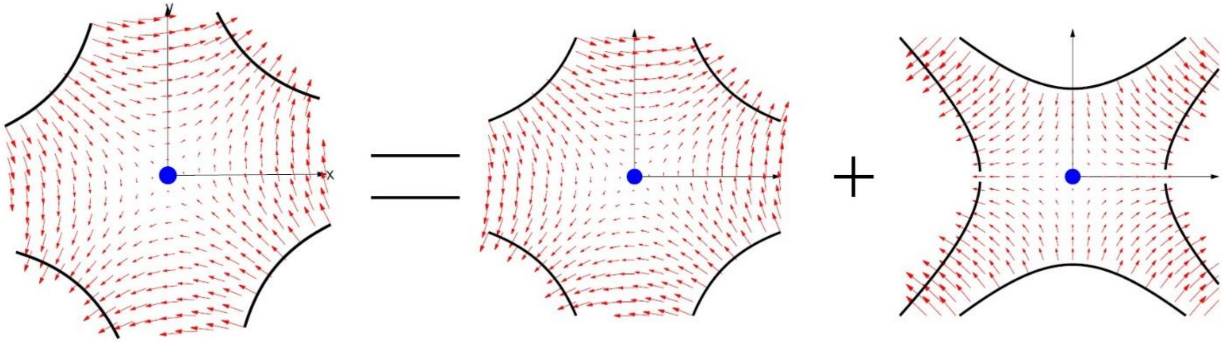


Quadrupole strength error: Beta-beating (2/2)

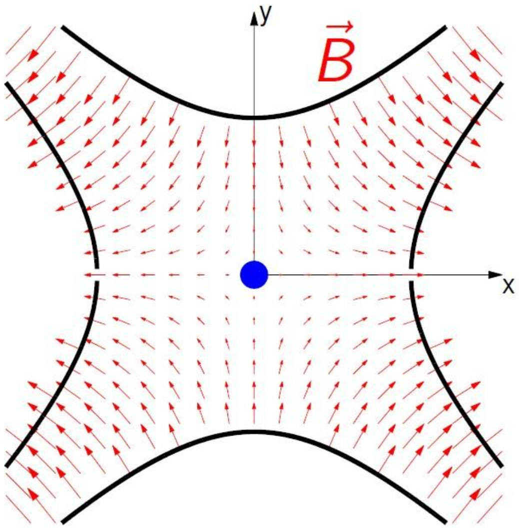


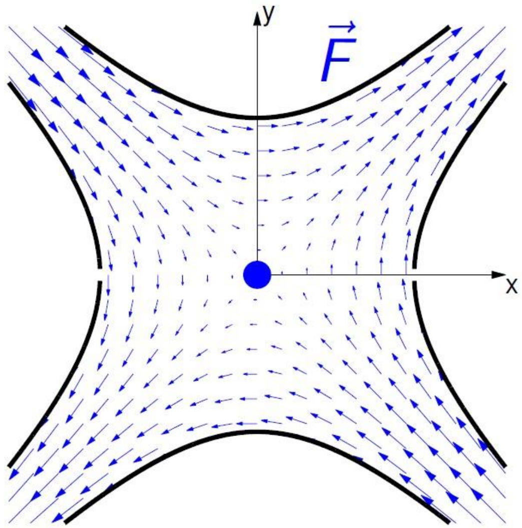
β functions change (β-beating$=\frac{\Delta\beta}{\beta} = \frac{\beta_{pert}-\beta_0}{\beta_0}$).

Quadrupole Errors 3/3



Any tilted quadrupole is seen as a normal quadrupole plus another quadrupole tilted by 45^0. (skew quad)





Note that in a skew quad
$F_x = k_s y$ and $F_y = k_s x$ produce coupling between the x and y planes

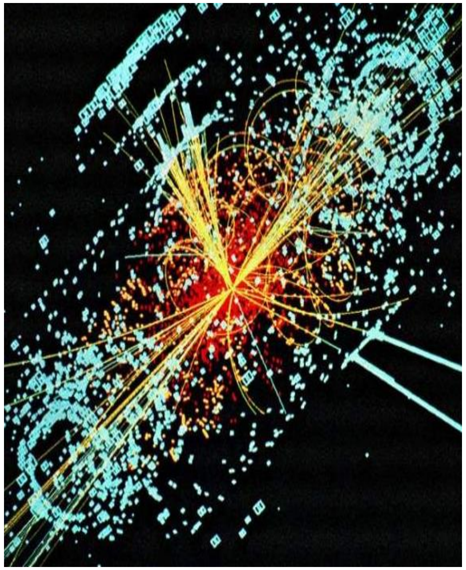
Additional skew quads in an accelerator are used to compensate coupling

Quadrupoles

- Main purpose: Transverse Focusing
- FODO lattice or more involved acromat layouts for circular lightsources
- Important design parameters:
  - gradient [Tm],
  - beam aperture (in the defocusing plane the beam is biggest in the quadrupole)
- in the arcs often powered in series → testbenches, sorting…
  additional "trombone" quadrupoles for corrections (with individual powersupplies)
- Operational daily procedure: Change betatron tunes of accelerator by changing strength of quadrupoles
- Insertion quadrupoles in colliders → next slides

Particle Collider figures of merit:

1. **c.m.s. energy**: higher energy means particles with higher masses can be produced
2. **Luminosity**: A number characterizing a collider to produce a certain number of events of a given process in a given time→



- **First: The cross section of a physics process:**

The cross-section σ_{ev} expresses the likelihood of a process to be produced by a particle interaction. Each production channel has its own cross-section.

- σ_{ev} can be understood as an “area” hit by the beam.
- Unit for cross-section: [m^2]
- in nuclear- and high energy physics we need smaller units: = barn (1 b = 10^{-24} cm^2)

definition: Luminosity (L)

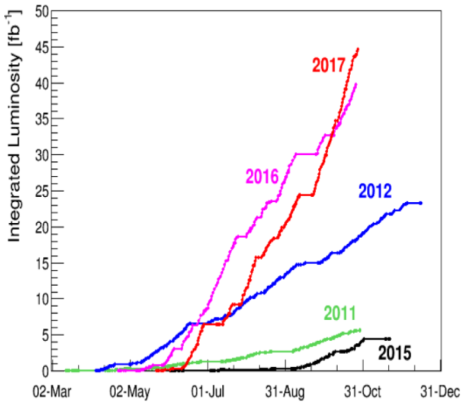
$$R = \frac{dN_{ev}}{dt} = L(t)\sigma_{ev}$$

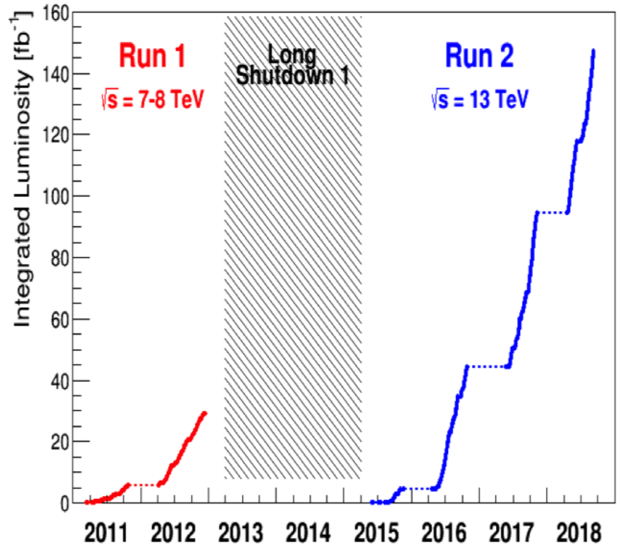
$$N_{ev} = \sigma_{ev}\int L(t)\,dt$$

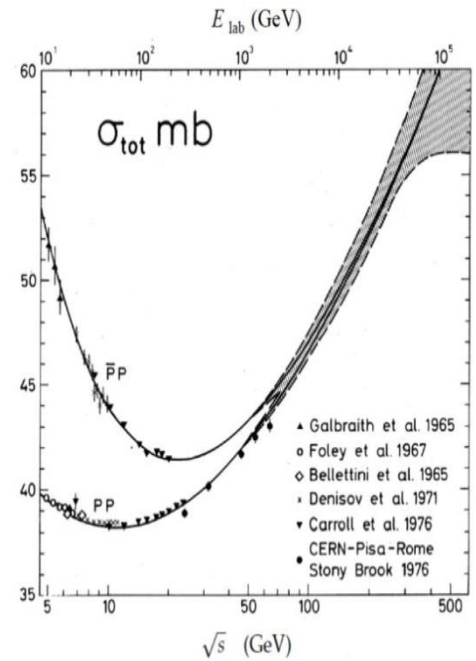
- luminosity L relates cross-section σ and event rate $R = dN_{ev}/dt$ at time t:
  - quantifies performance of collider
  - relativistic invariant and independent of physical reaction
- accelerator operation aims at maximizing the total number of events N_{ev} for the experiments
  - σ_{ev} is fixed by Nature for every event type
  - aim at maximizing $\int L(t)dt$

- Luminosity unit : [m^{-2} s^{-1}]
- The **integrated luminosity** $\int Ldt$ is frequently expressed as the inverse of a cross section $pb^{-1} = 10^{36}$ cm^{-2} or $fb^{-1} = 10^{39}$ cm^{-2}

Example: LHC







Total integrated luminosity LHC Run 2: 150 fb^{-1}

Total cross section pp collisions: 100 mb

→ $N_{collisions}$ = 150 * 10^{12} mb^{-1} * 100 mb = 15 * 10^{15} events !!!

→ On average a bit less than 100 charged tracks per event!

→ Only a small fraction gets recorded….still Pbytes of data

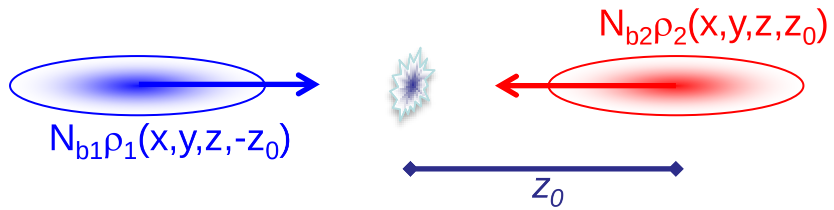
→ Total cross section for Higgs production: About 60 pb → About 9 * 10^{6} Higgs produced

→ Higgs cross-section for Diphoton-decay: About 60 fb → 9000 events to analyse

L from machine parameters -1-

- intuitively: more L if there are more protons and they more tightly packed

$$L \propto N_{b1} N_{b2} \Omega_{x,y}$$



$$L \propto N_{b1} N_{b2} K \int_{x,y,z,z_0} \rho_1(x,y,z,-z_0)\,\rho_2(x,y,z,z_0)\,dx\,dy\,dz\,dz_0$$

- K = kinematic factor (CAS lecture, *"Kinematics of Particle Beams I - Relativity"*)
- N_{b1}, N_{b2}: bunch population
- $\rho_{1,2}$: density distribution of the particles (normalized to 1)
- x,y: transverse coordinates
- z: longitudinal coordinate
- z_0: "time variable", $z_0 = c\,t$
- $\Omega_{x,y}$: overlap integral

L from machine parameters -2-

- for a circular machine can reuse the beams f times per second (storage ring)
- for n_b colliding bunch pairs per beam
- for uncorrelated densities in all planes: $\rho(x,y,z,t) = \rho_x(x)\rho_y(y)\rho_z(z - vt)$

$$L = 2 f n_b N_{b1} N_{b2} \int_{x,y,z,z_0} \rho_{1x}(x)\,\rho_{1y}(y)\,\rho_{1z}(z - z_0)\,\rho_{2x}(x)\,\rho_{2y}(y)\,\rho_{2z}(z + z_0)\,dx\,dy\,dz\,dz_0$$

- for Gaussian bunches: $\rho_u(u) = \frac{1}{\sigma_u\sqrt{2\pi}} \exp\left\{ -\frac{(u - u_0)^2}{2\sigma_u^2} \right\}; \quad u = x,y$ $\qquad \int_{-\infty}^{+\infty} e^{-at^2} = \sqrt{\frac{\pi}{a}}$

- for equal beams in x or y: $\sigma_{1x} = \sigma_{2x}$, $\sigma_{1y} = \sigma_{2y}$

- can derive a closed expression:

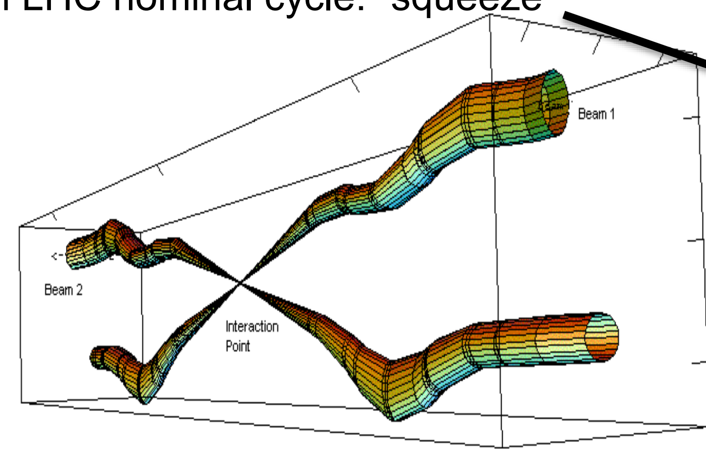
$$L = \frac{n_b N_{b1} N_{b2} f}{4\pi \sigma_x \sigma_y}$$

  - f: revolution frequency
  - n_b: number of colliding bunch pairs at that Interaction Point (IP)
  - N_{b1}, N_{b2}: bunch population
  - $\sigma_{x,y}$: transverse beam size at the collision point

LHC
n_b = 2808
N_{b1},N_{b2} = 1.15 10^{11} ppb
f = 11.25 kHz
σ_x, σ_y = 16.6 μm
L = 1.2 10^{34} $cm^{-2}s^{-1}$

need for small β*

- expand physical beam size $\sigma_{x,y}$: $\sigma_x^* = \sigma_y^* = \sqrt{\dfrac{\beta^* \varepsilon}{\gamma_r}}$ ➔ $L = \dfrac{n_b N_{b1} N_{b2} f \gamma_r}{4\pi \beta^* \varepsilon}$
  - * means “at the IP”
- try and conserve low ε from injectors
  - In addition explicit dependence on energy ($1/\gamma_r$)
- intensity N_b pays more than ε and β*
- **→design low β* insertions**
  - limits by triplet aperture, protection by collimators
  - in LHC nominal cycle: “squeeze”



Relative beam sizes around IP1 (Atlas) in collision

J. Jowett

LHC
β* = 18 ➔ 0.55 m
ε = 3.75 μm
γ_r = 7463
$\sigma_{x,y}$ = 16.6 μm

Example: Propagation of twiss parameters along s between two focusing quadrupoles

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0} \quad M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

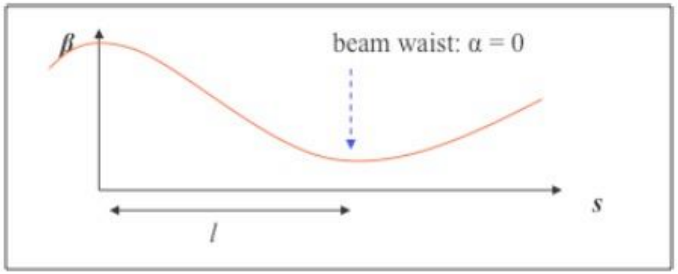
And in Matrix-Annotation:

$$A_{s_0} = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \rightarrow A_s = M^T A_{s_0} M$$

$$\beta_s = C^2\beta_0 - 2SC\,\alpha_0 + S^2\gamma_0 = \beta_0 + {s^2}/{\beta_0}$$

Example: Beta function between two strong focusing

Drift $M = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$



$$A_{s_0} = \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} = \begin{pmatrix} \gamma_0 & 0 \\ 0 & \beta_0 \end{pmatrix} = \begin{pmatrix} {1}/{\beta_0} & 0 \\ 0 & \beta_0 \end{pmatrix}$$

Starting from waist $\alpha = 0$

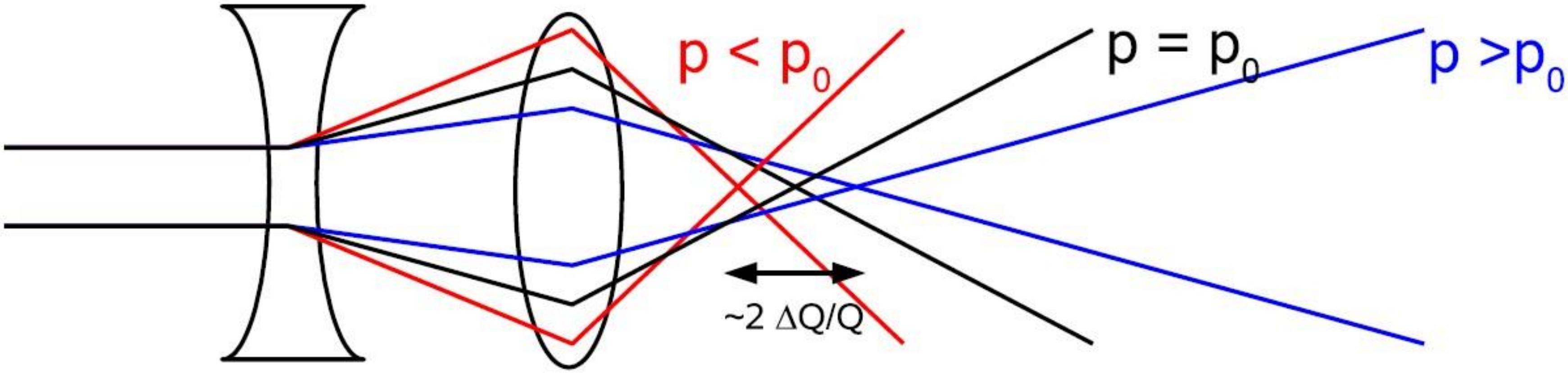
Using: $\beta\gamma - \alpha^2 = 1$

$$A_s = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \cdot \begin{pmatrix} {1}/{\beta_0} & 0 \\ 0 & \beta_0 \end{pmatrix} \cdot \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} {1}/{\beta_0} & {s}/{\beta_0} \\ {s}/{\beta_0} & \beta_0 + {s^2}/{\beta_0} \end{pmatrix}$$

$$\beta_s = \beta_0 + {s^2}/{\beta_0}$$

Sextupoles: A first taste of non-linearities (1/4)

- So far we have completely neglected the longitudinal plane
- Also in the longitudinal plane the beam has an emittance, which means a spread in momentum and a finite length of the bunches.
- We have "off momentum particles" with a longitudinal momentum $\frac{\Delta p}{p_0} \neq 0$.
- We already defined the Dispersion function, which describes the resulting change in orbit
- Now we look at what happens to the focusing in the quadrupoles:



Sextupoles: A first taste of non-linearities (2/4)

- Due to the change in focusing strength of the quadrupoles with varying momentum, particles have different betatron-tunes:

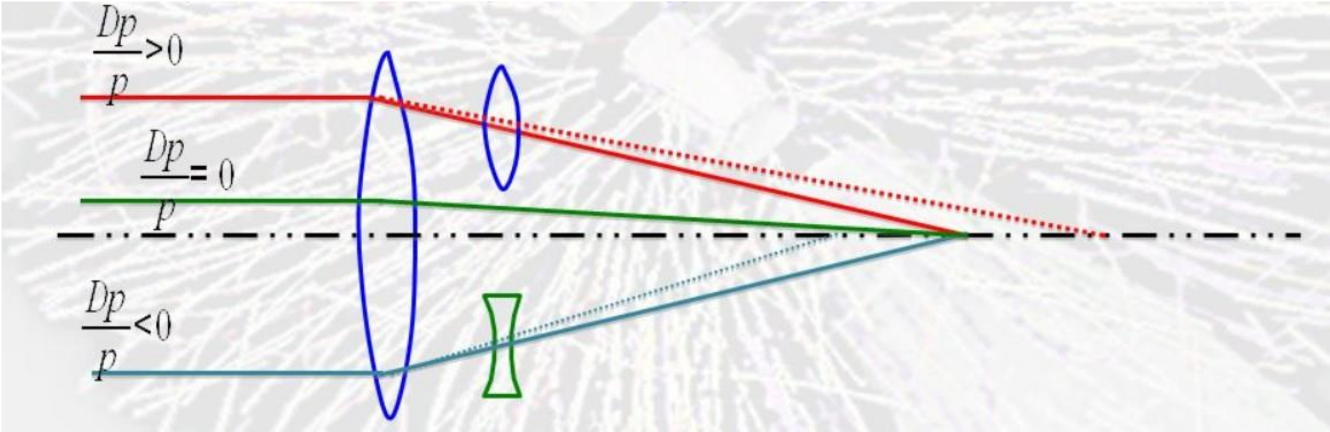
Definition: Chromaticity (H,V) := Dependence of tune on momentum

$\Delta Q = Q' \frac{\Delta p}{p}$ or relative chromaticity $\xi = \frac{Q'}{Q}$

- Is this bad? : Yes, the working point gets a "working blob"
- We need to correct. How?

  i) Inserting a magnetic element where we have dispersion (this separates in space particles with lower and higher momenta

  ii) Having there a "quadrupole", for which the strength grows for larger distances from the centre: a sextupole



Sextupoles: A first taste of non-linearities (3/4)

We will have a high price to pay for this chromaticity correction!
→ we have introduced the first non-linear element into our accelerator

The map M (no longer a matrix) of a single sextupole represents a "kick" in the transverse momentum:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

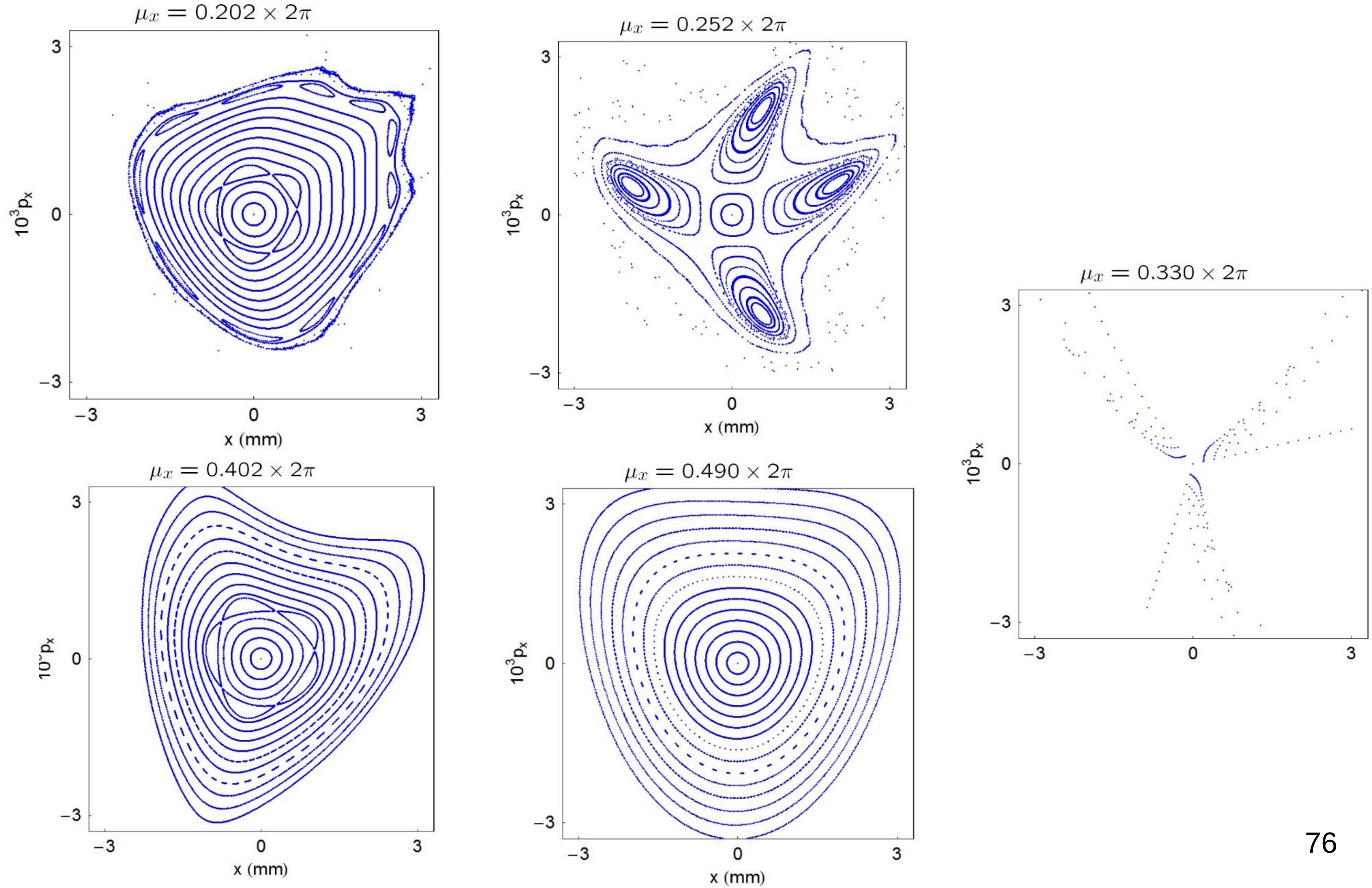
$$\begin{aligned} x &\mapsto x, \\ p_x &\mapsto p_x - \frac{1}{2} k_2 L x^2 \end{aligned}$$

We choose a fixed value $k_2L = -600\ m^{-2}$ and we construct phase space portraits after repeated application of the map.

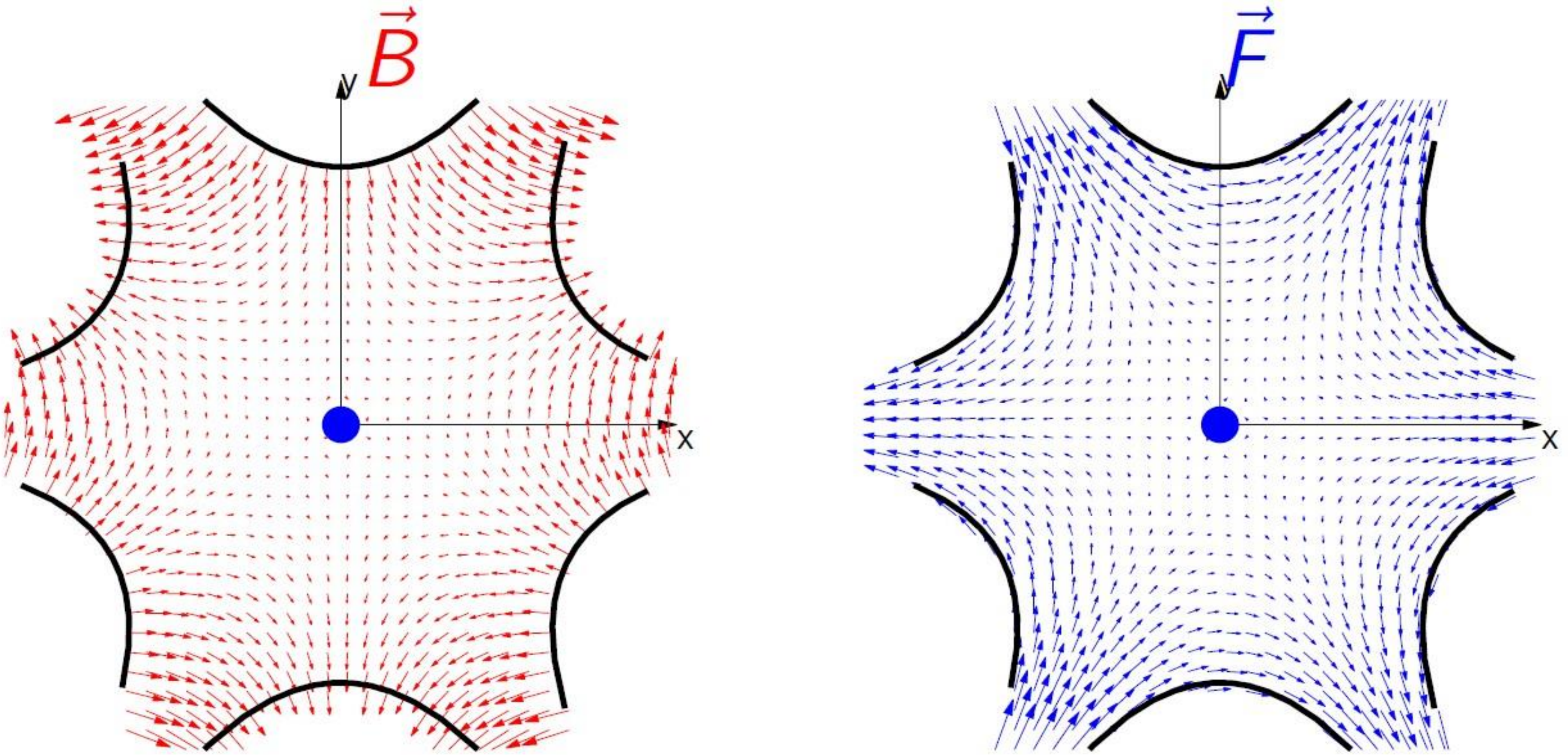
We vary the phase advance per turn (fractional part of the tune) from

$$0.2 \cdot 2\pi \ \ to \ 0.5 \cdot 2\,\pi$$

Sextupoles: A first taste of non-linearities (4/4)



Last not least: Sextupole errors (1/2)

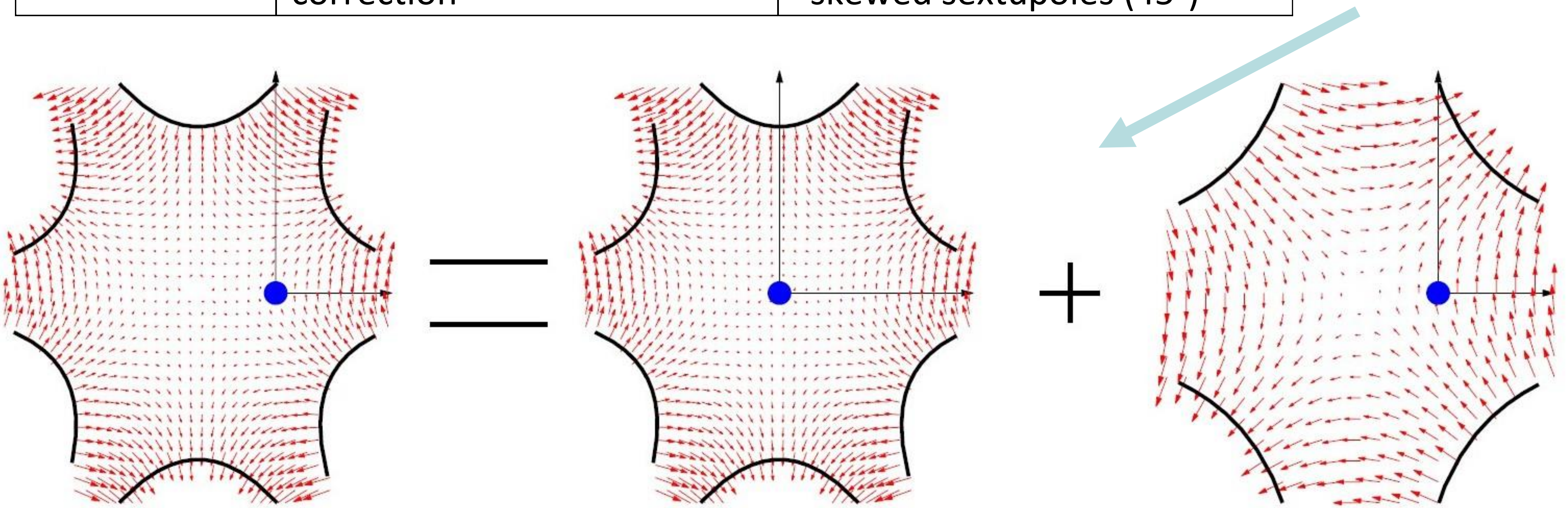


$$F_x = \frac{1}{2}K_2(x^2 - y^2)\ , \quad F_y = -K_2 xy$$

Last not least: Sextupole errors (2/2)

Error type	effect on beam	correction(s)
strength	Change in chromaticity correction, beta-beating	Change excitation current, Repair/Replace magnet
Lateral shift	Extra quadrupole and skew quadrupole, beat-beating, tune change, coupling	Compensation with quadrupoles and skew quadrupoles, realignment
tilt	Error in the chromaticity correction	Excitation of a additional “skewed sextupoles (45^0)

A horizontally (vertically) displaced sextupole is seen as a centred sextupole plus an offset quadrupole (skew quadrupole)



Higher order-pole magnets

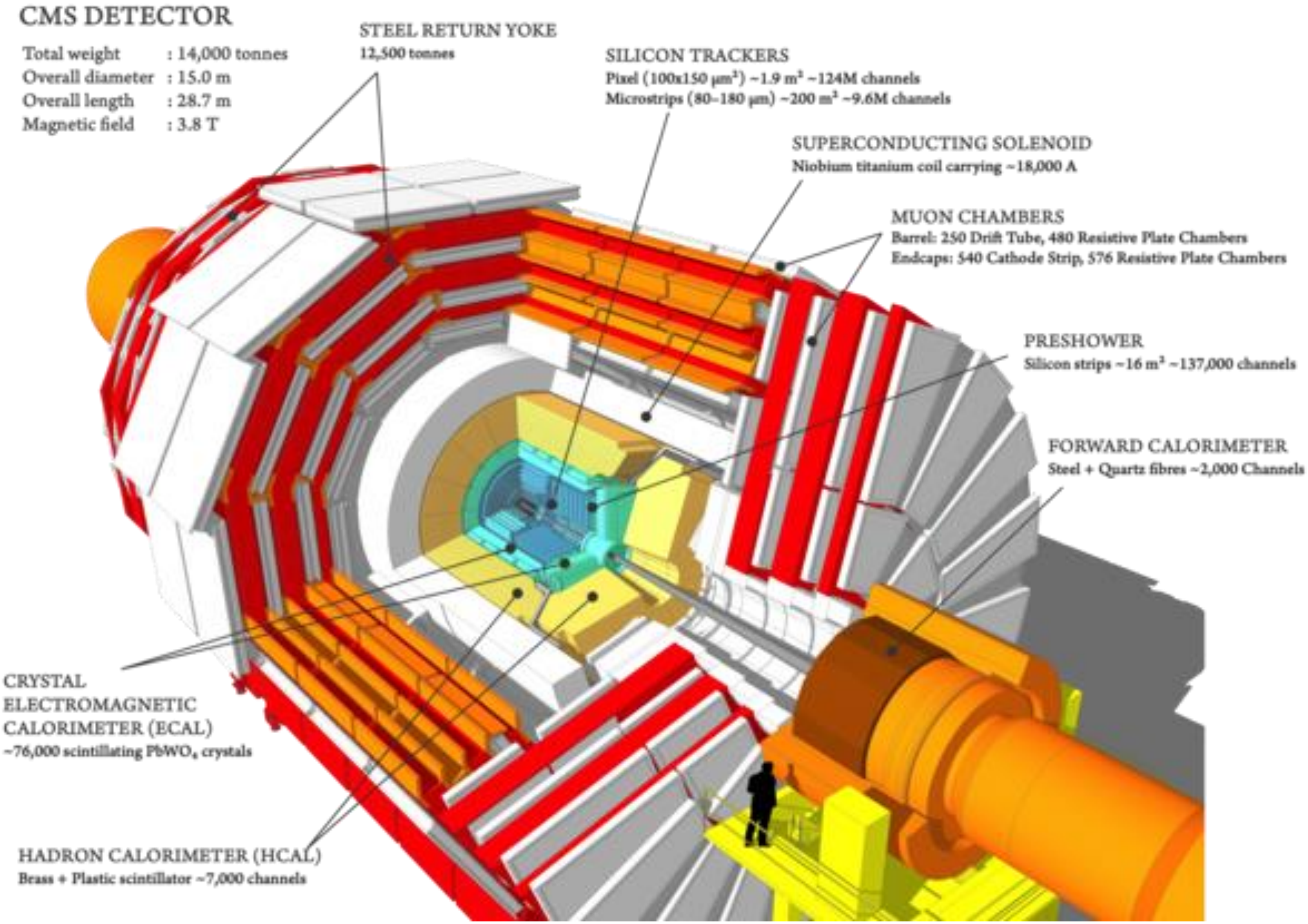
- Octupoles, Decapoles…
- Increasingly stronger non-linearities
- Used in several accelerators to compensate non-linear beam effects
- Beyond the scope of this presentation
- But example: Landau damping

  - take a forest with all trees of same length
  - during a storm resonant behaviour → trees fall
  - make length of trees different → no resonance
  - for beams: make betatron tunes for high amplitude particles different → beam more stable
- → need highly non-linear magnets

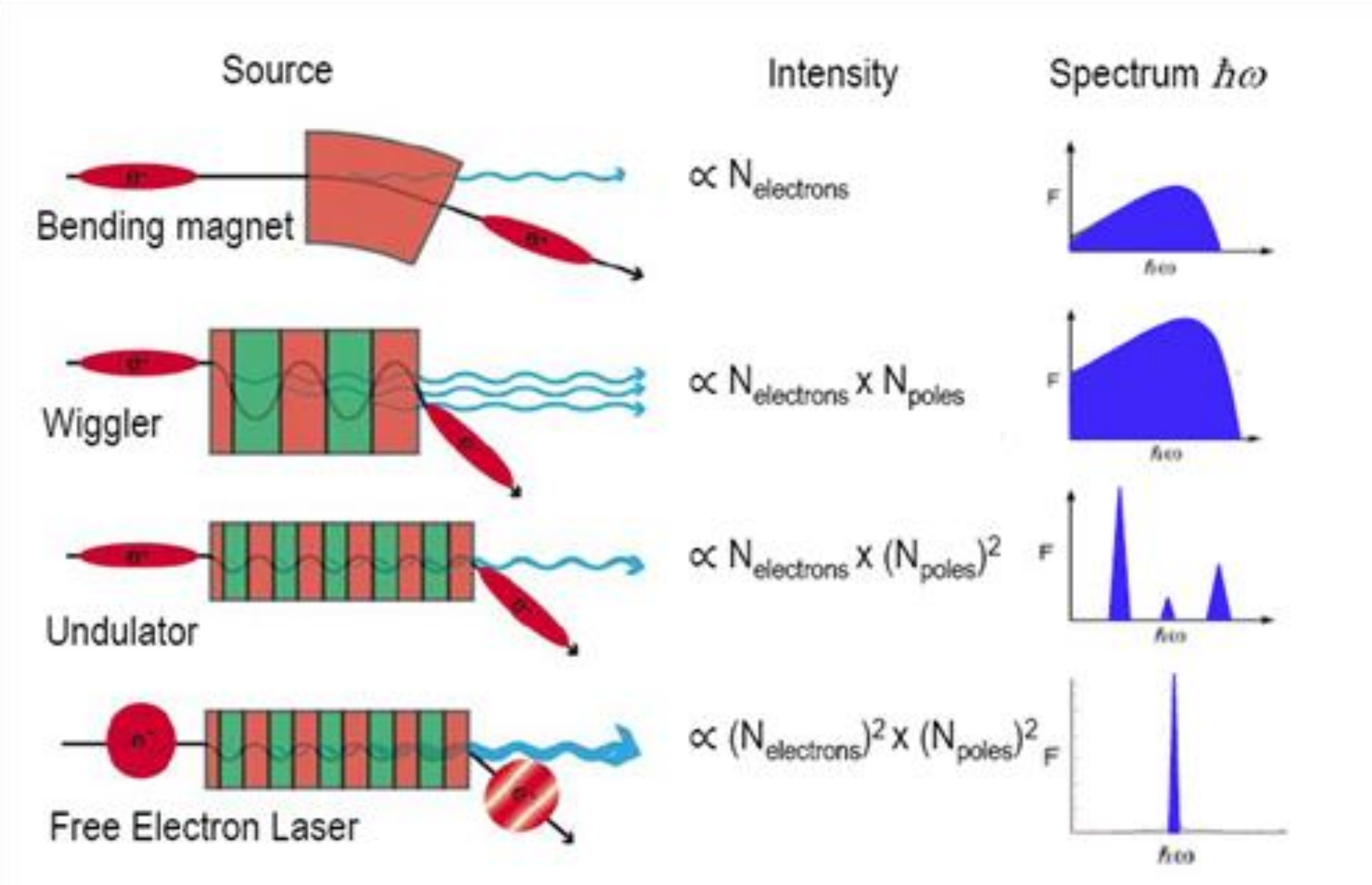
“Other magnets”

- Solenoids
  - Helmholtz coils in beam instrumentation for beam imaging
  - huge detector magnets in colliders for the identification of secondary particles
  (need skewed quadrupoles to compensate coupling introduced by the solenoid field)
- Magnet assemblies
  (sequence of small dipoles, often Permanent magnets) for light production
- Acromats (lightsources)
- Combined function magnets

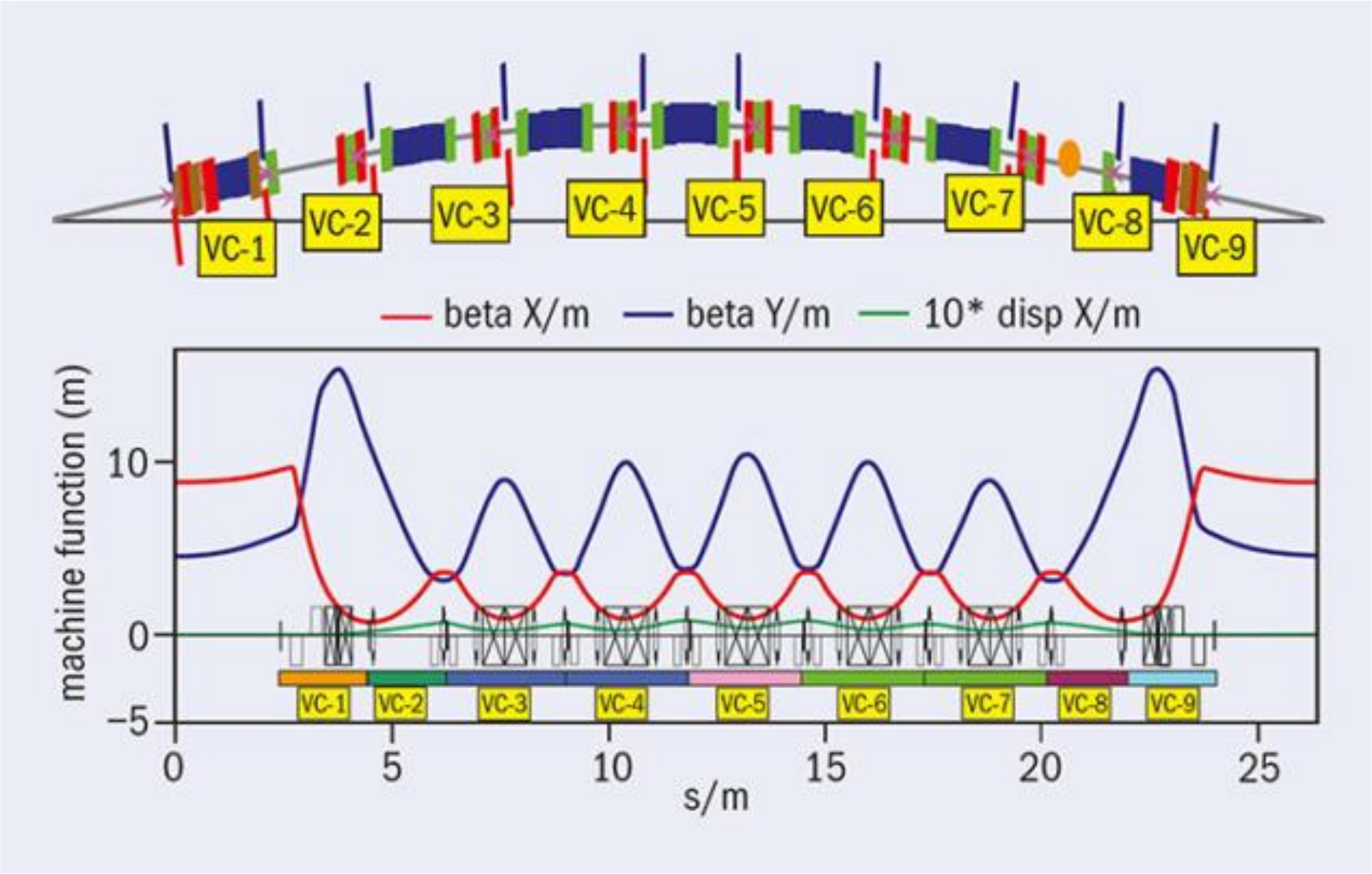
CMS detector at the LHC



Magnets for light production



Max IV : arc layout



Max IV multifunction magnet block

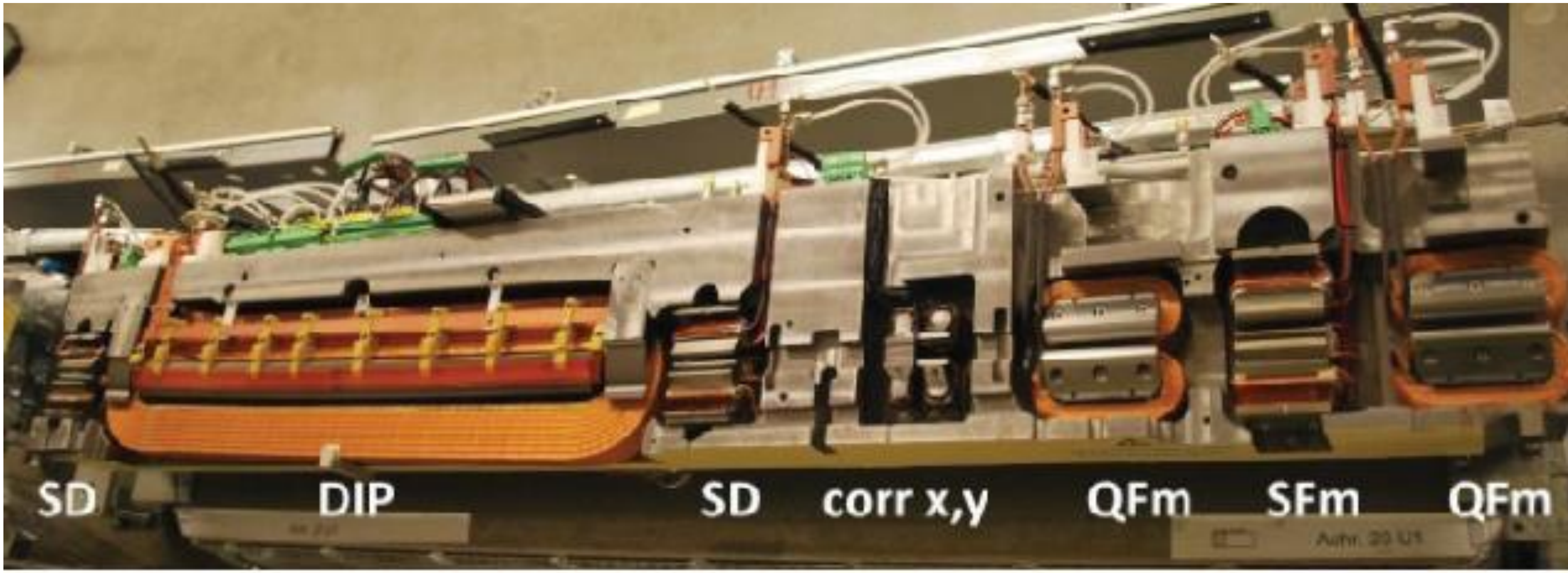
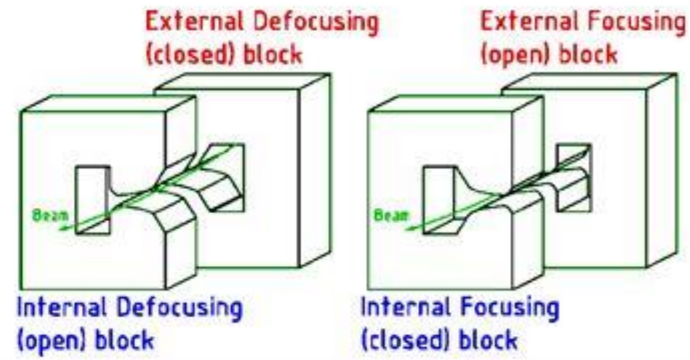


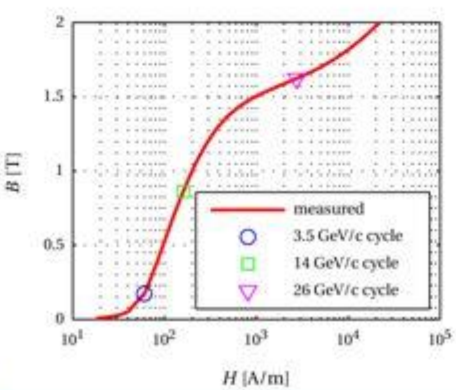
Figure 1: U1 magnet block bottom half.

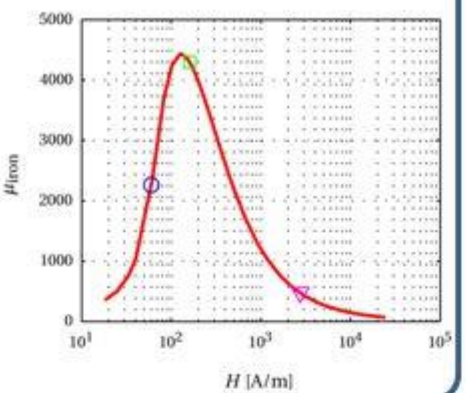
PS main magnetic unit

- Combined-function magnet with hyperbolic pole shape (4 types)



- Saturation of iron magnetization





- Complex geometry of coils system