Overview: Design of Accelerator Magnets

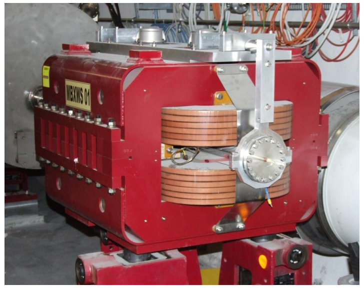
Stephan Russenschuck, CAS, 2023

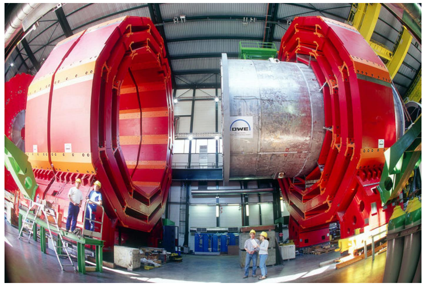
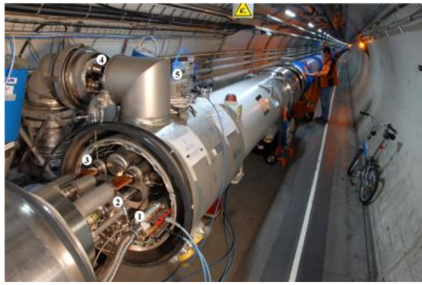
Magnet Types

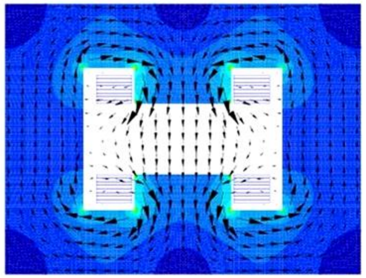
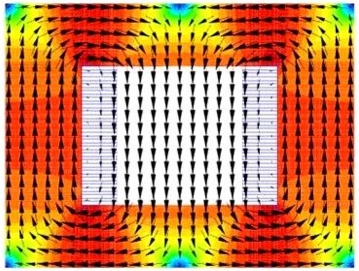
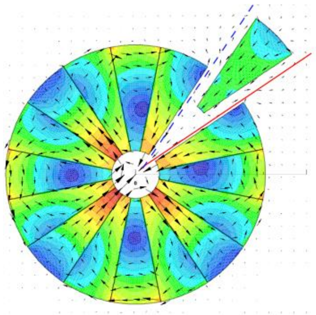
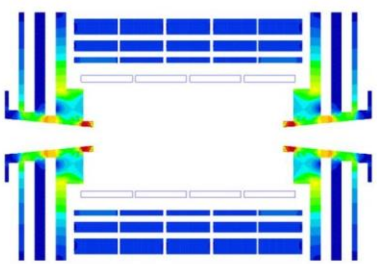
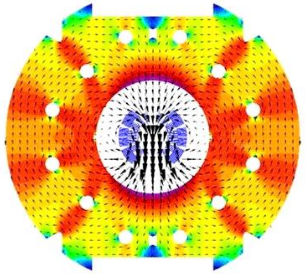
Iron dominated

Coil dominated

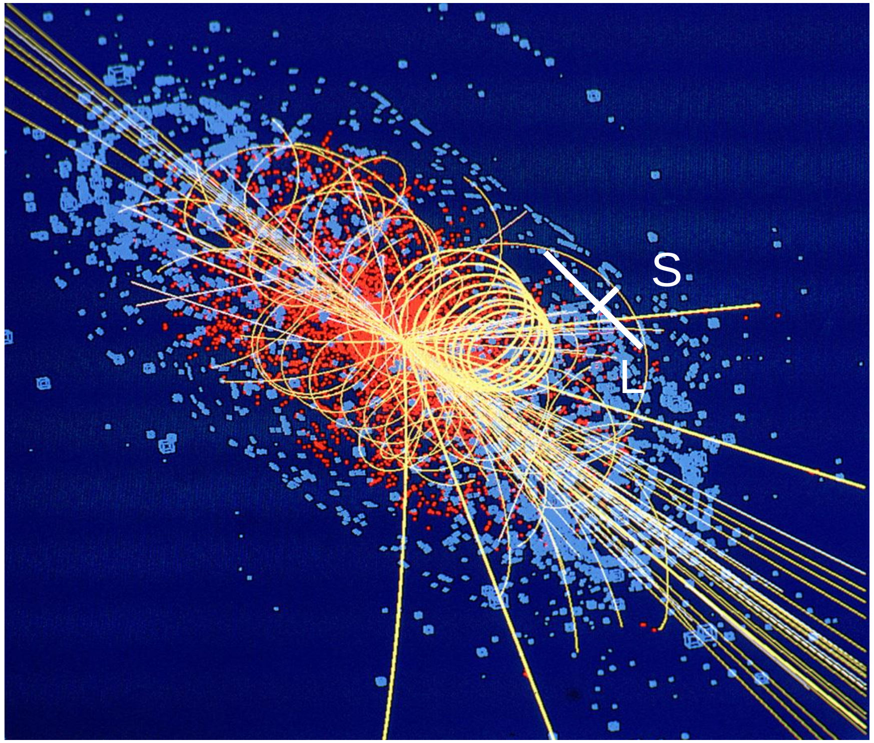
Normal conducting	Superferric	Permanent magnet	Class 1 large area, “medium” field	Class 2 Small area high **B**, high **J**



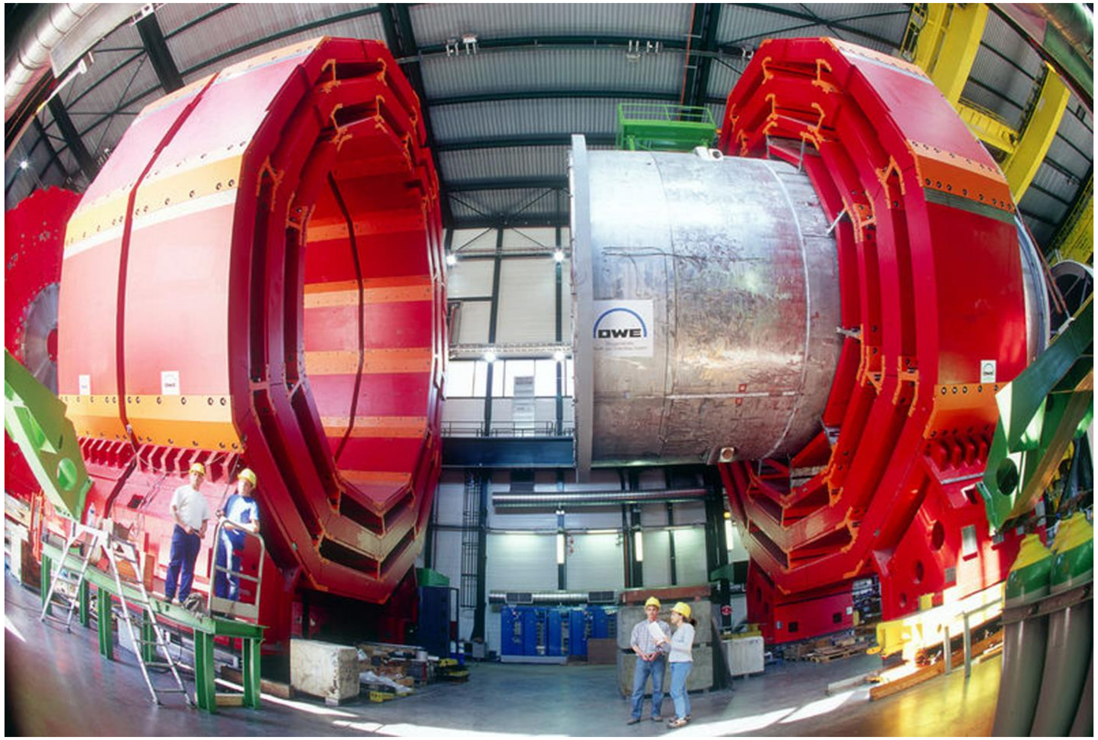




Solenoidal Magnet System for CMS (Superconducting Class 1 Magnet)

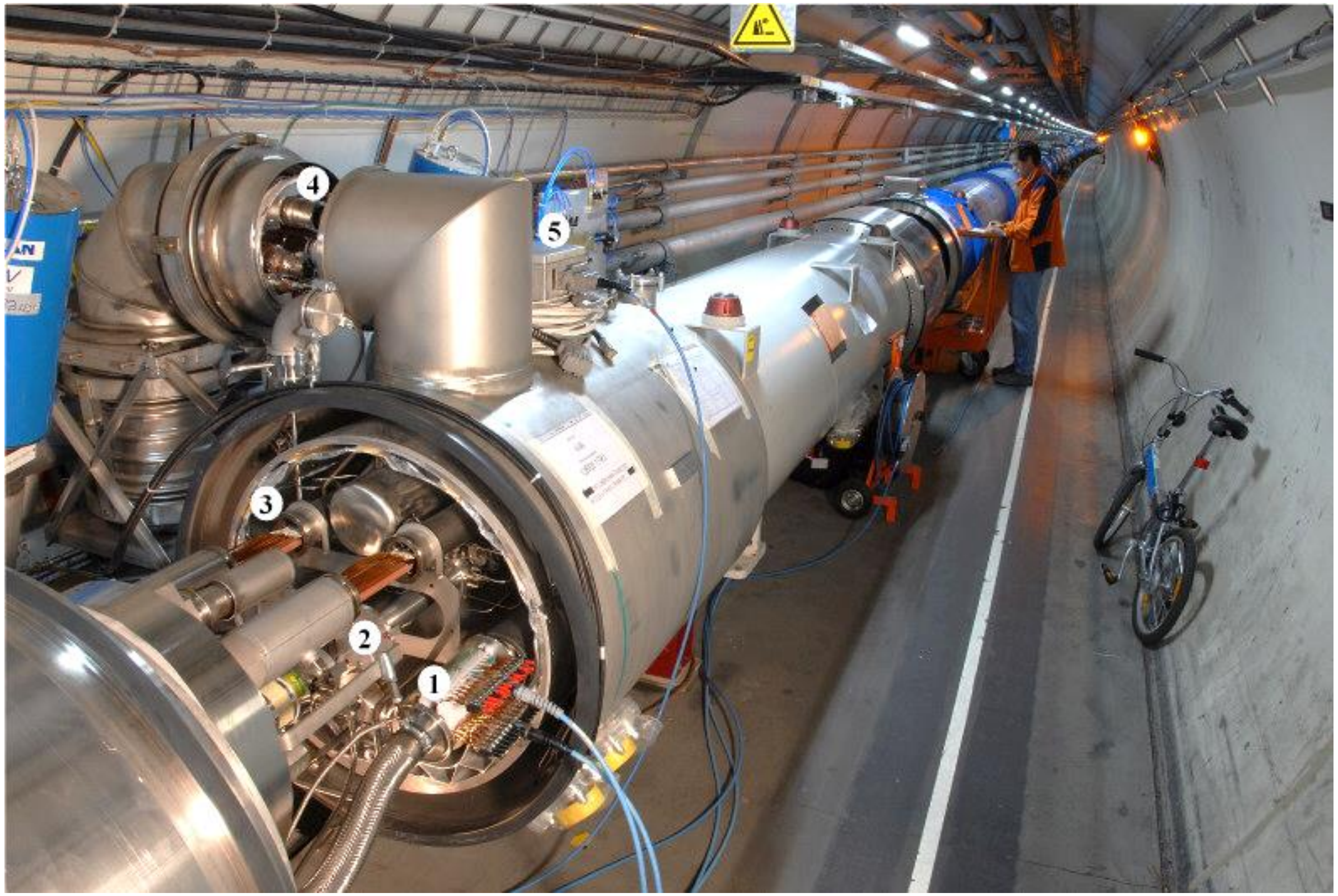


$$S = R\left[1 - \cos\left(\frac{\phi}{2}\right)\right] \approx \frac{R\phi^2}{8} = \frac{L^2}{8R} = \frac{eB_0L^2}{8p}$$



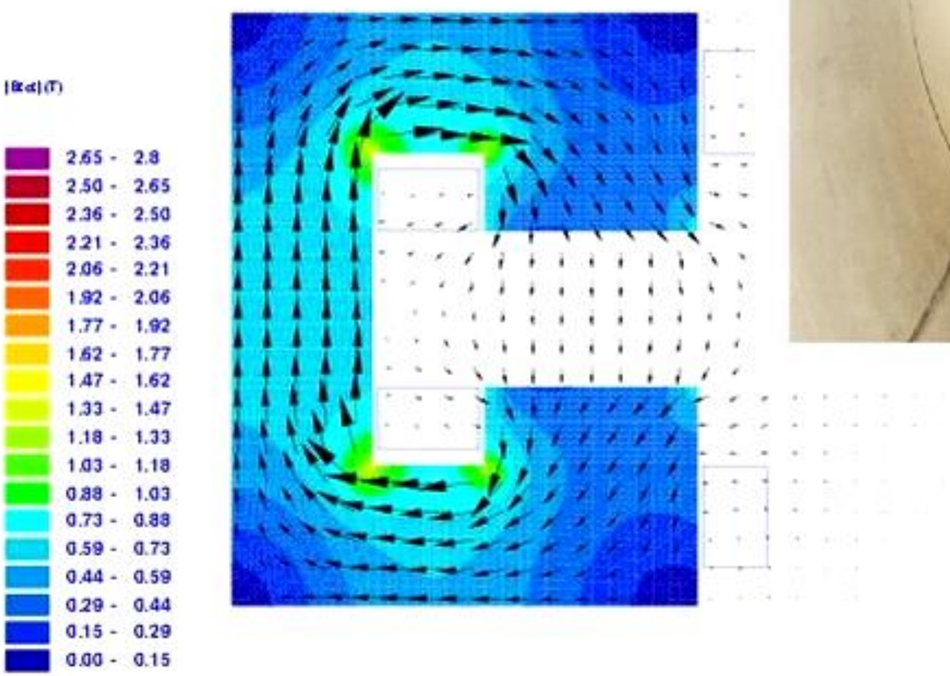
String of LHC Magnets in the Tunnel (Class 2 Magnets)

$$\{p\}_{\mathrm{GeV/c}} \approx 0.3\{Q\}_e\{R\}_\mathrm{m}\{B_0\}_\mathrm{T}$$



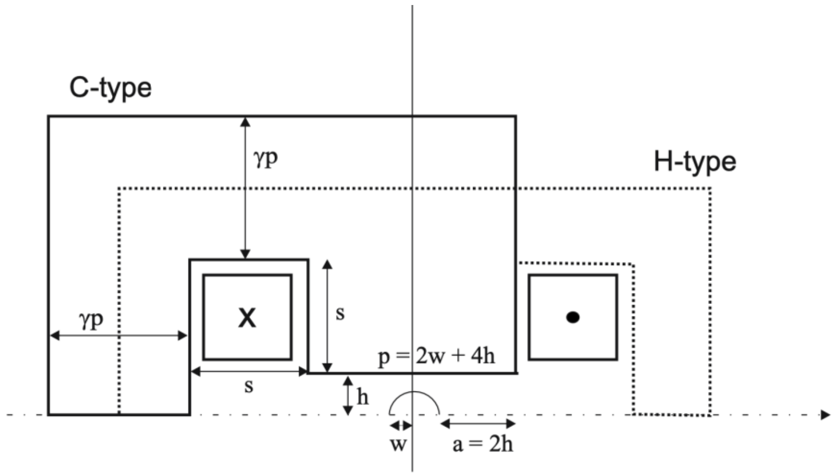
High field and high current density

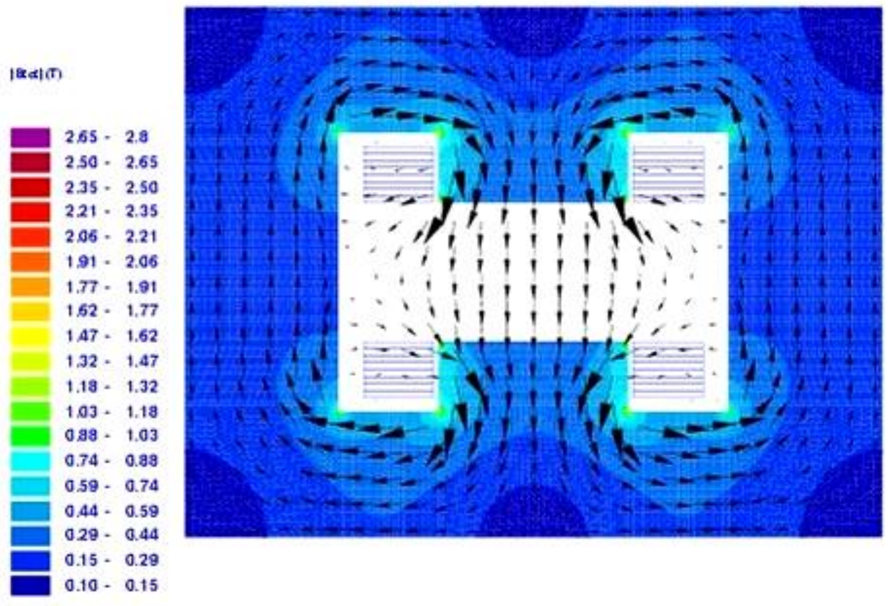
LEP Dipole (Iron Dominated Magnet)

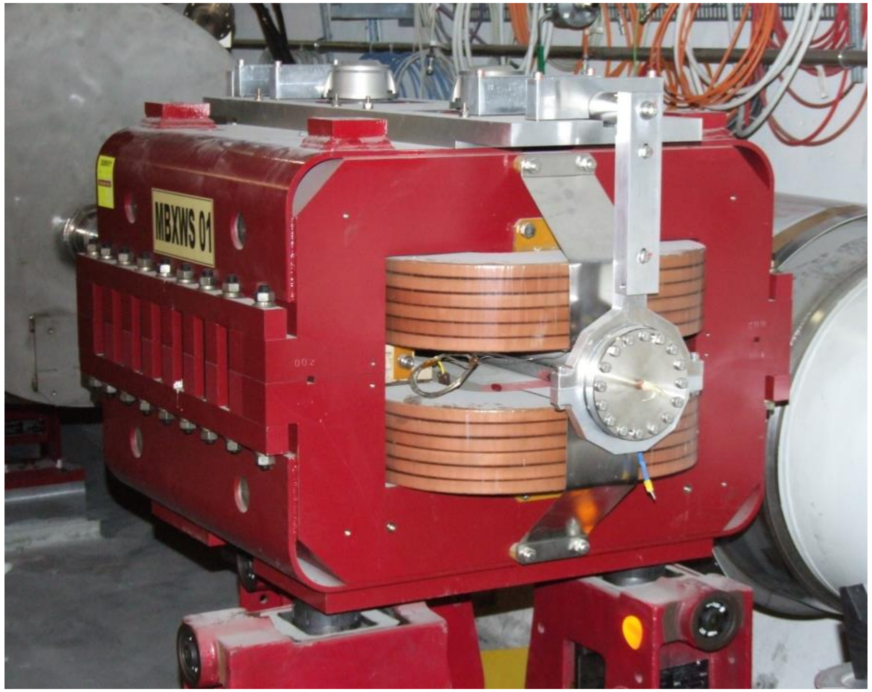


$N \cdot I = 4480$ A $B_1 = 0{,}13$ T $B_S = 0{,}042$ T Fill.fac. 0.27

H Magnet (LHC Transfer Line)

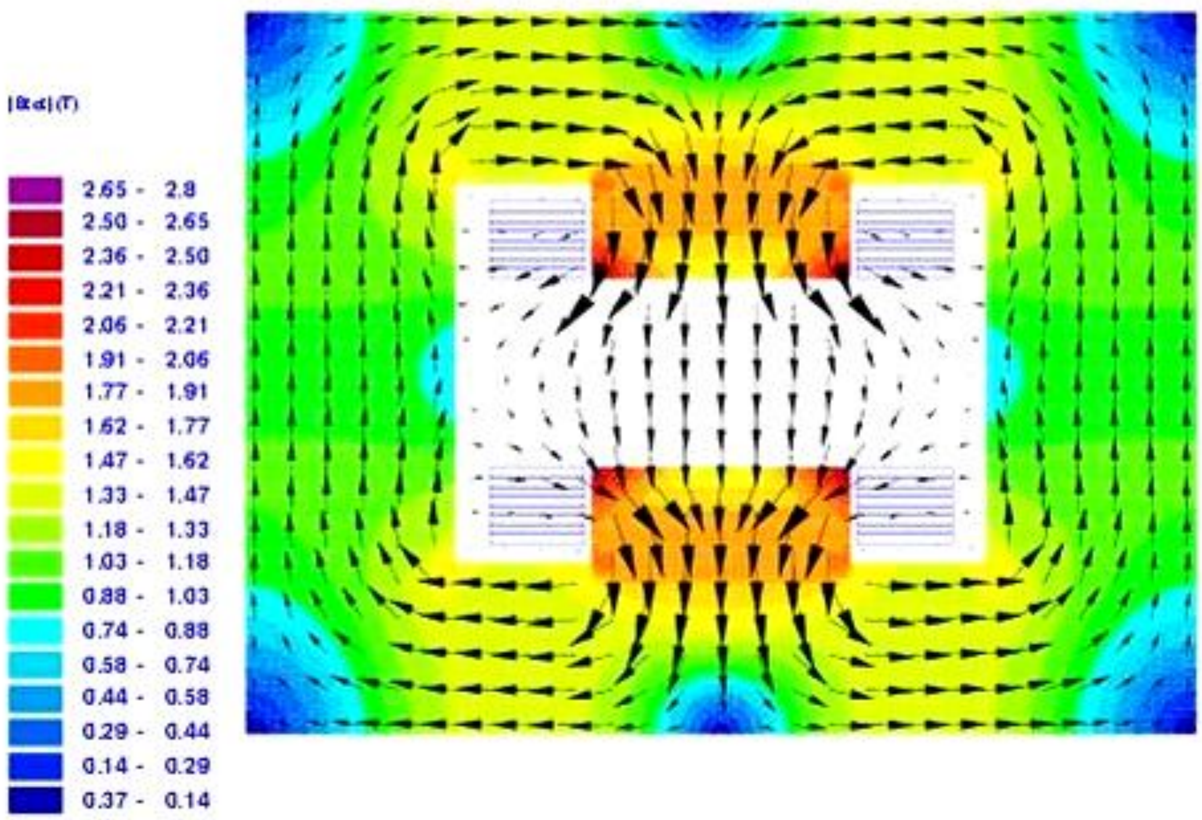






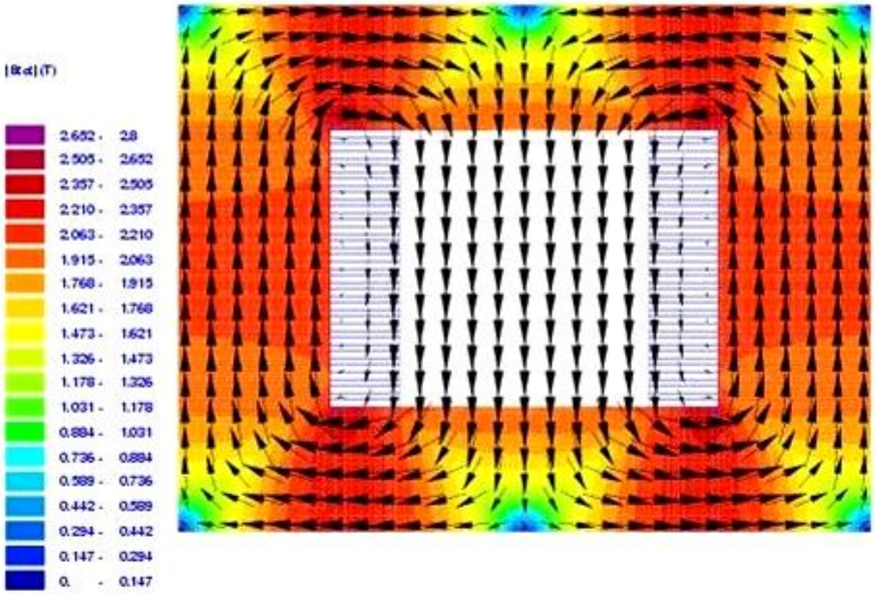
$N \cdot I = 24000$ A $B_1 = 0.3$ T $B_s = 0.065$ T Fill.fac. 0.98

Super-Ferric H Magnet

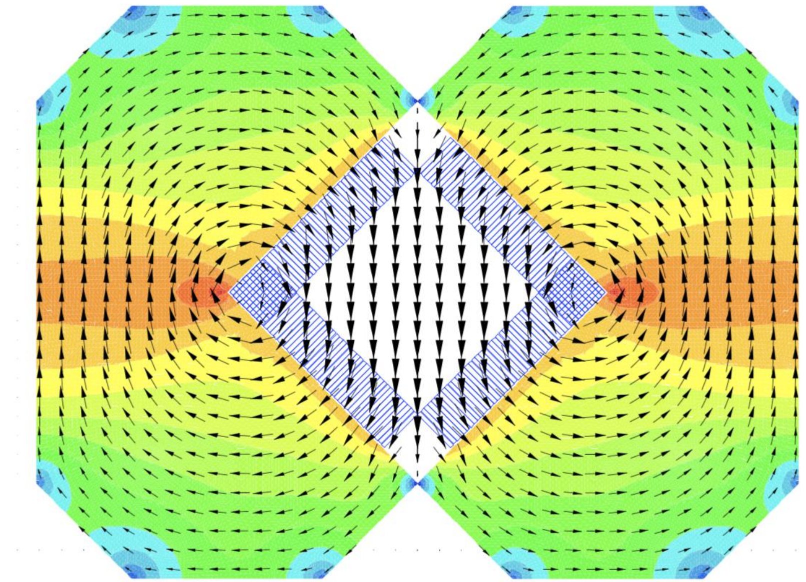


$N \cdot I = 96000$ **A** $\quad B_1 = 1.18$ **T** $\quad B_S = 0.26$ **T**

Window Frame Magnet

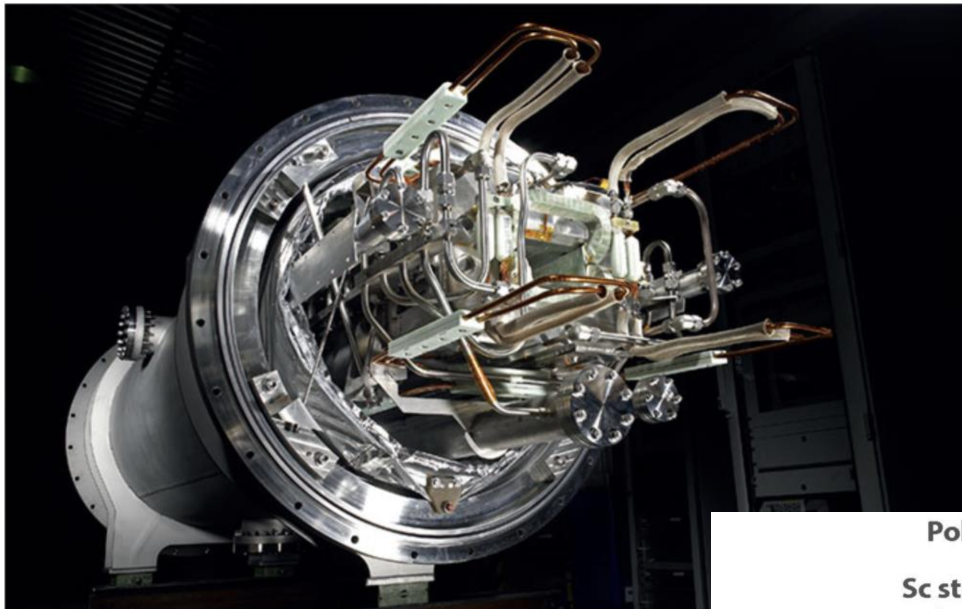


$N \cdot I = 360$ kA, $B_t = 2.0$ T, $B_s = 1.04$ T



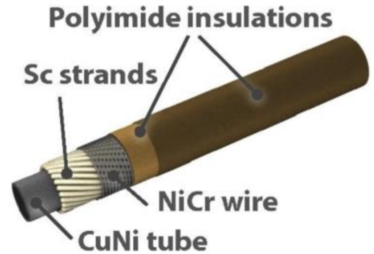
$N \cdot I = 625$ kA, $B_t = 2.38$ T, $B_s = 1.36$ T

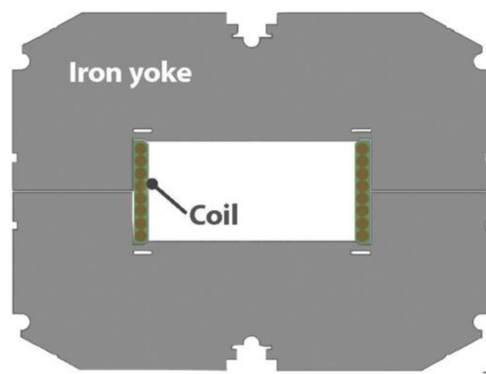
Example: SIS 100 Magnets

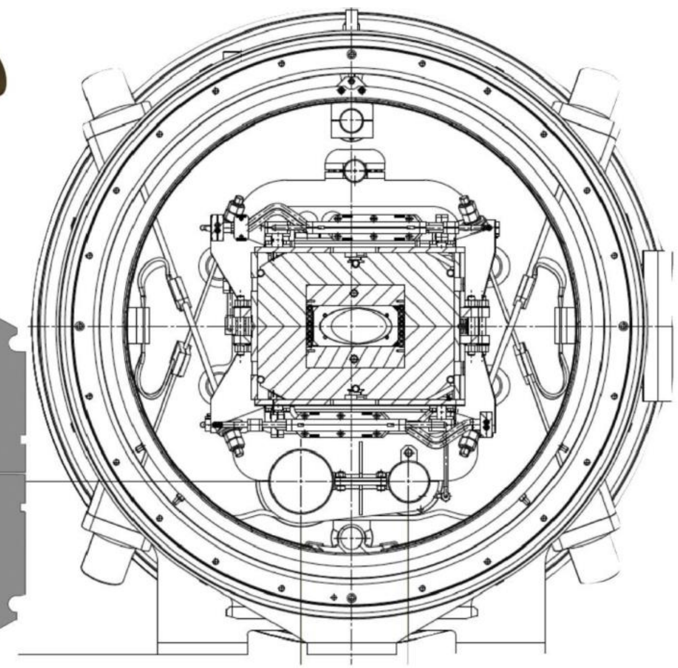


The newly developed fast-cycling superconductiong dipole magnet
for FAIR's SIS100 synchrotron.
Image credit: Babcock Noell GmbH.

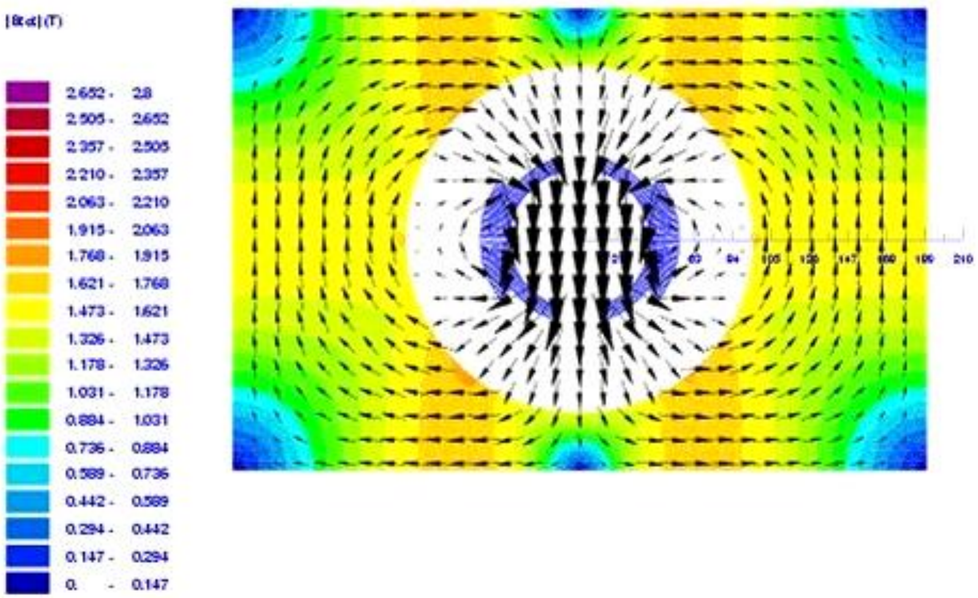
Parameter	Unit	Value
Max. field strength B_1	T	1.9
Max. current	kA	13.1
Ramp rate	T/s	4
Magnetic field quality		$\pm 6 \times 10^{-4}$

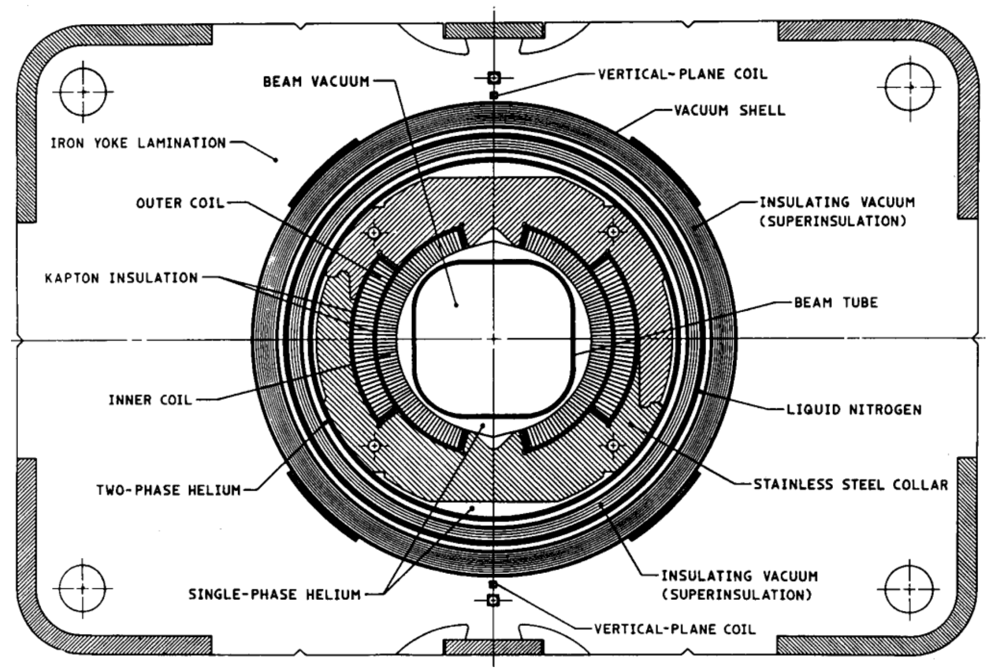






Cos θ (Warm iron yoke) - Tevatron Dipole (Coil Dominated Magnet)

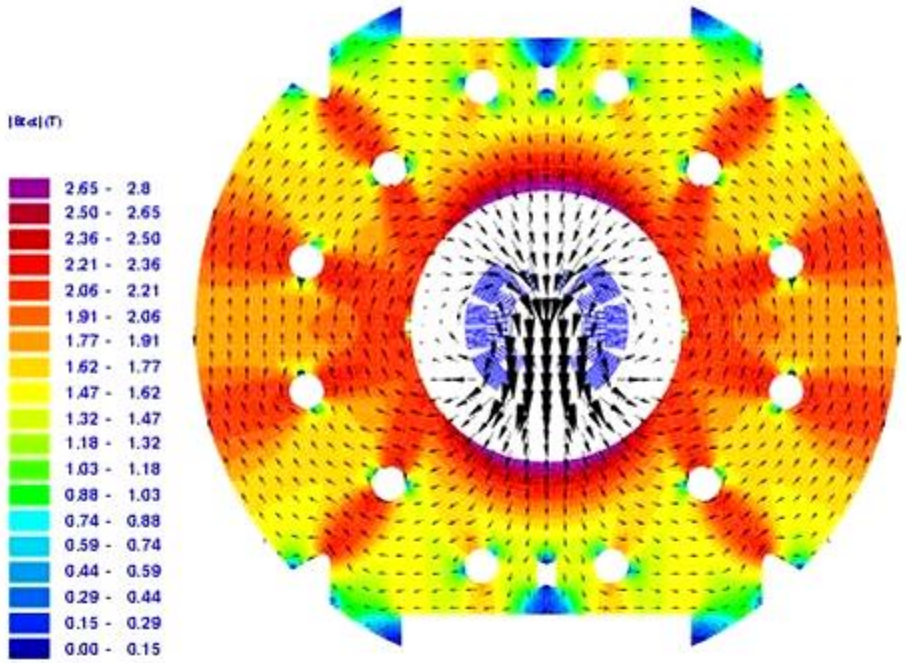




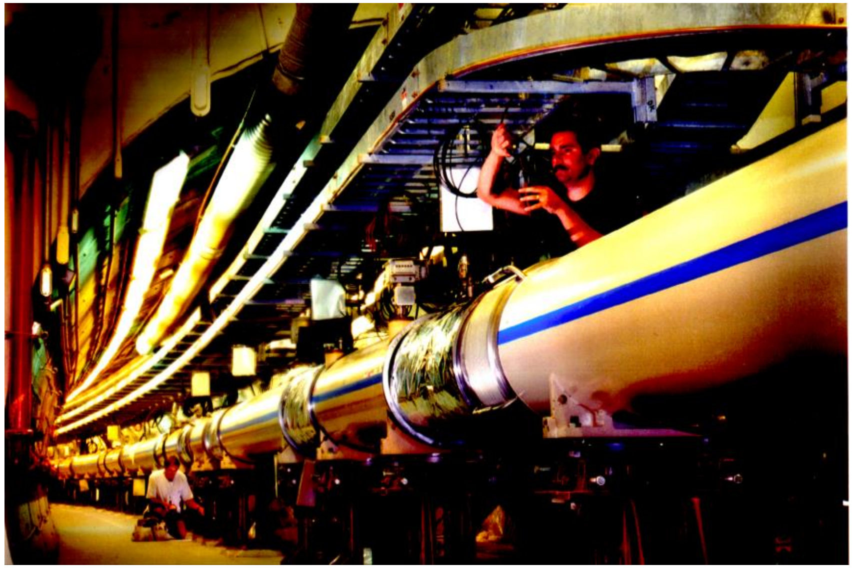
$N \cdot I = 471000$ A $\quad B_1 = 4.16$ T $\quad B_S = 3.39$ T

Notice the lower field in the iron yoke compared to the window frame

LHC Coil Test Facility for LHC (Based on HERA/RHIC Magnet Technology)

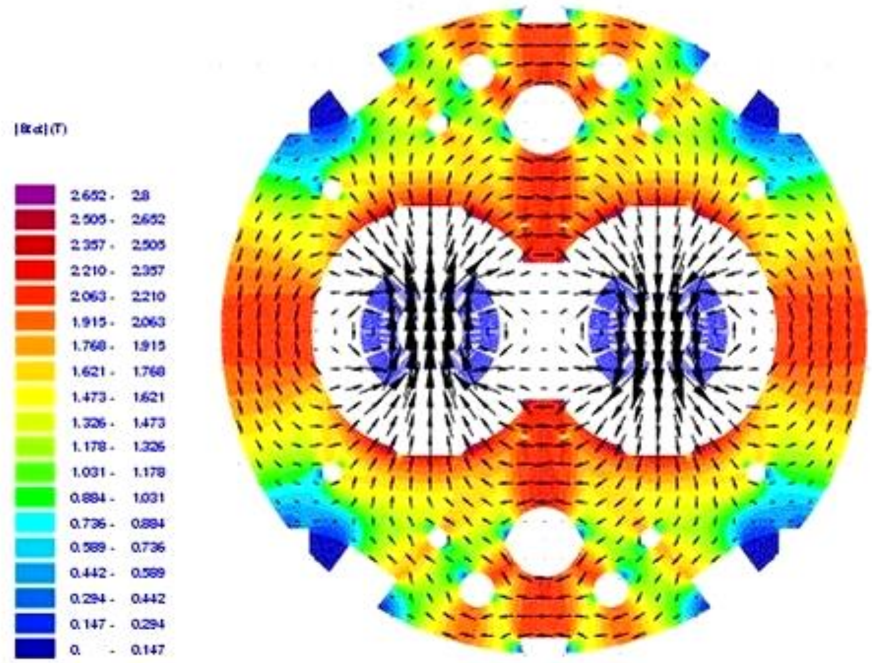


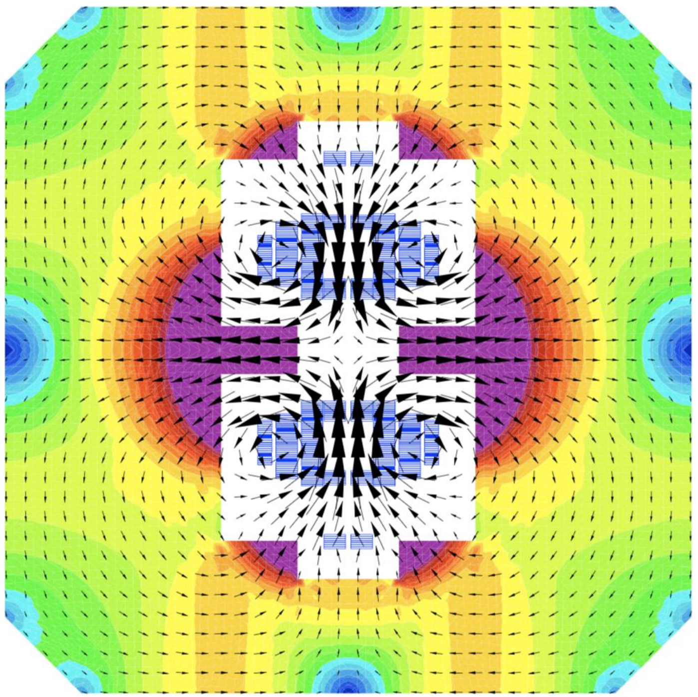
$N \cdot I = 960000$ A $\qquad B_1 = 8.33$ T $\qquad B_s = 7.77$ T



Two-In-One Dipoles

$$N \cdot I = 2 \cdot 944 \text{ kA}, B_t = 8.32 \text{ T}, B_s = 7.44 \text{ T}$$





$$N \cdot I = 2 \cdot 1034 \text{ kA}, B_t = 8.34 \text{ T}, B_s = 7.35 \text{ T}$$

Conventional and Superconducting Accelerators Magnets

- Normal conducting magnets
  - Important Ohmic losses require (water) cooling
  - The field is defined by the iron pole shape (max 1.5 T)
  - Easy electrical and beam-vacuum interconnections
  - The voltage drop over one coil of the MBW magnets = 22 V

- Superconducting magnets
  - The field is dominated by the coil layout
  - The maximum field is limited to 10 T (Nb-Ti), 14 T (Nb_3Sn)
  - Strong electromagnetic forces (400 tons/m in MB for LHC)
  - Quench detection and magnet protection system is required
  - Cryogenic installation (1.8 K)
  - Electrical interconnections in cryo-lines
  - Voltage drop on LHC magnet string at nominal ramp rate (154 MB) 155 V

A Multiphysics Problem

- Beam physics – H. Schmickler
- Material science: superconducting cable, steel, insulation
  - M. Eisener (technical SC LTS), A. Ballariono (technical SC HTS), R. Piccin (dielectric insulation), S. Sgobba (steel), G. Le Bec (permanent magnets)
- Mechanics and large-scale mechanical engineering
  - F. Toral (SC magnet design – mechanical), S. Izquierdo Bermudez (fabrication)
- Vacuum technology
- Heat transfer – R. van Weelderen
- Metrology and alignment – P. Bestmann
- Field measurements
  - M. Buzio (overview), M. Liebsch (mapping techniques), L. Fiscarelli (coil magnetometers)

A Multiphysics Problem

- Electrical engineering (power supplies, leads, buswork, quench detection, and magnet protection)
  - S. Yammine (powering infrastructure), E. Todesco (quench)
- Analytical and numerical field computation
  - H. De Gersem (field computation using FEM), S. Farinon (SC magnet design), A. Milanese (normal conducting magnet design), E. Todesco (hysteresis and dynamic effects)
- Bringing it all together
  - A. Bernhard (insertion devices), G. Le Bec (permanent magnets), L. Bottura (collider magnets), J. Borbough (injection and extraction), F. Toral (low emittance rings)

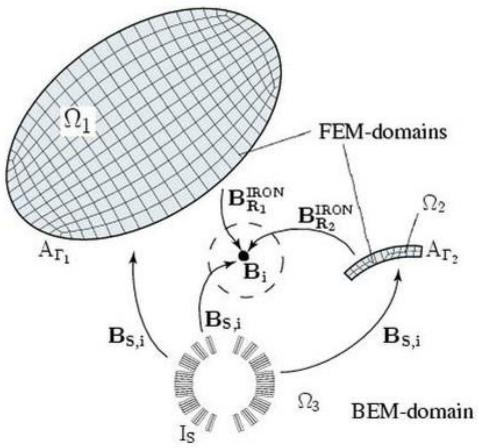
Comparison NC and SC Magnets (EM-Design)

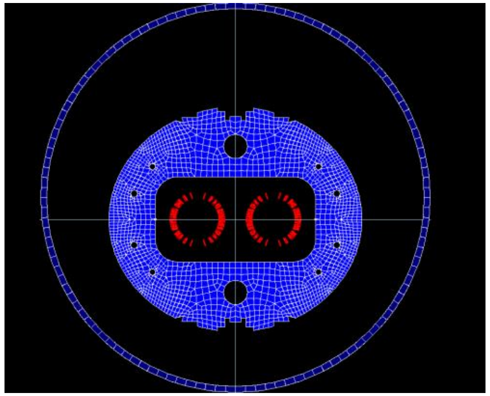
- **Normal conducting (iron-dominated) magnets**
  - Ideal pole shape is known from potential theory
  - One-dimensional (analytical) field computation for the main field
  - Commercial FEM software can be used as a black box
  - Hysteresis modeling, eddy currents, and combined 3D effects

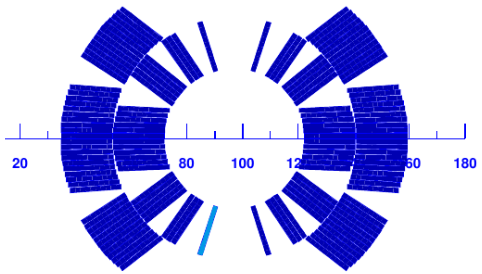
- **Superconducting (coil dominated) magnets**
  - Accuracy of the field solution
  - Modeling of the coils
  - Decoupling of coil and yoke optimization
  - Filament magnetization
  - Dynamic effects (interfilament and interstrand coupling currents)
  - Quench simulations

The CERN Field Computation Program ROXIE

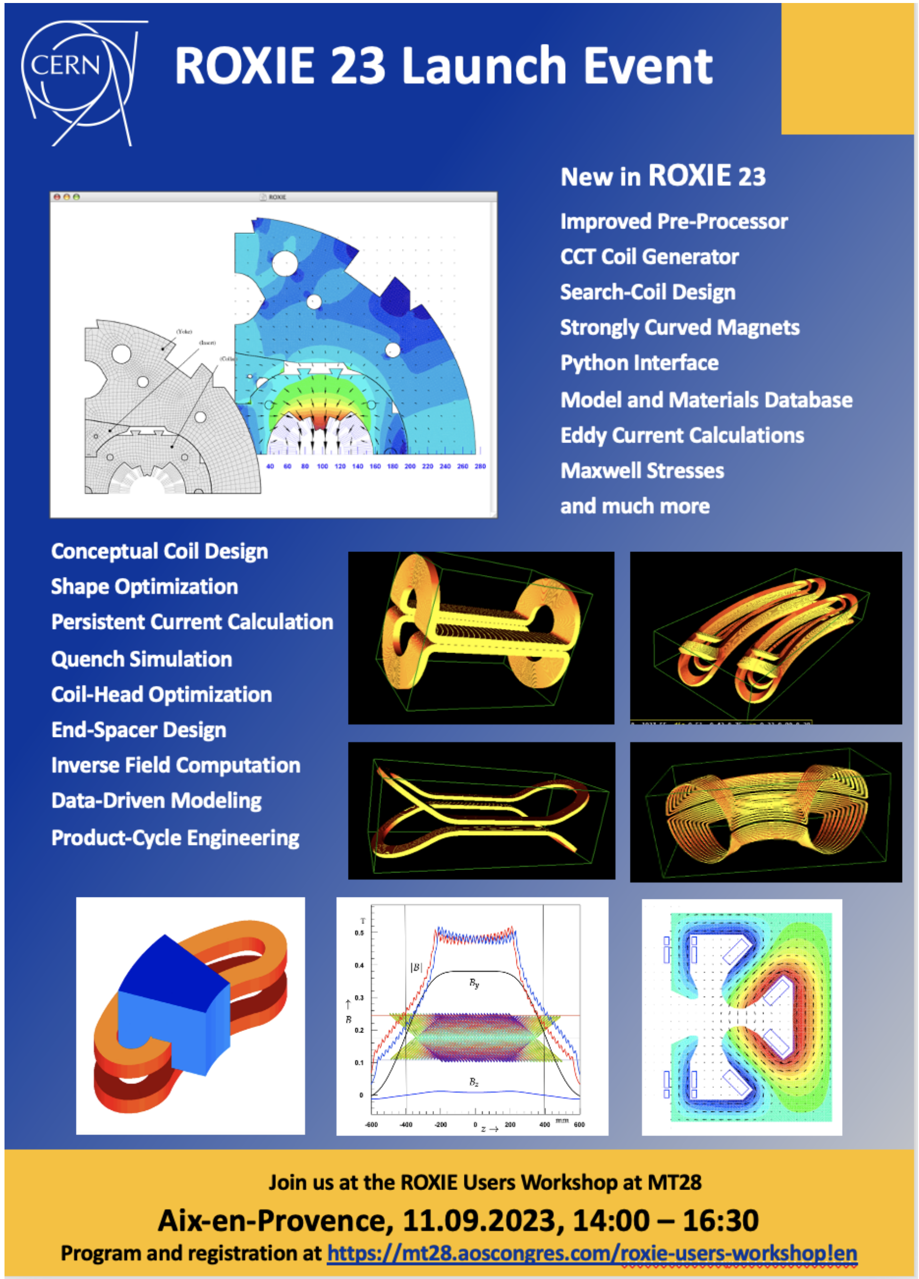
- **Automatic generation of coil and yoke geometries**
  - Feature-based design
- **Field computation especially suited for magnet design (BEM-FEM)**
  - No meshing of the coil, no artificial boundary conditions
  - Higher order quadrilateral meshes, parametric mesh generator, morphing
  - Modeling of superconductor magnetization
  - Permanent magnets
  - Quench simulation of long accelerator magnets (2.5 D)
- **Mathematical optimization techniques**
  - Genetic optimization, Pareto optimization, Search algorithms
- **Simulation of magnetic measurements**
- CAD/CAM interfaces







New Version 2023



CCT Magnets

Search-Coil Design

Curved Magnets

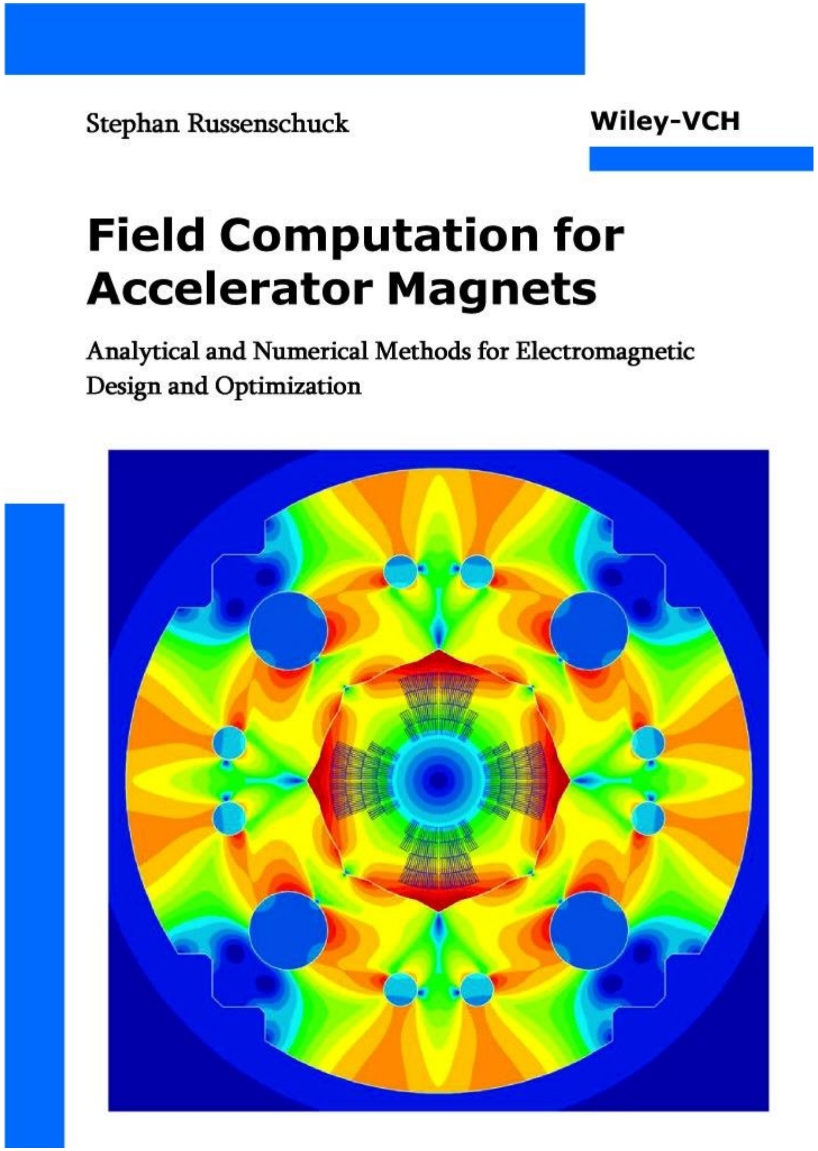
Maxwell-Stresses

Improved Pre-Processor

Eddy-Current Solver

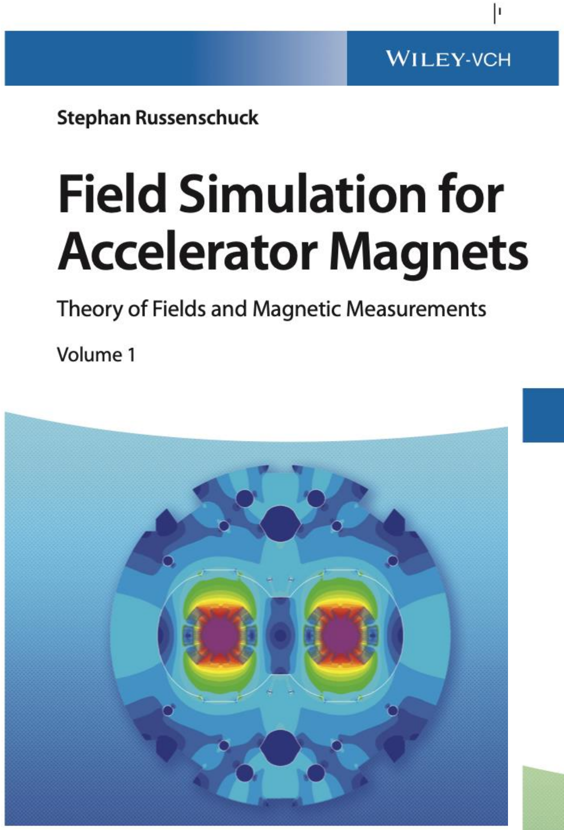
Analytical and Numerical Field Computation

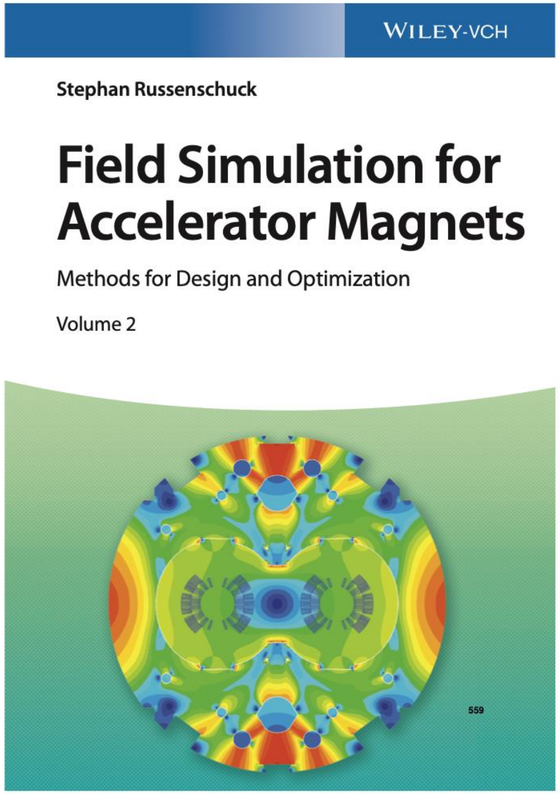
- Linear algebra
- Vector analysis
- Harmonic fields
- Green's functions and the method of images
- Complex analysis
- Differential geometry
- Numerical field computation
- Hysteresis modeling
- Coupled (thermal, magnetic, electric) systems
- Mathematical optimization



New Edition, Summer 2024

- ➔ Field harmonics
  - Toroidal harmonics
  - Pseudo-multipoles
- ➔ Coil Magnetometers
- ➔ Stretched-Wire Measurements
- ➔ Synchrotron Radiation
- ➔ Faraday Paradoxes
- ➔ Iron-dominated magnets
  - Wigglers and Undulators
- ➔ Coil-dominated magnets
  - CCT Magnets
  - Strongly curved magnets





Mathematical Foundations of Magnet Design

Maxwell Equations

Integral Form

- Lumped circuit calc. of NC magnets

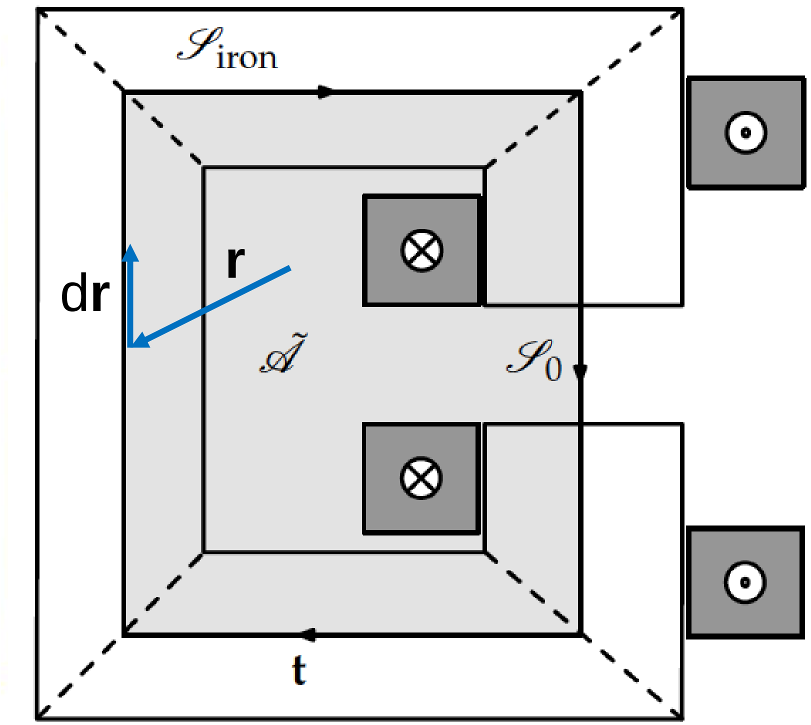
Local Form

- Laplace's Equation
  - Harmonic Fields
    - Field quality in Accelerator magnets
  - Green's Functions
    - The field of line-currents Coil-dominated magnets
- Curl-Curl Equation
  - Weak-Forms
    - FEM
  - Kichhoff's Theorem
    - BEM

Global Form

- DEM

Main Field in Normal Conducting Dipole



Ampere`s law

$$\int_{\partial\tilde{\mathscr{A}}} \mathbf{H}\cdot\mathrm{d}\mathbf{r} = \int_{\tilde{\mathscr{A}}} \mathbf{J}\cdot\mathbf{n}\,\mathrm{d}a\,,$$

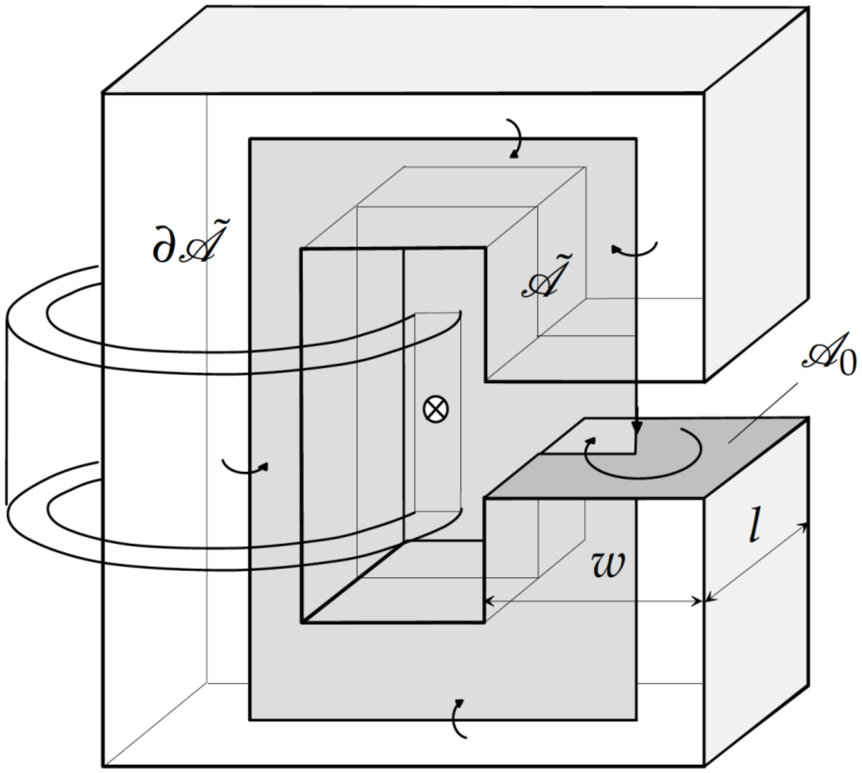
$$\int_{\mathscr{S}_{\text{iron}}} \mathbf{H}\cdot\mathrm{d}\mathbf{r} + \int_{\mathscr{S}_0} \mathbf{H}\cdot\mathrm{d}\mathbf{r} = \int_{\tilde{\mathscr{A}}_{\text{coil}}} \mathbf{J}\cdot\mathbf{n}\,\mathrm{d}a\,,$$

$$H_{\text{iron}}\, s_{\text{iron}} + H_0\, s_0 = N\, I\,,$$

$$\frac{1}{\mu_0\mu_{\text{r}}} B_{\text{iron}}\, s_{\text{iron}} + \frac{1}{\mu_0} B_0\, s_0 = N\, I\,,$$

$$B_0 = \frac{\mu_0 N\, I}{s_0}\,.$$

Dipole with Varying Cut-Section



$$\sum_{i=0}^{n} H_i s_i = N\,I$$

$$H_i = \frac{B_i}{\mu_i} = \frac{\Phi}{a_i\,\mu_i}$$

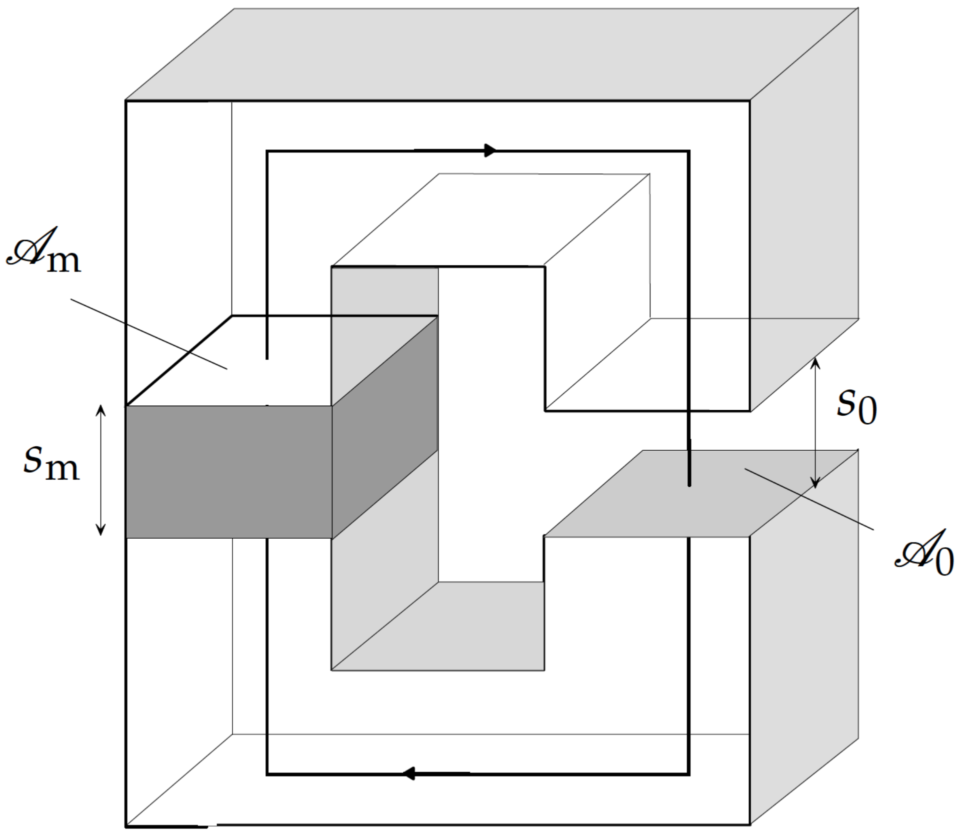
$$\Phi \sum_{i=0}^{n} \frac{s_i}{a_i\,\mu_i} = N\,I = V_\mathrm{m}$$

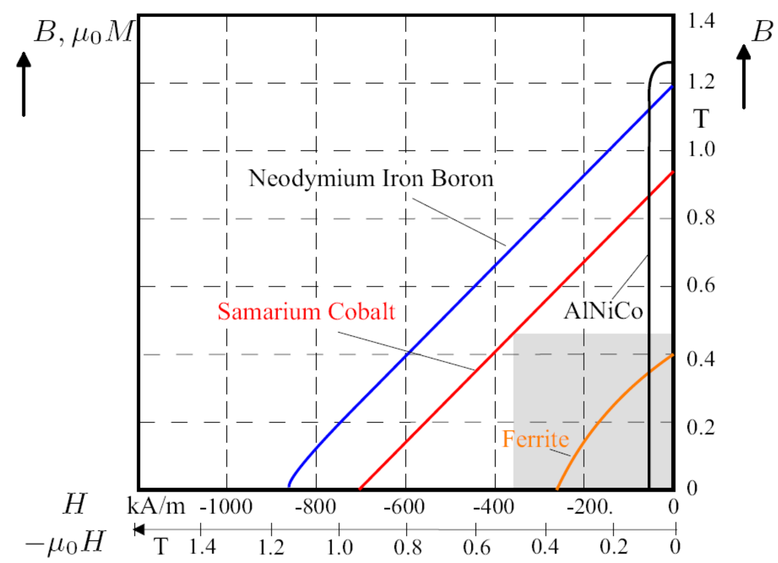
Ohm's law: $I \sum_{i=0}^{n} \frac{s_i}{a_i\,\kappa_i} = U$

$$N\,I = \Phi \sum_{i=0}^{n} \frac{s_i}{a_i\,\mu_i} = \Phi \left(\frac{s_0}{a_0\,\mu_0} + \sum_{i=1}^{n} \frac{s_i}{a_i\,\mu_i} \right)$$

Conclusion: Magnet with large air gap is stabilized against variations in permeability

Permanent Magnet Excitation





$$B_\mathrm{m} a_\mathrm{m} = B_0 a_0 = \mu_0 H_0 a_0$$

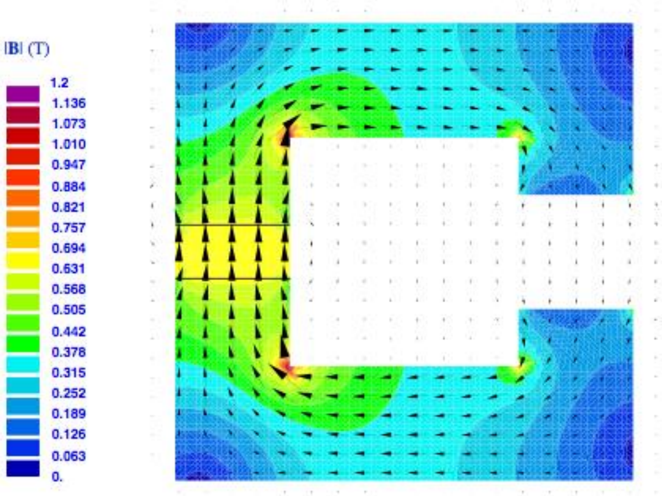
$$H_0 s_0 = -H_\mathrm{m} s_\mathrm{m}$$

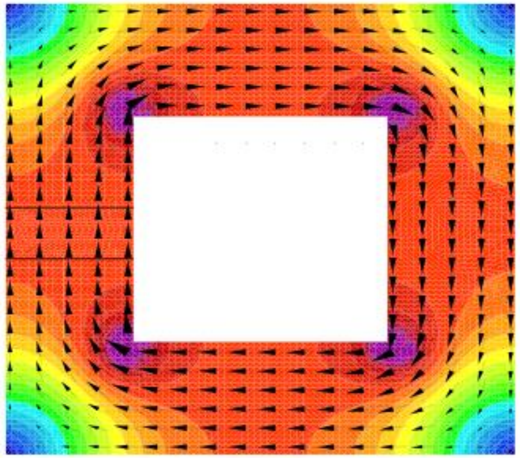
$$B_\mathrm{m} a_\mathrm{m} s_\mathrm{m} = \mu_0 H_0 a_0 \frac{-H_0 s_0}{H_\mathrm{m}}$$

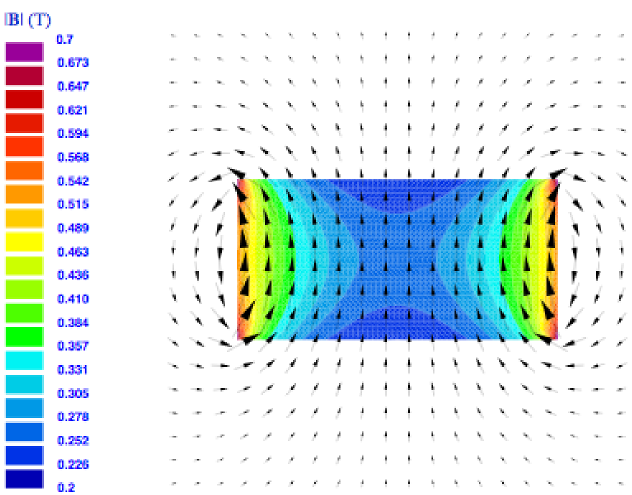
$$H_0 = \sqrt{\frac{(a_\mathrm{m} s_\mathrm{m})(-B_\mathrm{m} H_\mathrm{m})}{\mu_0 (a_0 s_0)}} = \sqrt{\frac{V_\mathrm{m}(-B_\mathrm{m} H_\mathrm{m})}{\mu_0 V_0}}$$

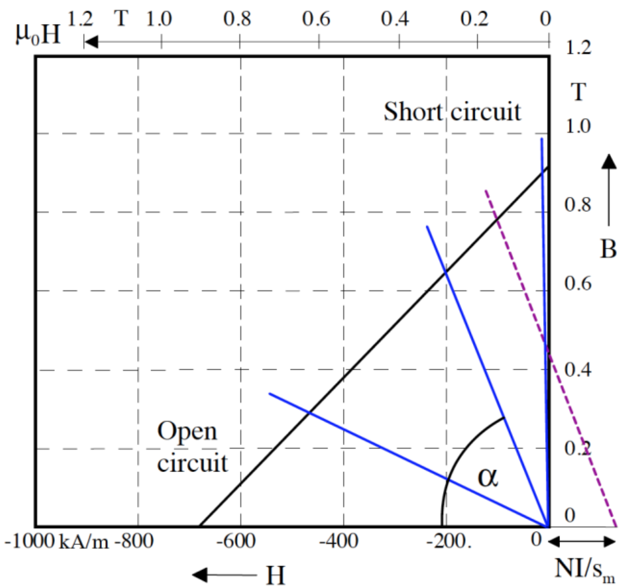
Conclusion: Operate PM in its $B_m H_m$ maximum

Permanent Magnet Circuits





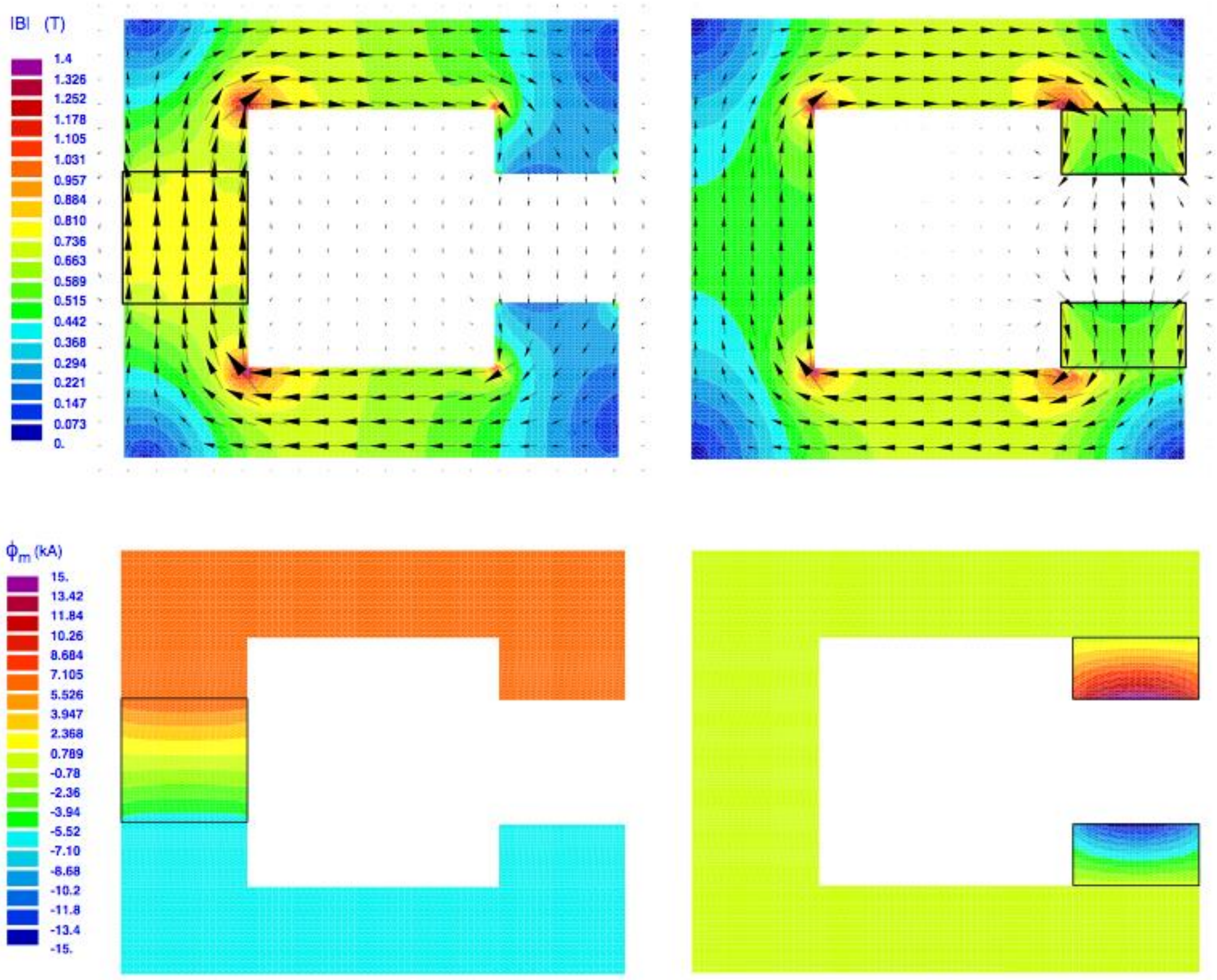




$$s = \frac{B_\mathrm{m}}{H_\mathrm{m}} = \mu_0 P = -\mu_0 \frac{s_\mathrm{m}}{s_0} \frac{a_0}{a_\mathrm{m}}$$

s = - tan α

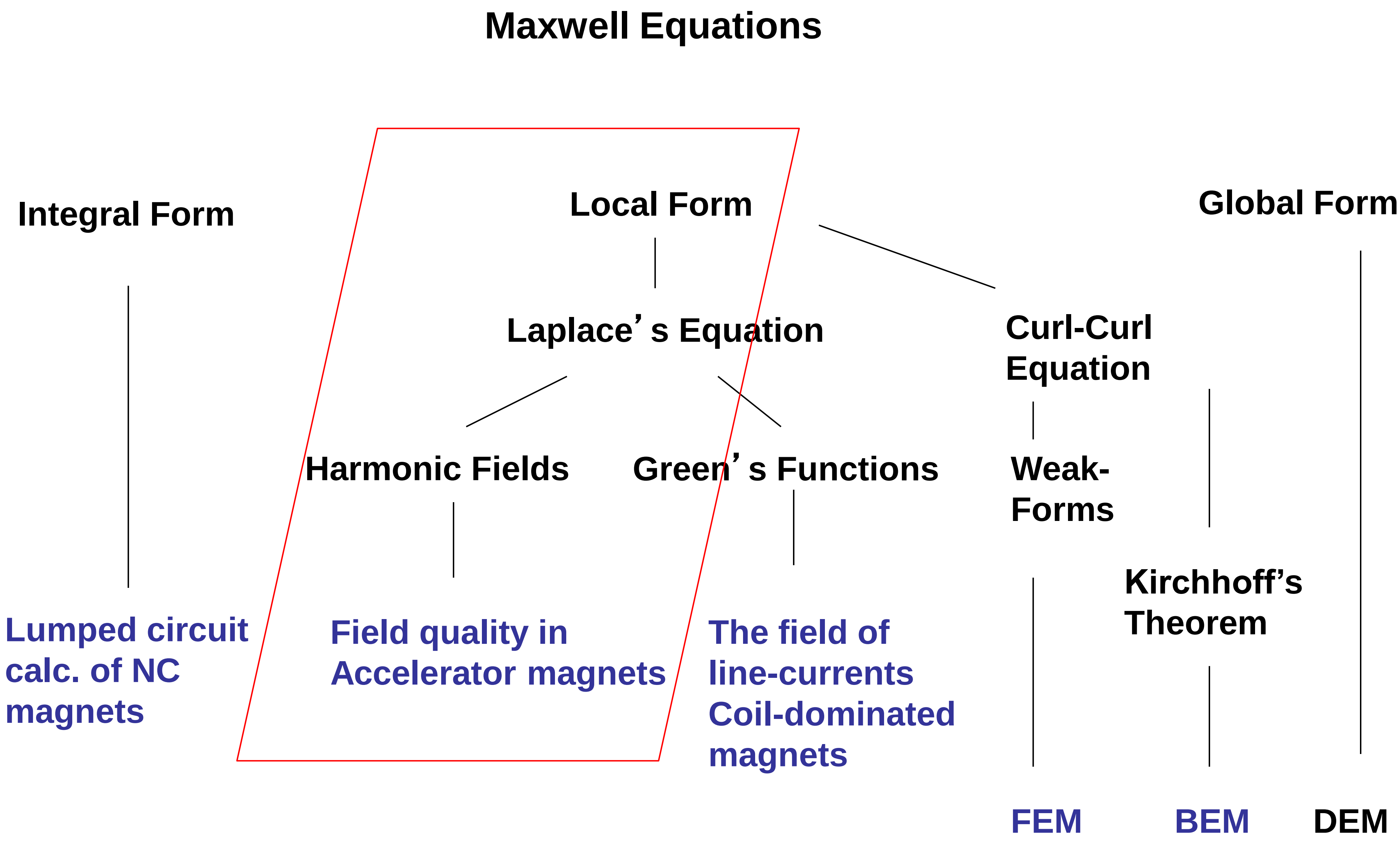
Optimal Position of Permanent Magnets



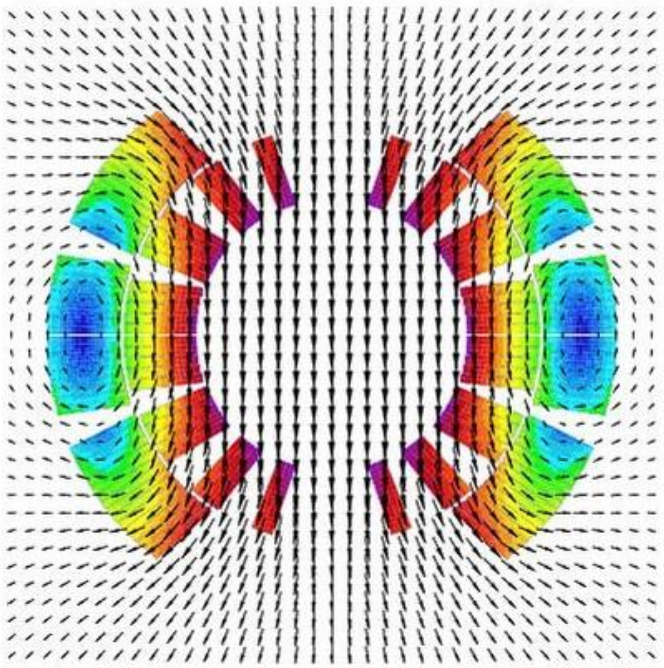
Magnetic Scalar Potential

Conclusion: Do not bother with reluctance models – use FEM codes

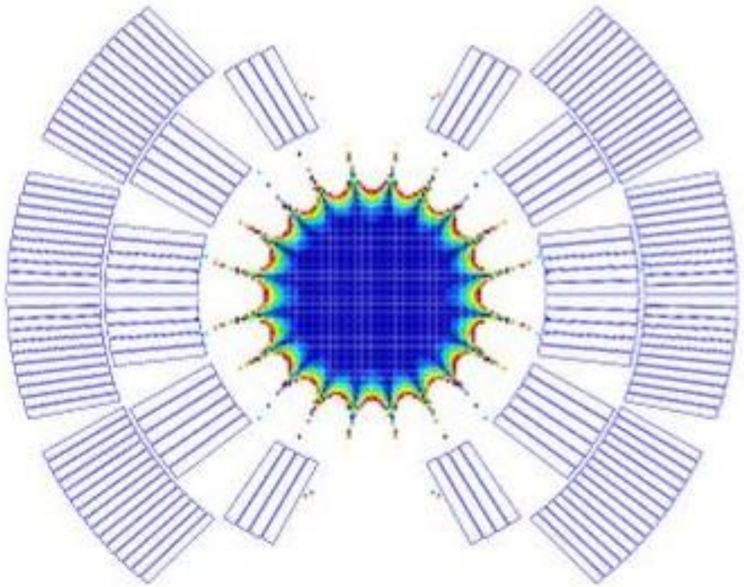
Mathematical Foundations of Magnet Design



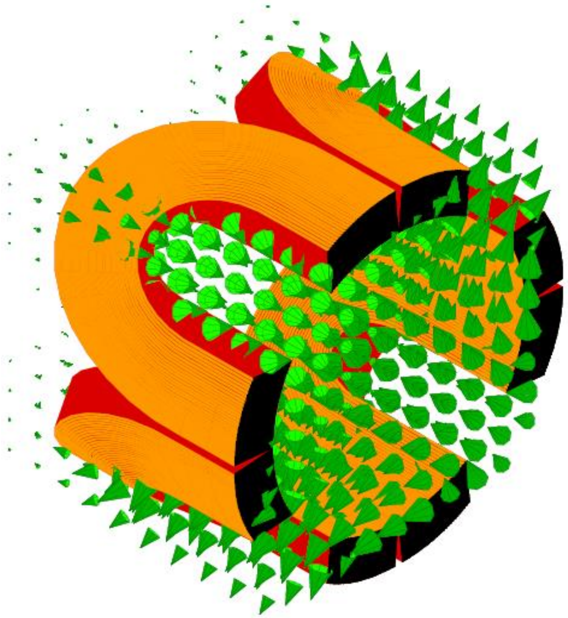
Field Quality



2D Field map



Good field region



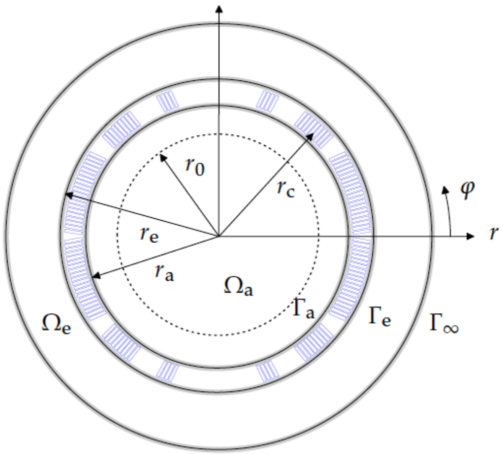
3D Field map

Solving of Boundary Value Problems

1. Governing equation in the air domain

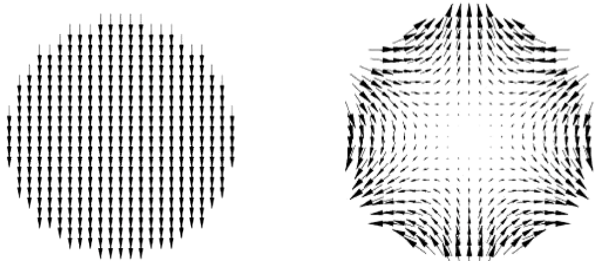
$$\nabla^2 A_z = 0,$$

2. Chose a **suitable** coordinate system



$$r^2 \frac{\partial^2 A_z}{\partial r^2} + r \frac{\partial A_z}{\partial r} + \frac{\partial^2 A_z}{\partial \varphi^2} = 0,$$

3. Find eigenfunctions and incorporate some knowledge. Coefficients are not known yet



$$A_z(r,\varphi) = \sum_{n=1}^{\infty} r^n (\mathcal{A}_n \sin n\varphi + \mathcal{B}_n \cos n\varphi).$$

4. Calculate the field components

$$B_r(r,\varphi) = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1} (\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi),$$

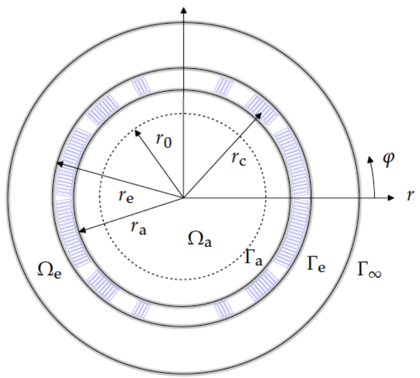
$$B_\varphi(r,\varphi) = -\frac{\partial A_z}{\partial r} = -\sum_{n=1}^{\infty} n r^{n-1} (\mathcal{A}_n \sin n\varphi + \mathcal{B}_n \cos n\varphi),$$

Solving of Boundary Value Problems

$$B_r(r,\varphi) = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1} (\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi),$$

5. Measure the field on a reference radius and perform Fourier analysis (develop into the eigenfunctions). Coefficients are known here.

$$B_r(r_0,\varphi) = \sum_{n=1}^{\infty} (B_n(r_0)\sin n\varphi + A_n(r_0)\cos n\varphi),$$



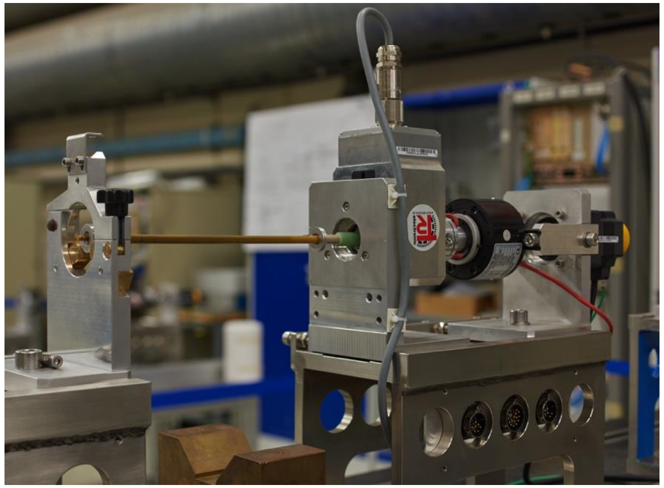
6: Equate the known and unknown coefficients

$$\mathcal{A}_n = \frac{1}{n\, r_0^{n-1}} A_n(r_0)\,, \qquad \mathcal{B}_n = \frac{-1}{n\, r_0^{n-1}} B_n(r_0)\,.$$

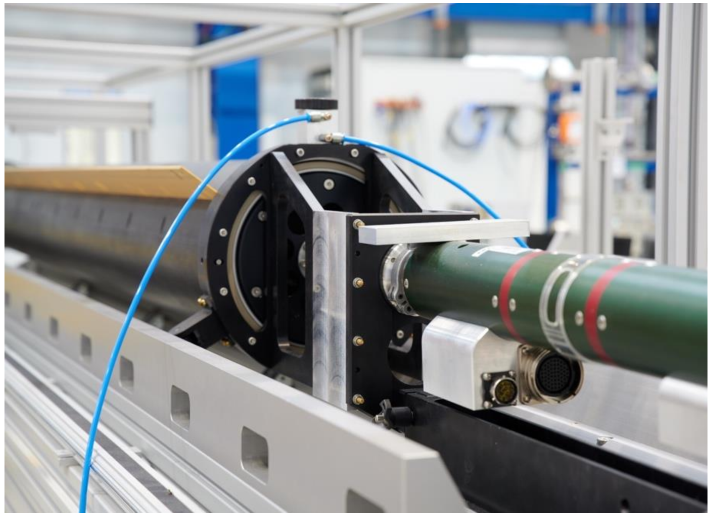
7. Put this into the original solution for the entire air domain

$$A_z(r,\varphi) = -\sum_{n=1}^{\infty} \frac{r_0}{n}\left(\frac{r}{r_0}\right)^n (B_n(r_0)\cos n\varphi - A_n(r_0)\sin n\varphi).$$

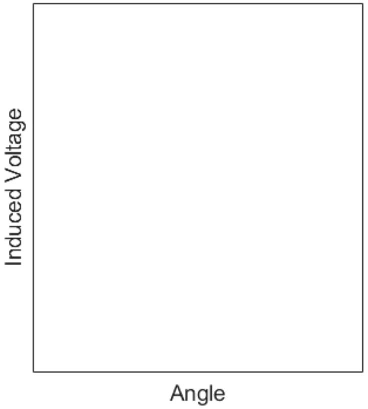
Rotating Coil Magnetometers

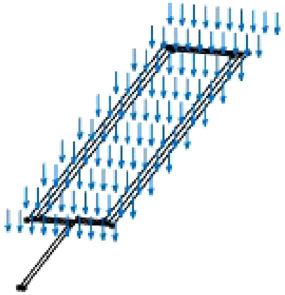


8 mm

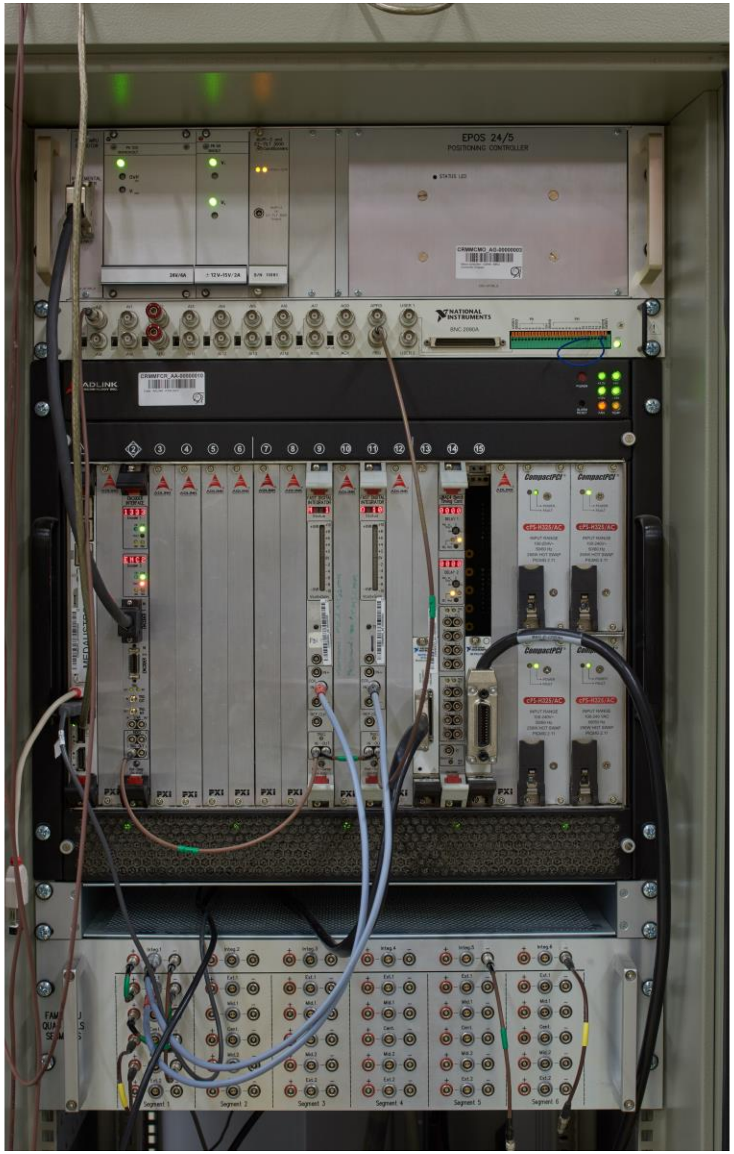


380 mm

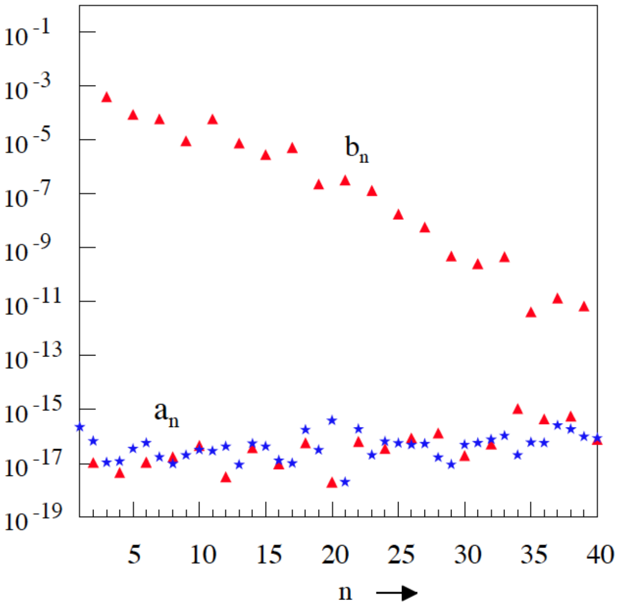




The Electronic Rack

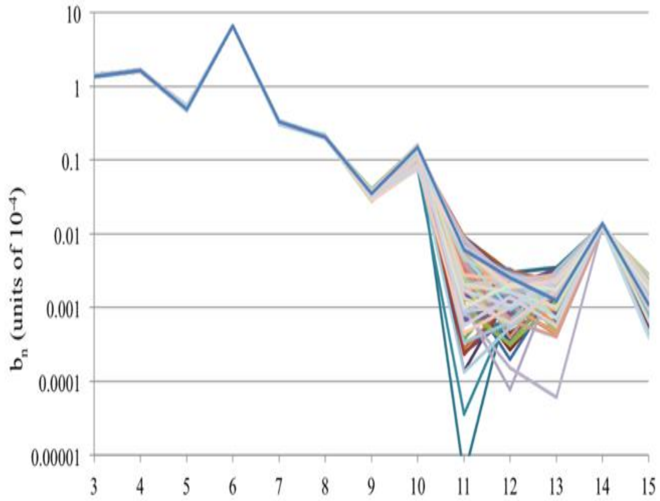


Power converters, DAC
(angular encoder)

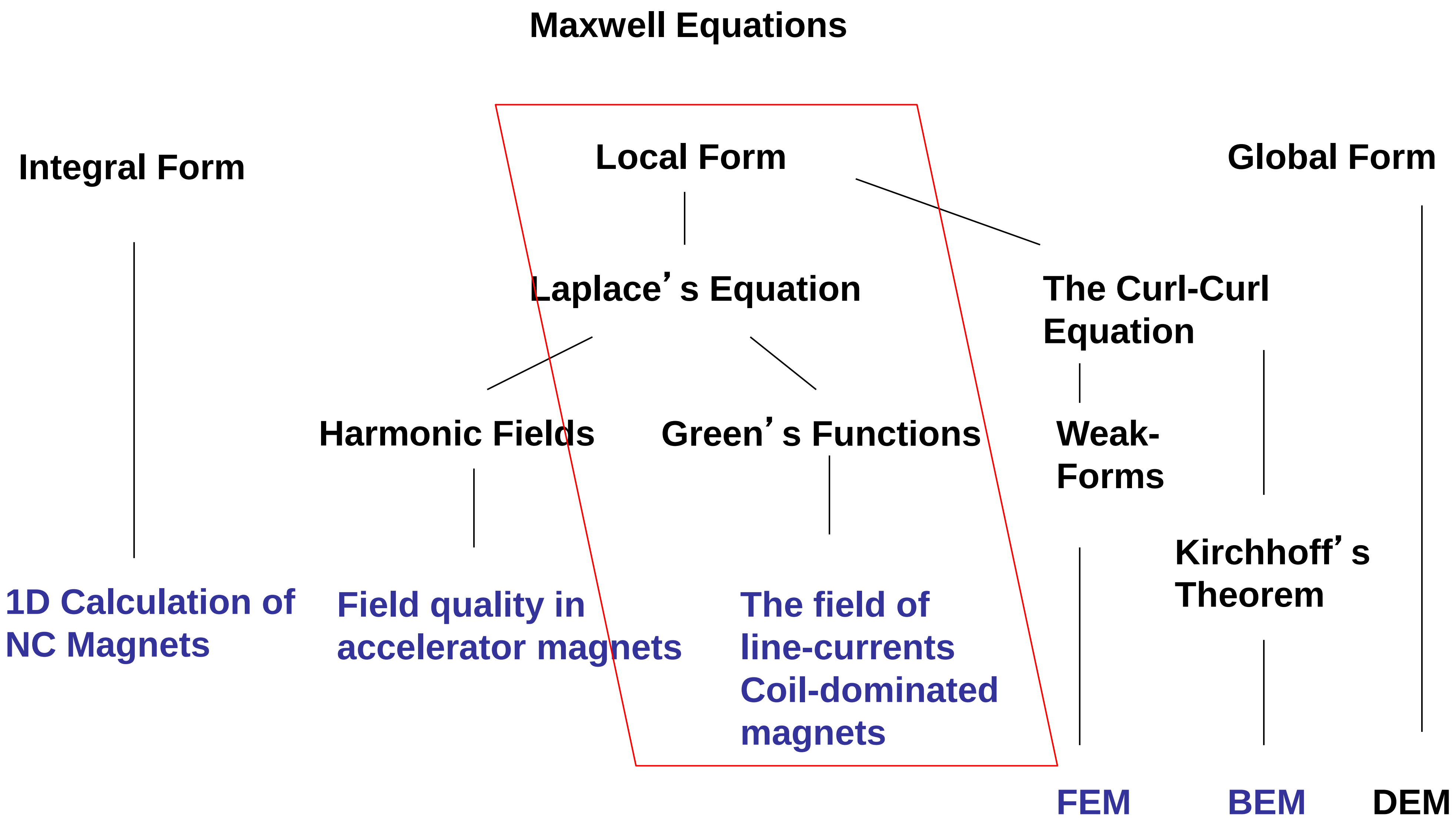


Integrators, PC
(re-parametrization to arc length)

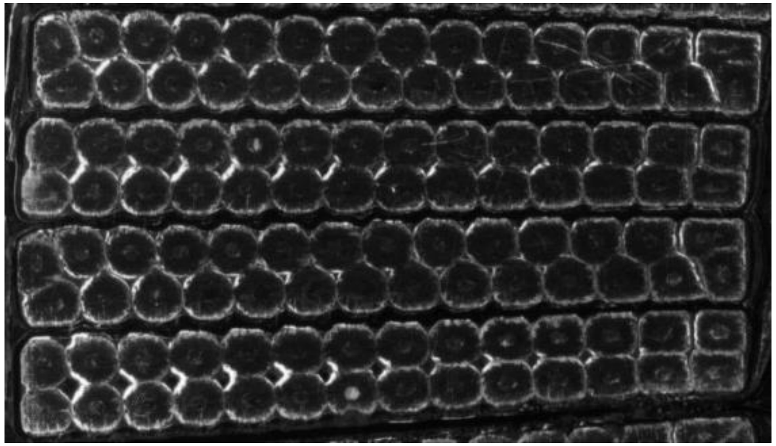
Patch panel
(compensation of signals)

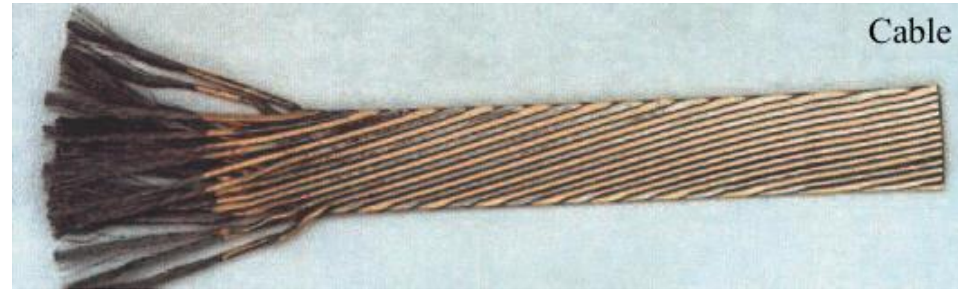


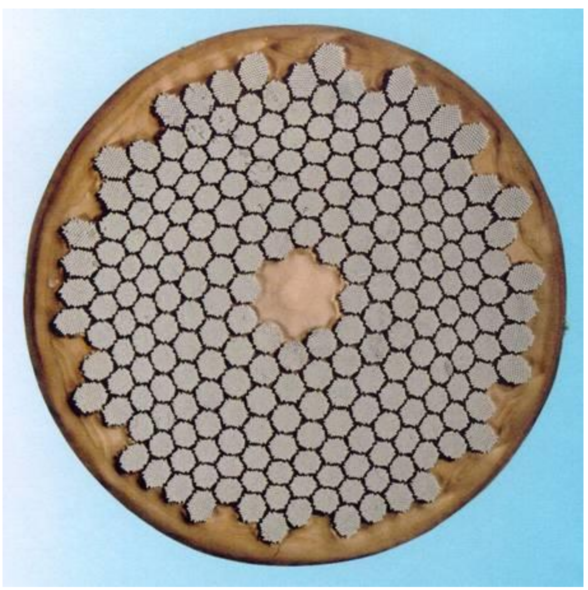
Mathematical Foundations of Magnet Design



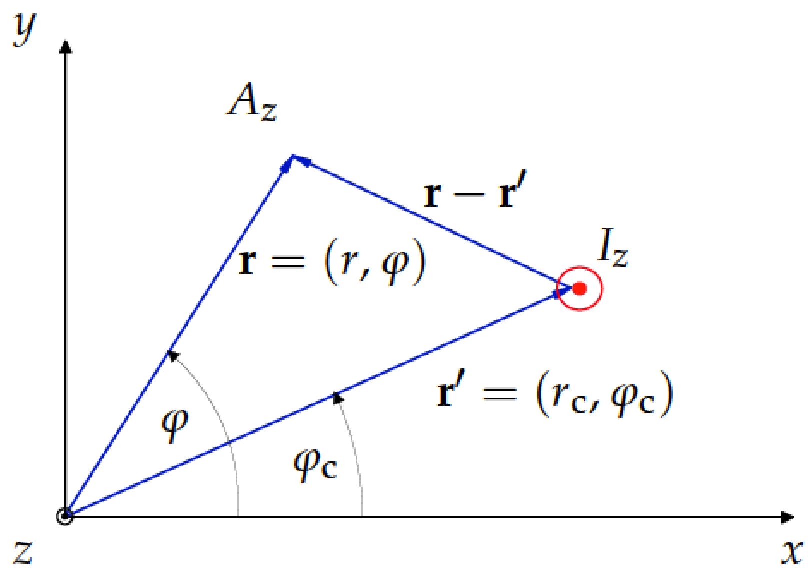
Rutherford (Roebel) Kabel, Strand, Nb-Ti Filament







Expanding the Green Function



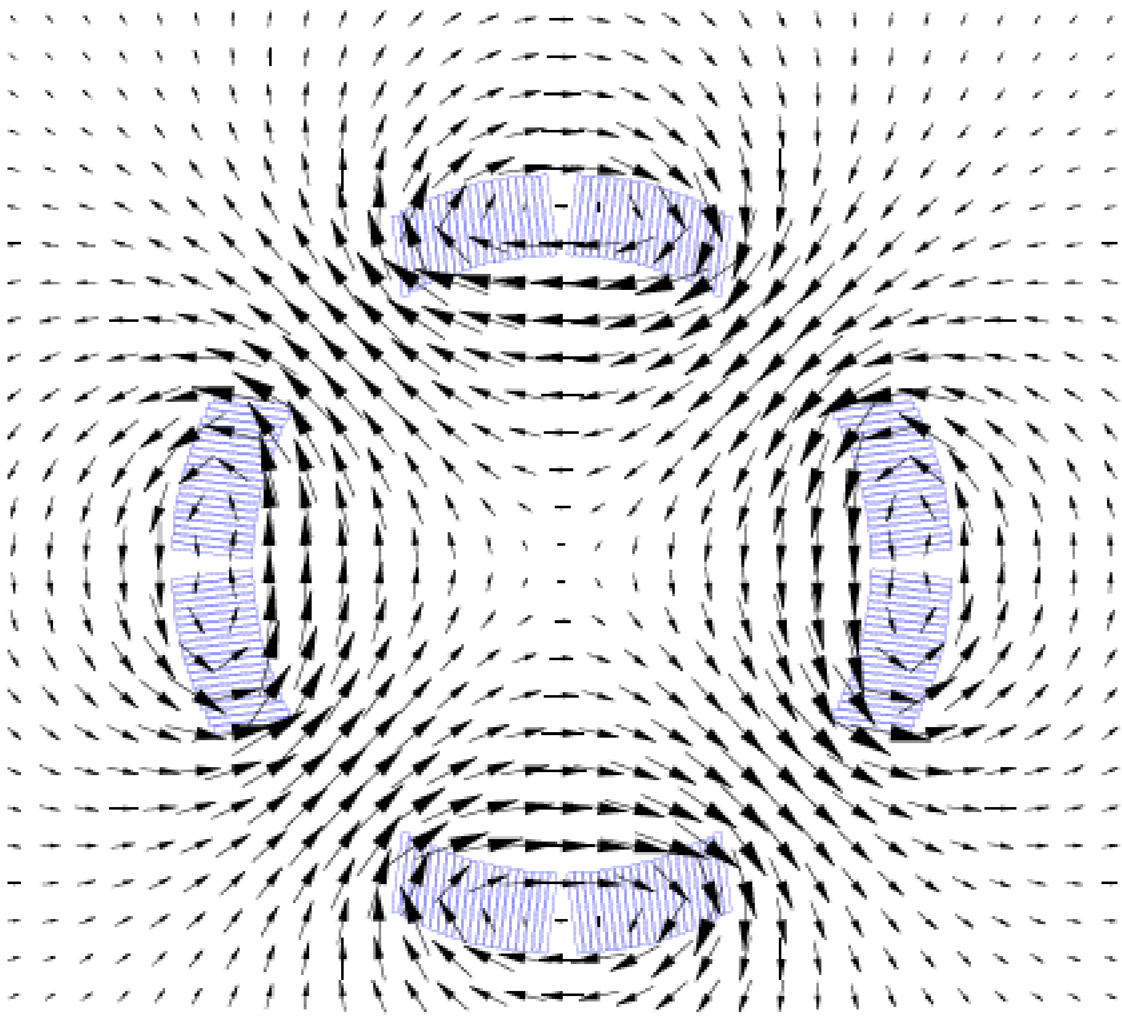
$$A_z(\mathbf{r}) = -\frac{\mu_0 I}{2\pi}\ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{r_{\mathrm{ref}}}\right)$$

This is a H[1] cohomology solution: curl-free but cannot be expressed as the gradient of a scalar potential

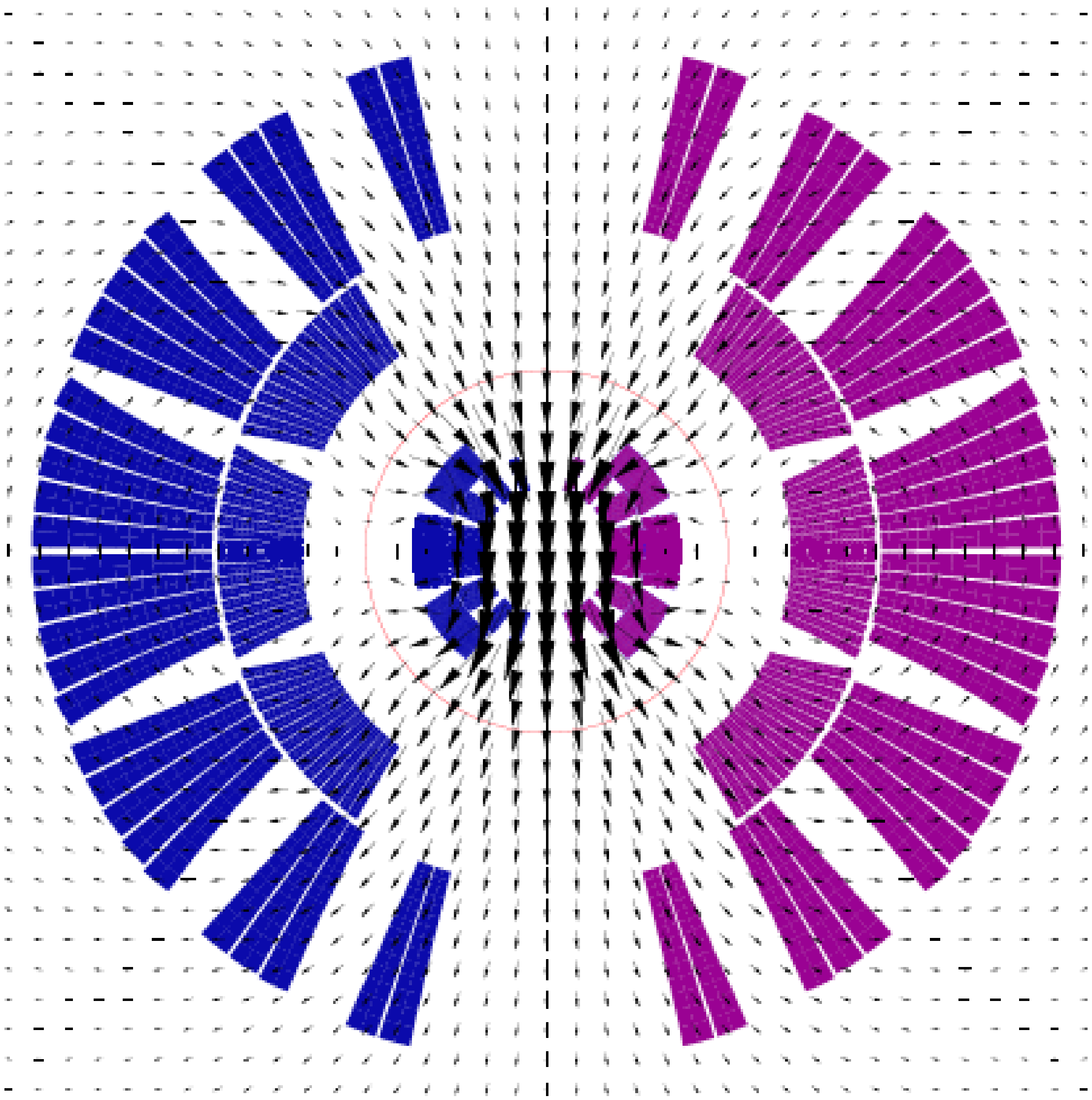
$$A_z(r,\varphi) = -\frac{\mu_0 I}{2\pi}\ln\left(\frac{r_c}{r_{\mathrm{ref}}}\right) + \frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{r_c}\right)^n \cos n(\varphi-\varphi_c)$$

$$B_n(r_0) = -\frac{\mu_0 I}{2\pi r_c}\left(\frac{r_0}{r_c}\right)^{n-1}\cos n\varphi_c, \qquad A_n(r_0) = \frac{\mu_0 I}{2\pi r_c}\left(\frac{r_0}{r_c}\right)^{n-1}\sin n\varphi_c.$$

The Imaging Current Method

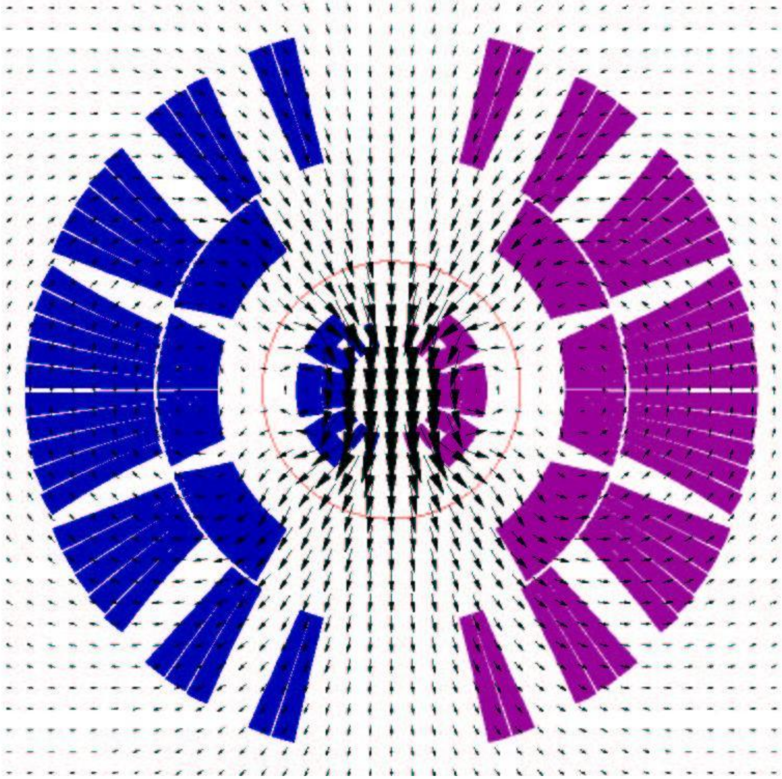


The Image Current Method



The Image Current Method

$$B_n(r_0) = -\sum_{k=1}^{K} \frac{\mu_0 I_k}{2\pi} \frac{r_0^{n-1}}{r_{c,k}^n} \left(1 + \lambda_\mu \left(\frac{r_{c,k}}{r_y}\right)^{2n}\right) \cos n\varphi_{c,k}$$



$$\lambda_\mu I := \frac{\mu_r - 1}{\mu_r + 1} I$$

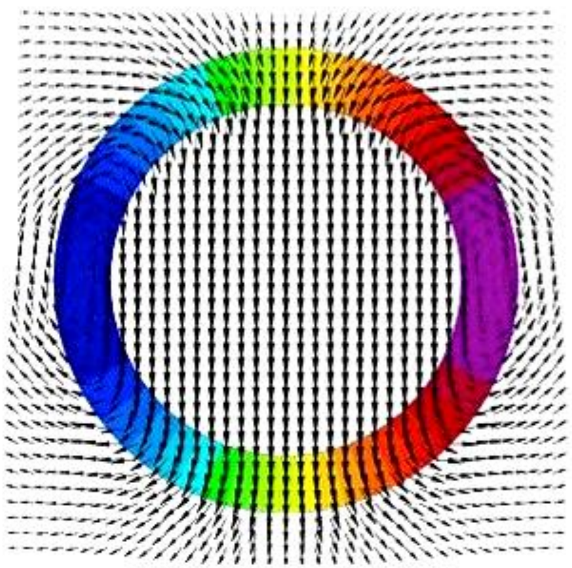
$$\frac{B_N^{\text{imag}}}{B_N + B_N^{\text{imag}}} \approx \left(1 + \left(\frac{r_y}{r}\right)^{2N}\right)^{-1}$$

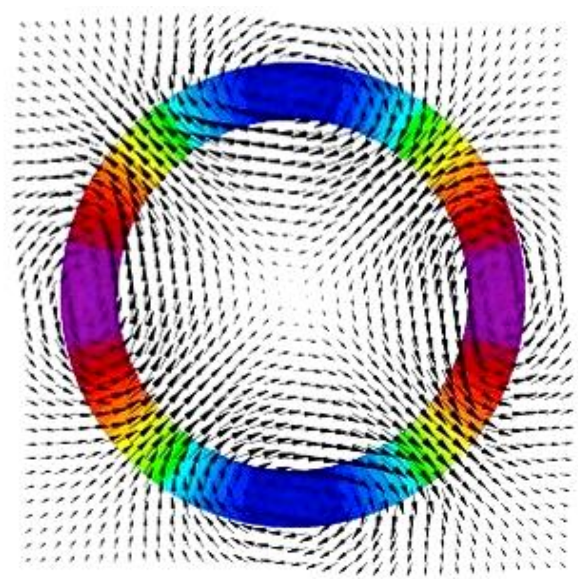
Conclusion: Iron yoke contributes about 20% to the dipole field (and much less to the higher order harmonics)

Ideal Current Distributions

$$B_n(r_0) = \int_{r_a}^{r_e}\int_0^{2\pi} -\frac{\mu_0 J_E r_0^{n-1}}{2\pi r_c^n}\left(1+\lambda_\mu\left(\frac{r_c}{r_y}\right)^{2n}\right)\cos m\varphi_c \cos n\varphi_c \, r_c \, d\varphi_c \, dr_c$$

$$B_1(r_0) = -\frac{\mu_0 J_E}{2}\left((r_e - r_a) + \lambda_\mu \frac{1}{r_y^2}\frac{1}{3}\left(r_e^3 - r_a^3\right)\right)$$



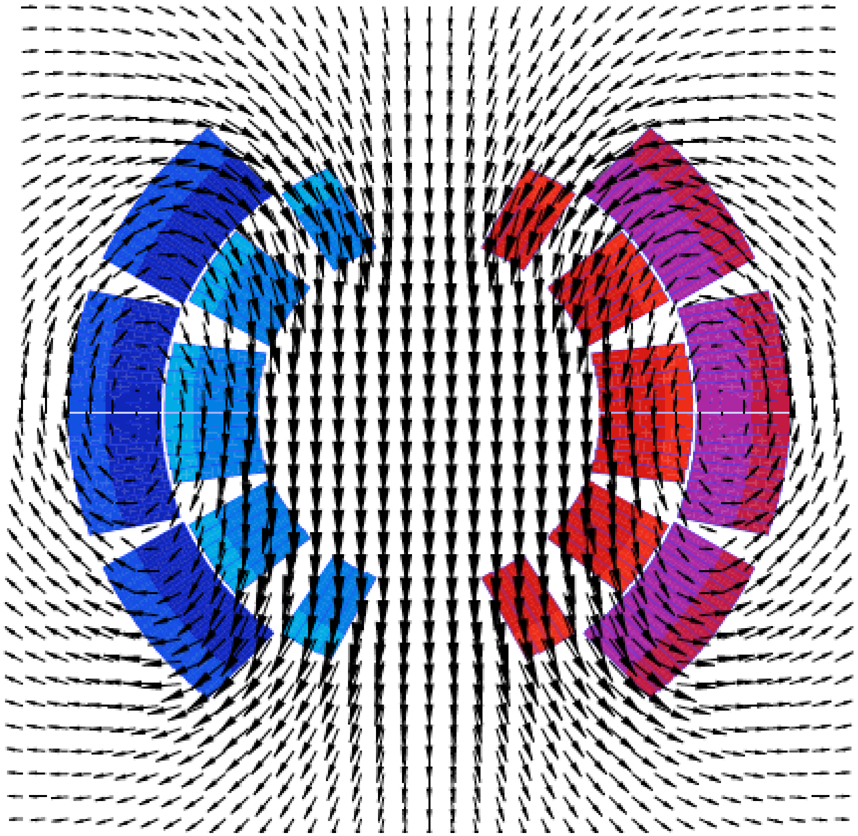


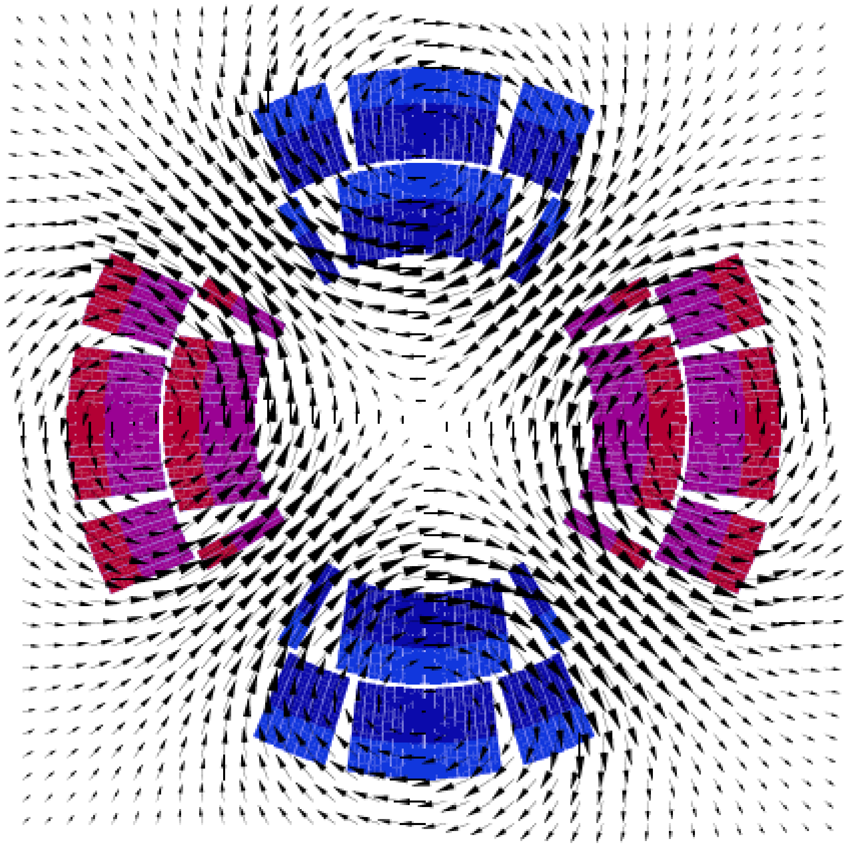
“Allowed” Multipoles for fully symmetric magnets

B_1, b_3, b_5, b_7, …

B_2, b_6, b_{10}, b_{14}…

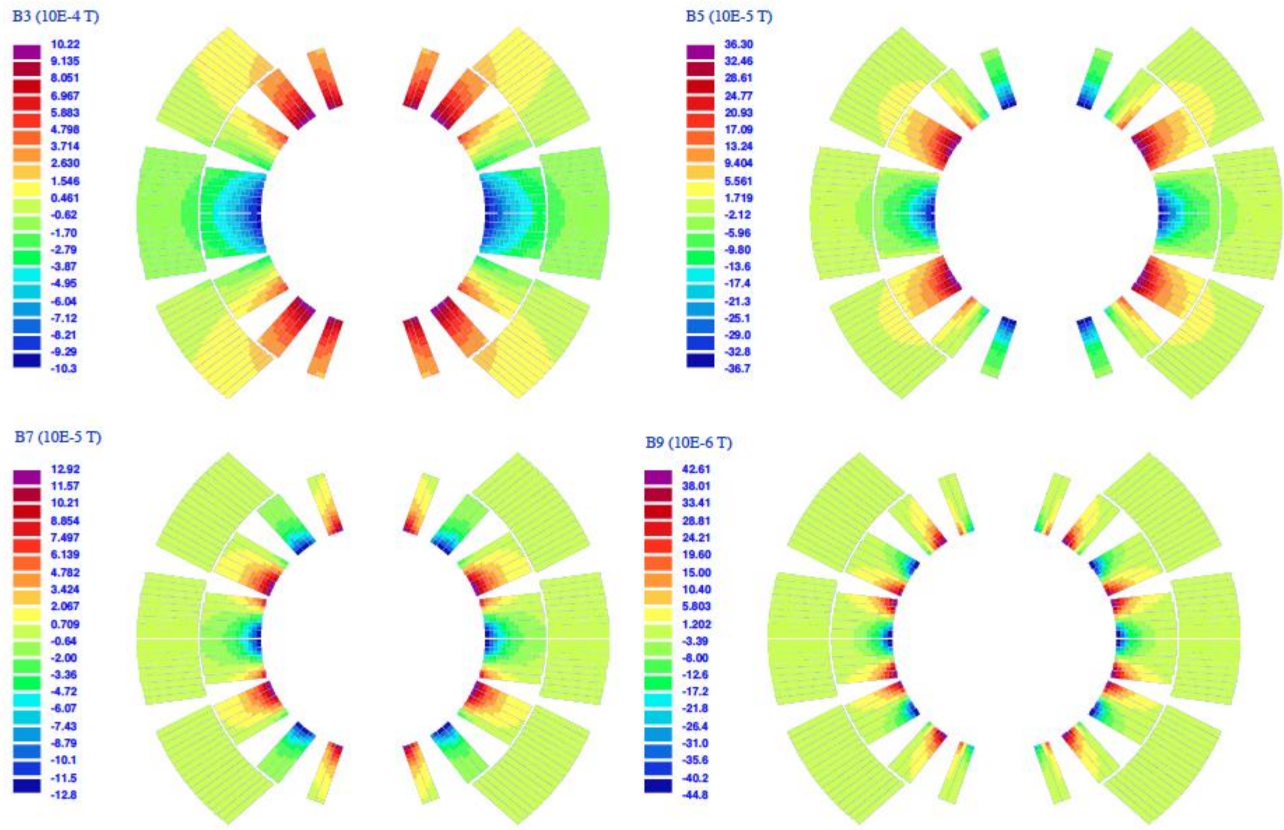
Coil-Block Approximations





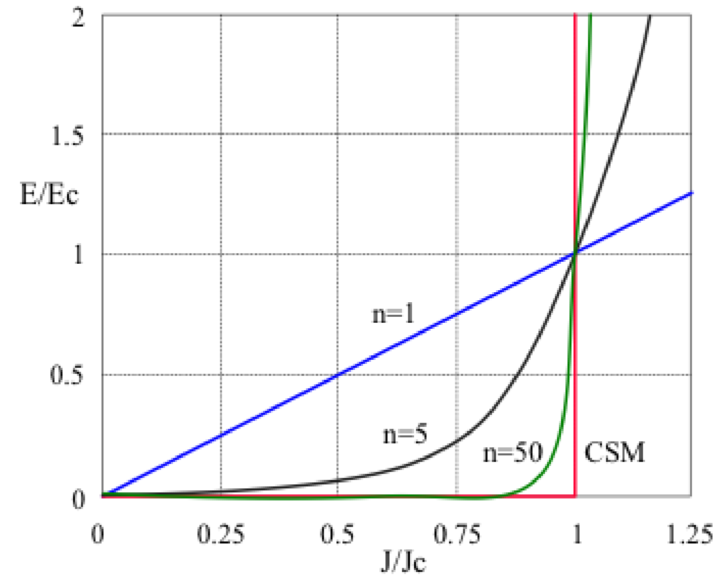
Sources of Multipole Field Errors

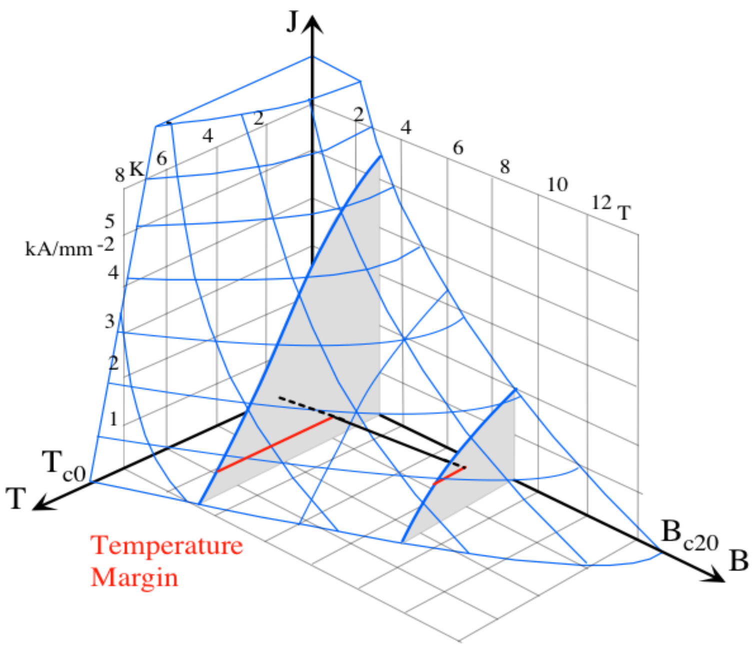
$$B_n(r_0) = -\sum_{k=1}^{K} \frac{\mu_0 I_k}{2\pi} \frac{r_0^{n-1}}{r_{c,k}^n} \left(1 + \lambda_\mu \left(\frac{r_{c,k}}{r_y}\right)^{2n}\right) \cos n\varphi_{c,k} \,,$$



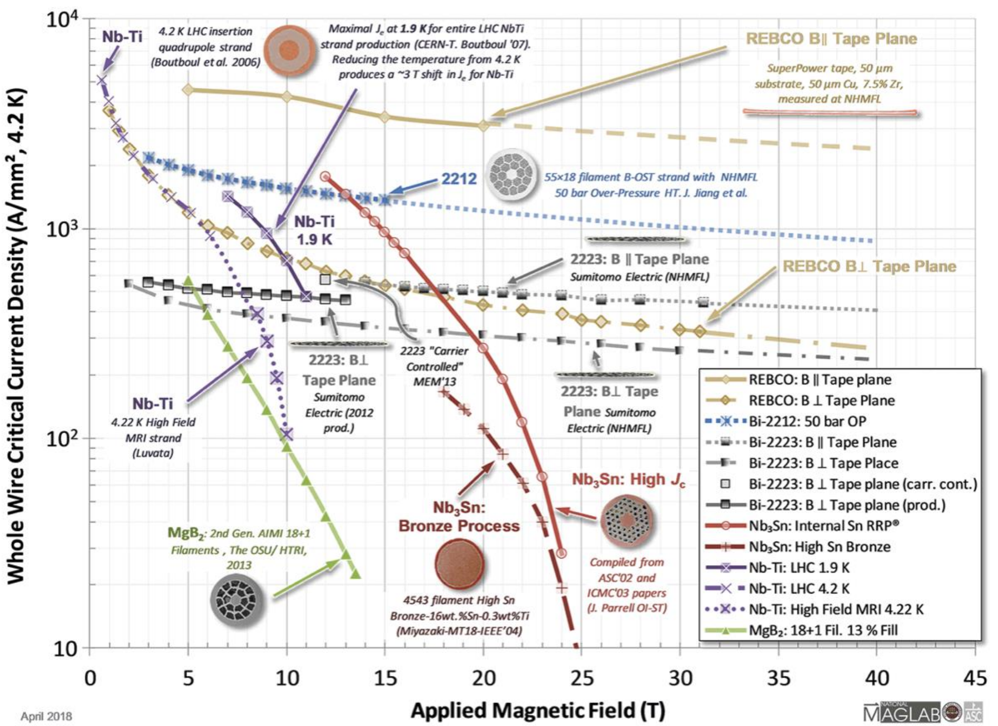
Superconductor Properties

- Hard Superconductors (Type 2)
  - Magnetic field can penetrate
  - Magnetization with hysteresis
- Critical current density J_c
  - Current density at spec. electric field (E_c = 1 µV/cm)
- Critical surface
  - Dependence of J_c on T and B



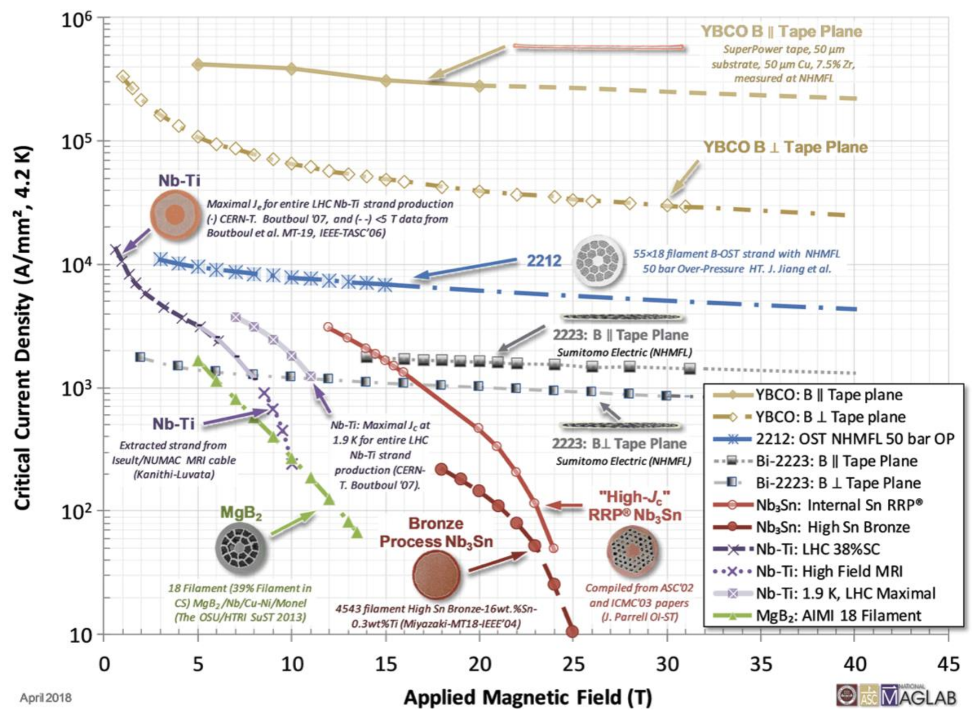


Peter Lee`s Jc Tables https://fs.magnet.fsu.edu/~lee/plot/plot.htm



Be careful with the definition of engineering current density

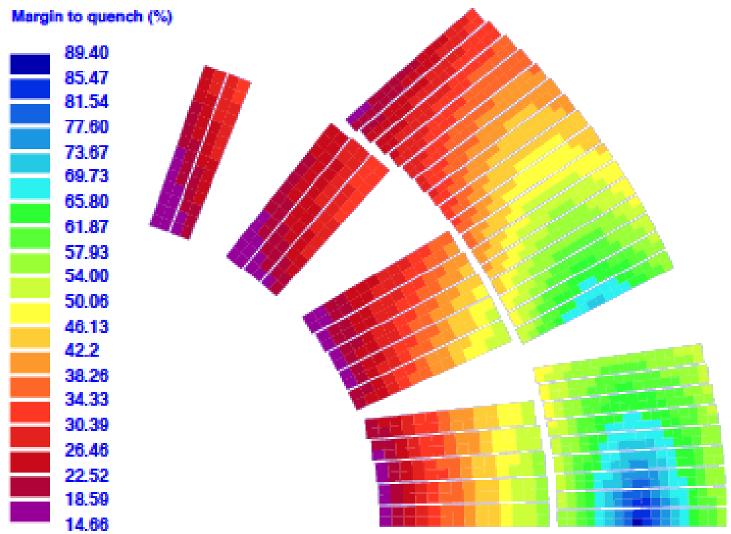
Non-stabilized

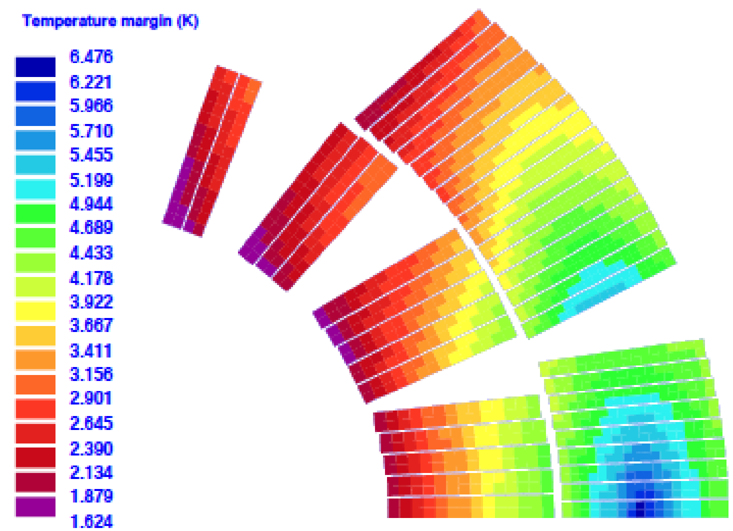


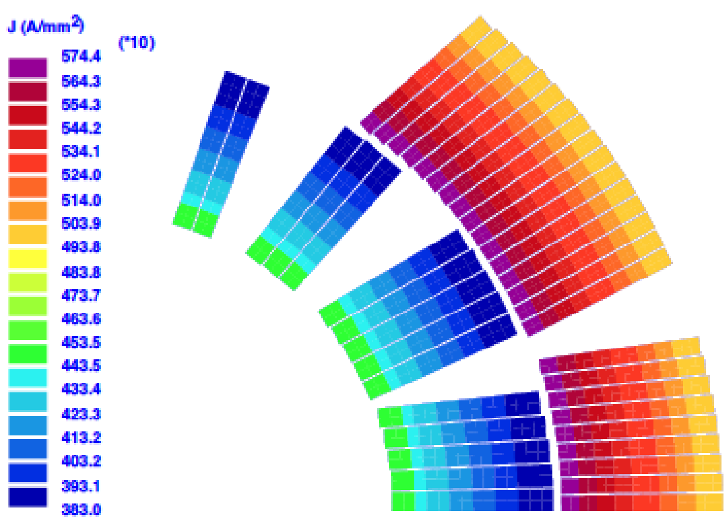
$$J_\mathrm{E} = \lambda_\mathrm{SC}\,\lambda_\mathrm{c}\,\lambda_\mathrm{i}\,J_\mathrm{c} =: \lambda_\mathrm{tot} J_\mathrm{c}$$

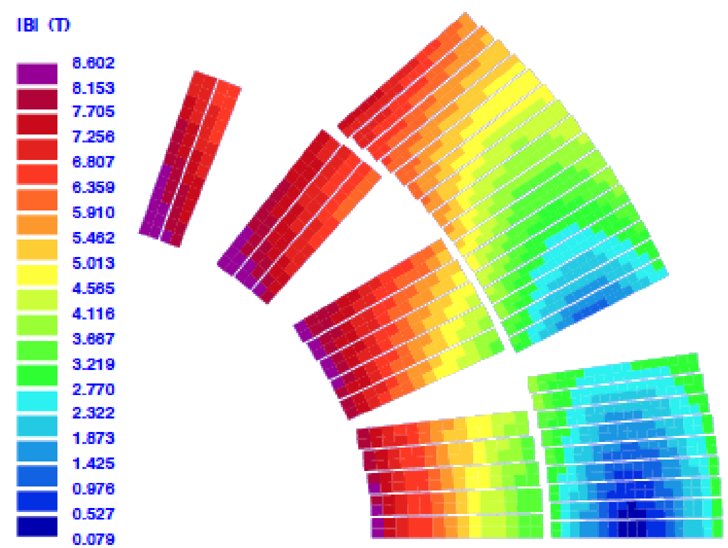
$$\lambda_\mathrm{c} := \frac{n\pi d_\mathrm{s}^2}{2w_\mathrm{c}(t_\mathrm{n}+t_\mathrm{w})\cos(\psi)}$$

Margins


Margin to quench (%)
89.40
85.47
81.54
77.60
73.67
69.73
65.80
61.87
57.93
54.00
50.06
46.13
42.2
38.26
34.33
30.39
26.46
22.52
18.59
14.66

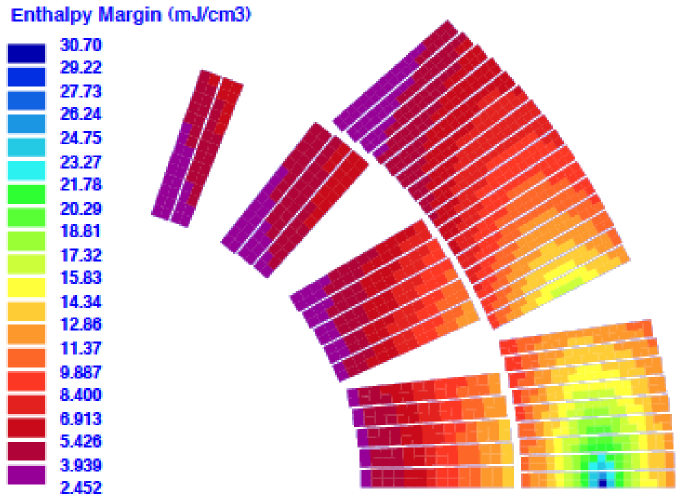

Temperature margin (K)
6.476
6.221
5.966
5.710
5.455
5.199
4.944
4.689
4.433
4.178
3.922
3.667
3.411
3.156
2.901
2.645
2.390
2.134
1.879
1.624

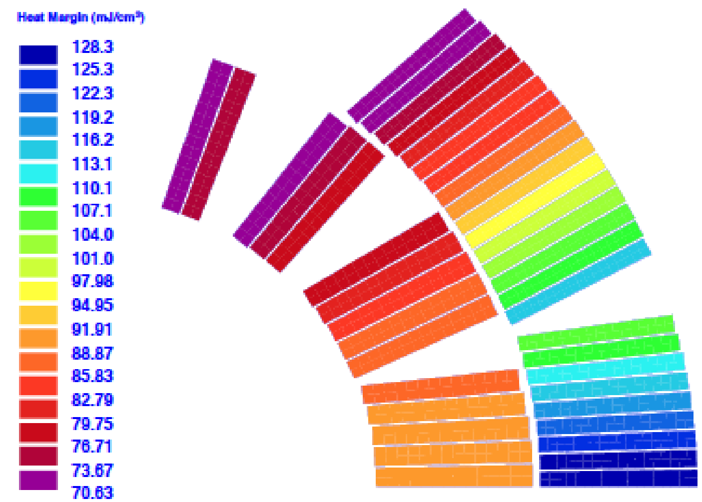

J (A/mm2)
(*10)
574.4
564.3
554.3
544.2
534.1
524.0
514.0
503.9
493.8
483.8
473.7
463.6
453.5
443.5
433.4
423.3
413.2
403.2
393.1
383.0


IBI (T)
8.602
8.153
7.705
7.256
6.807
6.359
5.910
5.462
5.013
4.565
4.116
3.667
3.219
2.770
2.322
1.873
1.425
0.976
0.527
0.079

Margins

$$\Delta h := \int_{T_b}^{T_c(J,B)} \rho\, c_p(T)\, \mathrm{d}T\,,$$

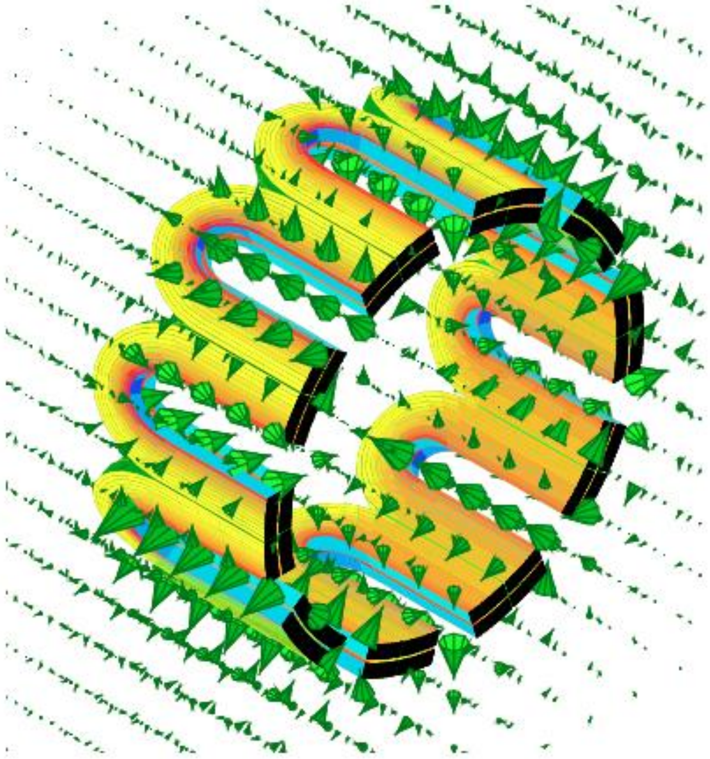


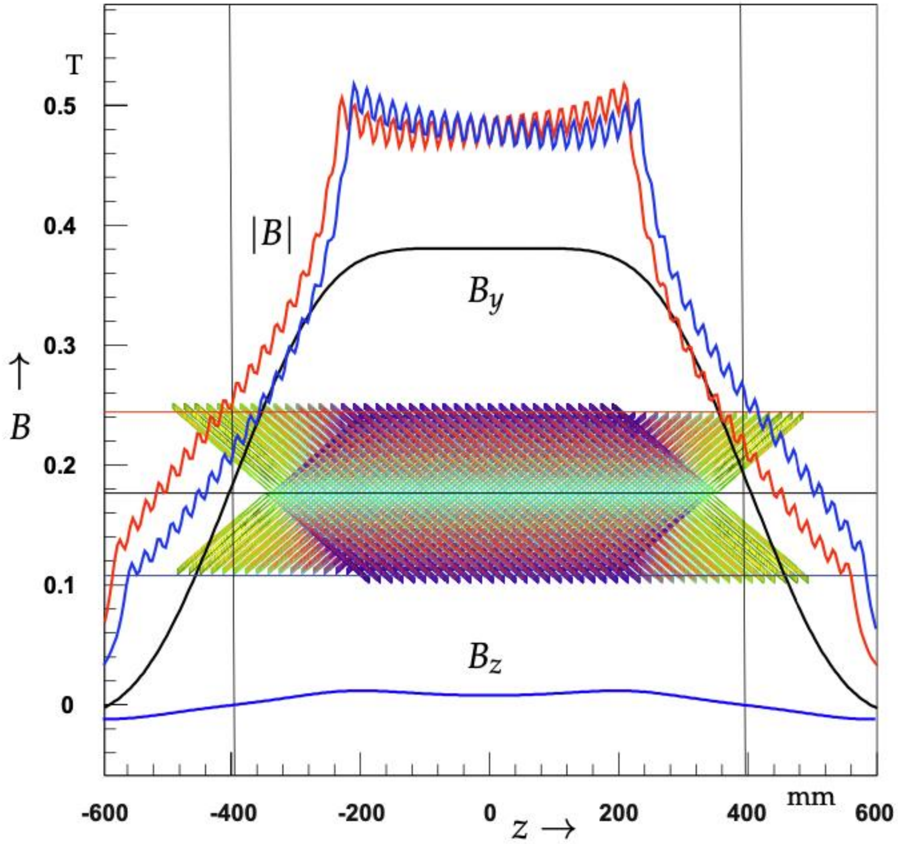


Enthalpy reserve
Heat capacity of the copper/SC strand until quench

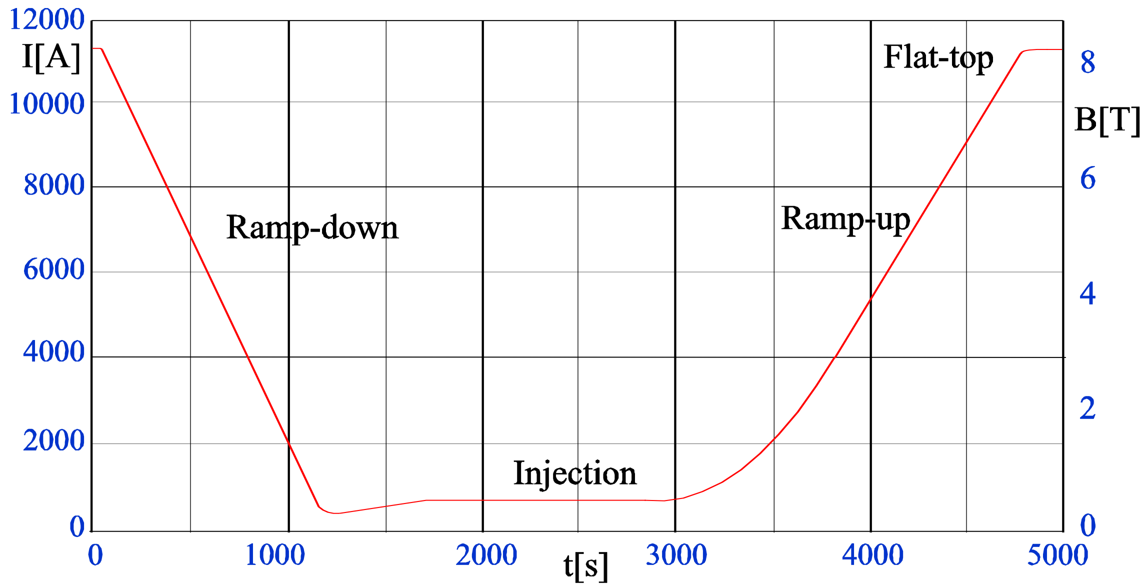
Average heat reserve (copper, SC and helium) in the cable (slow losses). No heat transfer.

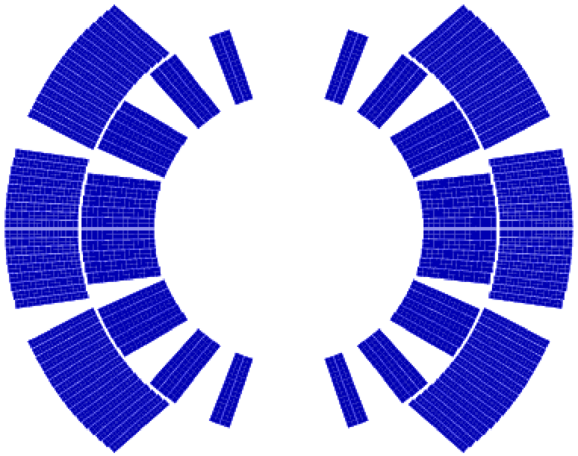
Margins

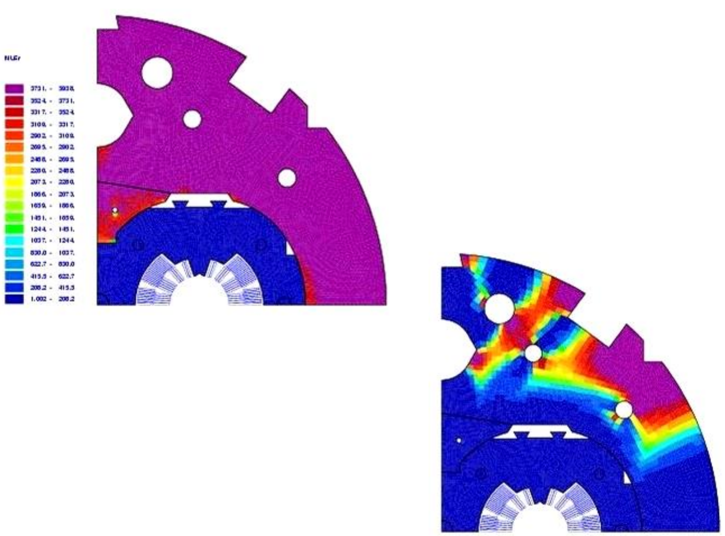


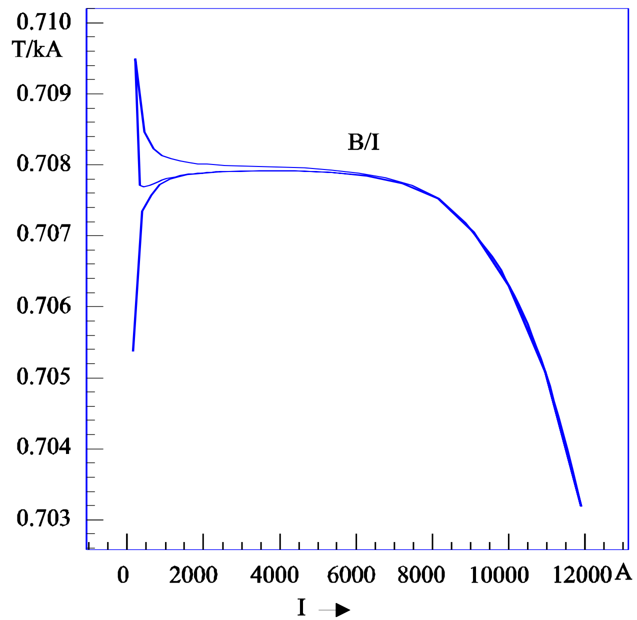


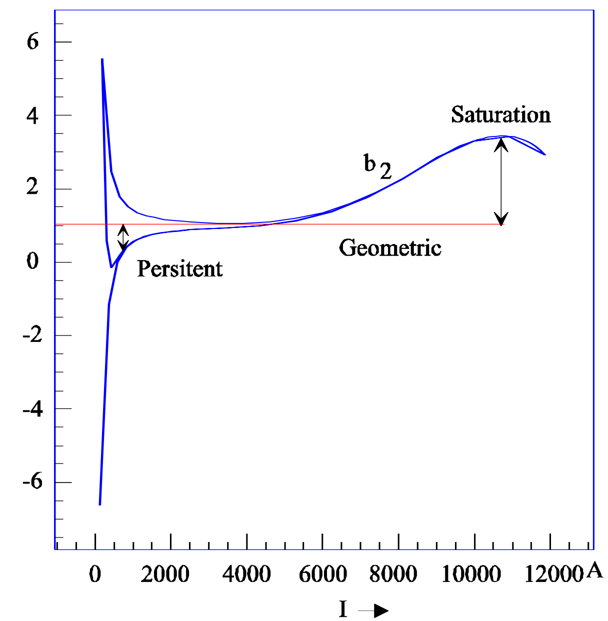
Excitation Cycle



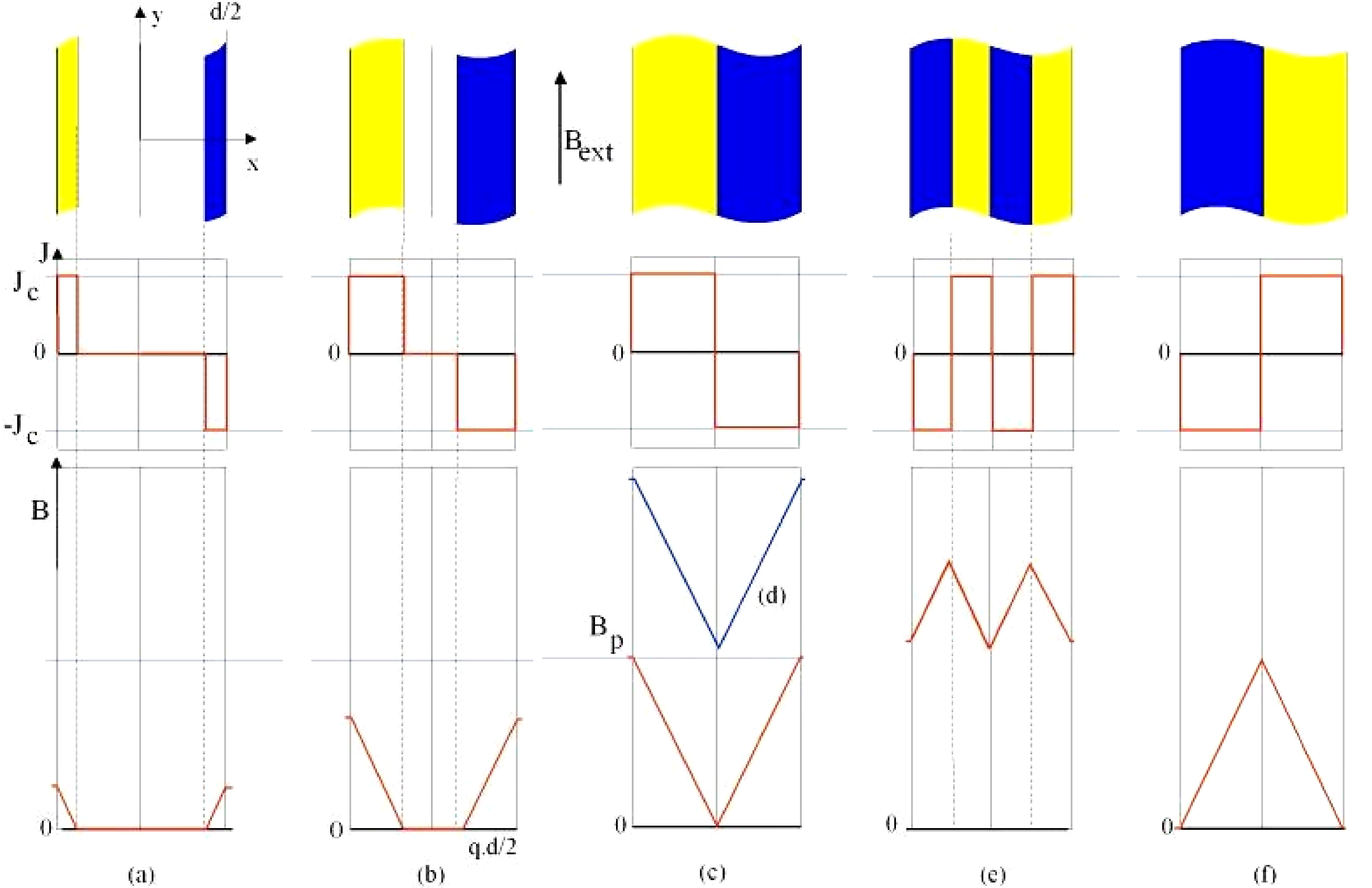






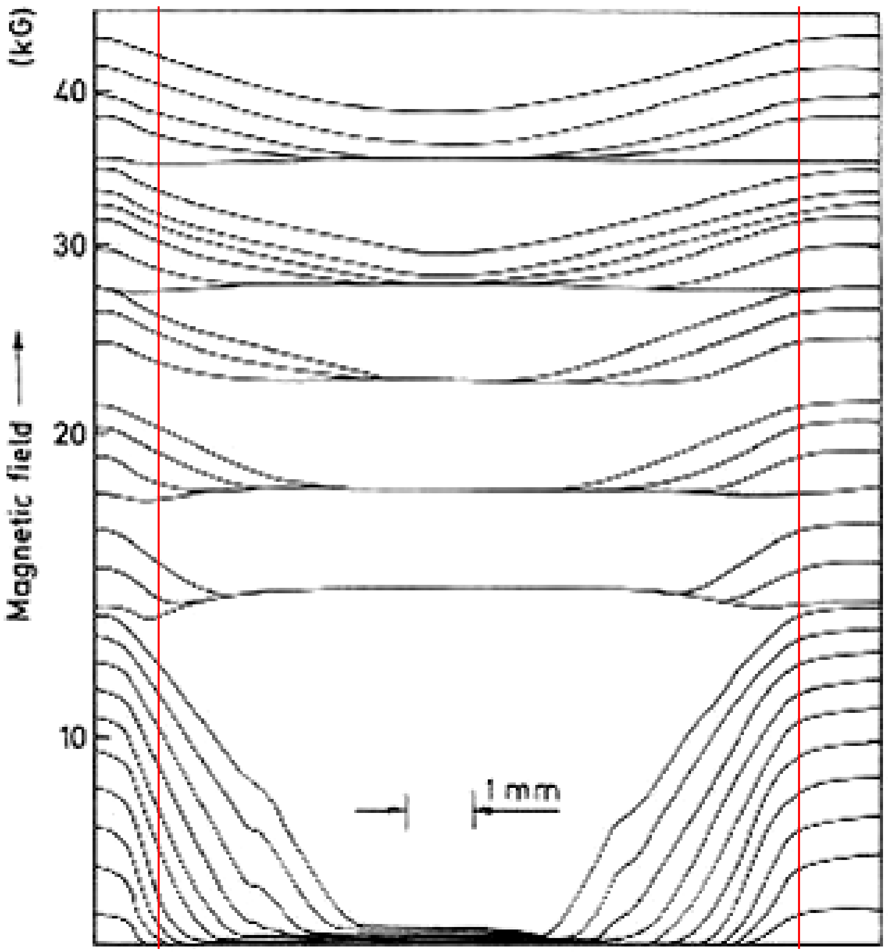


Bean’s Critical State Model (CSM)



Screening Field in a Slab

Coffey, 1967



$$q^* := \frac{B_{\text{ext}}}{\mu_0 J_c}\frac{2}{d}$$

$$B_p = \mu_0 J_c \frac{d}{2}$$

$$M = \int_{\frac{d}{2}(1-q^*)}^{\frac{d}{2}} -\frac{J_c\, 2x}{d}\,\mathrm{d}x$$

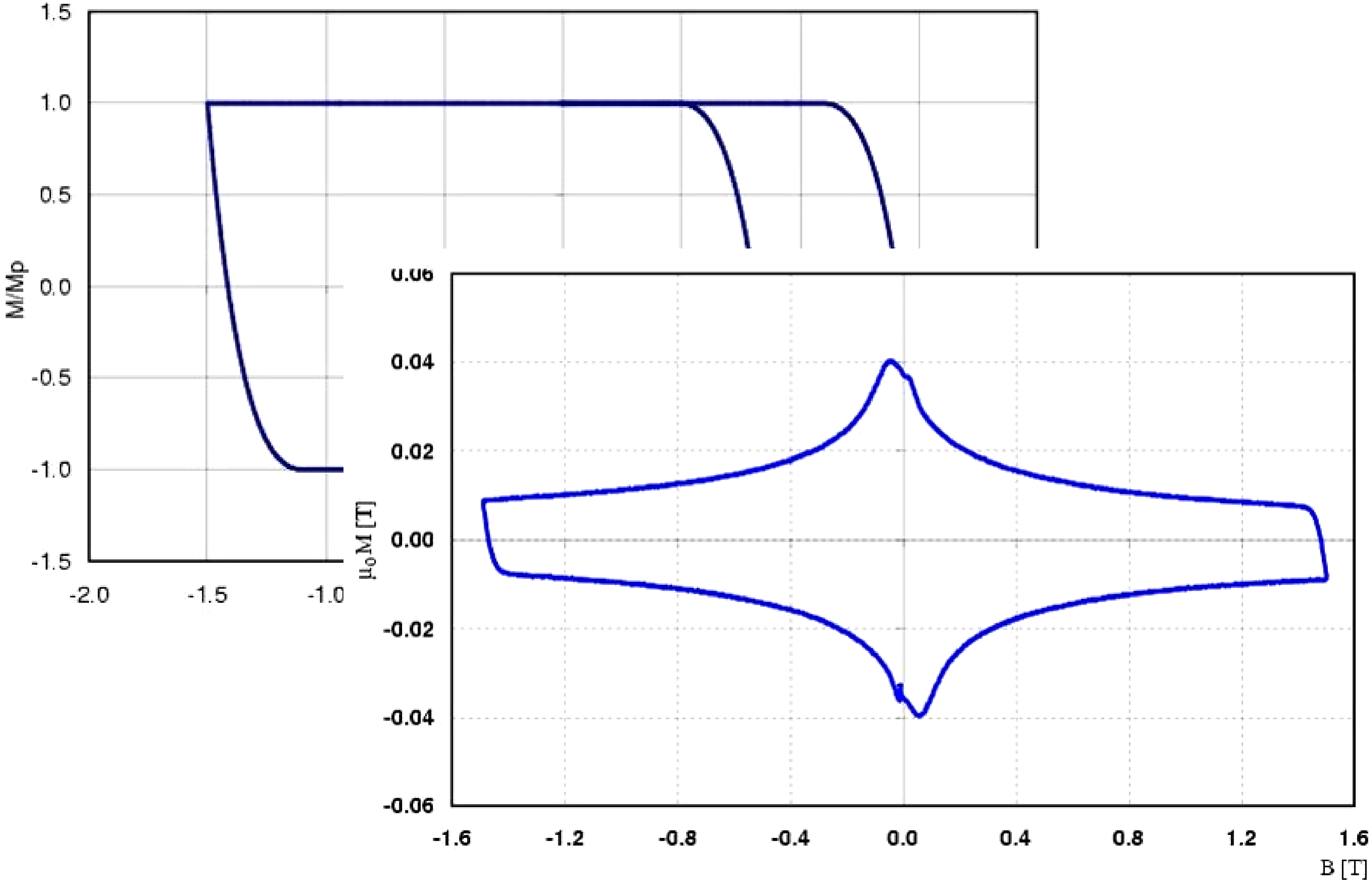
$$M_p = -J_c \frac{d}{4}$$

q =1 center of filament

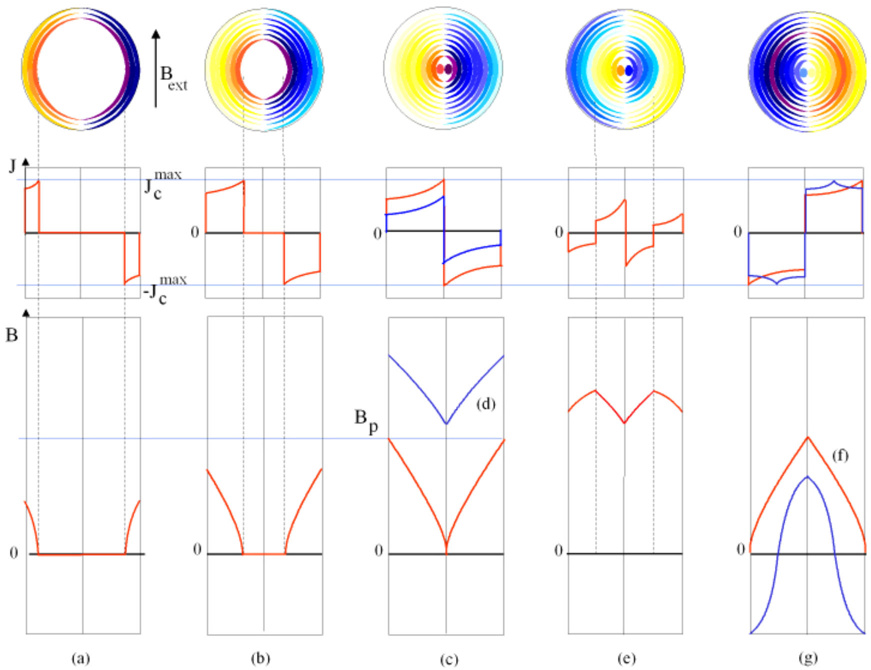
t = modulus of shielding field

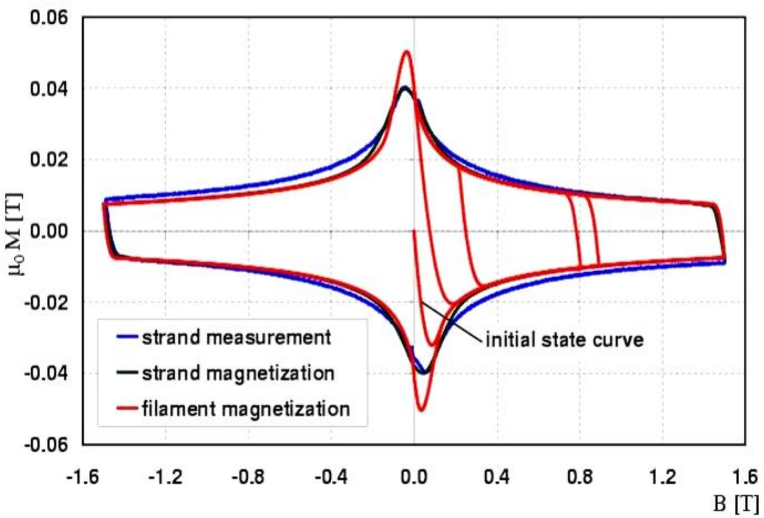
$$B_y = B_{\text{ext}} - t = B_{\text{ext}} - \mu_0 J_c \frac{d}{2} q$$

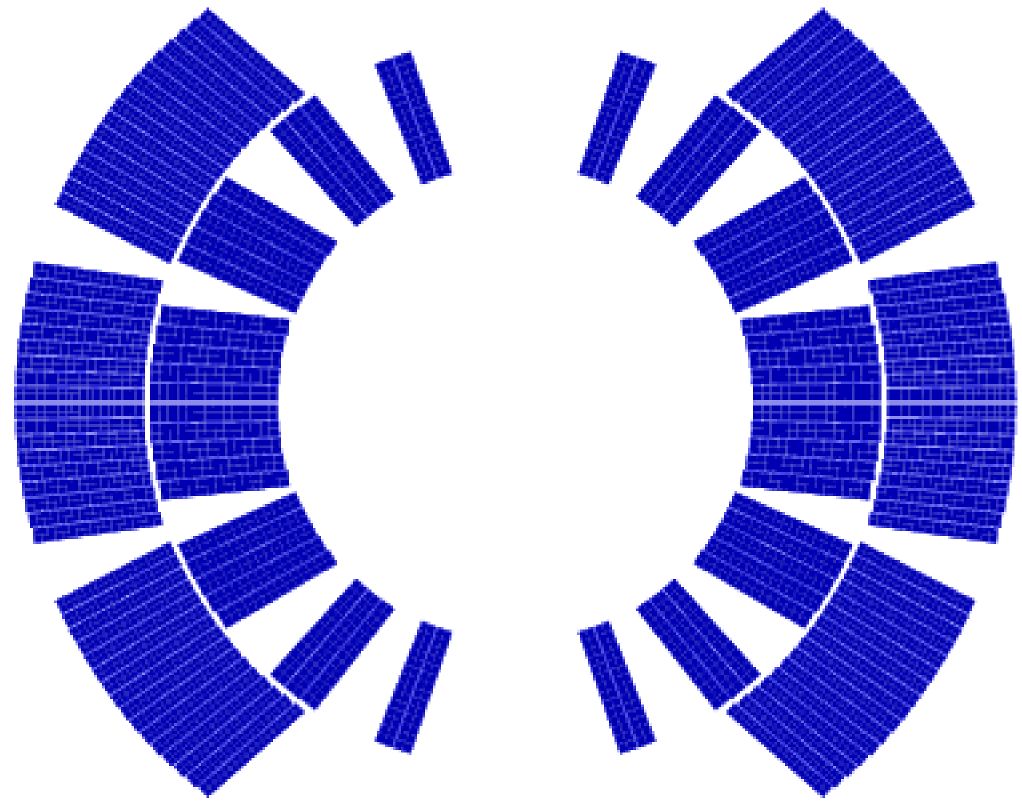
Magnetization in SC Slab, Measured in LHC Strands



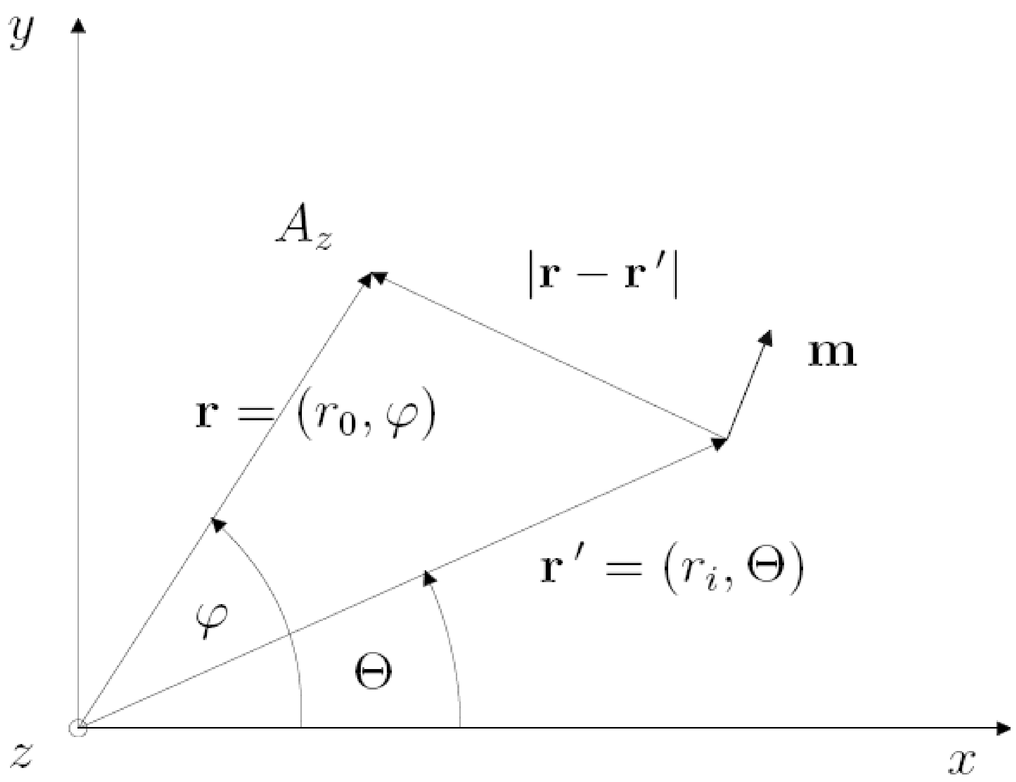
Superconducting Magnetization (Hysteresis Model)







Field Quality Calculation from Magnetization

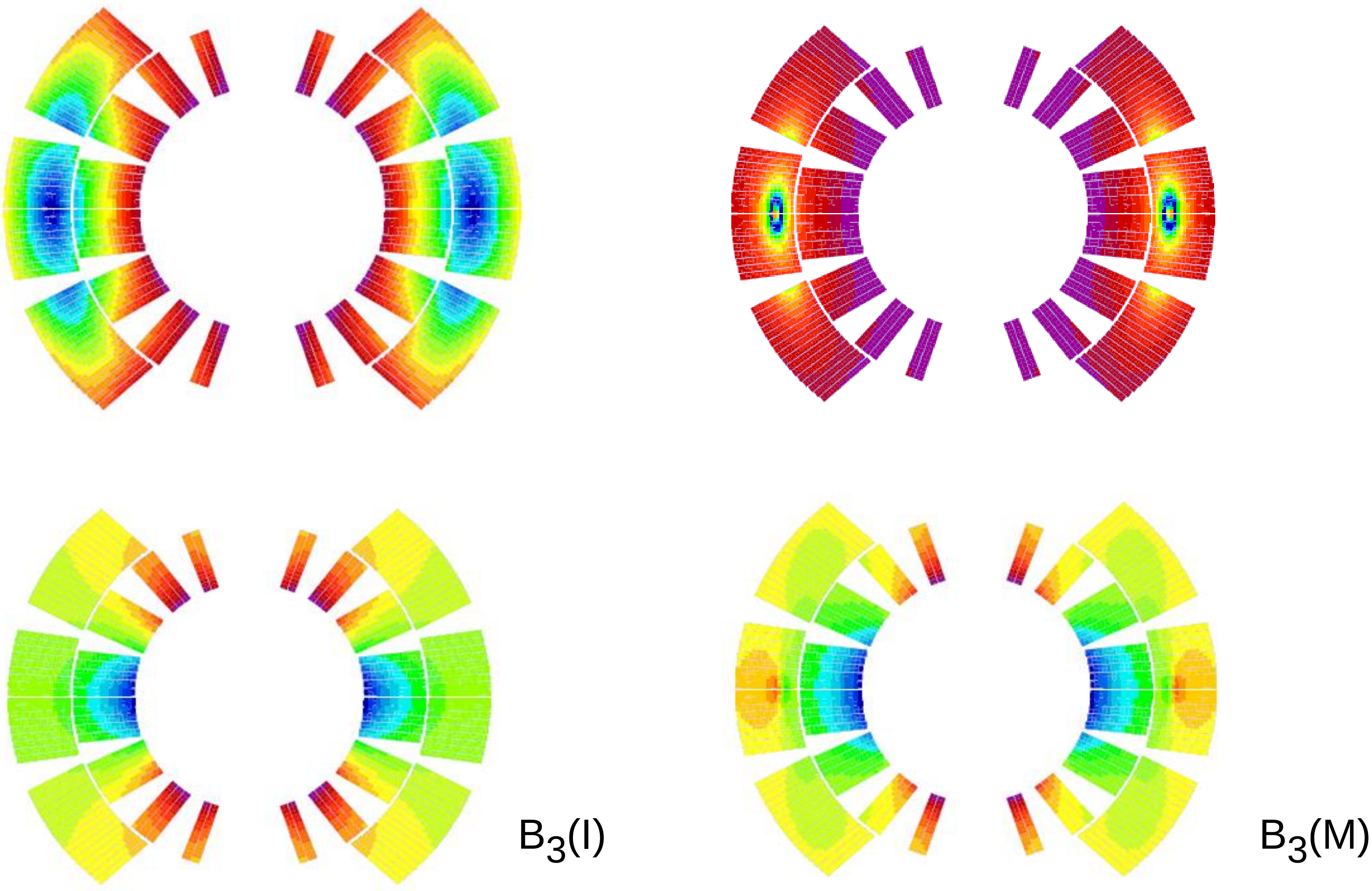


$$A_z(\mathbf{r}) = \frac{\mu_0 \mathbf{m}}{2\pi} \times \operatorname{grad}_{\mathbf{r}'} \ln \left(\frac{|\mathbf{r}-\mathbf{r}'|}{R_{\mathrm{ref}}} \right)$$

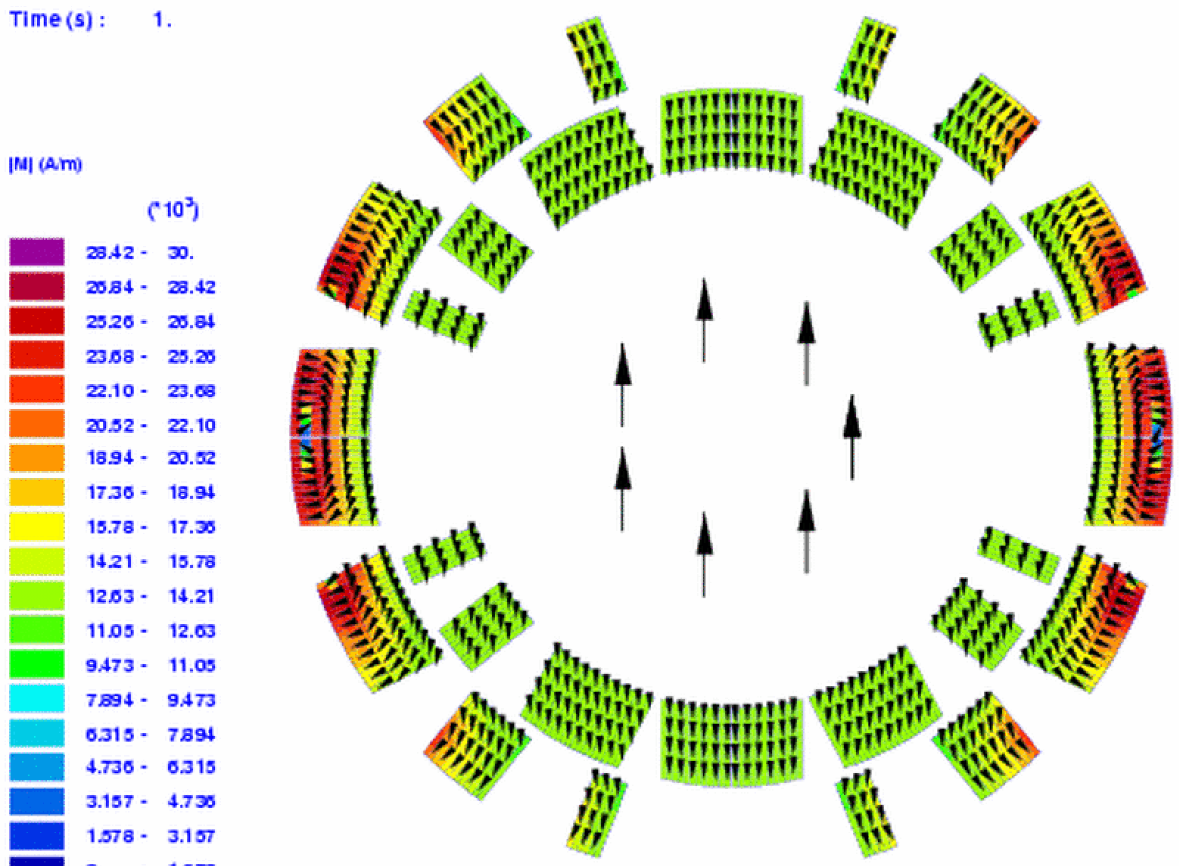
$$A_n = \frac{\mu_0}{2\pi} \frac{r_0^{n-1}}{r_i^{n+1}} n (m_{\mathbf{r}'} \cos n\theta - m_\theta \sin n\theta)$$

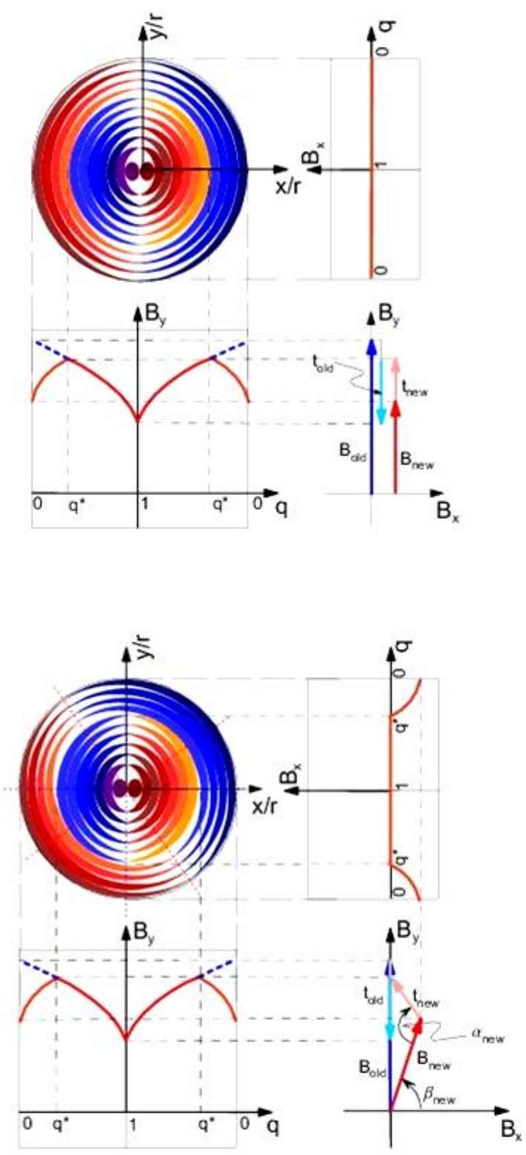
$$B_n = \frac{\mu_0}{2\pi} \frac{r_0^{n-1}}{r_i^{n+1}} n (m_{\mathbf{r}'} \sin n\theta + m_\theta \cos n\theta)$$

Field and SC Magnetization

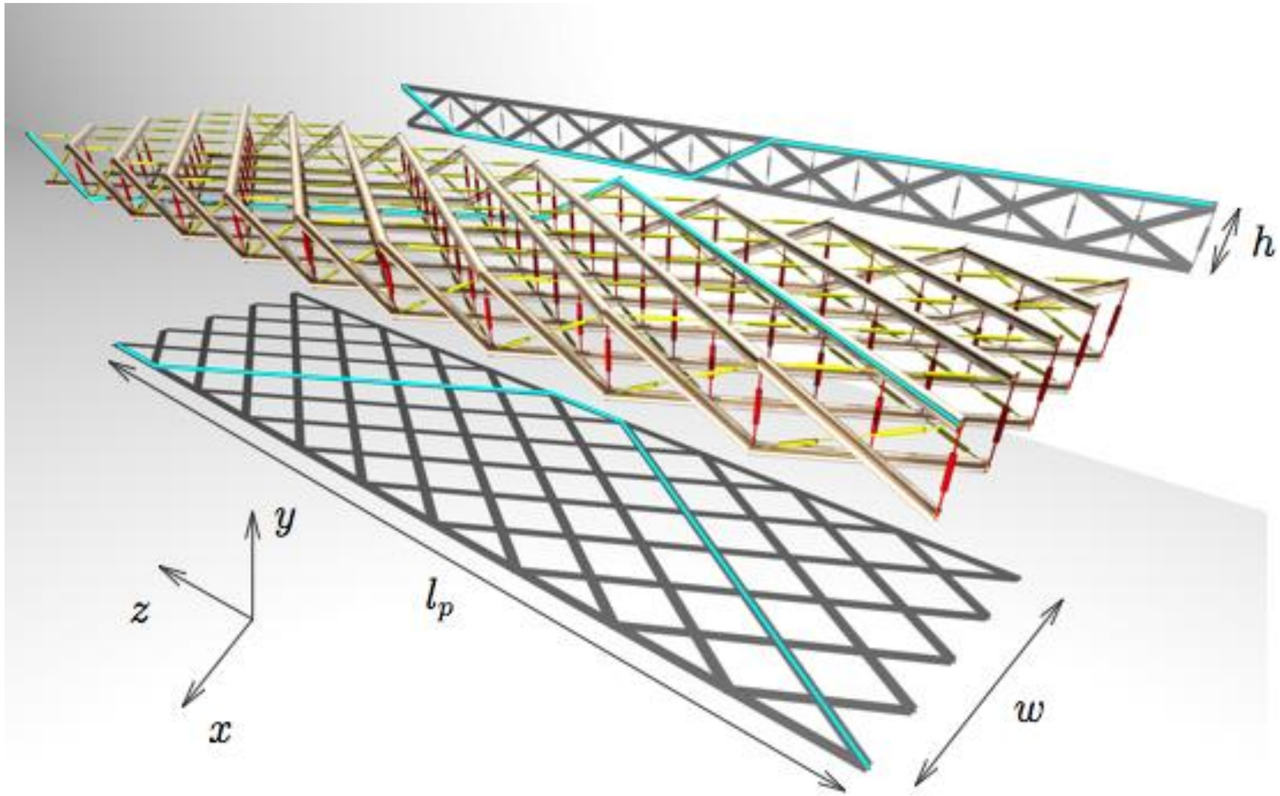


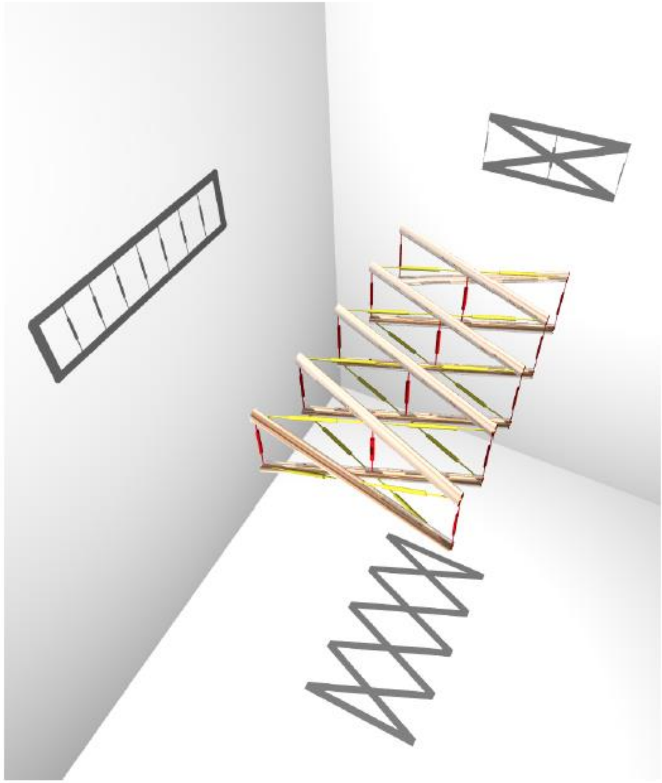
Vector-Hysteresis in Combined Function Magnet (MCBX)





Eddy Currents in Rutherford Cables

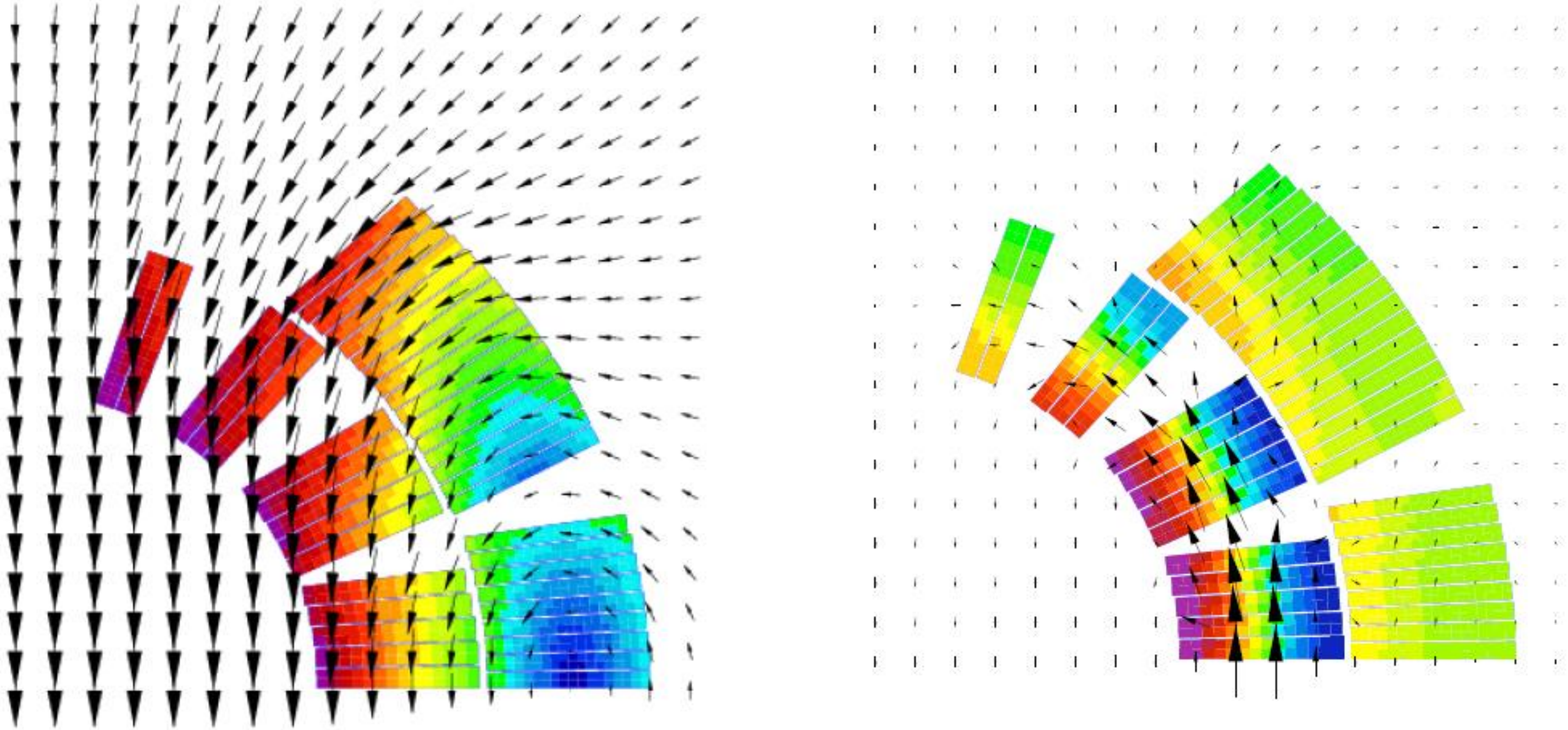




Nodes: 2 Ns * Nb + Ns
Nb = L / pitch * Ns

Pitch = 100 mm
Ns = 36
Nodes = 30 000 / meter
LHC main dipole = 4.2 km cable
126 Million Nodes

Field Generated by ISCC



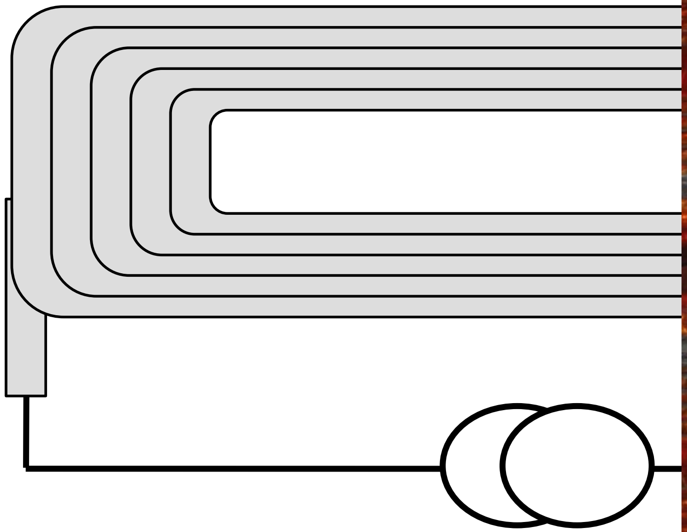
Computation relying on empirical parameters such as RRR, and adjacent/transversal contact resistances in the cable

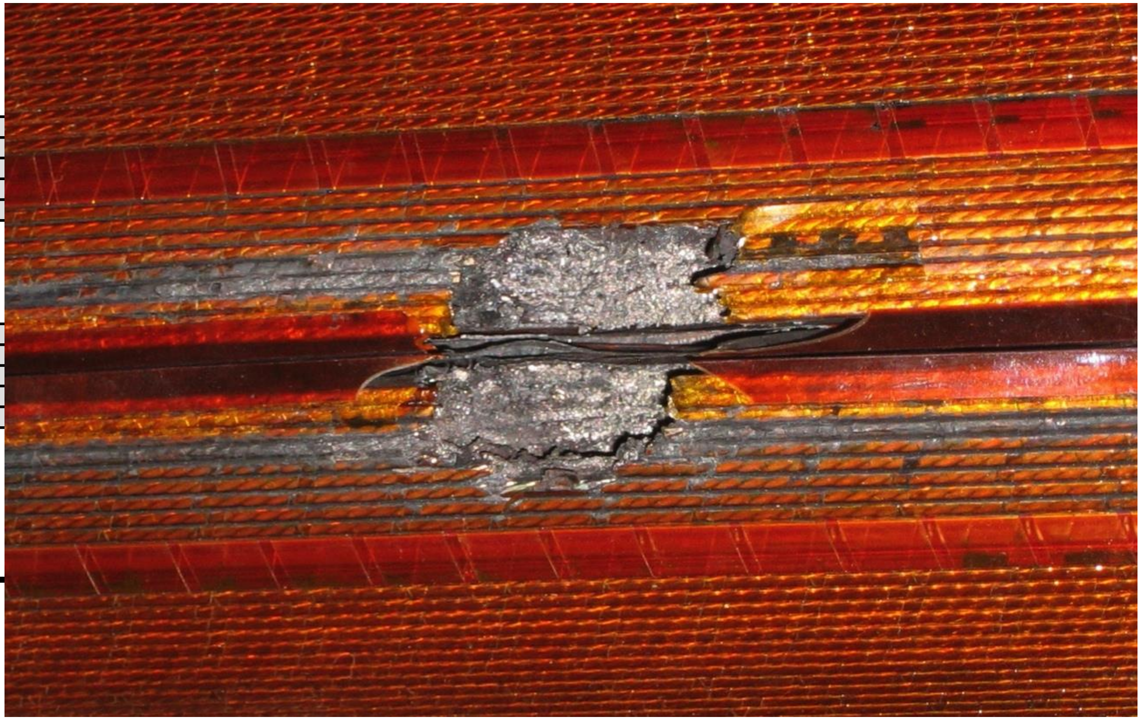
Quench

- ➔ Quench: Transition from SC to normal conducting state caused by

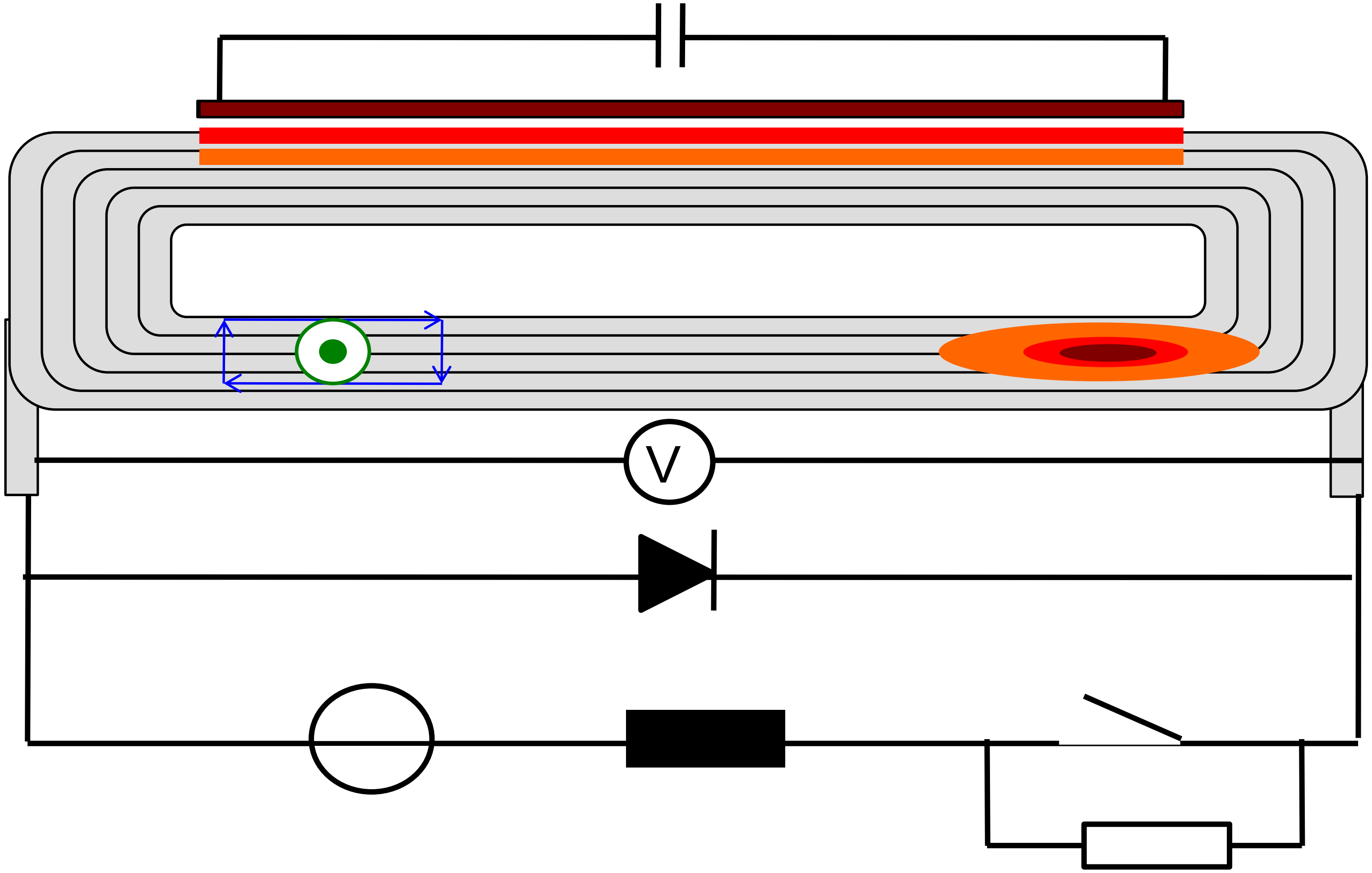
  beam losses, conductor movement, eddy currents etc.

- ➔ Propagation:
  - Normal conducting zone generates Ohmic heat
  - Quench- und temperature distribution determined by loss-mechanisms and cooling capacity

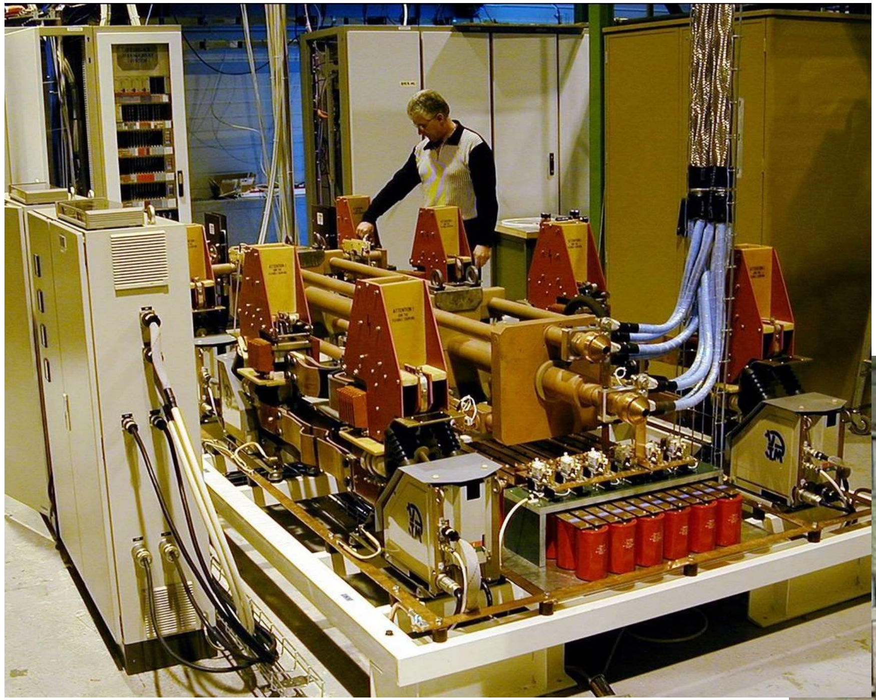




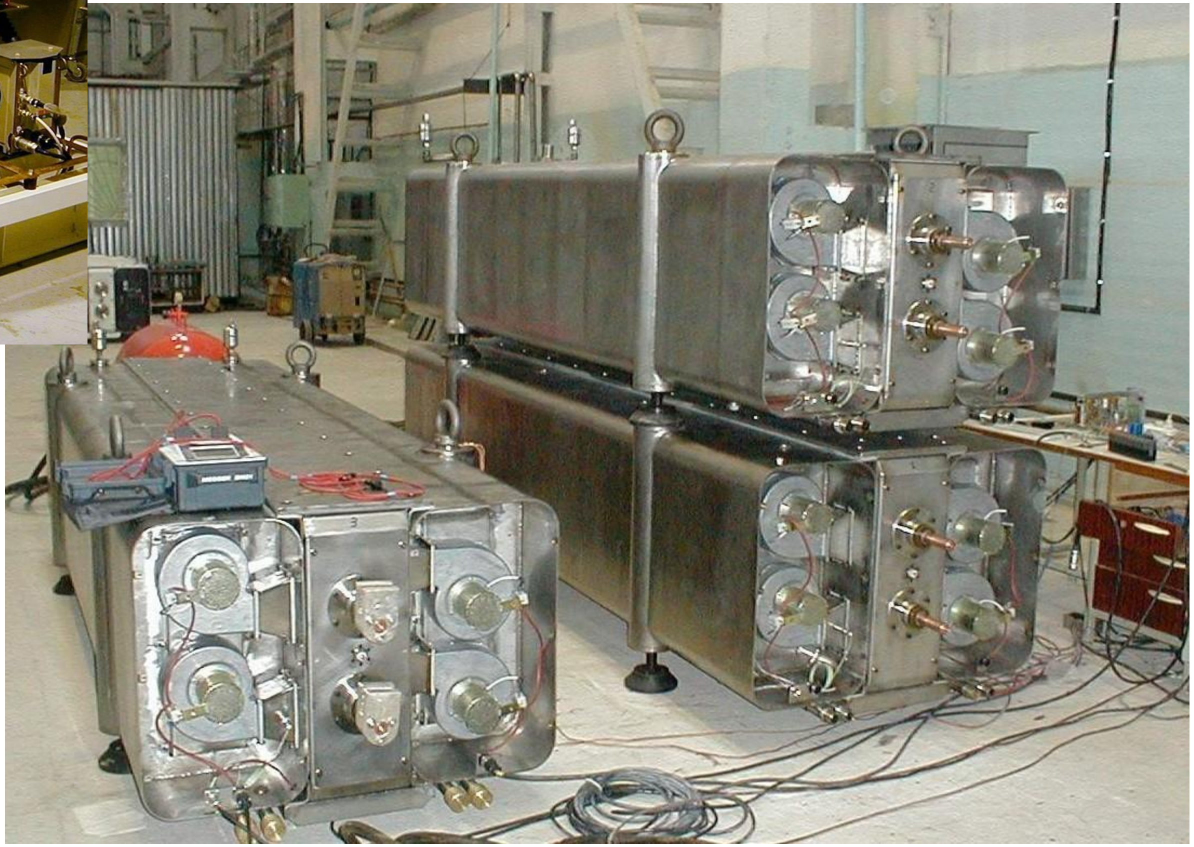
Quench Mechanism and Magnet Protection



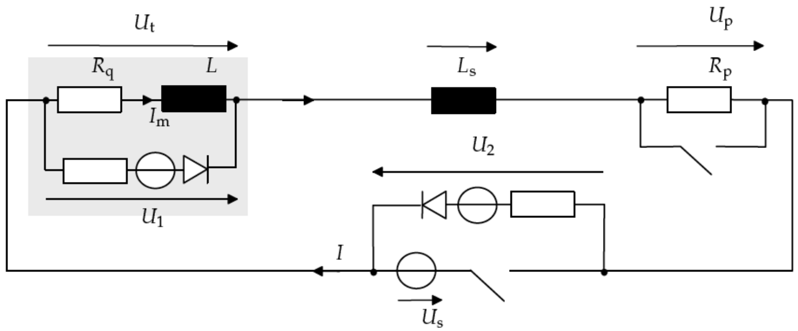
Switches and Dump Resistors

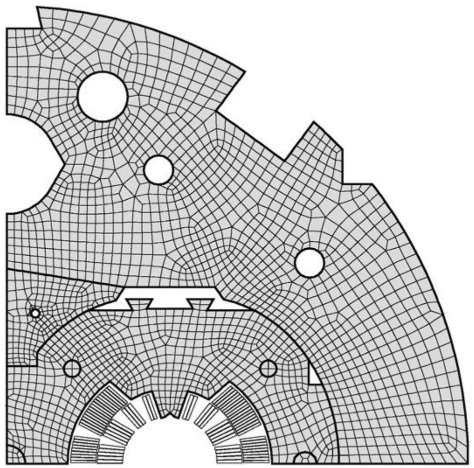


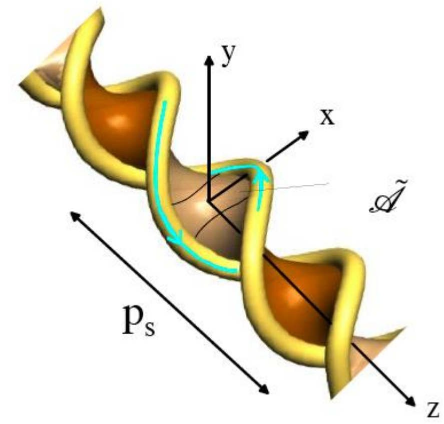
Heater failure and switch failure are not permitted
Anekdote: Dynamite switch

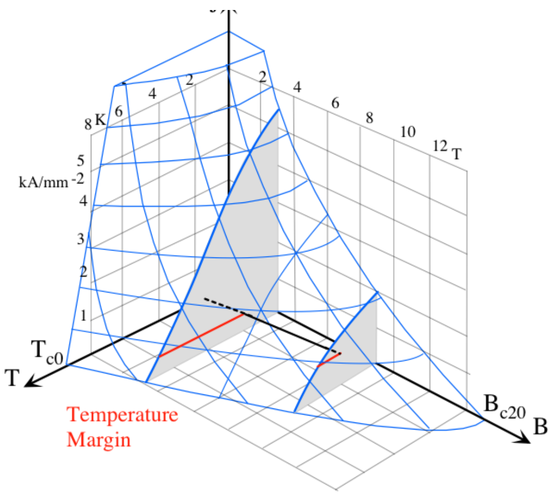


Quench Simulation (Multi-Physics, Multi-Scale)

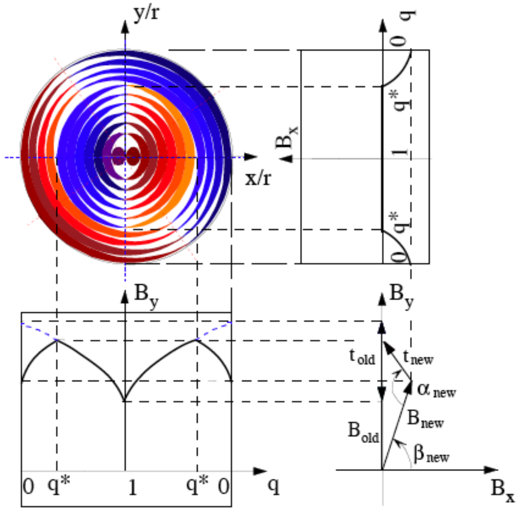


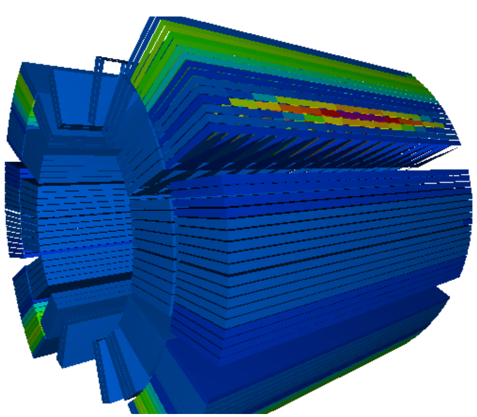


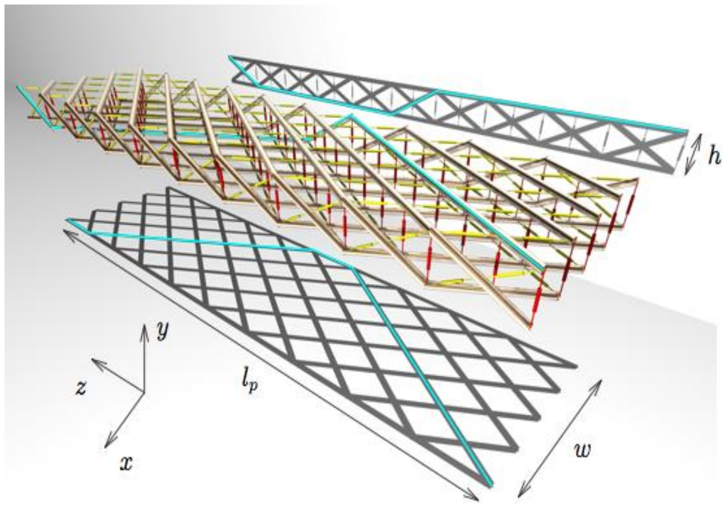




Quench Simulation

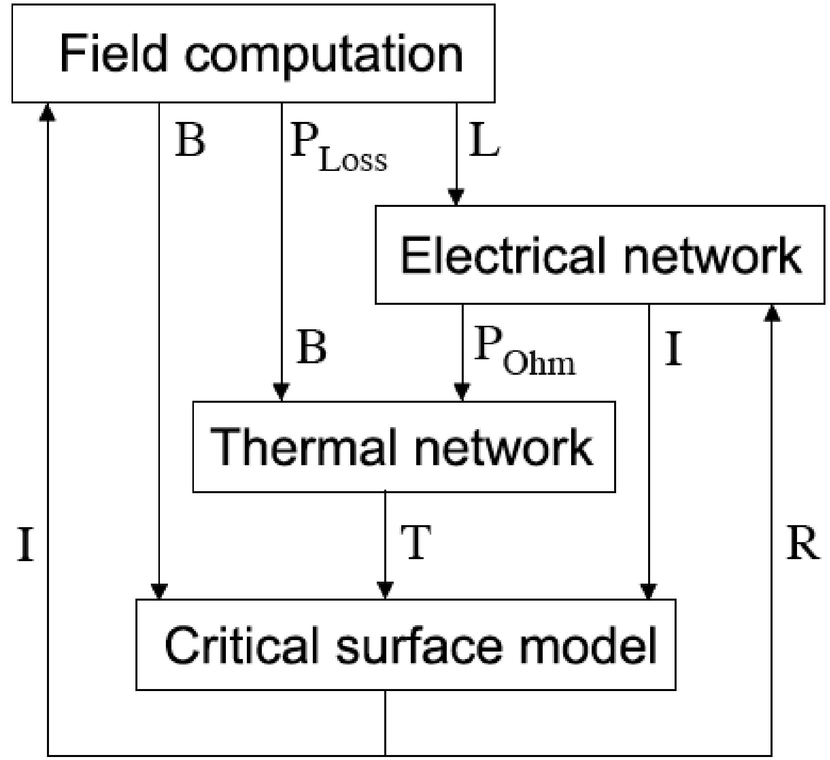




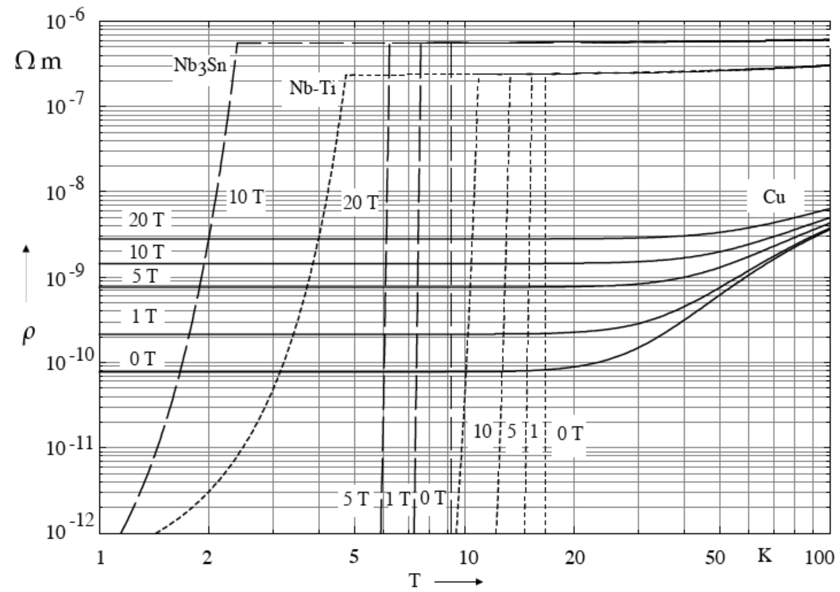


Challenges I

- Multiscale
  - Filaments 6 µm
  - Strands 1 mm
  - Cable 0.1 m
  - Magnet 10 m
  - String 3.2 km
- Multiphysics
- The smallest time constant determines the Runge-Kutta step



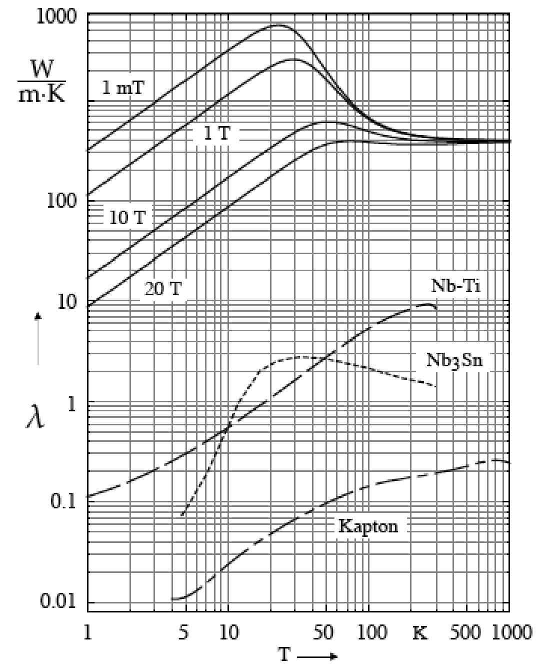
Challenges II: Material Parameters

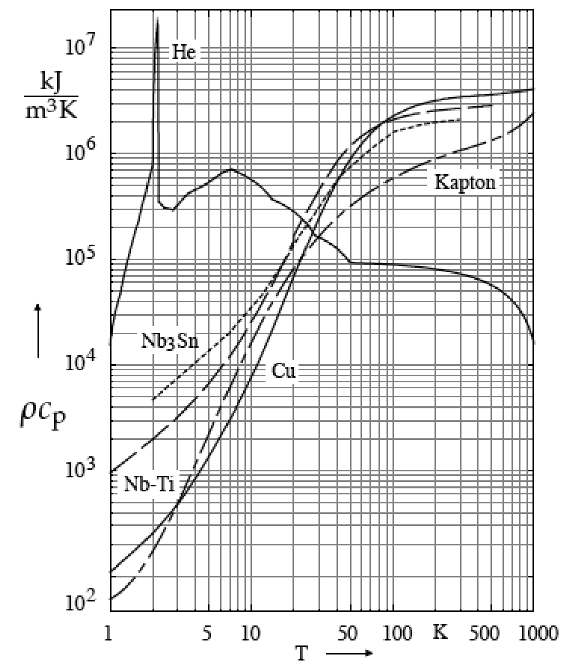


Electrical resistivity

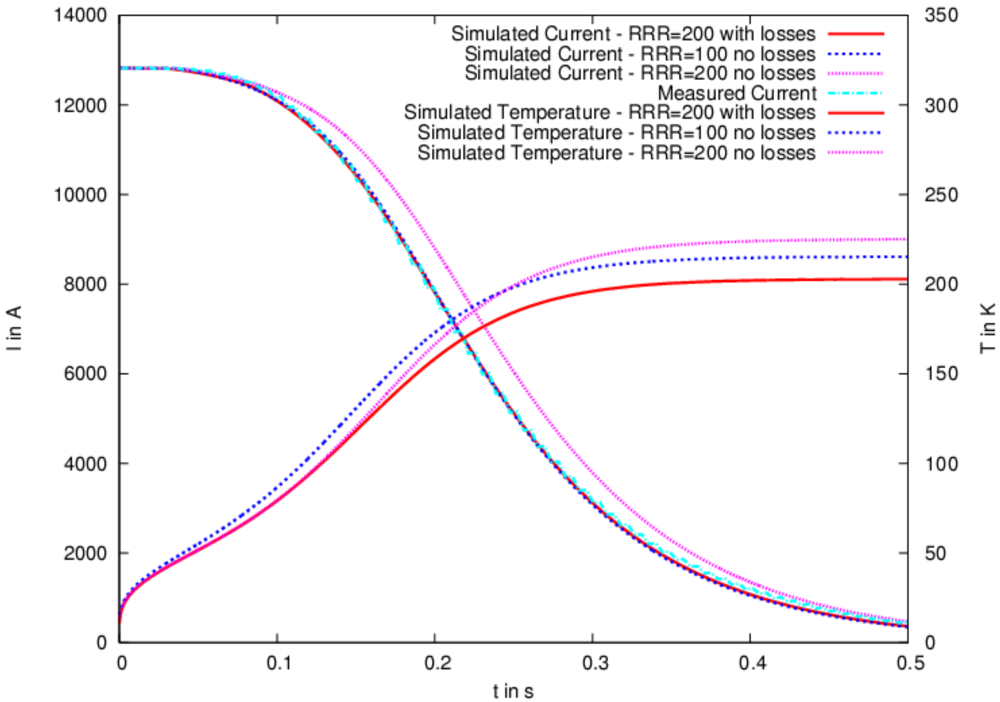
Volumetric heat capacity

Thermal conductivity





Validation



Empirical parameters:

- RRR
- Ra/Rc
- IFCC effective res.
- heat conductivity
- heat capacity

➔ Different families of parameters yield exactly the same observable I(t)

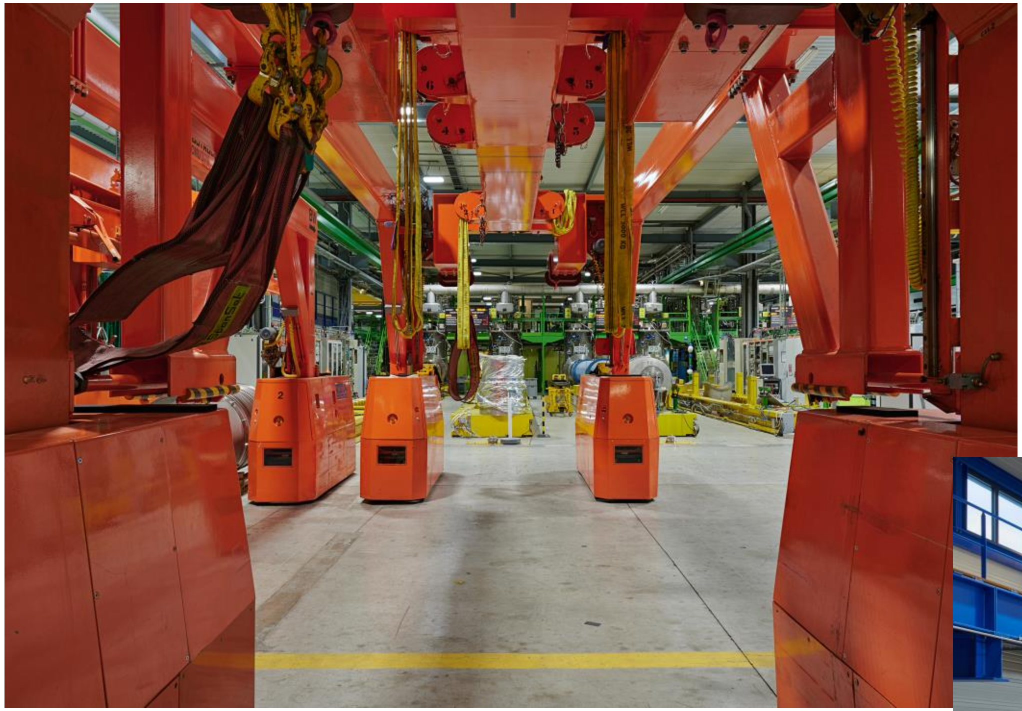
➔ More than one solution exists

➔ Care must be taken to model

– all relevant phenomena

– using realistic material parameters

Our Test and Measurement Environment (SM18 and 311)



Nobody believes in the field simulations, but the field computation expert himself.

Everybody believes in the measurements, but the measurement engineer himself.

Challenge

Model all relevant physical phenomena with adequate accuracy so that we can be confident to simulate the internal states of a quenching magnet and understand its behavior.

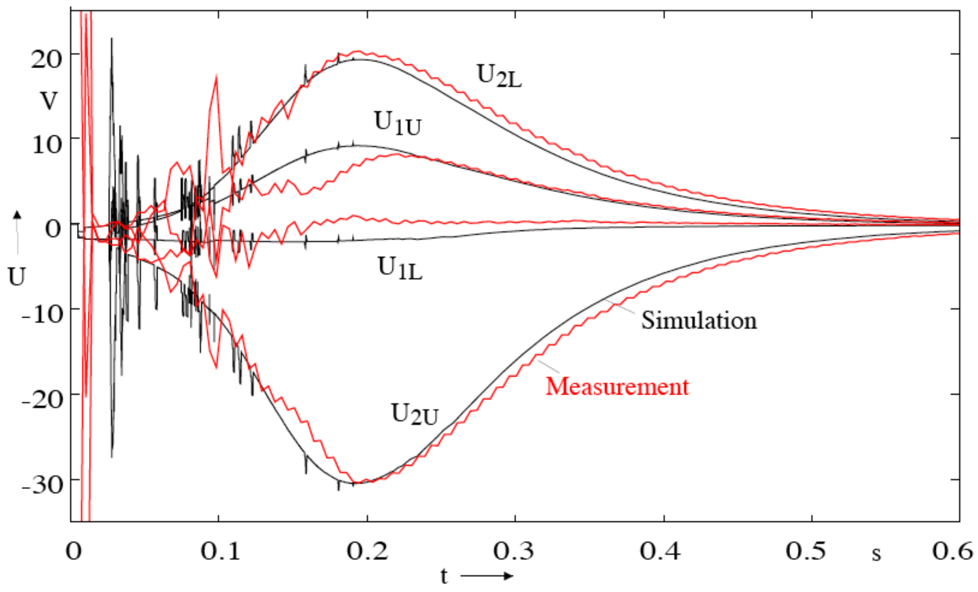
Validation

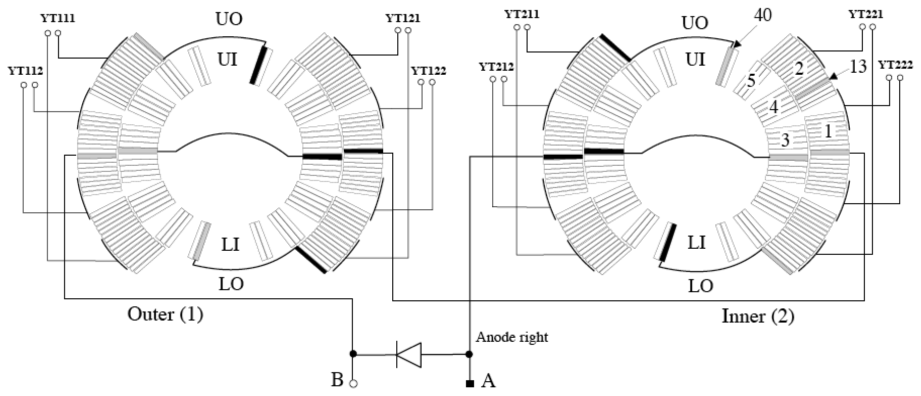
Measured quantities can be reproduced with all material- and model-parameters within their range of uncertainty.

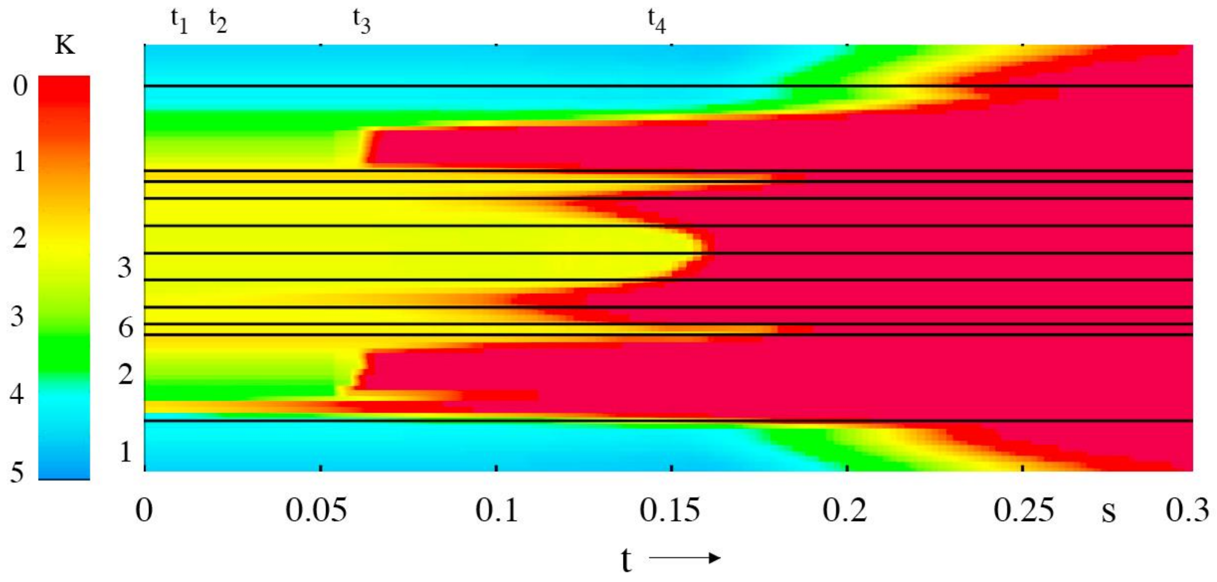
Extrapolation and Introspection

If the above criteria are reached, extrapolated results will match measurements without adaptation of material- and model parameters. It is then also possible to simulate the internal states of the magnet that escape measurements.

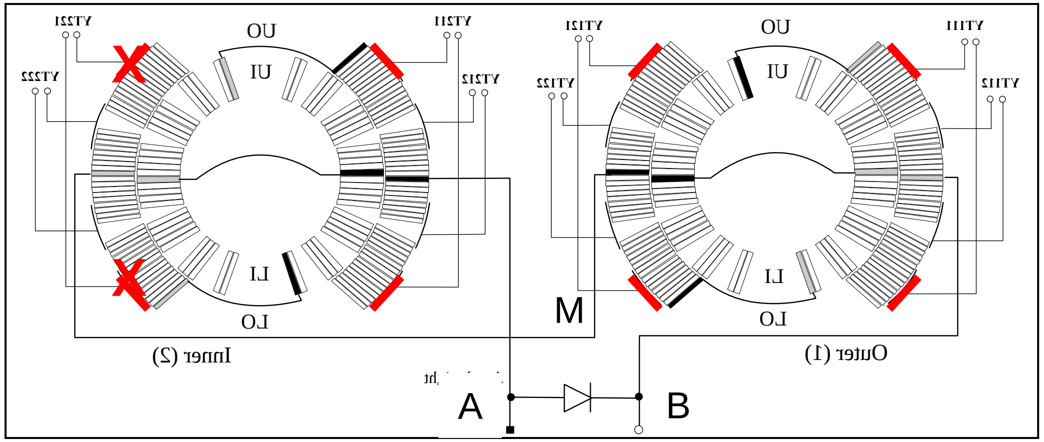
Introspection (Voltage Ripples)

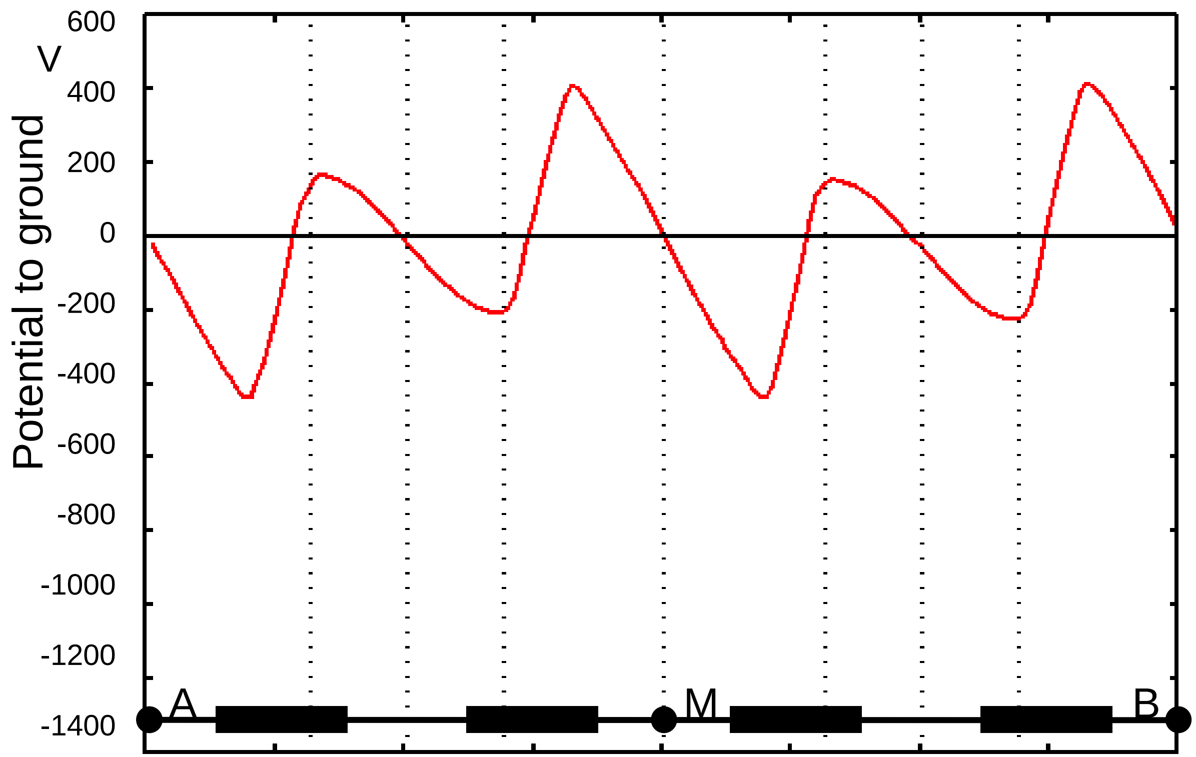




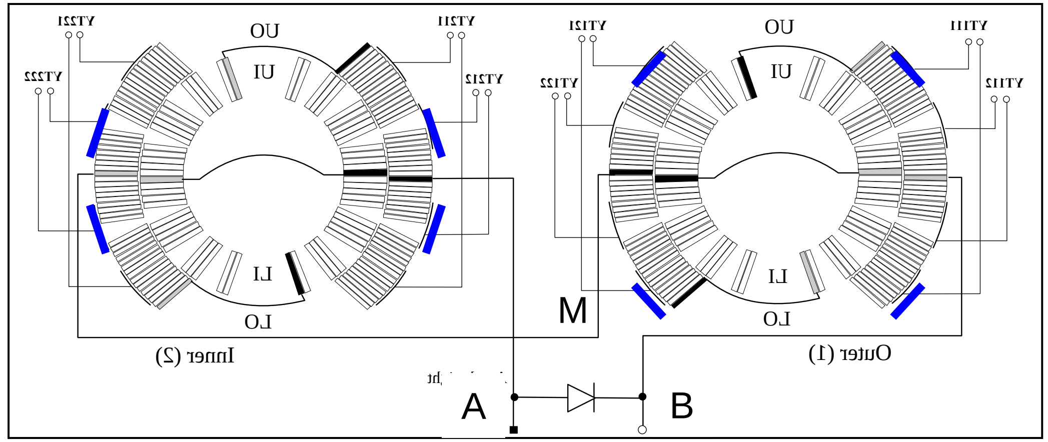


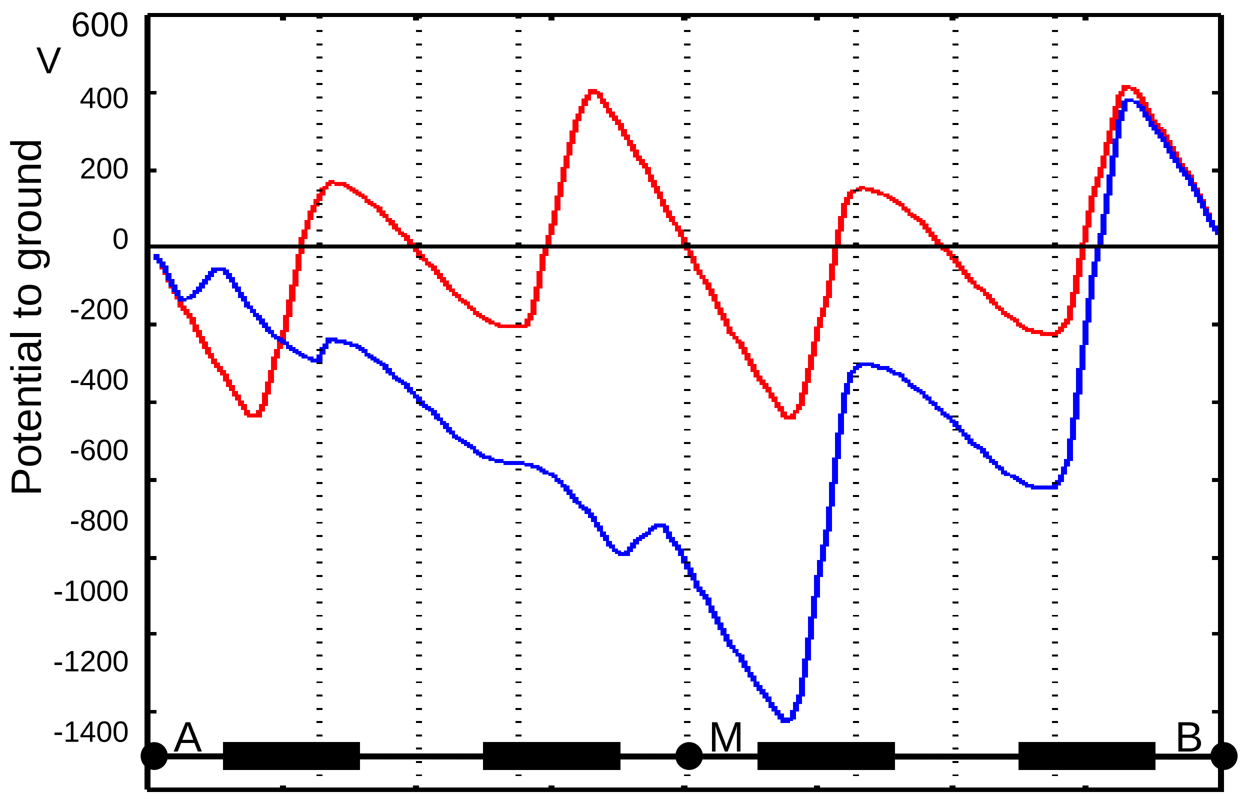
Defect on Quench Heater Circuit



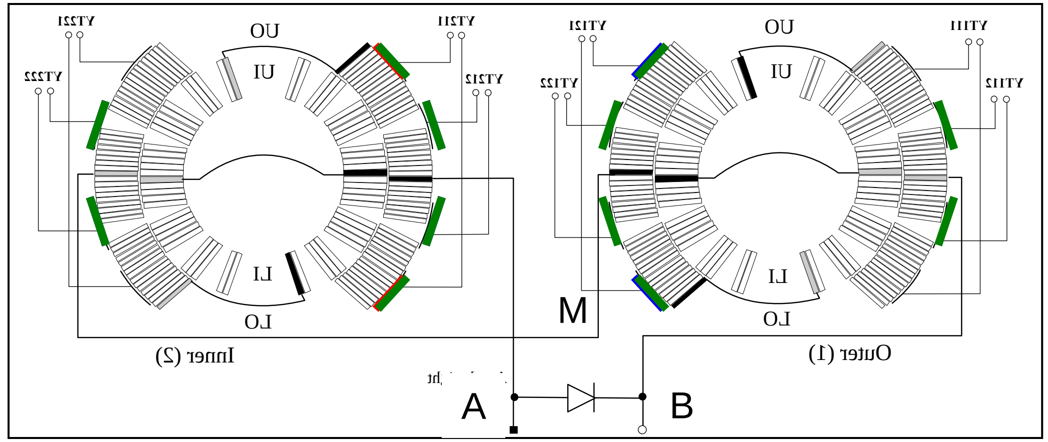


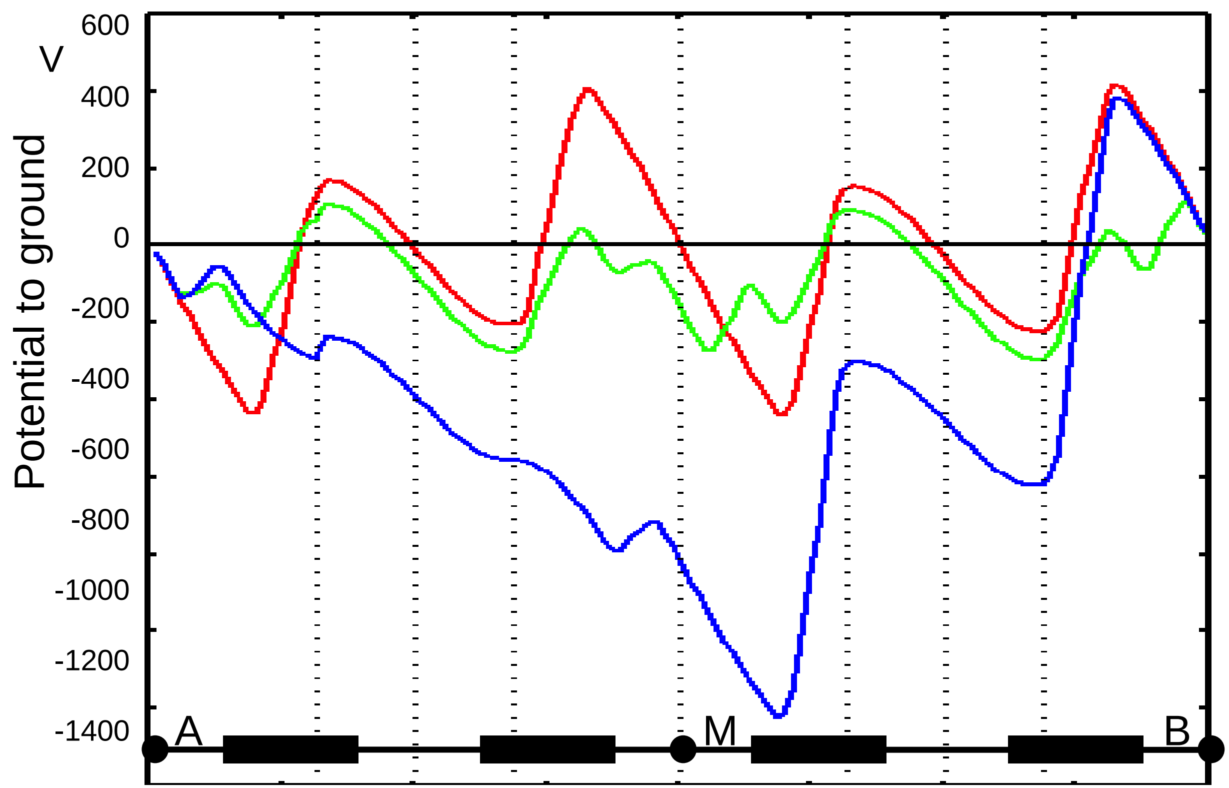
Defect on Quench Heater Circuit



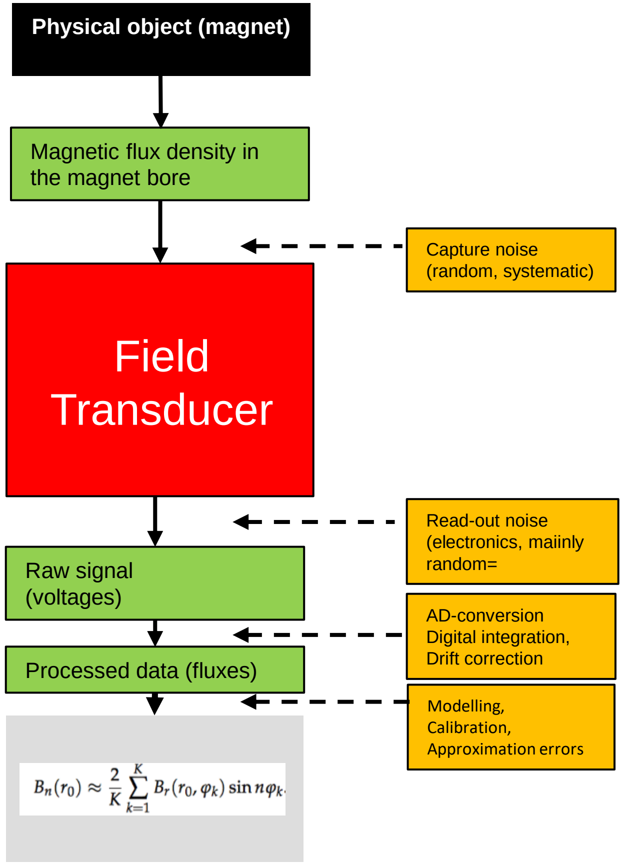


Defect on Quench Heater Circuit

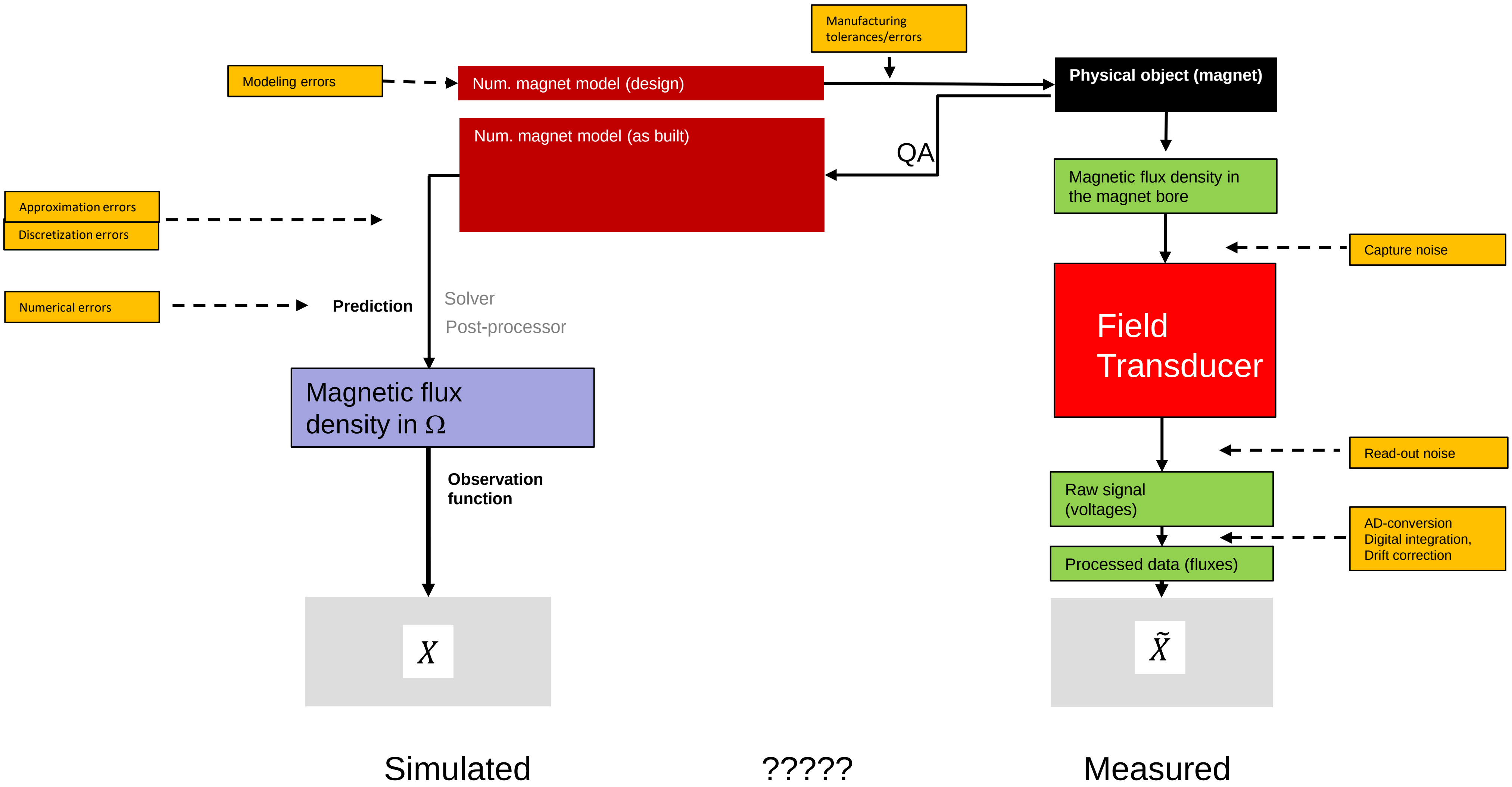




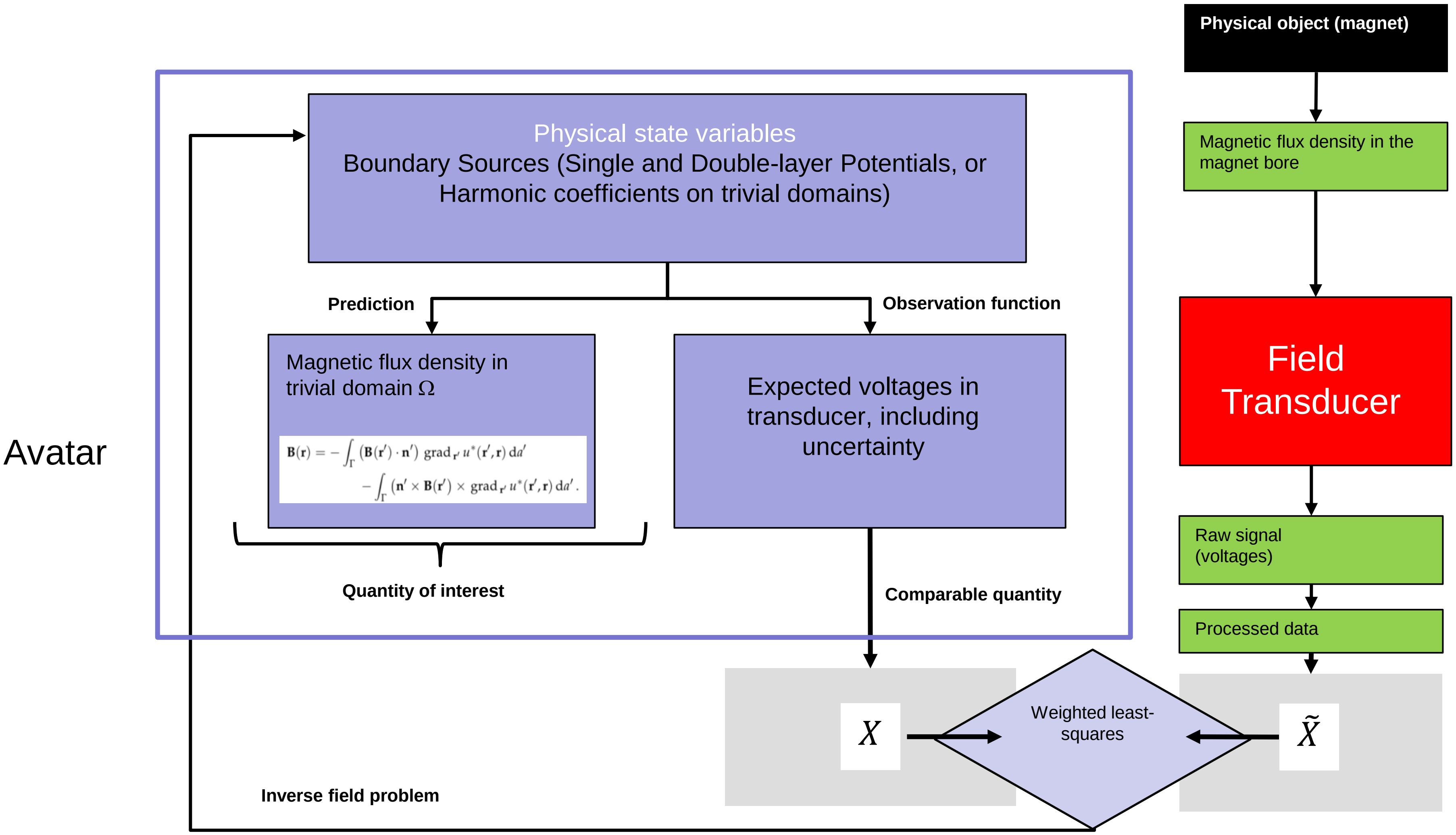
Bringing together Simulations Tests and Magnetic Measurements



Bringing together Simulations and Magnetic Measurements



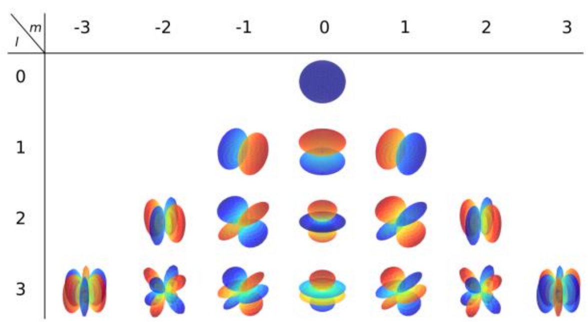
Bringing together Simulations and Magnetic Measurements

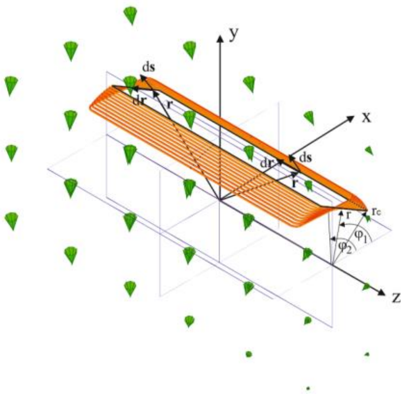


The Observation Function

- The **observation function** $s: \boldsymbol{B}(\boldsymbol{r},t) \rightarrow U(\boldsymbol{r},t)$ is determined by modelling the magnetic measurement technique which allows including calibration and the sources of uncertainty:
  - Modelling errors (neglect of temperature dependent
  - Approximation errors (coil parameters approximated by surface and radius)
  - Calibration errors (e.g., errors in the surface and radius measurements)

- The inverse observation function $s^{-1}: U(\boldsymbol{r},t) \rightarrow \boldsymbol{B}(\boldsymbol{r},t)$ may **not** exist

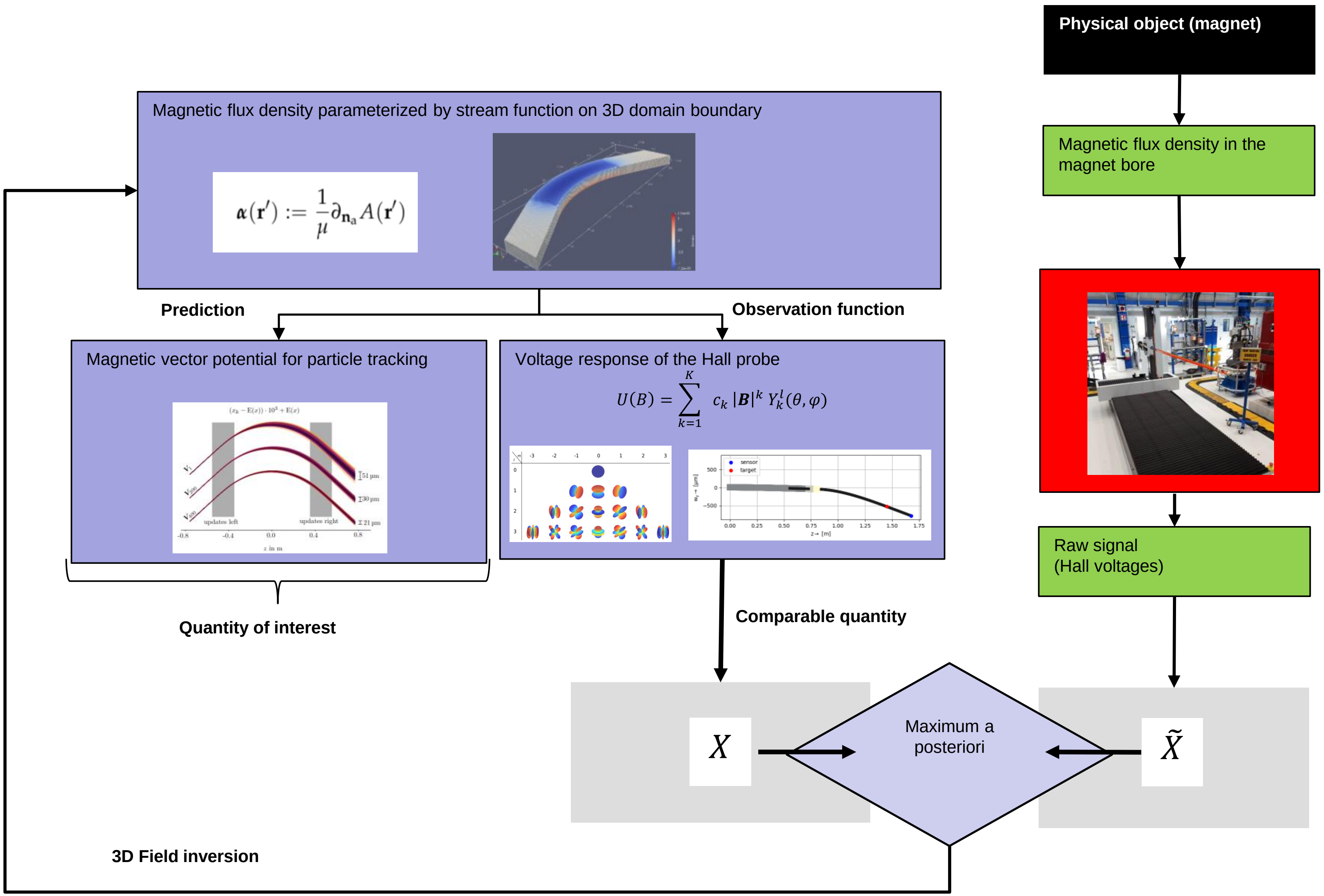




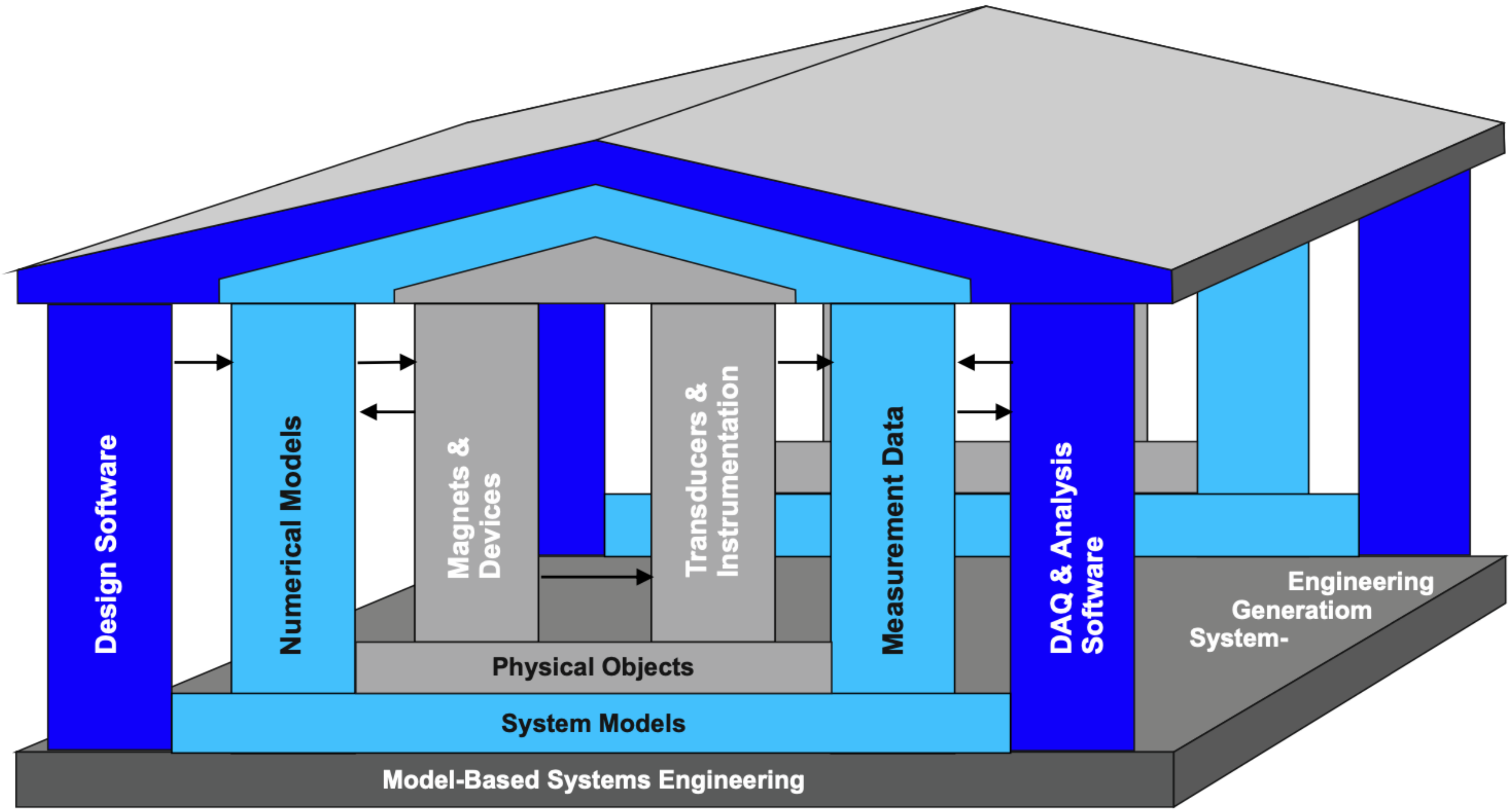
Coil couples with B_r and B_z

- The observation function allows the combination of different transducers (sensor fusion)

Bringing together Simulations and Magnetic Measurements



Model-Based Systems Engineering

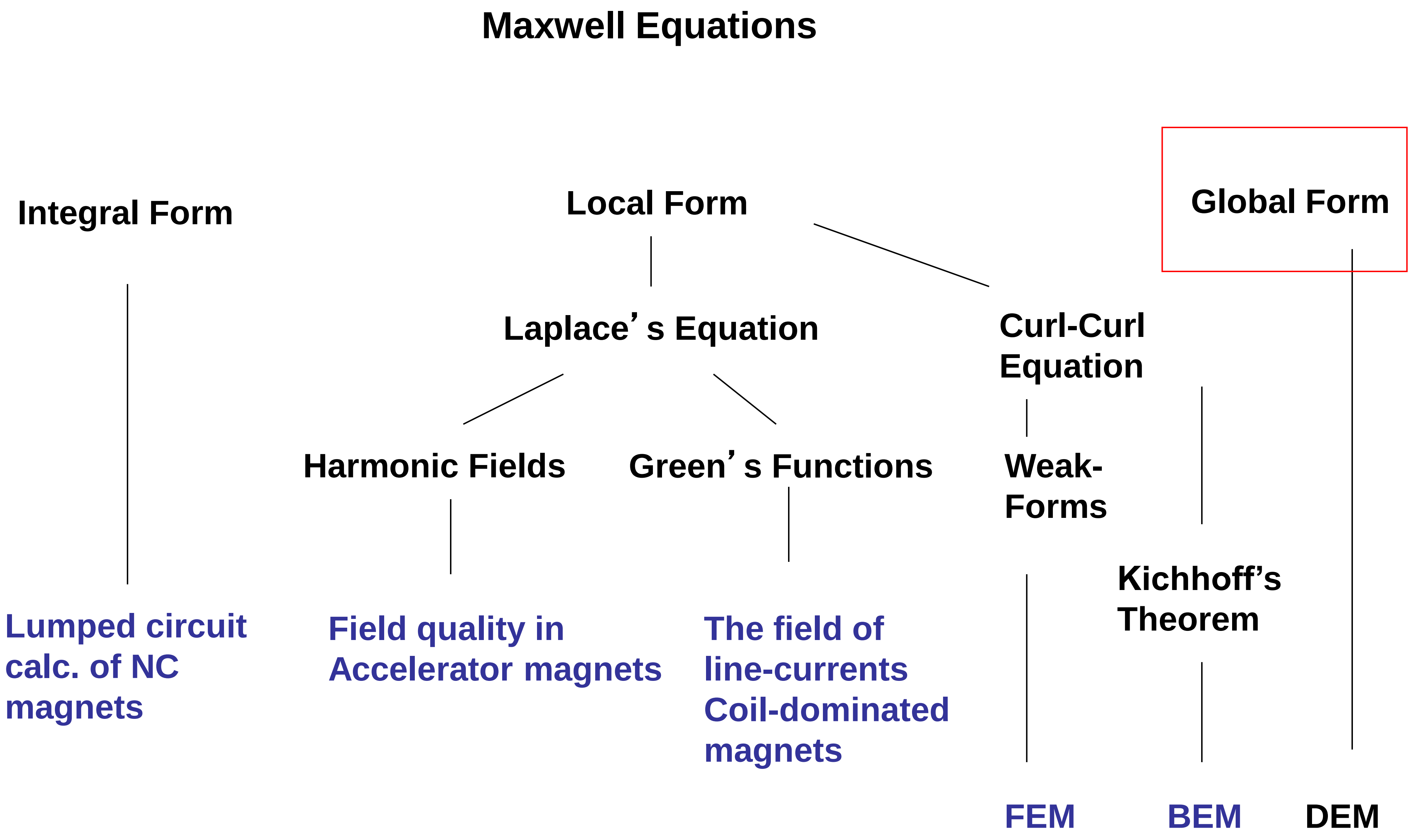


Field Description I-III

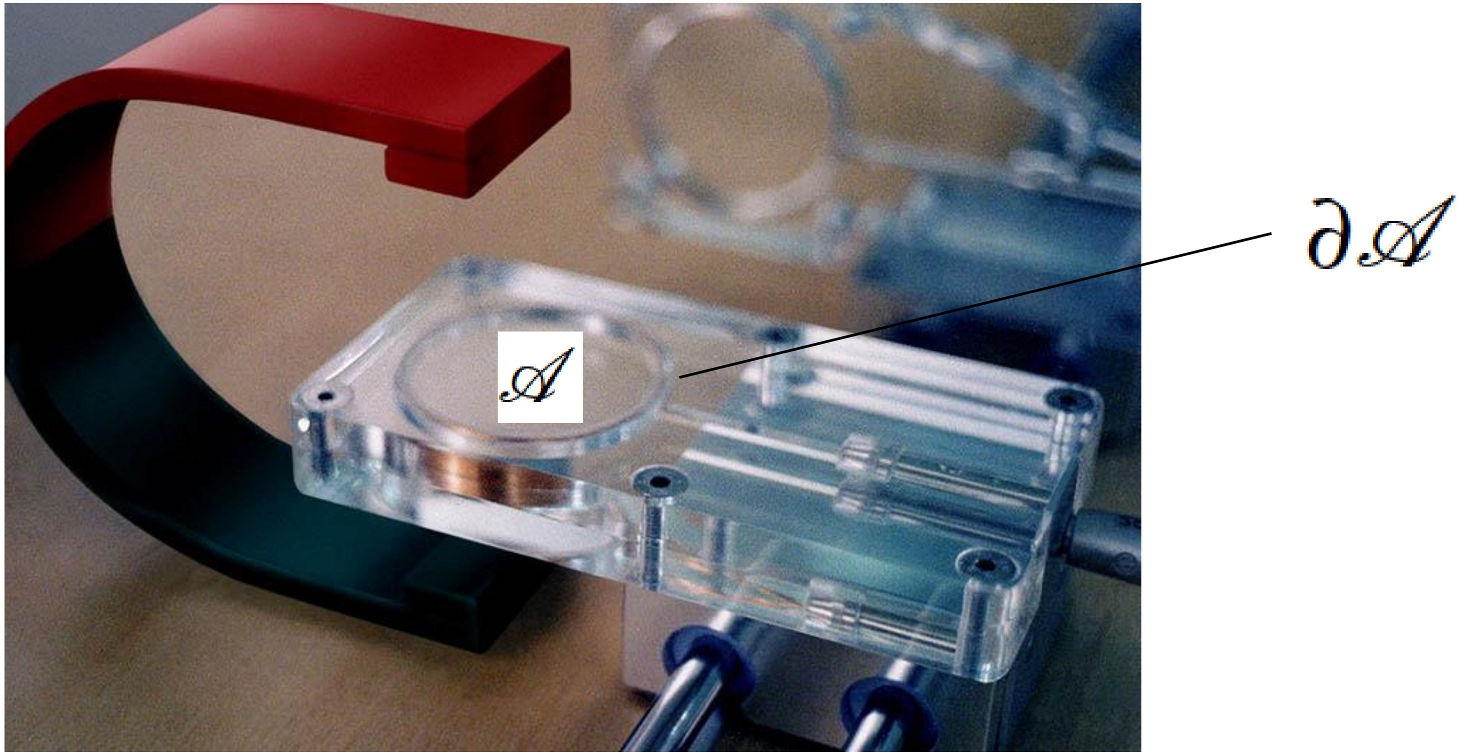
Stephan Russenschuck, CAS, 2023

A Little Self-Test

- What is a vector? An arrow, a tuple of numbers, a quantity having direction and magnitude, a solution of a linear equation system, a contravariant tensor?
- What is the difference between coefficients, components, and coordinates?
- We know (from school) how to add vectors represented as arrows by means of the parallelogram law. Why not add the position vector (units of meter) and the force vector (units of newton), represented by an arrow at the tip of the position vector?
- What are field lines
- What **are** magnetic fields
- We say that fields are linear. So is the field in a sextupole non-linear?


Maxwell Equations
Integral Form
Local Form
Global Form
Laplace's Equation
Curl-Curl Equation
Harmonic Fields
Green's Functions
Weak-Forms
Kichhoff's Theorem
Lumped circuit calc. of NC magnets
Field quality in Accelerator magnets
The field of line-currents Coil-dominated magnets
FEM
BEM
DEM

Faraday' s Law (Inner Oriented Surface, Voltage along its Rim)

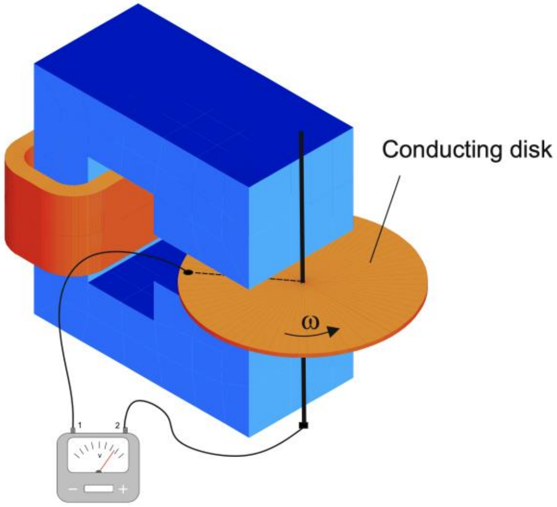


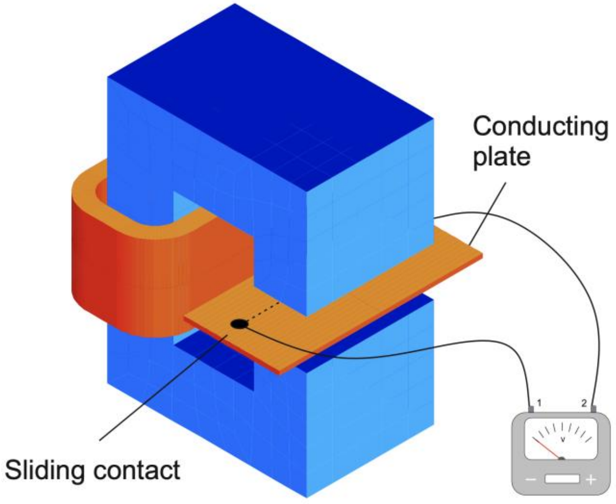
$$U(\partial\mathscr{A}) = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mathscr{A})$$

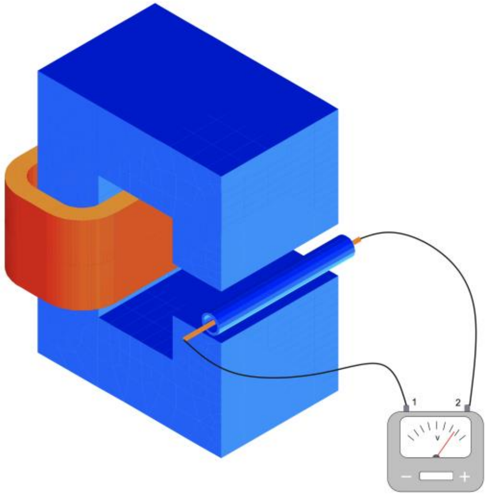
The potential to induce a voltage (electro-motive force)

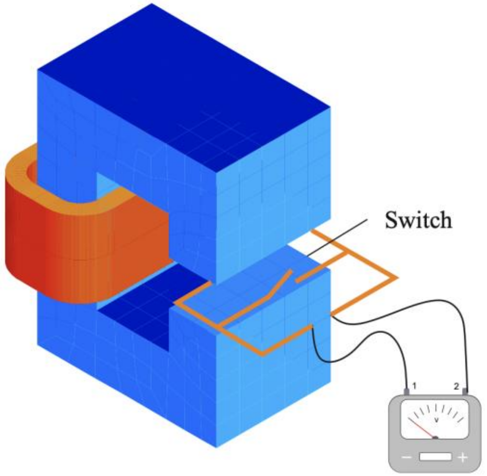
B. Auchmann, S. Kurz and S. Russenschuck, "A Note on Faraday Paradoxes," in *IEEE Transactions on Magnetics*, vol. 50, no. 2, Feb. 2014

Faraday Paradoxes









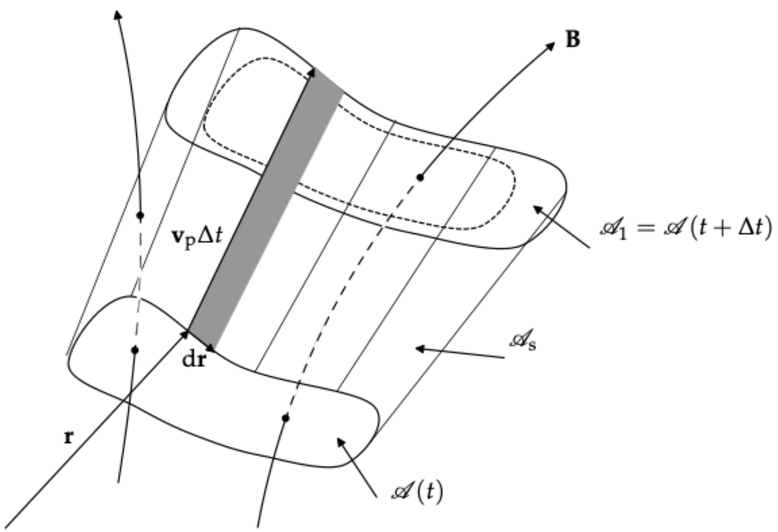
Convective derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x}(\mathbf{r},t)) = \frac{\partial \mathbf{x}}{\partial t} + (\mathbf{v} \cdot \operatorname{grad})\mathbf{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mathscr{A}(t)) = \int_{\mathscr{A}(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{a} + \int_{\mathscr{A}(t)} (\mathbf{v} \cdot \operatorname{grad}_{\mathrm{p}})\mathbf{B} \cdot \mathrm{d}\mathbf{a}\,.$$

$$= -\int_{\mathscr{A}(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{a} + \int_{\mathscr{A}(t)} \operatorname{curl}_{\mathrm{p}} (\mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{a}$$

$$= -\int_{\mathscr{A}} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{a} + \int_{\mathscr{C}} (\mathbf{v}_{\mathrm{p}} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{r}$$

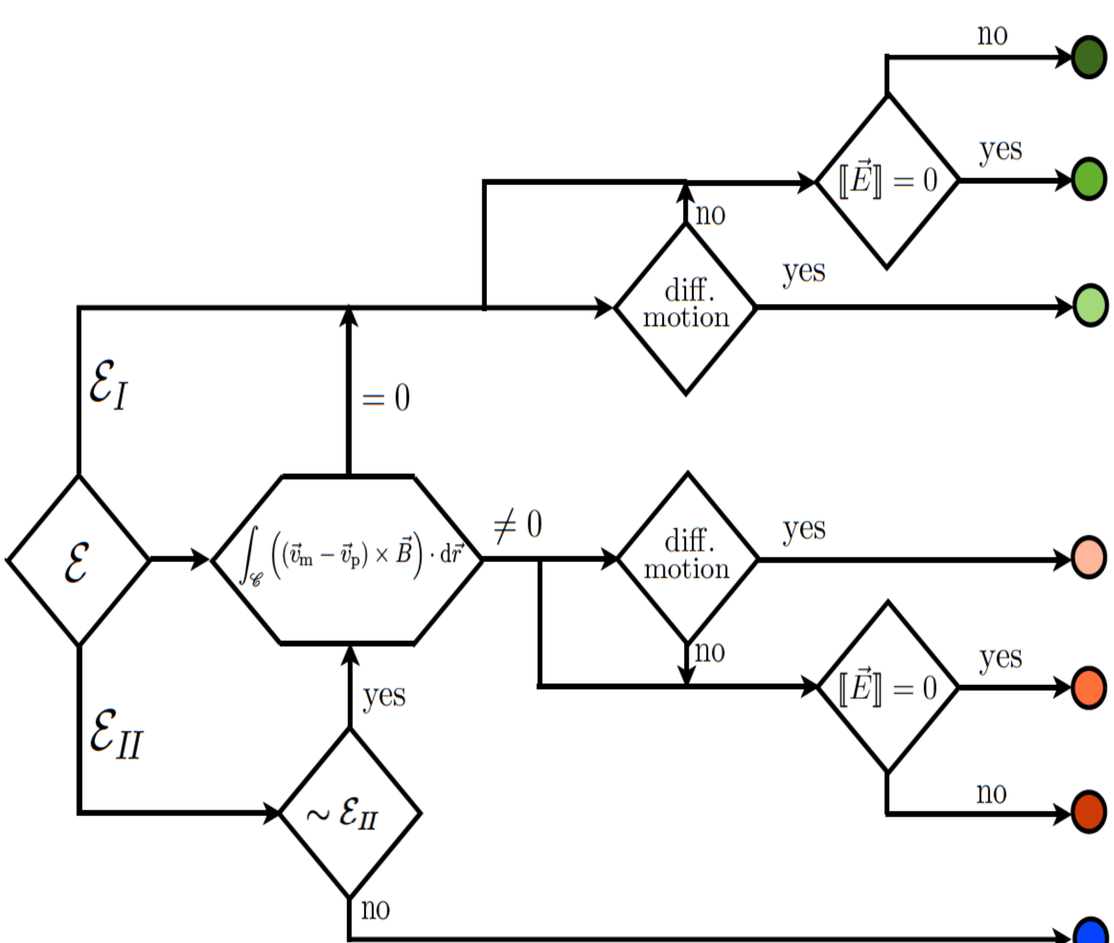


Ohms law for moving media

$$\mathcal{E}_{II} = -\int_{\mathscr{A}} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{a} + \int_{\mathscr{C}} (\mathbf{v}_{\mathrm{m}} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{r},$$

Terminal voltage (measured)

$$U_{12} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} + \int_{\mathscr{C}} ((\mathbf{v}_{\mathrm{m}} - \mathbf{v}_{\mathrm{p}}) \times \mathbf{B}) \cdot \mathrm{d}\mathbf{r}.$$



$$-\int_{\mathscr{A}}\frac{\partial\vec{B}}{\partial t}\cdot{\rm d}\vec{a}-\int_{\mathscr{A}\cap\mathscr{S}}\vec{v}_{\rm j}\times[\![\vec{B}]\!]\cdot{\rm d}\vec{r}+\int_{\mathscr{C}}(\vec{v}_{\rm p}\times\vec{B})\cdot{\rm d}\vec{r}$$

$$-\int_{\mathscr{A}}\frac{\partial\vec{B}}{\partial t}\cdot{\rm d}\vec{a}+\int_{\mathscr{C}}(\vec{v}_{\rm p}\times\vec{B})\cdot{\rm d}\vec{r}$$

$$-\frac{{\rm d}\Phi}{{\rm d}t}$$

$$+\int_{\mathscr{C}}\left((\vec{v}_{\rm m}-\vec{v}_{\rm p})\times\vec{B}\right)\cdot{\rm d}\vec{r}$$

$$-\frac{{\rm d}\Phi}{{\rm d}t}+\int_{\mathscr{C}}\left((\vec{v}_{\rm m}-\vec{v}_{\rm p})\times\vec{B}\right)\cdot{\rm d}\vec{r}$$

$$-\int_{\mathscr{A}}\frac{\partial\vec{B}}{\partial t}\cdot{\rm d}\vec{a}+\int_{\mathscr{C}}(\vec{v}_{\rm m}\times\vec{B})\cdot{\rm d}\vec{r}$$

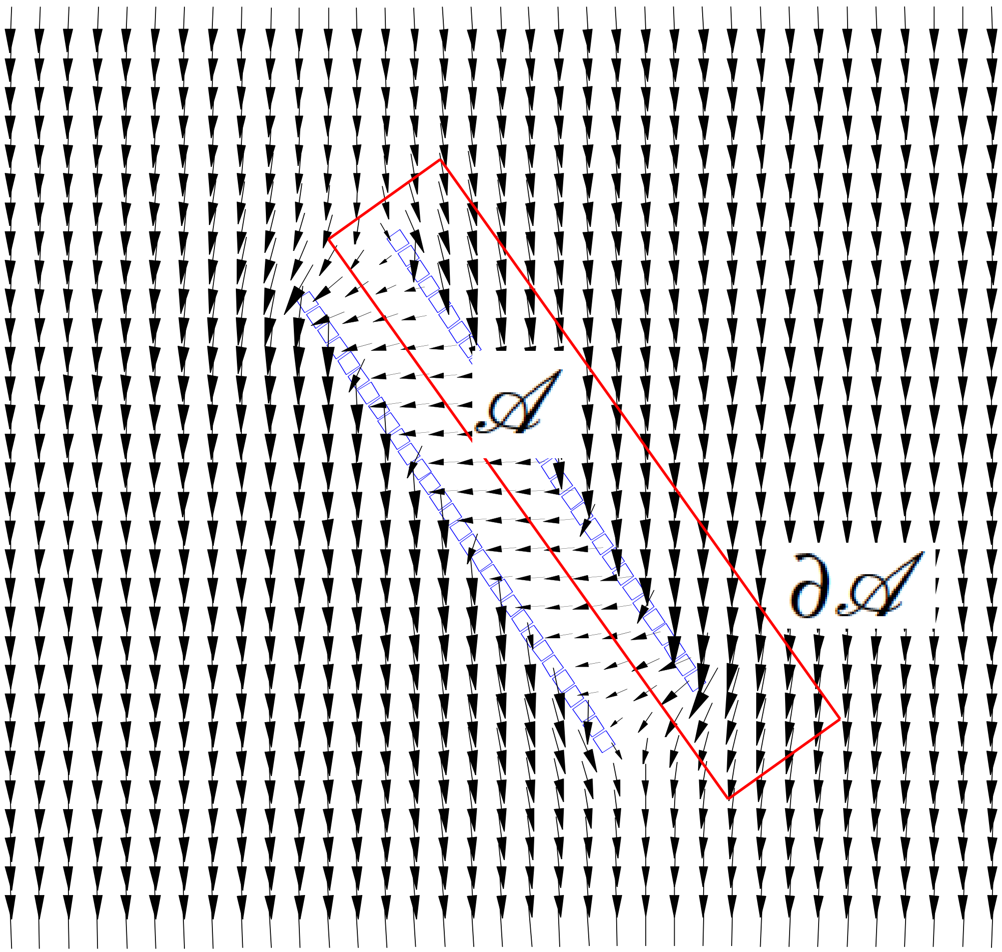
$$-\int_{\mathscr{A}}\frac{\partial\vec{B}}{\partial t}\cdot{\rm d}\vec{a}-\int_{\mathscr{A}\cap\mathscr{S}}\vec{v}_{\rm j}\times[\![\vec{B}]\!]\cdot{\rm d}\vec{r}+\int_{\mathscr{C}}(\vec{v}_{\rm m}\times\vec{B})\cdot{\rm d}\vec{r}$$

$$\vec{F}=Q(\vec{E}+\vec{v}\times\vec{B}),\quad {\rm curl}\,\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

When slip rings or bulk materials are present, always use the local Ohm's law. The symmetry of the device must allow to use "arbitrary" integration paths, otherwise a boundary value problem must be solved. The "naive" application of the "flux rule" works in cases where the loop is made of a thin wire.

Ampere's Law (Outer Oriented Surface - Current Crossing)

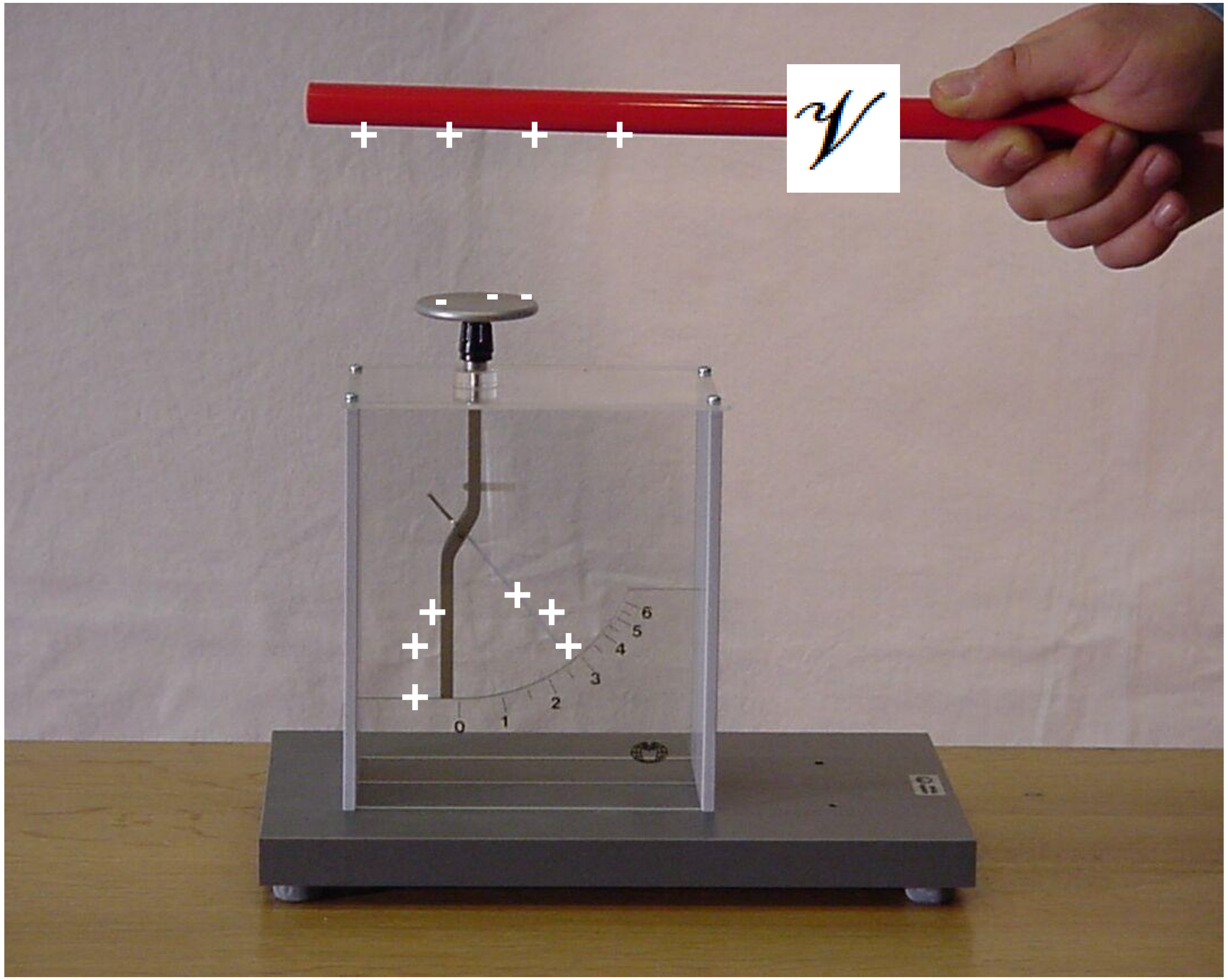
$$V_{\mathrm{m}}(\partial\mathscr{A}) = I(\mathscr{A})\,.$$



The current needed to cancel the longitudinal field component (magnetomotive force)

Gauss Law (Outer Oriented Volume - Electric Charge Influenced)

$$\Psi(\partial\mathscr{V}) = Q(\mathscr{V})$$

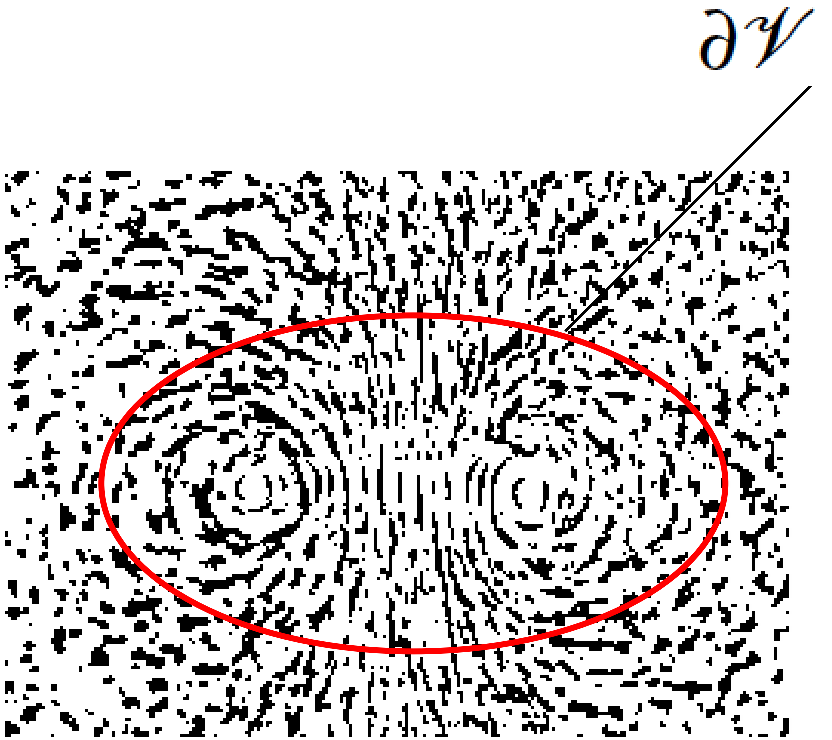


The capacity to induce charge

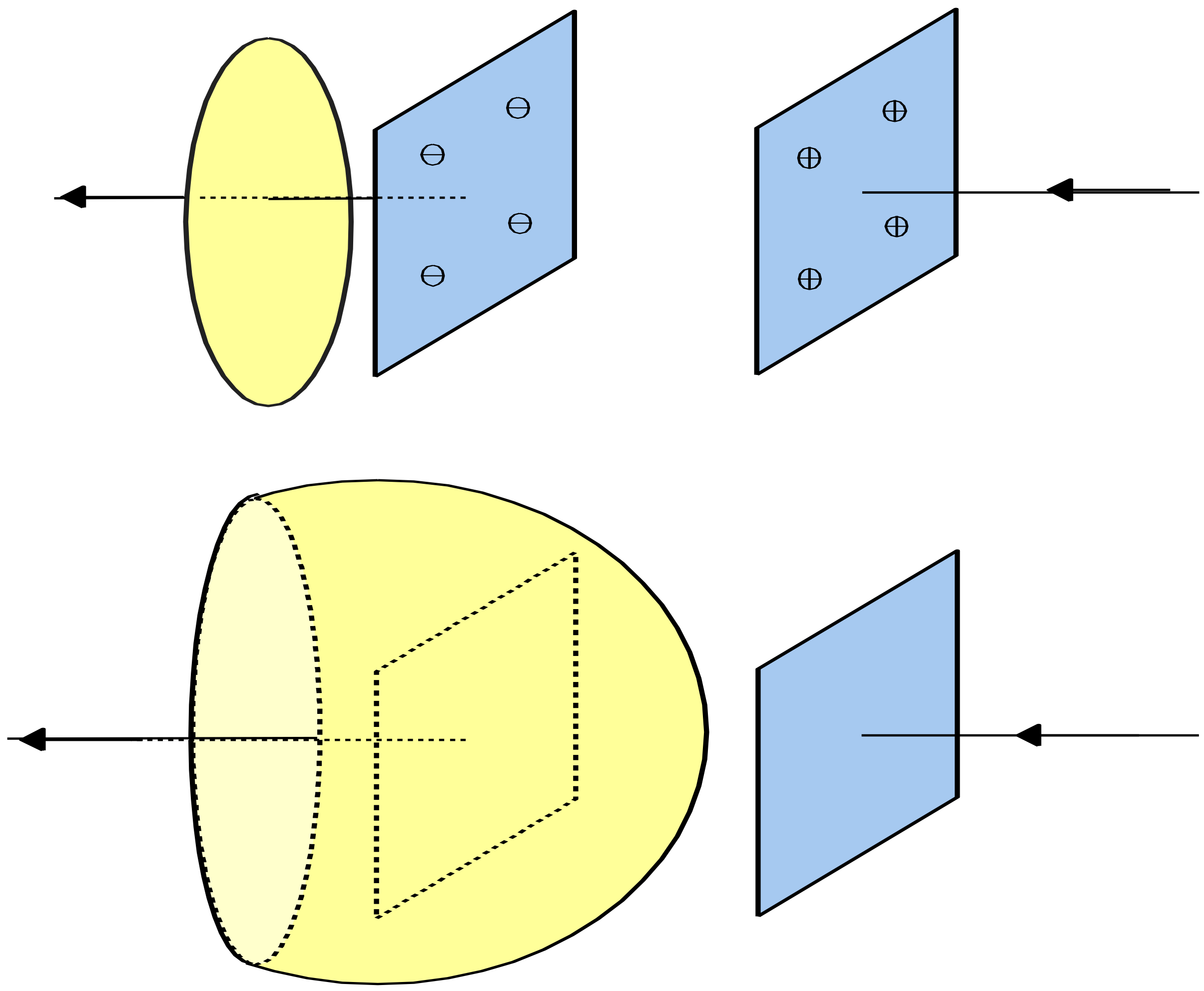
Magnetic Flux Conservation Law (Inner Oriented Volume)

Conservation of flux

$$\Phi(\partial\mathscr{V}) = 0$$



Maxwell's Extension



Ampere

$$V_{\mathrm{m}}(\partial\mathscr{A}) = I(\mathscr{A}) + \frac{\mathrm{d}}{\mathrm{d}t}\Psi(\mathscr{A})$$

Rate of change of charge

Maxwell's Equations in Global Form

Ampere	$V_{\mathrm{m}}(\partial a) = I(a) + \frac{\mathrm{d}}{\mathrm{d}t}\Psi(a)$
Faraday	$U(\partial a) = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(a)$
Flux conservation	$\Phi(\partial V) = 0$
Gauss	$\Psi(\partial V) = Q(V)$

Conservation of charge / Kirchhoff law

$$V_{\mathrm{m}}(\partial(\partial V)) = 0 = I(\partial V) + \tfrac{\mathrm{d}}{\mathrm{d}t}Q(V)$$

The current exiting a volume is equal to the negative rate of the charge in that volume

For it is unworthy of excellent men to lose hours like slaves in the labor of calculation which could safely be regulated to anyone else if machines were used.
Gottfried Wilhelm Leibniz (1646–1716)

Maxwell Equations

Integral Form

Lumped circuit calc. of NC magnets

Local Form

Laplace's Equation

Harmonic Fields

Field quality in Accelerator magnets

Green's Functions

The field of line-currents Coil-dominated magnets

Curl-Curl Equation

Weak-Forms

FEM

Kirchhoff's Theorem

BEM

Global Form

DEM

Maxwell's Equations in Integral Form

$$\int_{\partial\mathcal{A}} \mathbf{H}\cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{A}} \mathbf{J}\cdot \mathrm{d}\mathbf{a} + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{A}} \mathbf{D}\cdot \mathrm{d}\mathbf{a},$$

$$\int_{\partial\mathcal{A}} \mathbf{E}\cdot \mathrm{d}\mathbf{r} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{A}} \mathbf{B}\cdot \mathrm{d}\mathbf{a},$$

$$\int_{\partial\mathcal{V}} \mathbf{B}\cdot \mathrm{d}\mathbf{a} = 0,$$

$$\int_{\partial\mathcal{V}} \mathbf{D}\cdot \mathrm{d}\mathbf{a} = \int_{\mathcal{V}} \rho\, \mathrm{d}V.$$

$$V_{\mathrm{m}}(\partial\mathcal{A}) = I(\mathcal{A}) + \frac{\mathrm{d}}{\mathrm{d}t}\Psi(\mathcal{A}),$$

$$U(\partial\mathcal{A}) = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mathcal{A}),$$

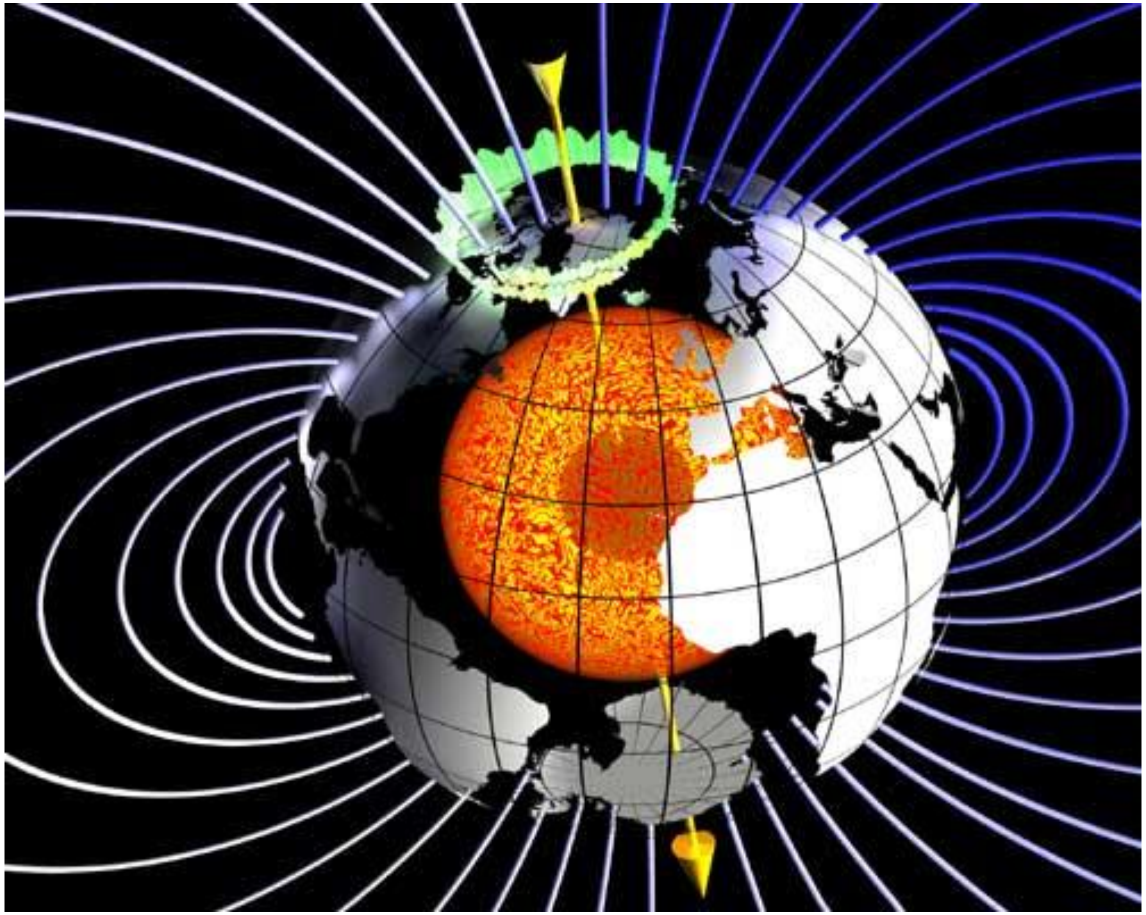
$$\Phi(\partial\mathcal{V}) = 0,$$

$$\Psi(\partial\mathcal{V}) = Q(\mathcal{V}).$$

Electromagnetic Fields

Global quantity	SI unit	Relation			SI unit	Field
MMF	$1\,\mathrm{A}$	$V_\mathrm{m}(\mathscr{S})$	$=$	$\int_{\mathscr{S}} \mathbf{H} \cdot \mathrm{d}\mathbf{r}$	$1\,\mathrm{A\,m^{-1}}$	Magnetic field
Electric voltage	$1\,\mathrm{V}$	$U(\mathscr{S})$	$=$	$\int_{\mathscr{S}} \mathbf{E} \cdot \mathrm{d}\mathbf{r}$	$1\,\mathrm{V\,m^{-1}}$	Electric field
Magnetic flux	$1\,\mathrm{V\,s}$	$\Phi(\mathscr{A})$	$=$	$\int_{\mathscr{A}} \mathbf{B} \cdot \mathrm{d}\mathbf{a}$	$1\,\mathrm{V\,s\,m^{-2}}$	Magnetic flux density
Electric flux	$1\,\mathrm{A\,s}$	$\Psi(\mathscr{A})$	$=$	$\int_{\mathscr{A}} \mathbf{D} \cdot \mathrm{d}\mathbf{a}$	$1\,\mathrm{A\,s\,m^{-2}}$	Electric flux density
Electric current	$1\,\mathrm{A}$	$I(\mathscr{A})$	$=$	$\int_{\mathscr{A}} \mathbf{J} \cdot \mathrm{d}\mathbf{a}$	$1\,\mathrm{A\,m^{-2}}$	Electric current density
Electric charge	$1\,\mathrm{A\,s}$	$Q(\mathscr{V})$	$=$	$\int_{\mathscr{V}} \rho \cdot \mathrm{d}V$	$1\,\mathrm{A\,s\,m^{-3}}$	Electric charge density

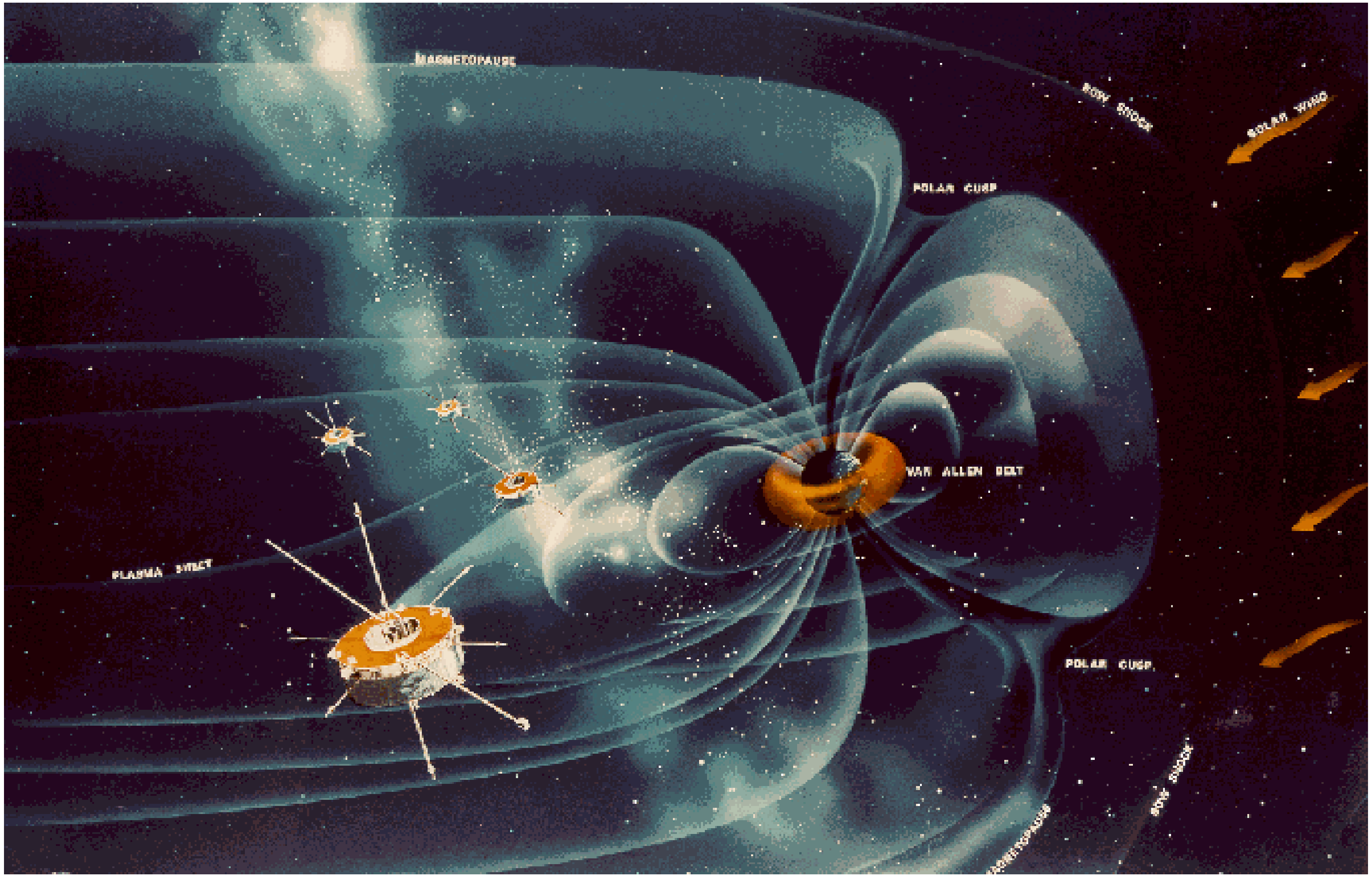
Flux Tubes of Mother Earth



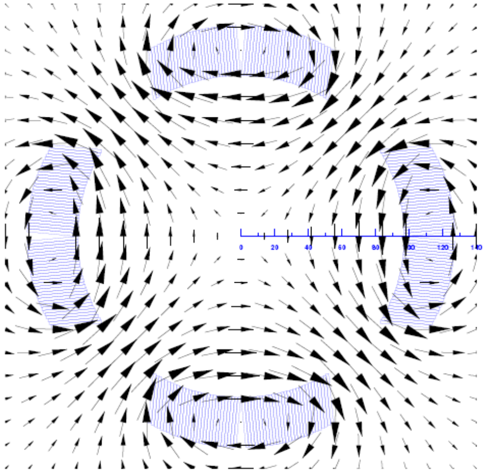
What is a magnetic field?
What is more fundamental H or B?

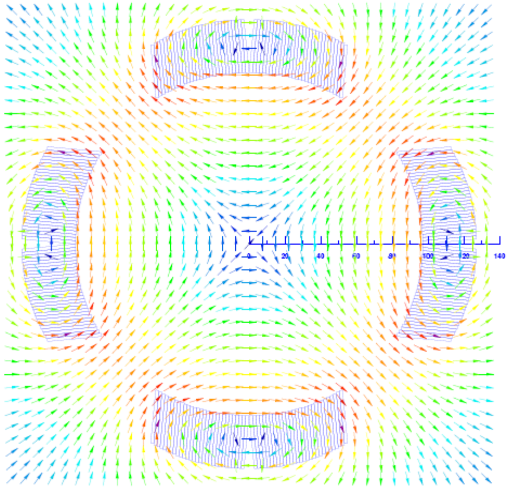
Earth Magnetic Field (better representation)

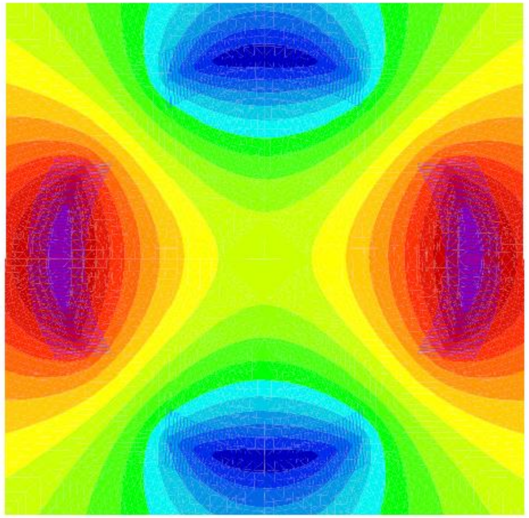
Erdmagnetfeld

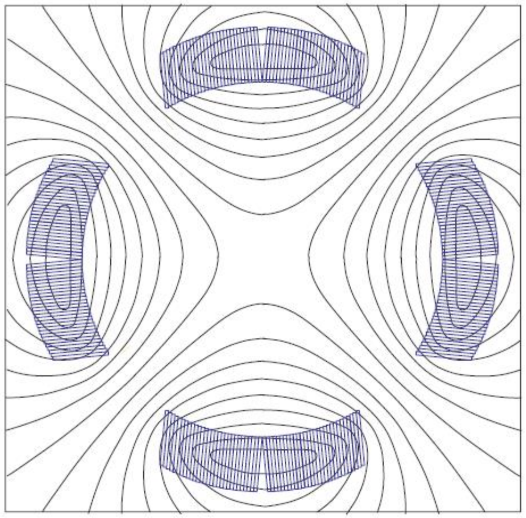


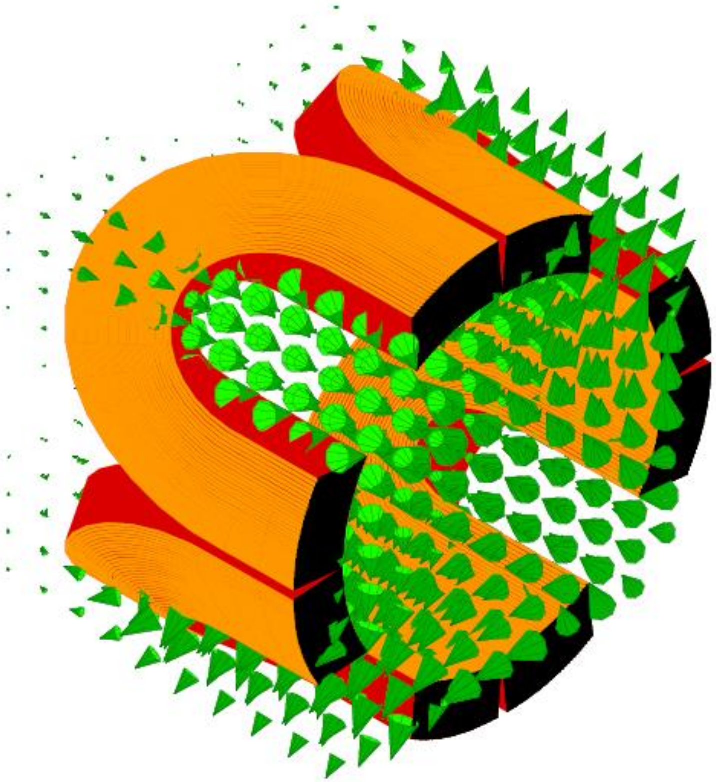
Different Renderings of the Same Vector Field





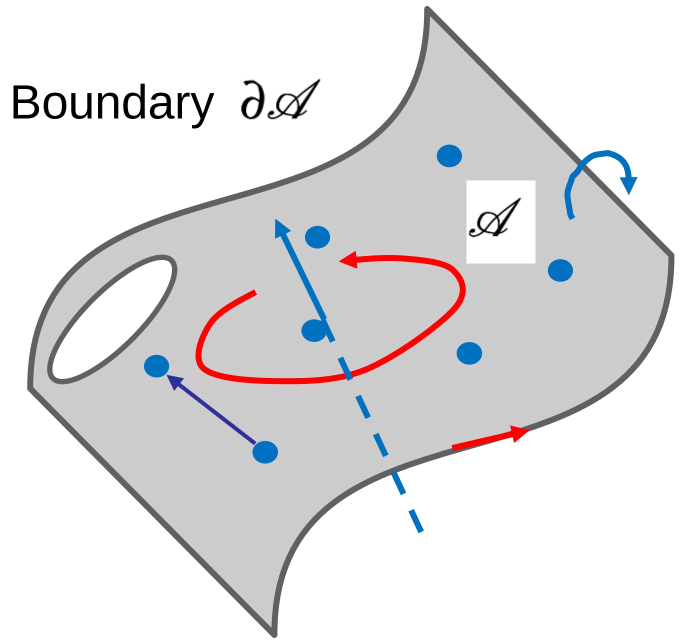




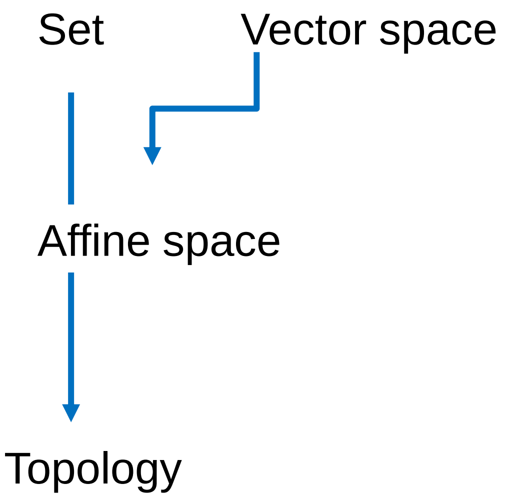


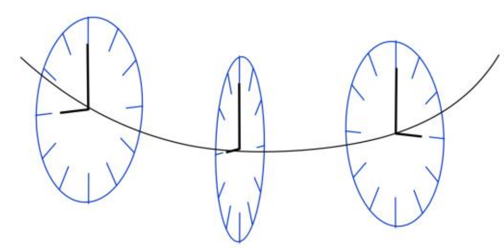
Oriented Manifolds

Contractable
(or not)



Inner and outer orientation

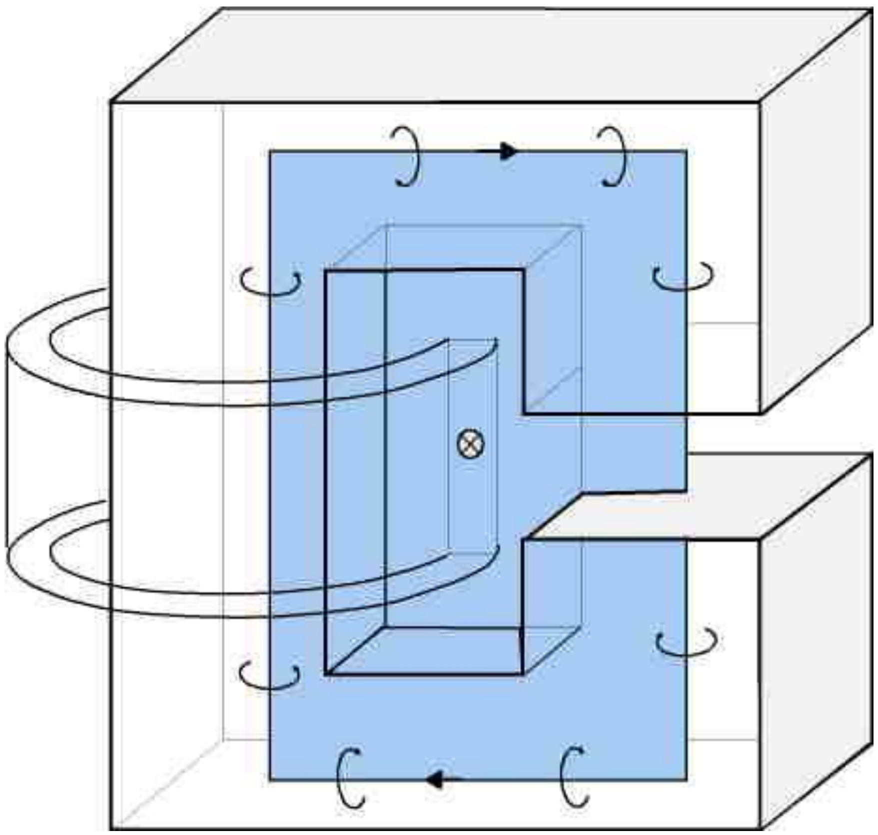




09:00 or 15:00 ?

Inner and Outer Oriented Surfaces

$$\int_{\partial\tilde{\mathscr{A}}} \mathbf{H}\cdot \mathrm{d}\mathbf{r} = \int_{\partial\tilde{\mathscr{A}}} \mathbf{H}\cdot \mathbf{t}\,\mathrm{d}s = \int_{\tilde{\mathscr{A}}} \mathbf{J}\cdot \mathbf{n}\,\mathrm{d}a$$

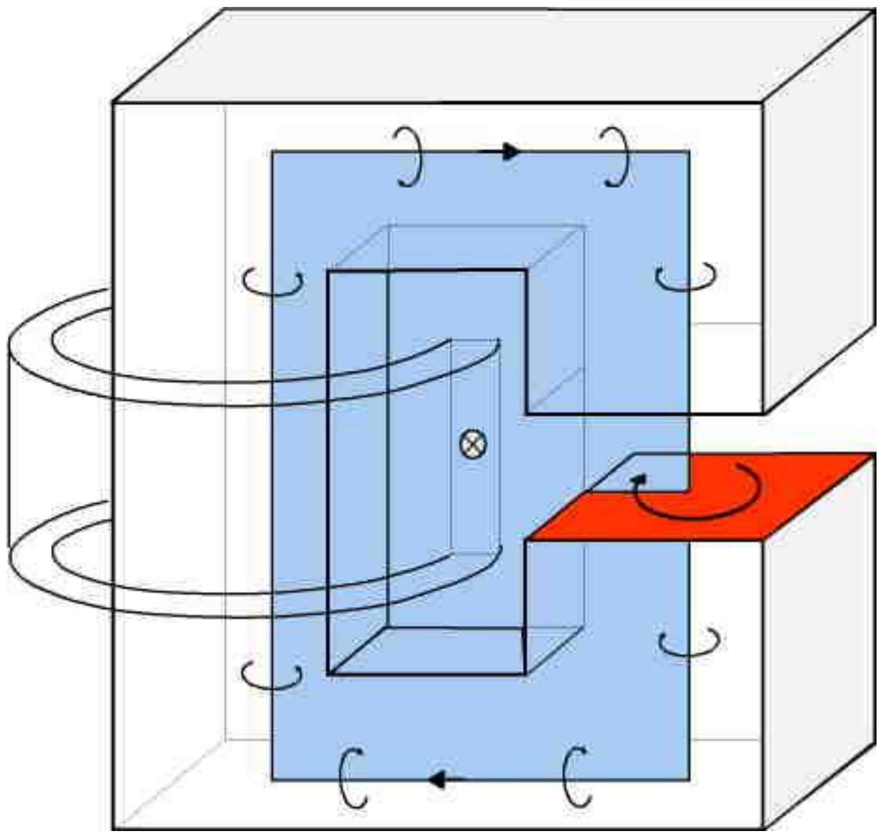


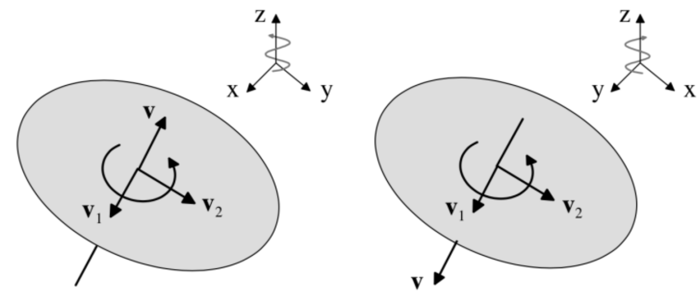
Outer oriented
by the current

$$\Phi_i = \int_{\mathscr{A}_i} \mathbf{B}\cdot \mathbf{n}\,\mathrm{d}a$$

Inner and Outer Oriented Surfaces

$$\int_{\partial\tilde{\mathscr{A}}} \mathbf{H}\cdot \mathrm{d}\mathbf{r} = \int_{\partial\tilde{\mathscr{A}}} \mathbf{H}\cdot\mathbf{t}\,\mathrm{d}s = \int_{\tilde{\mathscr{A}}} \mathbf{J}\cdot\mathbf{n}\,\mathrm{d}a$$



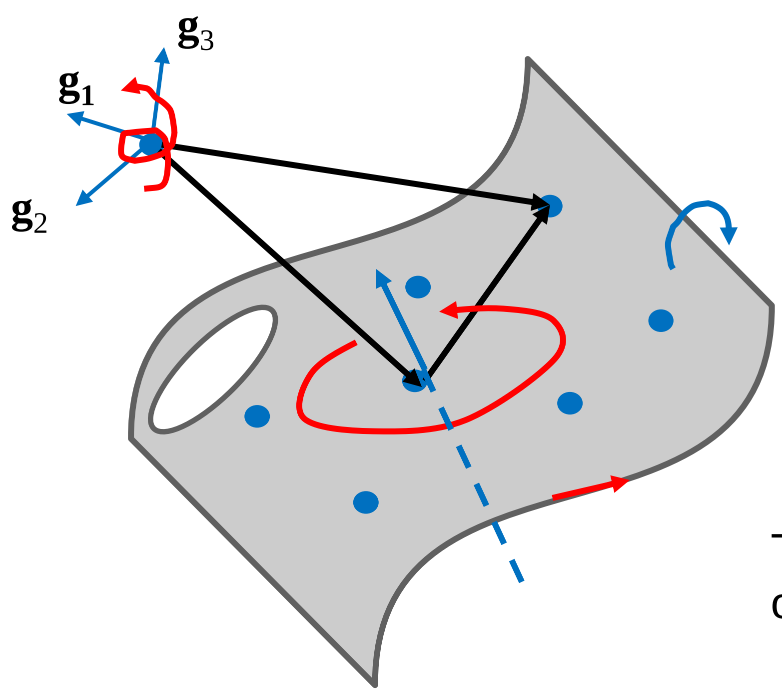


Inner oriented, because flux is a measure for the voltage that can be generated on the rim

Embedding into oriented ambient space (Origin, Basis)

$$\Phi_i = \int_{\mathscr{A}_i} \mathbf{B}\cdot\mathbf{n}\,\mathrm{d}a$$

Consistently-Oriented and Embedded Manifolds



Set

Vector space

Affine space + Origin + Basis

Topology, orientation

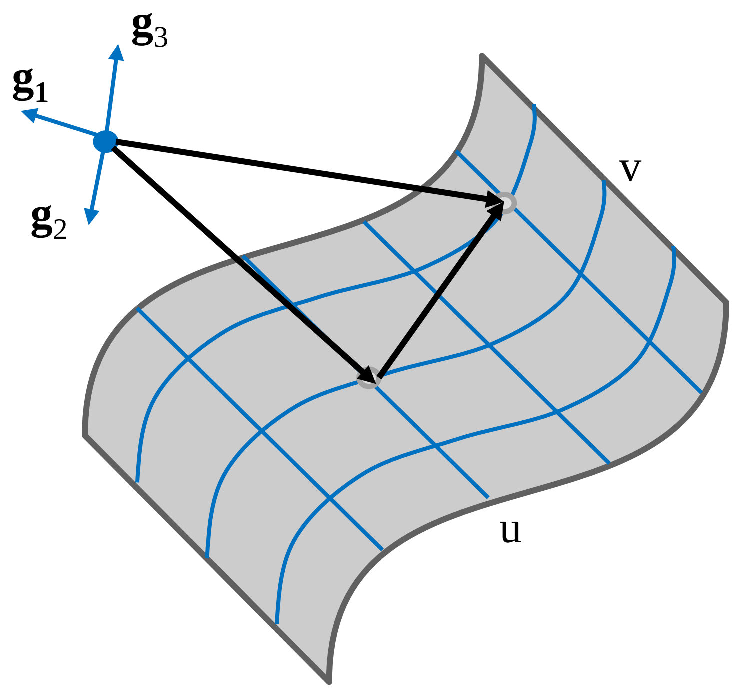
Embedded, consistently inner and outer oriented

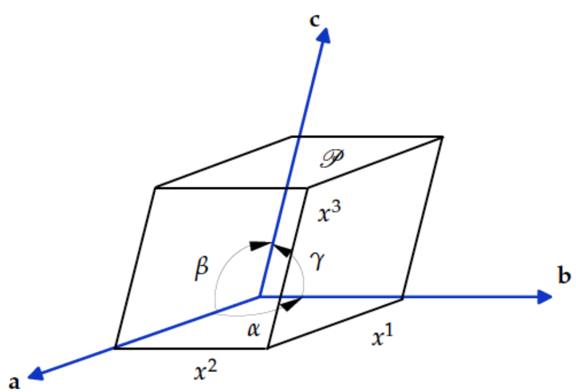
Bruno Touschek (1921-1978)

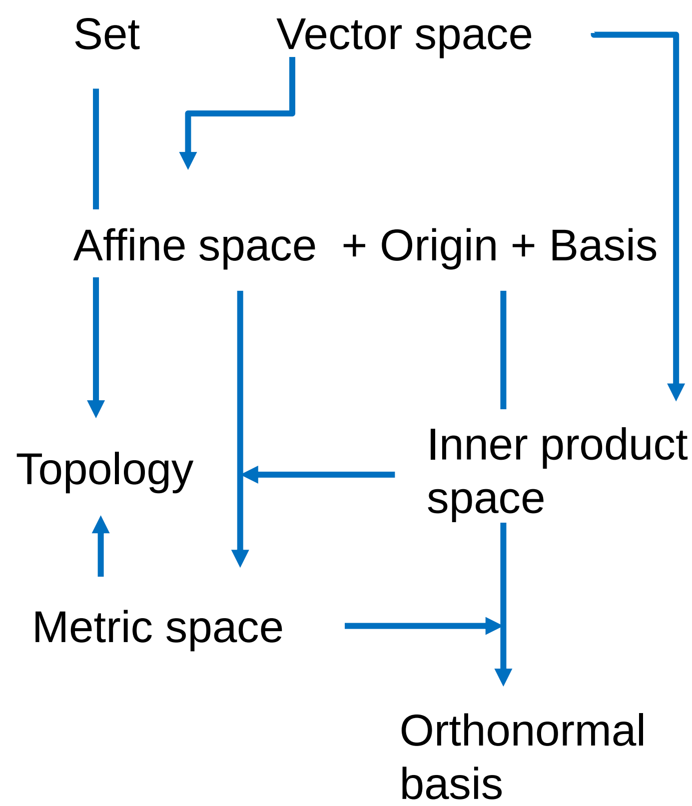
Coordinates and Metric

$$\mathscr{P} \in A_n \xrightarrow{\text{Origin}} \mathbf{r} \in V_n \xrightarrow{\text{Basis}} (x^1, \dots, x^n) \in \mathbb{R}^n .$$

$$[G] = \begin{pmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{pmatrix}$$

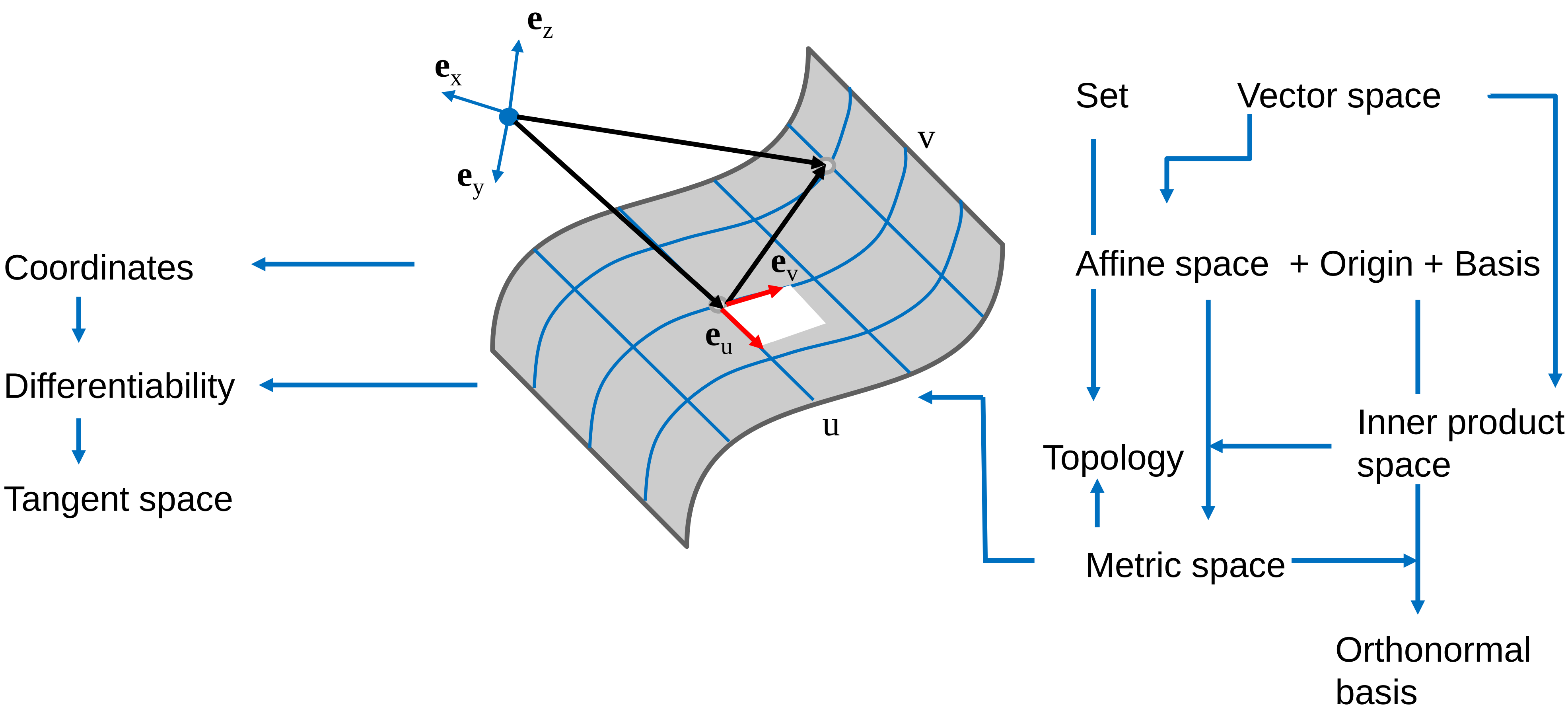






Applications: Calibration of Helmholtz coils, calibration of 3-axis displacement stages and robots

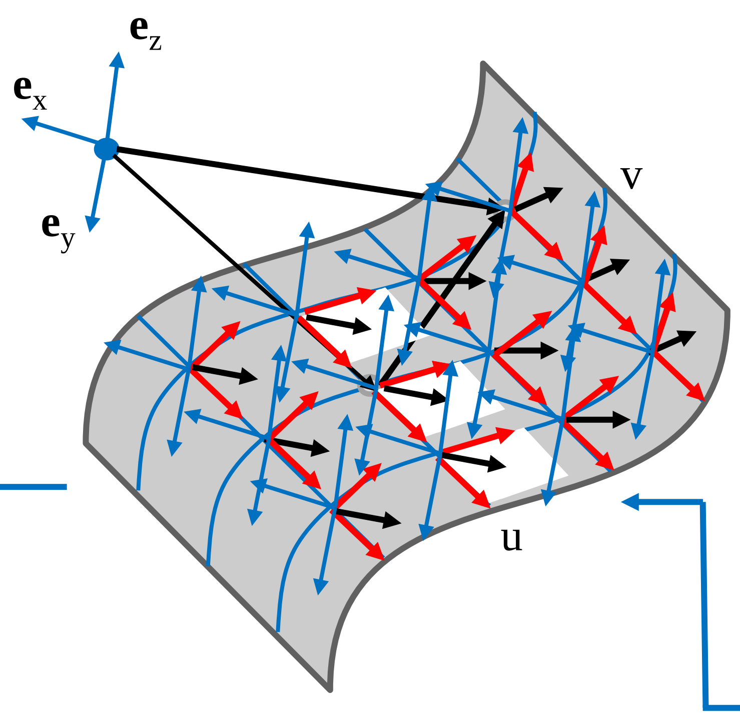
Tangent Space on Differential Surfaces



The Required Mathematical Structure of Vector Fields

Required: Orientable manifolds, origin, coordinate frame, metric, smoothness
No switches, no Moebius strips, no internal boundaries

E_3
Oriented
Euclidean
Affine Space



Coordinates → Differentiability → Tangent space → Vector fields → Basis field

Set → Affine space → Topology

Vector space → Affine space + Origin + Basis

Vector space → Inner product space

Inner product space → Metric space → Topology

Metric space → Orthonormal basis

Orthonormal basis → Basis field

$$\mathbf{a} : \Omega \to \mathbb{R}^3 : \mathbf{r} \mapsto \mathbf{a}(\mathbf{r}) : \mathbf{a}(\mathbf{r}) = (a^1(\mathbf{r}), a^2(\mathbf{r}), a^3(\mathbf{r}))$$

Constitutive Equations

$$\mathbf{B} = \mu \mathbf{H}, \qquad \mathbf{D} = \varepsilon \mathbf{E}, \qquad \mathbf{J} = \varkappa \mathbf{E},$$

Permeability: $[\mu] = 1\,\mathrm{V\,s\,A^{-1}\,m^{-1}} = 1\,\mathrm{H\,m^{-1}},$

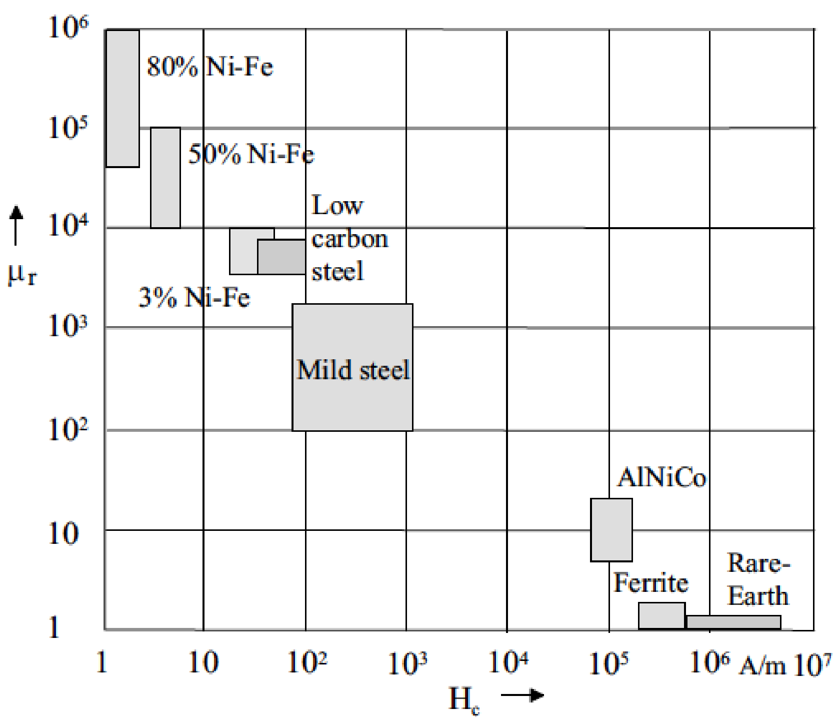
Permittivity: $[\varepsilon] = 1\,\mathrm{A\,s\,V^{-1}\,m^{-1}},$

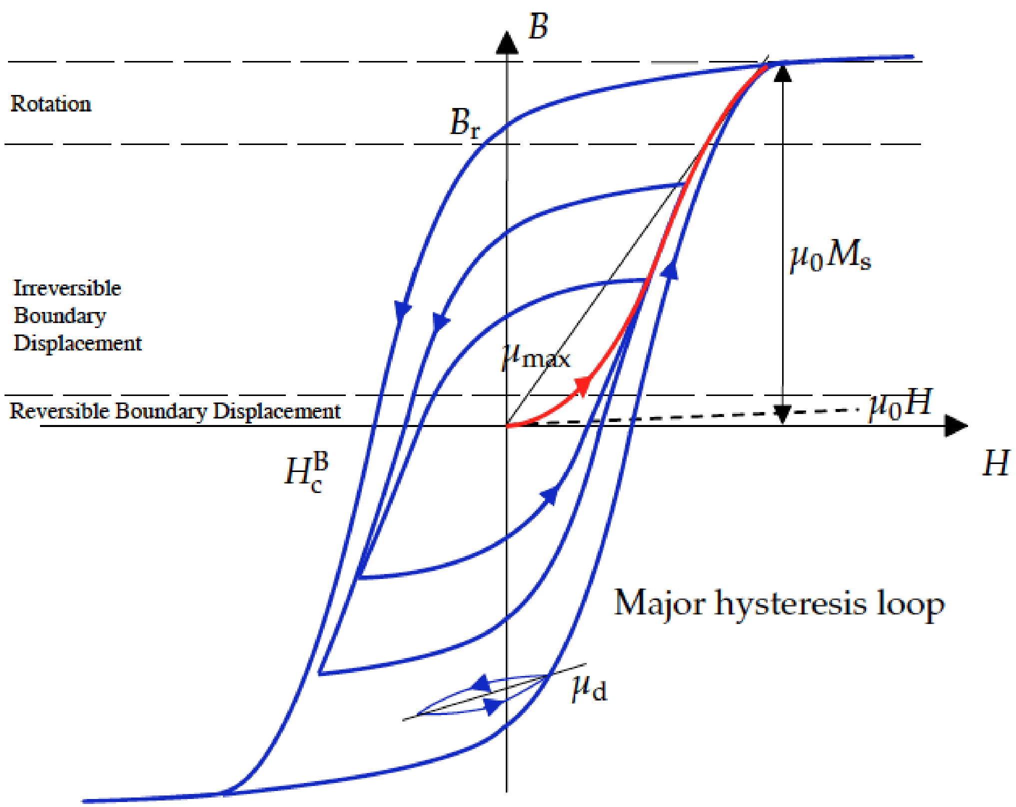
Conductivity: $[\varkappa] = 1\,\mathrm{A\,V^{-1}m^{-1}} = 1\,\Omega^{-1}\,\mathrm{m^{-1}}.$

Linear (field independent), homogeneous (position independent), lossless, isotropic (direction independent), stationary

$$\mu = \mu_r \mu_0, \qquad \varepsilon = \varepsilon_r \varepsilon_0,$$

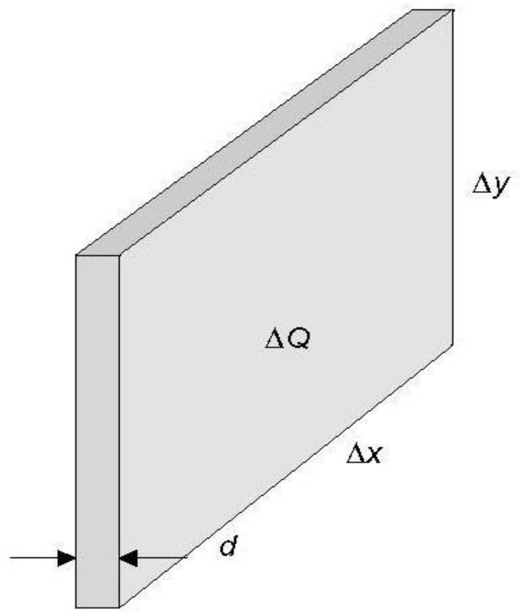
$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}, \qquad \varepsilon_0 = 8.8542\ldots \times 10^{-12}\,\mathrm{F\,m^{-1}},$$

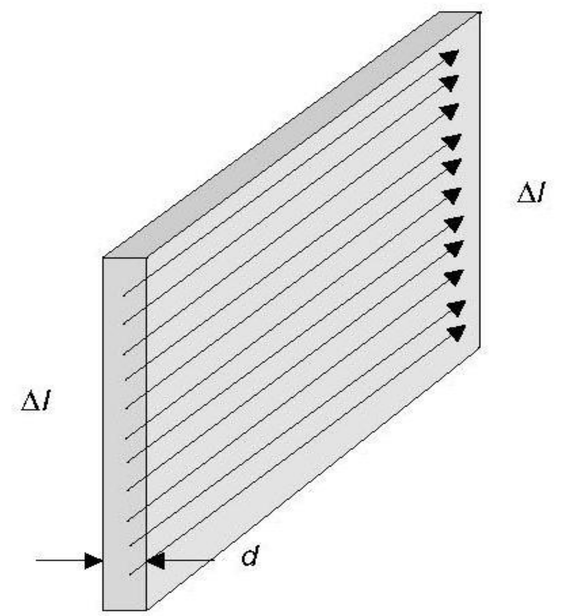




$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{P}_m(\mathbf{H}) = \mu_0(\mathbf{H} + \mathbf{M}(\mathbf{H})),$$

Surface Charge and (Fictitious) Surface Current





Thin layer with ρ_{mag}

$\Delta Q = \Delta x \Delta y d \rho_{\text{mag}}$

$\rho_{\text{mag}} \to \infty$ and $d \to 0$

$\sigma_{\text{mag}} = d\rho_{\text{mag}}$

$[\sigma_{\text{mag}}] = 1$ V·s/m^2

Thin layer with J

$\Delta I = J d \Delta l$

$J \to \infty$ and $d \to 0$

$\boldsymbol{\alpha} = \mathbf{J} d$

$[\boldsymbol{\alpha}] = 1\ \text{A} \cdot \text{m}^{-1}$

Fictitious quantities to define boundary values

Continuity Conditions (1)

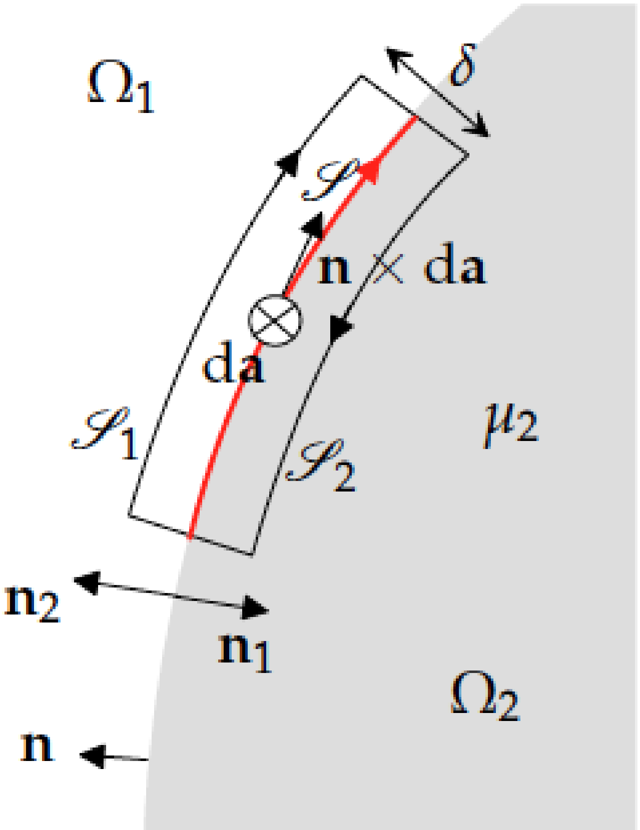
$$\int_{\mathscr{S}_2} \mathbf{H}_2 \cdot \mathrm{d}\mathbf{r} + \int_{\mathscr{S}_1} \mathbf{H}_1 \cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{S}} (\mathbf{H}_1 - \mathbf{H}_2) \cdot \mathrm{d}\mathbf{r} = -\int_{\mathscr{S}} (\mathbf{n} \times \boldsymbol{\alpha}) \cdot \mathrm{d}\mathbf{r}$$

$$(\mathbf{H}_1 - \mathbf{H}_2) + \lambda \mathbf{n} = -\mathbf{n} \times \boldsymbol{\alpha}\,.$$

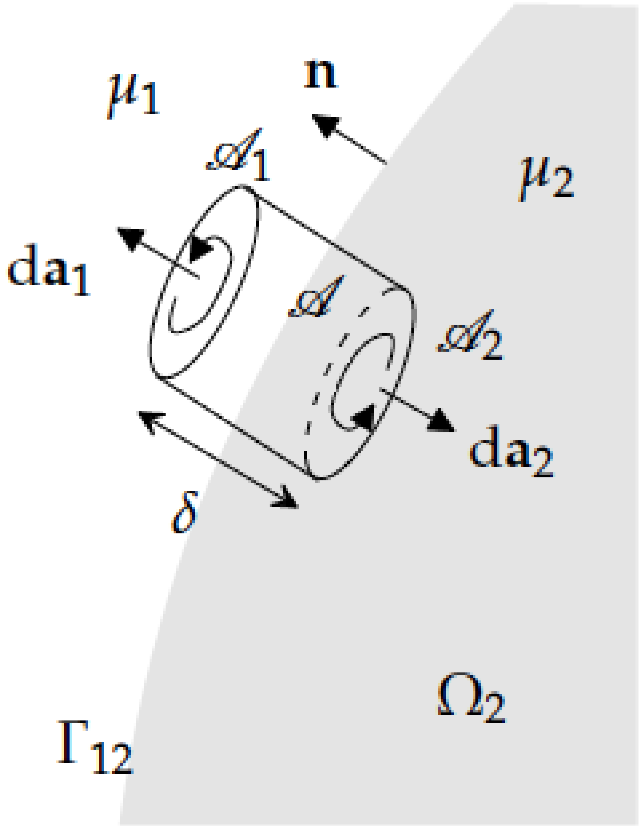
$$\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\alpha}) = (\mathbf{n} \cdot \boldsymbol{\alpha})\mathbf{n} - (\mathbf{n} \cdot \mathbf{n})\boldsymbol{\alpha} = -\boldsymbol{\alpha}$$

$$\boldsymbol{\alpha} = \mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{n} \times [\![\, \mathbf{H} \,]\!]_{12}$$

$$H_{t1} = H_{t2} \qquad \equiv \qquad \mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = 0$$



Continuity Conditions (2)



$$\int_{\partial V} \mathbf{B} \cdot \mathbf{da} = 0 \qquad \delta \to 0$$

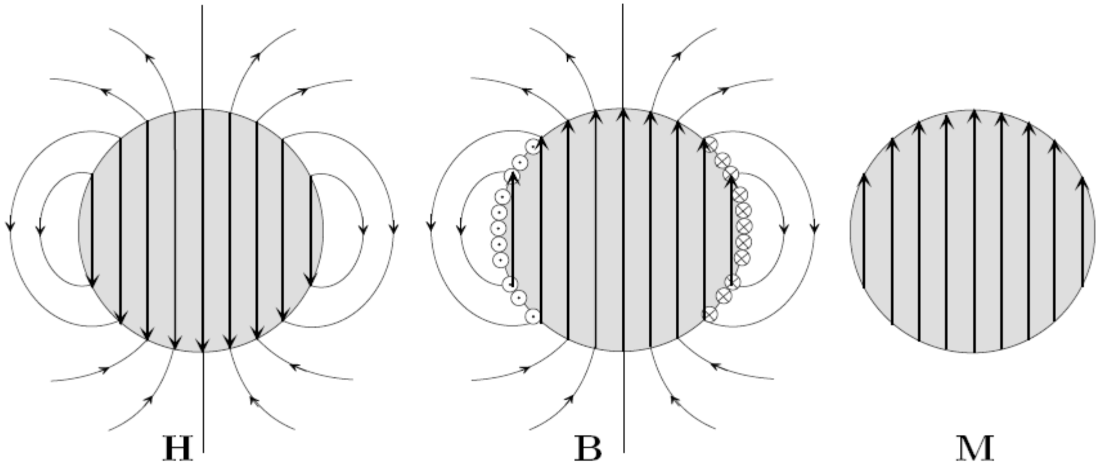
$$\int_a \sigma_{\text{mag}} \mathrm{d}a = \int_a \mathbf{B}_1 \cdot \mathbf{da}_1 + \mathbf{B}_2 \cdot \mathbf{da}_2 = \int_a (\mathbf{B}_1 - \mathbf{B}_2) \cdot \mathbf{n}_1 \mathrm{d}a$$

Holds for any surface a if

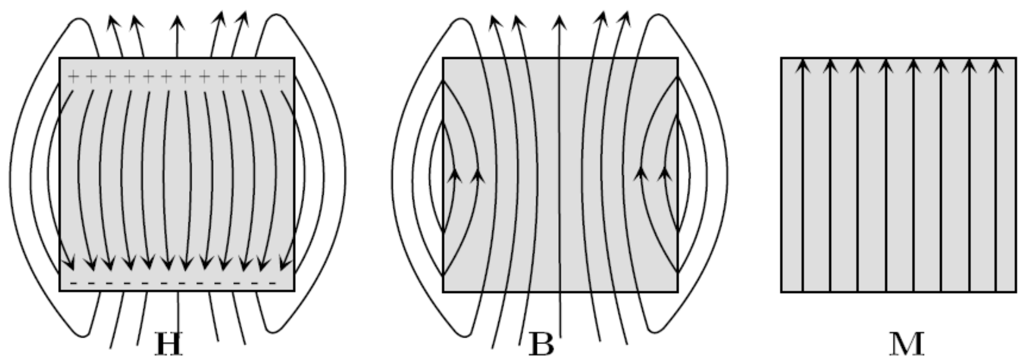
$$\sigma_{\text{mag}} = (\mathbf{B}_1 - \mathbf{B}_2) \cdot \mathbf{n} = [\mathbf{B} \cdot \mathbf{n}]_{12}$$

$$B_{\text{n}1} = B_{\text{n}2} \quad \equiv \quad (\mathbf{B}_1 - \mathbf{B}_2) \cdot \mathbf{n} = 0 \quad \equiv \quad [\mathbf{B} \cdot \mathbf{n}]_{12} = 0$$

Surface Current and Surface Charge



$$\lim_{c \to 0} \frac{\int_c \mathbf{H} \cdot \mathrm{d}\mathbf{s}}{c} = (\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{n} = -\boldsymbol{\alpha}$$

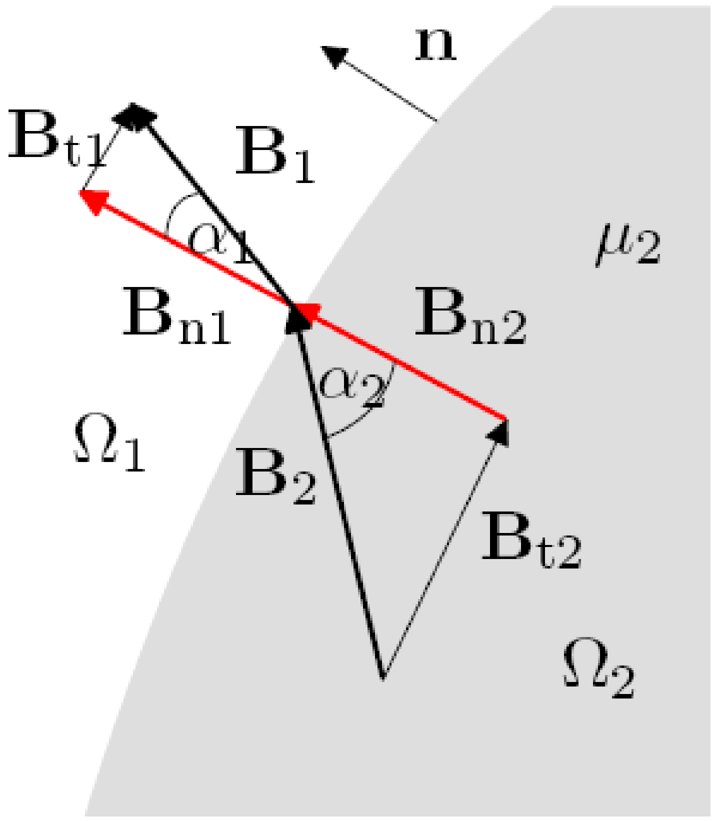


$$\lim_{a \to 0} \frac{\int_a \mathbf{B} \cdot \mathrm{d}\mathbf{a}}{a} = (\mathbf{B}_1 - \mathbf{B}_2) \cdot \mathbf{n} = \sigma_{\mathrm{mag}}$$

Continuity Conditions (3)

No surface currents:

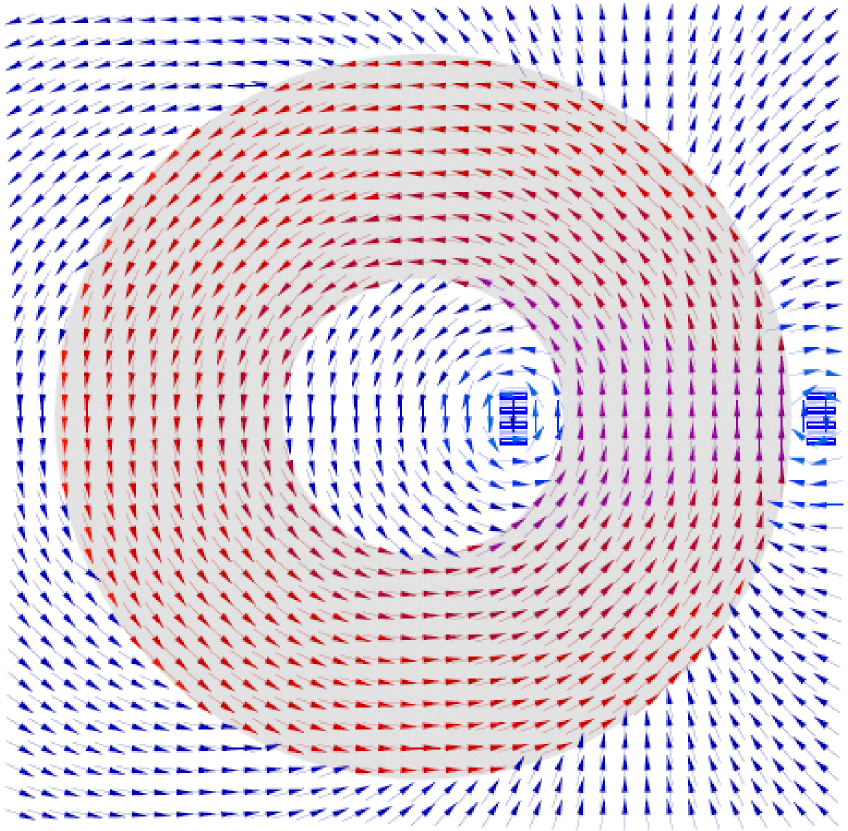
$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\frac{B_{t1}}{B_{n1}}}{\frac{B_{t2}}{B_{n2}}} = \frac{\mu_1 \frac{H_{t1}}{B_{n1}}}{\mu_2 \frac{H_{t2}}{B_{n2}}} = \frac{\mu_1 H_{t1}}{\mu_2 H_{t2}} = \frac{\mu_1}{\mu_2}$$

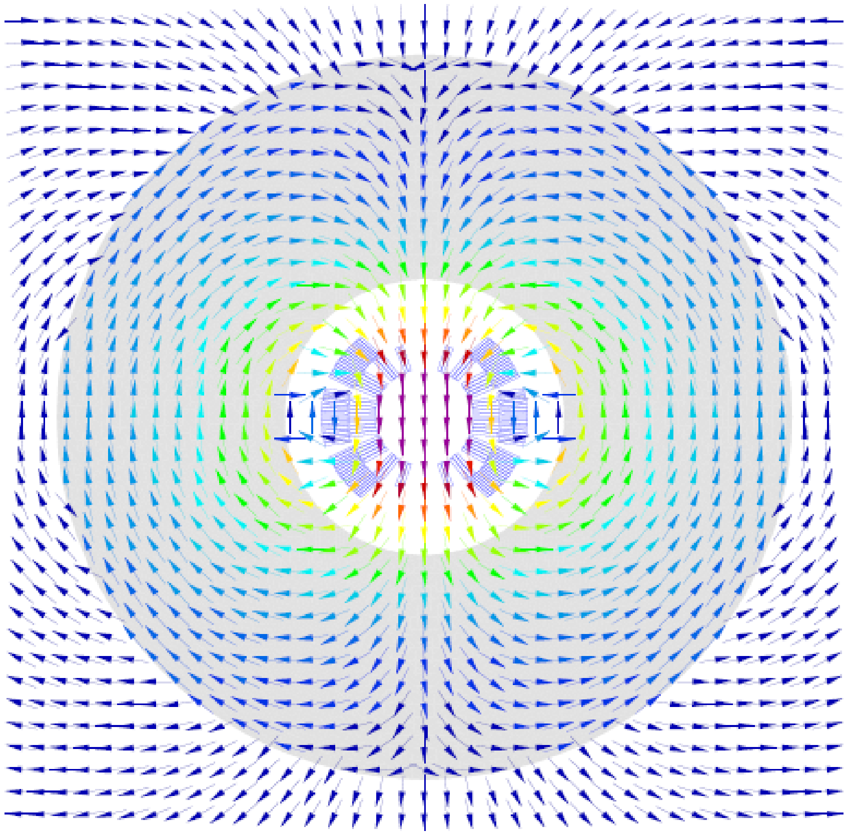


$$\mu_2 \gg \mu_1$$

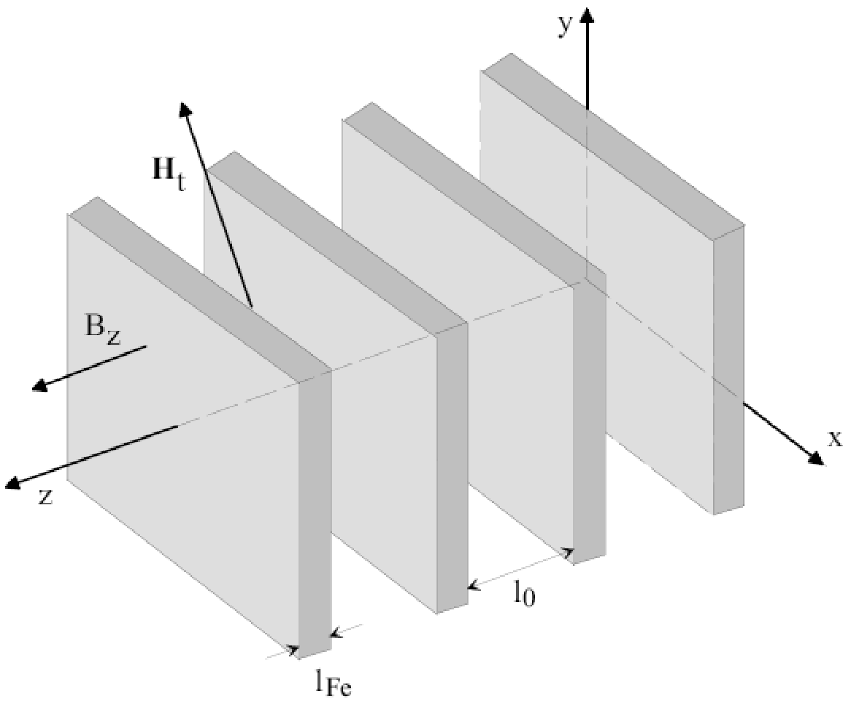
$\alpha_1 \approx 0,$ or $\alpha_2 \approx \pi/2,$

Continuity at Iron Boundaries





Stacking Factor for Yoke Laminations



$$\mathbf{H}_t^0 = \mathbf{H}_t^{Fe} = \overline{\mathbf{H}}_t$$

$$\overline{\mathbf{B}}_t = \frac{1}{l_{Fe} + l_0}\left(l_{Fe}\mu\overline{\mathbf{H}}_t + l_0\mu_0\overline{\mathbf{H}}_t\right)$$

$$B_z^0 = B_z^{Fe} = \overline{B}_z$$

$$\overline{H}_z = \frac{1}{l_{Fe} + l_0}\left(l_{Fe}\frac{\overline{B}_z}{\mu} + l_0\frac{\overline{B}_z}{\mu_0}\right)$$

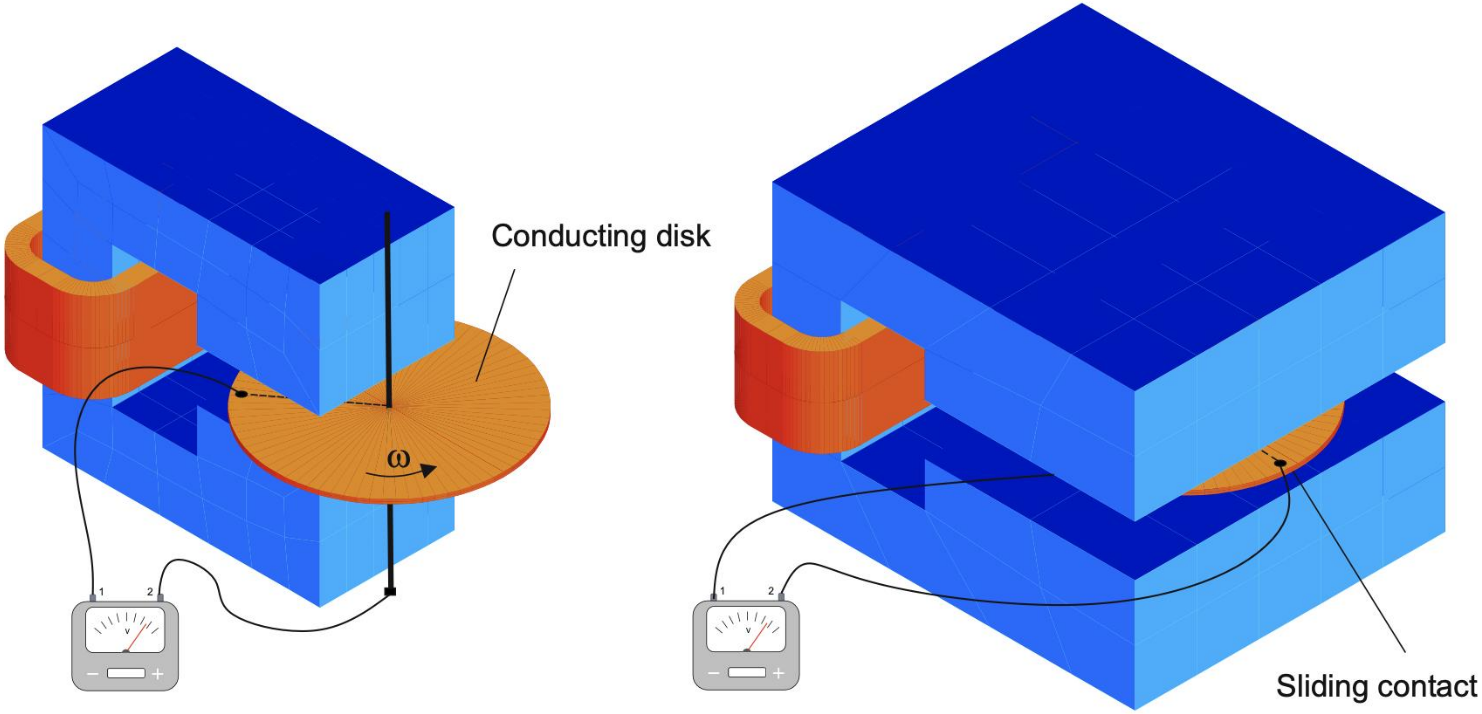
$$\lambda = \frac{l_{Fe}}{l_{Fe} + l_0}$$

$$\overline{\mu}_t = \lambda\mu + (1 - \lambda)\mu_0$$

$$\overline{\mu}_z = \left(\frac{\lambda}{\mu} + \frac{1 - \lambda}{\mu_0}\right)^{-1}$$

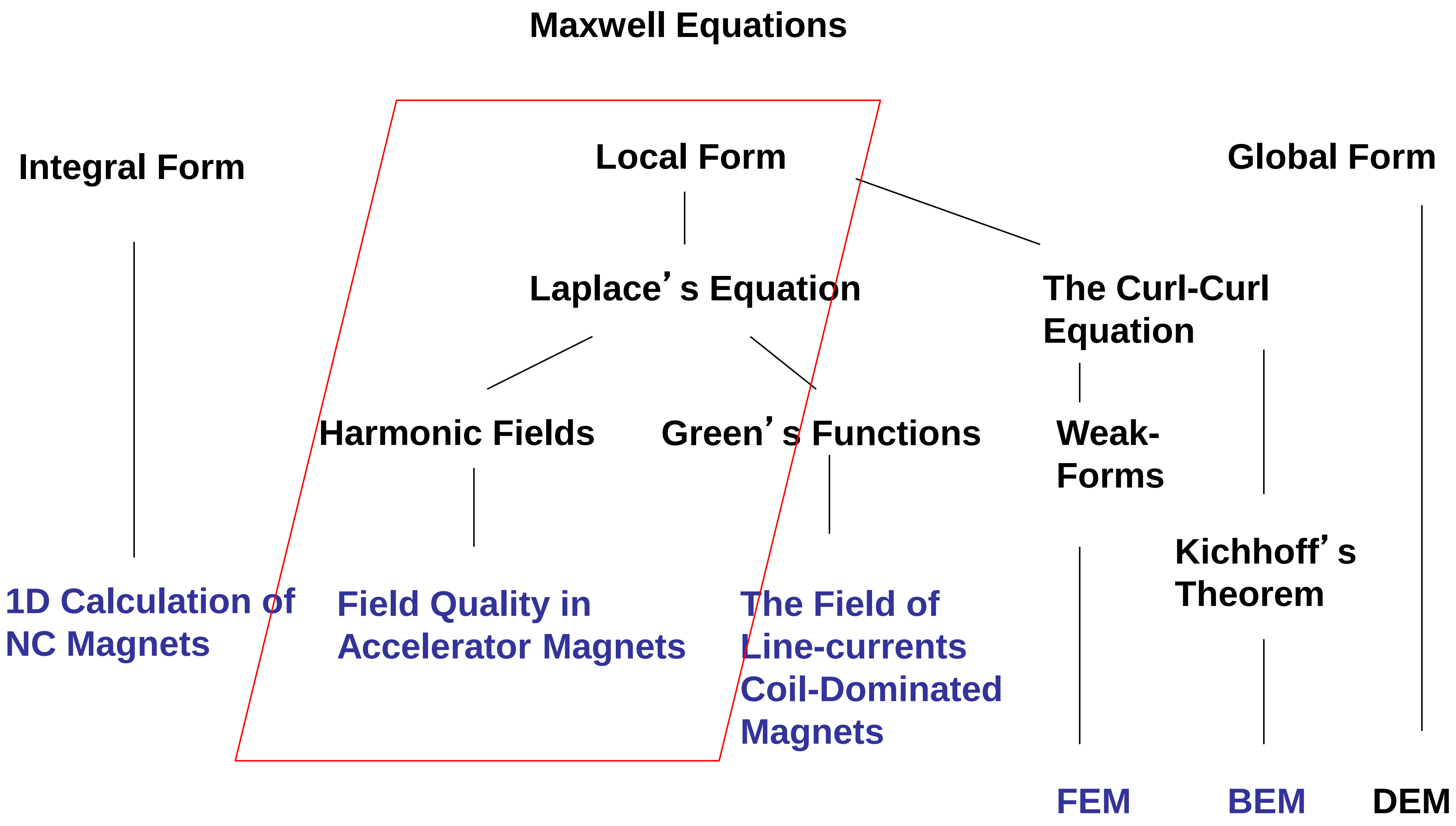
The Homopolar Generator

$d\mathbf{F} = I\, d\mathbf{r} \times \mathbf{B}$



Einstein: All physics is local

Mathematical Foundations of Magnet Design



Space Curves (as Mappings)

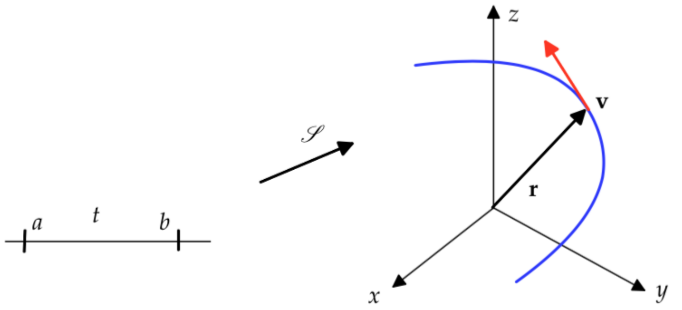
$$\mathscr{S} : I \to E_3 : t \mapsto \mathbf{r}(t)$$

$$\mathbf{r}(t) = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y + z(t)\mathbf{e}_z$$

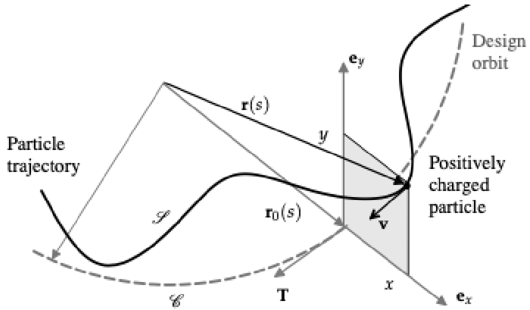
$$\frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\mathbf{e}_x + \frac{\mathrm{d}y}{\mathrm{d}t}\mathbf{e}_y + \frac{\mathrm{d}z}{\mathrm{d}t}\mathbf{e}_z$$

$$\mathbf{T}(t) := \frac{\mathbf{v}(t)}{v(t)} = T_x(t)\mathbf{e}_x + T_y(t)\mathbf{e}_y + T_z(t)\mathbf{e}_z$$

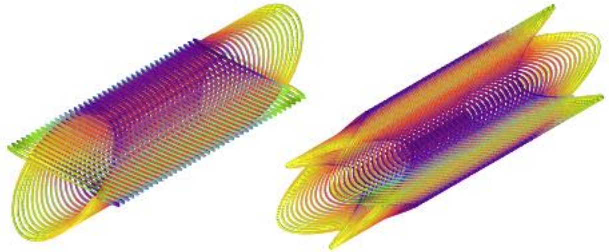
$$\frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} = \mathbf{a}(t) = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\mathbf{e}_x + \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\mathbf{e}_y + \frac{\mathrm{d}^2z}{\mathrm{d}t^2}\mathbf{e}_z$$



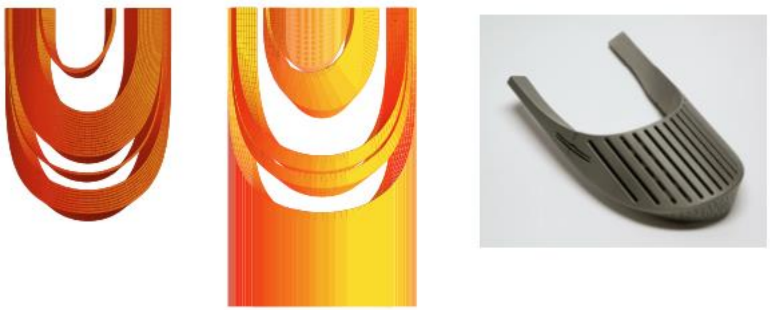
Beam orbit



CCT magnets

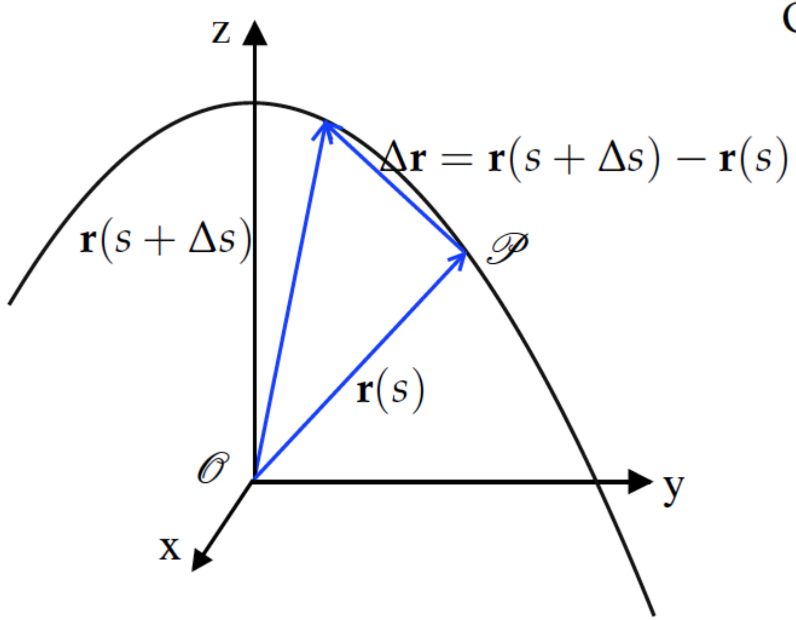


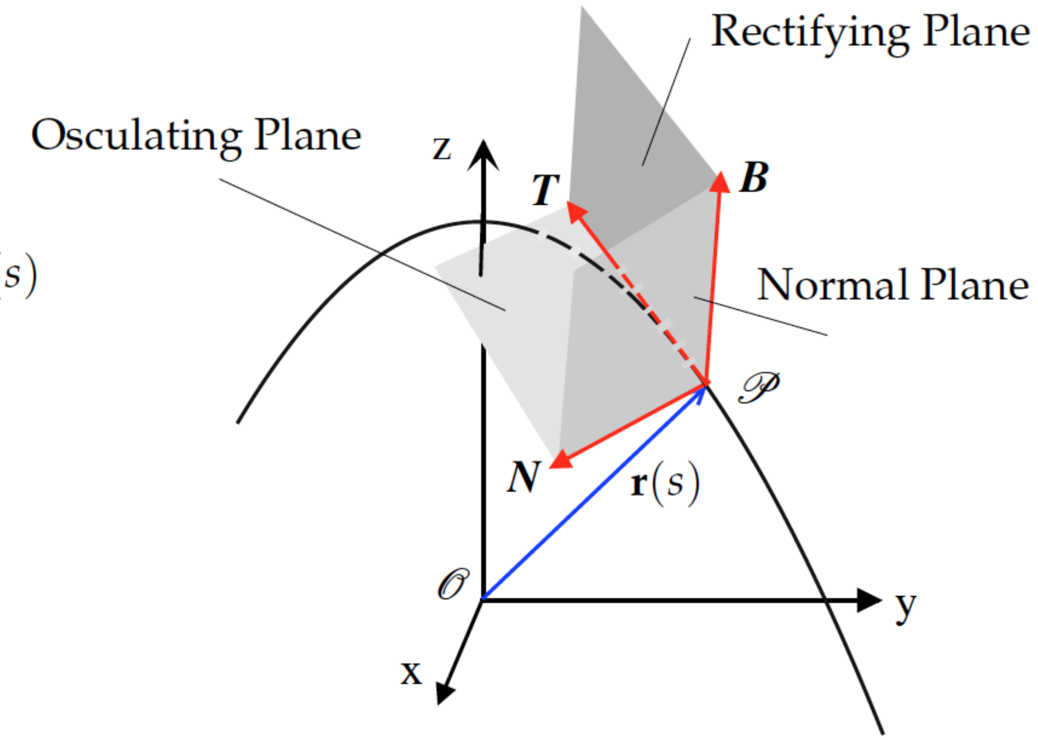
End spacers



Freney Frame of Space Curves

$$\mathbf{r}' \equiv d\mathbf{r}/ds$$



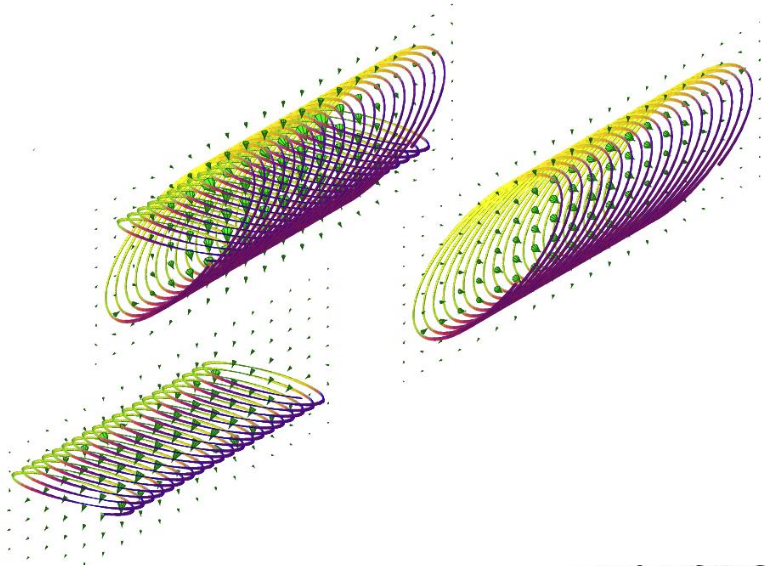


$$\mathbf{T}' = \mathbf{r}'' = |\mathbf{T}'|\,\mathbf{N} =: \kappa\,\mathbf{N}$$

$$\mathbf{N} = \frac{\mathbf{r}''}{\kappa} = \frac{1}{\kappa}\left(x''\mathbf{e}_x + y''\mathbf{e}_y + z''\mathbf{e}_z\right)$$

$$\mathbf{T}' = \kappa\,\mathbf{N}, \qquad \mathbf{N}' = \tau\,\mathbf{B} - \kappa\,\mathbf{T}, \qquad \mathbf{B}' = -\tau\,\mathbf{N}$$

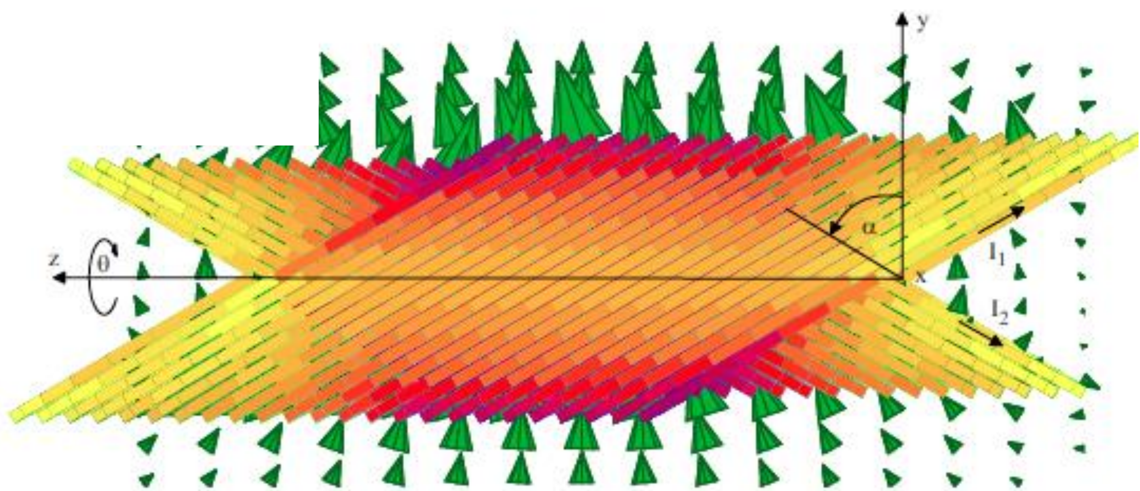
Nested Helices



$$\mathbf{T} = \frac{R\,\mathbf{e}_\varphi + z'(\varphi)\,\mathbf{e}_z}{\sqrt{R^2 + (z'(\varphi))^2}},$$

$$\mathbf{B} = \frac{Rz''(\varphi)\,\mathbf{e}_r - Rz'(\varphi)\,\mathbf{e}_\varphi + R^2\,\mathbf{e}_z}{\sqrt{(Rz''(\varphi))^2 + (Rz'(\varphi))^2 + R^4}},$$

$$\mathbf{N} = \frac{-R(z'(\varphi))^2 - R^3\,\mathbf{e}_r - Rz''(\varphi)\,z'(\varphi)\,\mathbf{e}_\varphi + R^2 z''(\varphi)\,\mathbf{e}_z}{\sqrt{(R^2 + (z'(\varphi))^2)\,(Rz''(\varphi))^2 + (Rz'(\varphi))^2 + R^4)}}$$



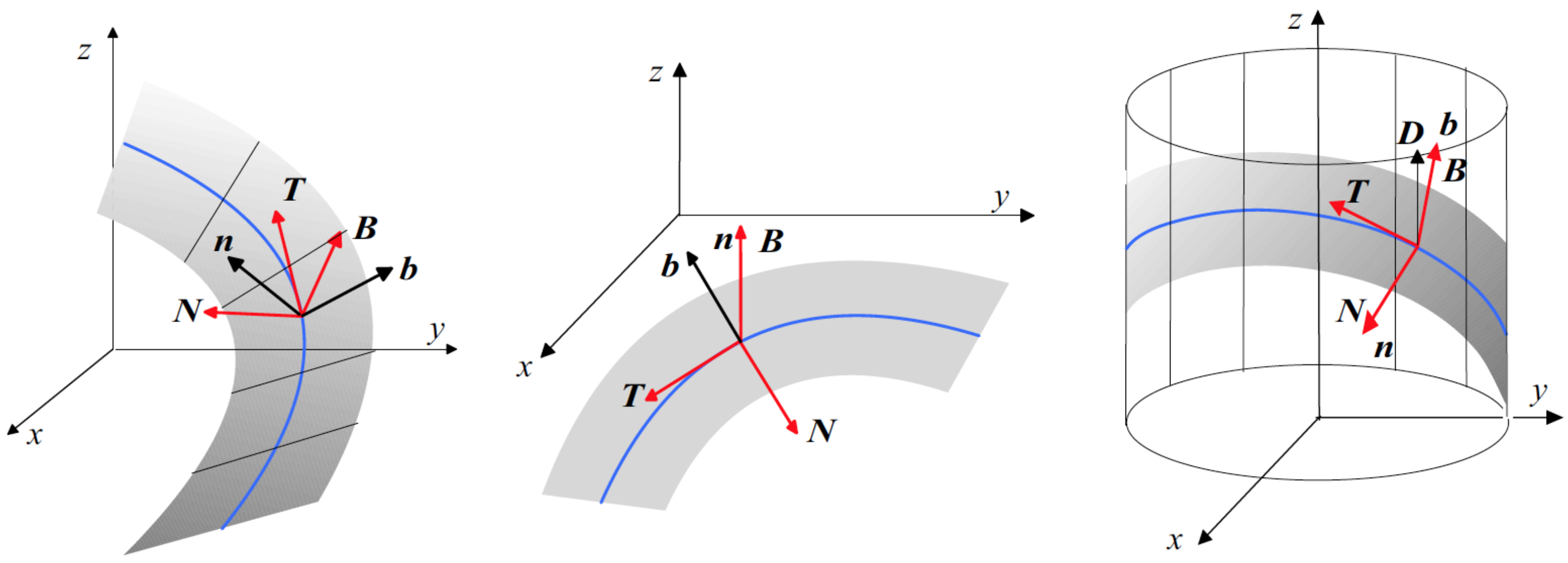
$$\mathbf{r}(\varphi) = R\cos(\varphi)\,\mathbf{e}_x + R\sin(\varphi)\,\mathbf{e}_y + z(\varphi)\,\mathbf{e}_z.$$

$$z(\varphi) = R\tan(\alpha)\sin(n\varphi) + \frac{p}{2\pi}\varphi$$

$$\mathbf{v}(\varphi) = R\,\mathbf{e}_\varphi + z'(\varphi)\,\mathbf{e}_z = R\,\mathbf{e}_\varphi + (nR\tan(\alpha)\cos(n\varphi) + q)\ \mathbf{e}_z$$

Geodesic Strips

$$\mathbf{D} = \tau\,\mathbf{T} + \kappa\,\mathbf{B}$$



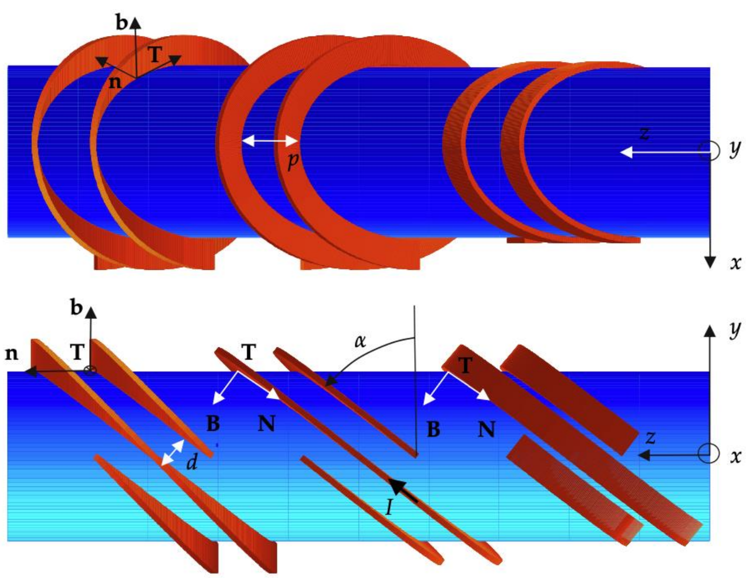
$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{n}' \\ \mathbf{b}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_\mathrm{n} & -\kappa_\mathrm{g} \\ -\kappa_\mathrm{n} & 0 & \tau \\ \kappa_\mathrm{g} & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}$$

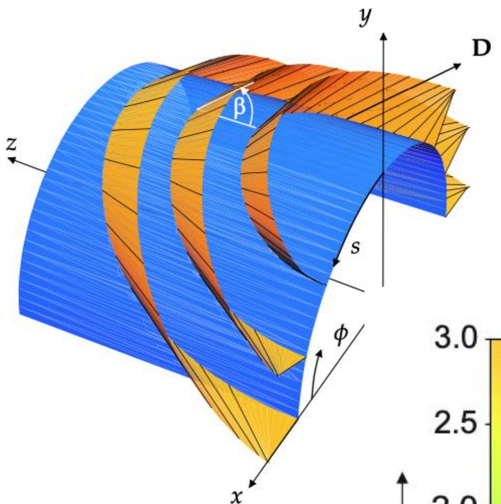
$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}$$

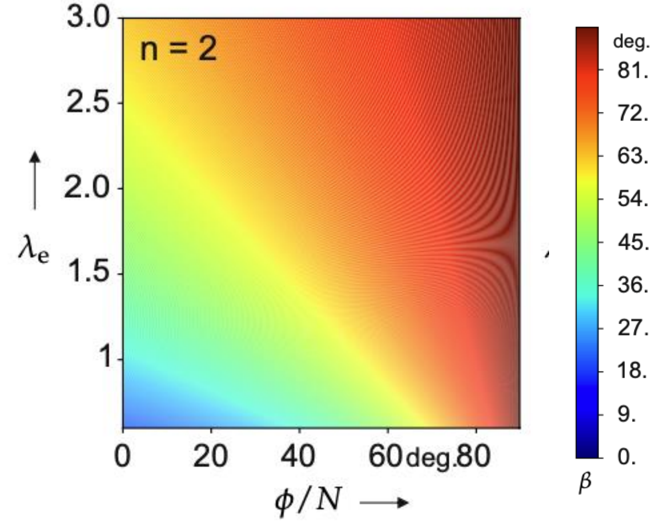
$$\mathbf{T}' = \mathbf{D} \times \mathbf{T}, \qquad \mathbf{N}' = \mathbf{D} \times \mathbf{N}, \qquad \mathbf{B}' = \mathbf{D} \times \mathbf{B}$$

Darboux and Frenet Frames

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\vartheta_{\mathbf{T}}) & -\sin(\vartheta_{\mathbf{T}}) \\ 0 & \sin(\vartheta_{\mathbf{T}}) & \cos(\vartheta_{\mathbf{T}}) \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}$$



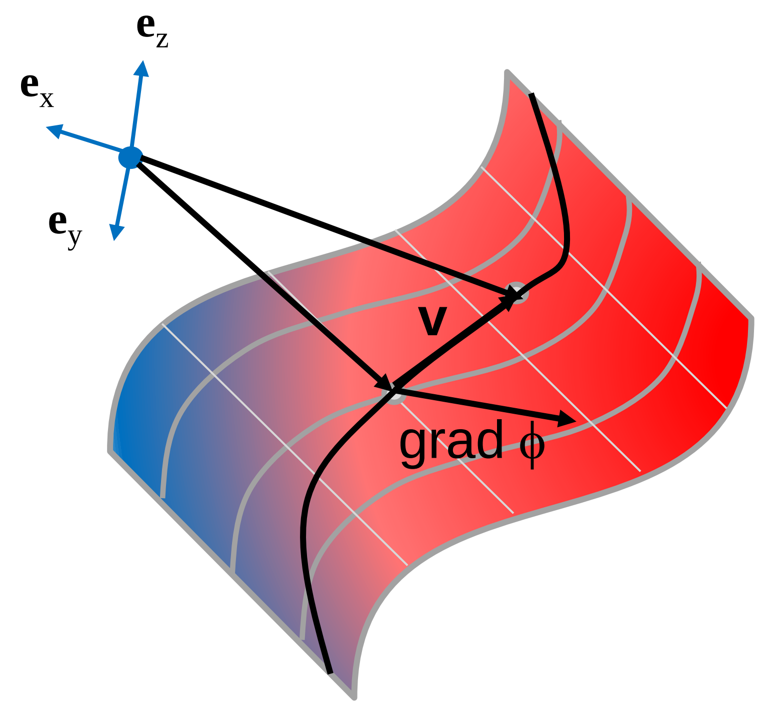




Directional Derivative and the Gradient

Space curve with $\mathbf{r}(t) = (x(t), y(t), z(t))$
parametrized such that $\mathbf{r}(0) = P$.

1-smooth scalar field $\phi : E_3 \to R : \mathbf{r} \mapsto \phi(\mathbf{r})$
expressed as $\phi(x, y, z)$, then $\phi(\mathbf{r}(t))$ at
parameter (time) t.



$$\partial_{\mathbf{v}}\phi = \frac{\partial \phi}{\partial v} = \frac{\mathrm{d}}{\mathrm{d}t}[\phi(\mathbf{r} + t\mathbf{v})]_{t=0} = \lim_{t \to 0} \frac{\phi(\mathbf{r} + t\mathbf{v}) - \phi(\mathbf{r})}{t}$$

$$\partial_{\mathbf{e_x}}\phi = \frac{\partial \phi(x, y, z)}{\partial x} = \lim_{\Delta x \to 0} \frac{\phi(x + \Delta x, y, z) - \phi(x, y, z)}{\Delta x}$$

$$\partial_{\mathbf{v}}\phi = \frac{\mathrm{d}}{\mathrm{d}t}\phi(\mathbf{r}(t)) = \frac{\partial \phi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \phi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = \text{grad}\,\phi \cdot \mathbf{v}$$

$$\text{grad}\phi = \frac{\partial \phi}{\partial x}\mathbf{e}_x + \frac{\partial \phi}{\partial y}\mathbf{e}_y + \frac{\partial \phi}{\partial z}\mathbf{e}_z$$

$$\frac{\partial \phi}{\partial \mathbf{n}}$$

Best linear approximation of ϕ over displacement distance d**r**

Grad, Curl and Div in Cartesian Coordinates

$$\operatorname{grad}\phi := \frac{\partial \phi}{\partial x}\mathbf{e}_x + \frac{\partial \phi}{\partial y}\mathbf{e}_y + \frac{\partial \phi}{\partial z}\mathbf{e}_z$$

$$\operatorname{curl}\mathbf{g} = \left(\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y}\right)\mathbf{e}_z .$$

$$\operatorname{div}\mathbf{g} = \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z} .$$

The First Lemma of Poincare

$$\begin{aligned}
\operatorname{curl}\operatorname{grad}\phi = {} & \operatorname{curl}\left[\frac{1}{h_1}\frac{\partial\phi}{\partial u^1}\mathbf{e}_{u^1} + \frac{1}{h_2}\frac{\partial\phi}{\partial u^2}\mathbf{e}_{u^2} + \frac{1}{h_3}\frac{\partial\phi}{\partial u^3}\mathbf{e}_{u^3}\right] \\
= {} & \frac{1}{h_2 h_3}\left(\frac{\partial^2\phi}{\partial u^2 \partial u^3} - \frac{\partial^2\phi}{\partial u^3 \partial u^2}\right)\mathbf{e}_{u^1} \\
& + \frac{1}{h_3 h_1}\left(\frac{\partial^2\phi}{\partial u^3 \partial u^1} - \frac{\partial^2\phi}{\partial u^1 \partial u^3}\right)\mathbf{e}_{u^2} \\
& + \frac{1}{h_1 h_2}\left(\frac{\partial^2\phi}{\partial u^1 \partial u^2} - \frac{\partial^2\phi}{\partial u^2 \partial u^1}\right)\mathbf{e}_{u^3} = 0\,,
\end{aligned}$$

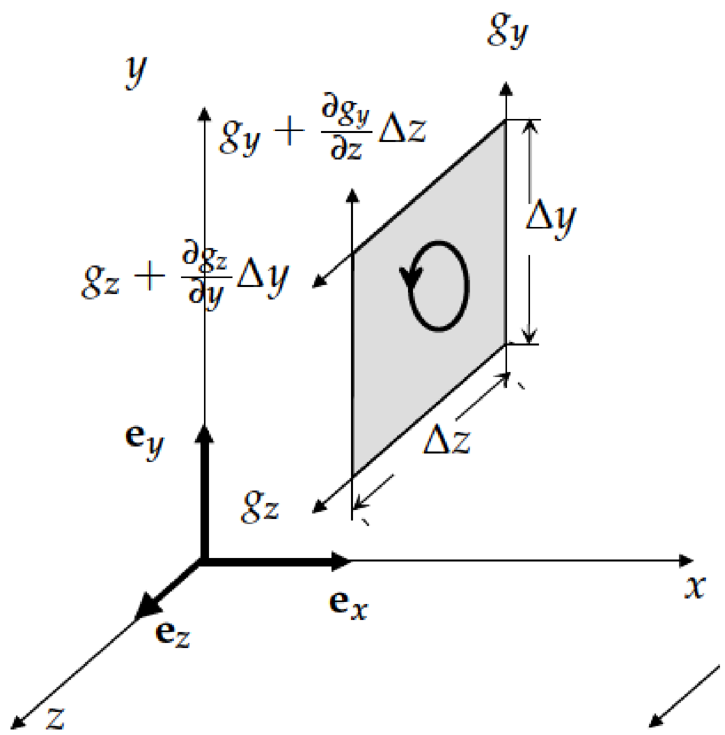
Ugly and not even a universal proof (orthogonality assumed)

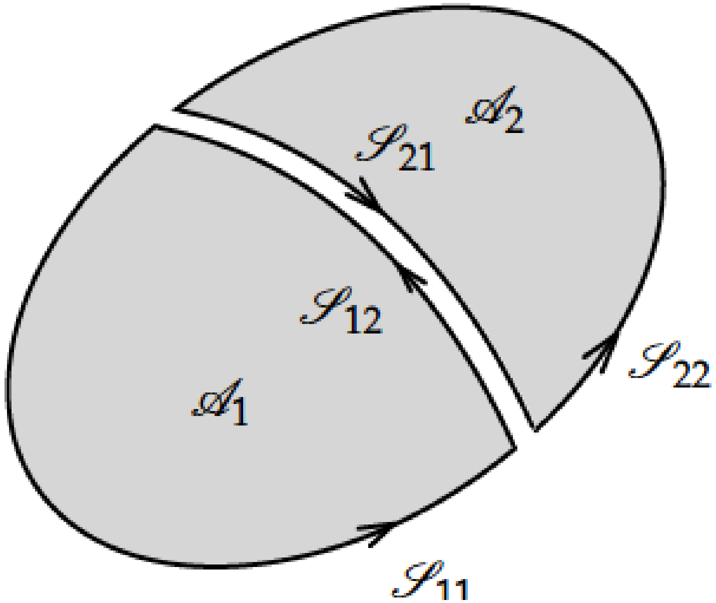
Coordinate Free Definition of Grad, Curl, and Div

$$\int_{\mathscr{P}_1}^{\mathscr{P}_2} \mathbf{a}\cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{P}_1}^{\mathscr{P}_2} \operatorname{grad}\phi \cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{P}_1}^{\mathscr{P}_2} \mathrm{d}\phi = \phi(\mathscr{P}_2) - \phi(\mathscr{P}_1),$$

$$\mathbf{n}\cdot \operatorname{curl}\mathbf{g} = \lim_{a\to 0} \frac{\int_{\partial\mathscr{A}} \mathbf{g}\cdot \mathrm{d}\mathbf{r}}{a},$$

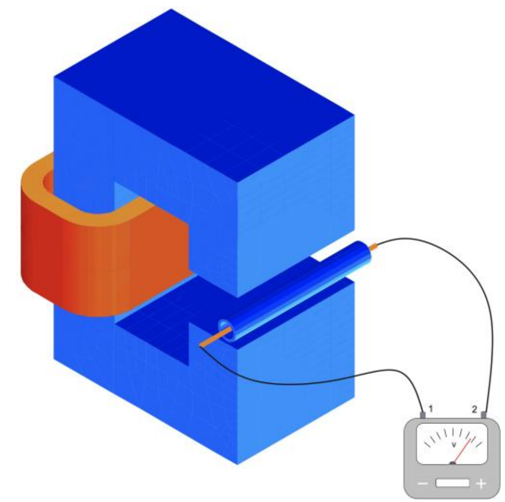
$$\operatorname{div}\mathbf{g} = \lim_{V\to 0} \frac{\int_{\partial\mathscr{V}} \mathbf{g}\cdot \mathrm{d}\mathbf{a}}{V},$$





Smooth vector fields, smooth surfaces with simply connected, closed, piecewise-smooth and consistently oriented boundaries, and volumes with piecewise-smooth, closed and consistently oriented surfaces.

No jump discontinuities (for example, co-moving shielding devices)

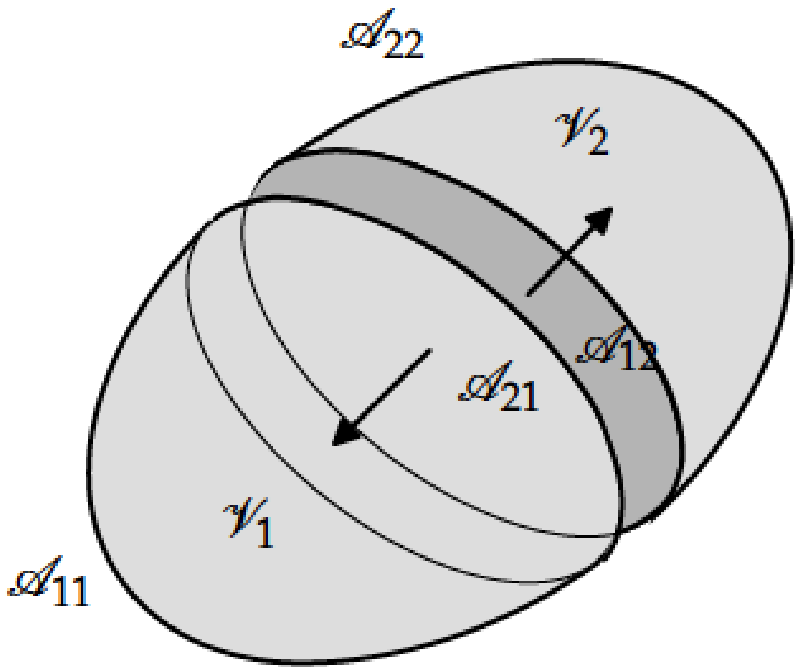


$$\int_{\partial\mathscr{A}} \mathbf{g}\cdot\mathrm{d}\mathbf{r} = \int_{\mathscr{S}_1} \mathbf{g}\cdot\mathrm{d}\mathbf{r} + \int_{\mathscr{S}_2} \mathbf{g}\cdot\mathrm{d}\mathbf{r} = \int_{\mathscr{S}_{11}} \mathbf{g}\cdot\mathrm{d}\mathbf{r} + \int_{\mathscr{S}_{22}} \mathbf{g}\cdot\mathrm{d}\mathbf{r},$$

$$\boxed{\int_{\partial\mathscr{A}} \mathbf{g}\cdot\mathrm{d}\mathbf{r}} = \lim_{I\to\infty}\sum_{i=1}^{I}\int_{\partial\mathscr{A}_i} \mathbf{g}\cdot\mathrm{d}\mathbf{r} = \lim_{I\to\infty}\sum_{i=1}^{I}\Delta a_i \frac{1}{\Delta a_i}\int_{\partial\mathscr{A}_i} \mathbf{g}\cdot\mathrm{d}\mathbf{r}$$

$$= \lim_{I\to\infty}\sum_{i=1}^{I}(\operatorname{curl}\mathbf{g})_i\cdot\mathbf{n}\,\Delta a_i = \boxed{\int_{\mathscr{A}} \operatorname{curl}\mathbf{g}\cdot\mathrm{d}\mathbf{a}\,.}$$

Gauss' Theorem



Smooth vector fields, smooth surfaces with simply connected, closed, piecewise-smooth and consistently oriented boundaries, and volumes with piecewise-smooth, closed and consistently oriented surfaces.

$$\int_{\partial\mathscr{V}} \mathbf{g}\cdot \mathrm{d}\mathbf{a} = \lim_{I\to\infty}\sum_{i=1}^{I}\int_{\partial\mathscr{V}_i}\mathbf{g}\cdot\mathrm{d}\mathbf{a} = \lim_{I\to\infty}\sum_{i=1}^{I}\Delta V_i \frac{1}{\Delta V_i}\int_{\partial\mathscr{V}_i}\mathbf{g}\cdot\mathrm{d}\mathbf{a}$$

$$= \lim_{I\to\infty}\sum_{i=1}^{I}(\operatorname{div}\mathbf{g})_i\,\Delta V_i = \int_{\mathscr{V}}\operatorname{div}\mathbf{g}\,\mathrm{d}V\,.$$

The Boundary Operator and the First Poincare Lemma

$$\partial(\partial\mathscr{V}) = \emptyset\,, \qquad \partial(\partial\mathscr{A}) = \emptyset\,,$$

$$\int_{\mathscr{V}} \operatorname{div}\,\operatorname{curl}\mathbf{g}\,\mathrm{d}V = \int_{\partial\mathscr{V}} \operatorname{curl}\mathbf{g}\cdot\mathrm{d}\mathbf{a} = \int_{\partial(\partial\mathscr{V})} \mathbf{g}\cdot\mathrm{d}\mathbf{r} = 0\,,$$

$$\int_{\mathscr{A}} \operatorname{curl}\,\operatorname{grad}\phi\cdot\mathrm{d}\mathbf{a} = \int_{\partial\mathscr{A}} \operatorname{grad}\phi\cdot\mathrm{d}\mathbf{r} = \phi|_{\partial(\partial\mathscr{A})} = 0\,,$$

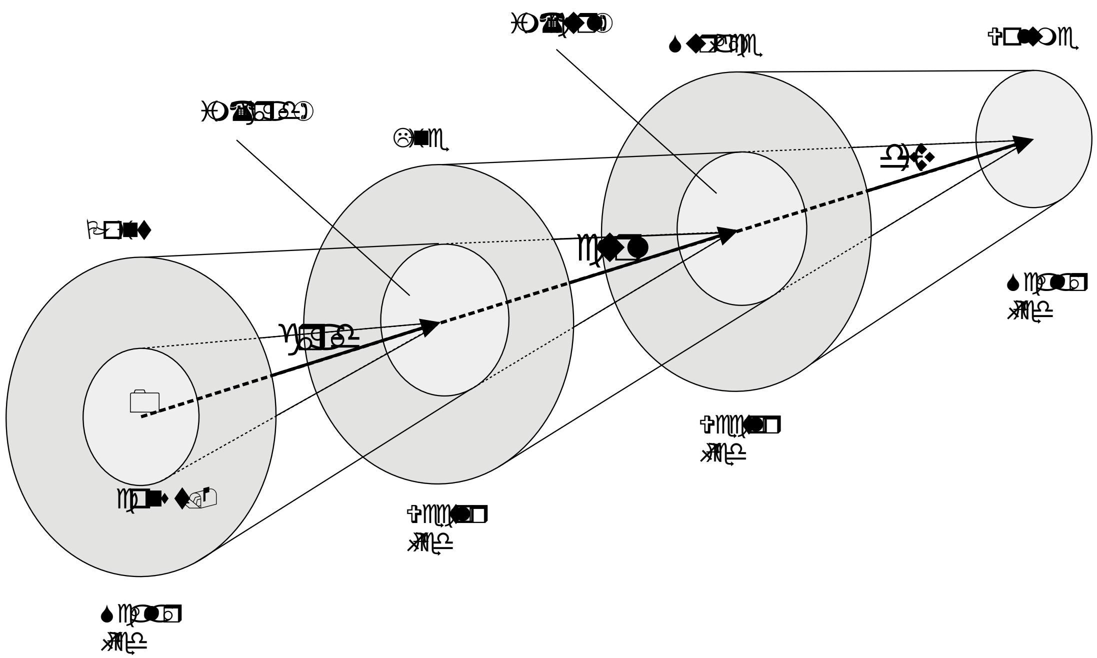
Much nicer than proving it in coordinates

Reversal of arguments yields two important statements (next)

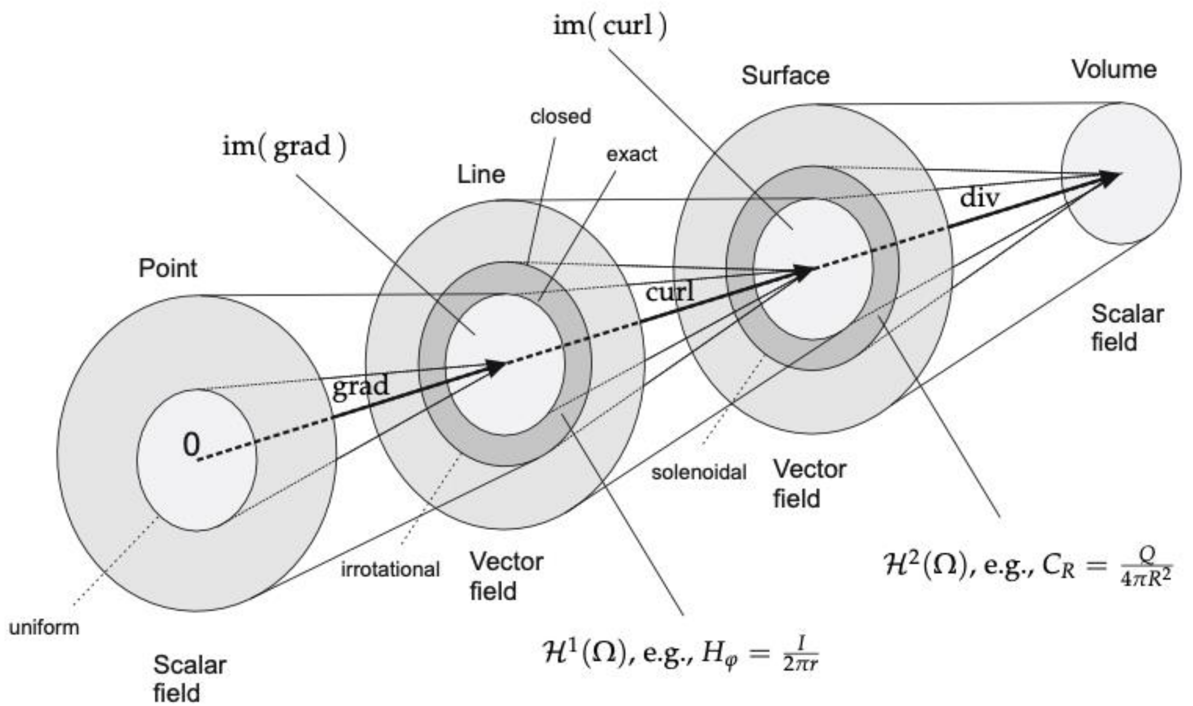
The Second Lemma of Poincare on Contractible Domains

$$\operatorname{div} \mathbf{b} = 0 \quad \rightarrow \quad \mathbf{b} = \operatorname{curl} \mathbf{a}.$$

$$\operatorname{curl} \mathbf{h} = 0 \quad \rightarrow \quad \mathbf{h} = \operatorname{grad} \phi.$$



Lemmata of Poincare on Non-Contractible Domains

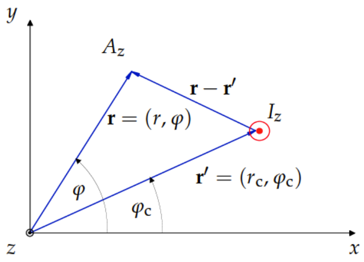


Toroidal domain Ω in a cylindrical coordinate system (r, φ, z):

$$H_\varphi = \frac{I}{2\pi r}$$

$$\operatorname{curl} \mathbf{H} = \frac{1}{r}\frac{\partial}{\partial r}(r H_\varphi) = 0$$

But $\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{s} = I$ and Ω, with $\oint_C \operatorname{grad} \phi \cdot \mathrm{d}\mathbf{s} = 0$



Domain Ω between two nested spheres centered at the origin.

$$D_R = \frac{Q}{4\pi R^2}\mathbf{e}_R$$

$$\operatorname{div} \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial R}(R^2 D_R) = 0$$

But $\oint_a \mathbf{D} \cdot \mathrm{d}\mathbf{a} = Q$ and $\oint_a \operatorname{curl} \mathbf{A} \cdot \mathrm{d}\mathbf{a} = 0$

Maxwell's Equations in Local Form

$$\int_{\mathscr{A}} \operatorname{curl} \mathbf{g} \cdot \mathrm{d}\mathbf{a} = \int_{\partial\mathscr{A}} \mathbf{g} \cdot \mathrm{d}\mathbf{r},$$

$$\int_{\mathscr{V}} \operatorname{div} \mathbf{g} \, \mathrm{d}V = \int_{\partial\mathscr{V}} \mathbf{g} \cdot \mathrm{d}\mathbf{a},$$

$$\int_{\partial\mathscr{A}} \mathbf{H} \cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{A}} \mathbf{J} \cdot \mathrm{d}\mathbf{a} + \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{A}} \mathbf{D} \cdot \mathrm{d}\mathbf{a},$$

$$\int_{\partial\mathscr{A}} \mathbf{E} \cdot \mathrm{d}\mathbf{r} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{A}} \mathbf{B} \cdot \mathrm{d}\mathbf{a},$$

$$\int_{\partial\mathscr{V}} \mathbf{B} \cdot \mathrm{d}\mathbf{a} = 0,$$

$$\int_{\partial\mathscr{V}} \mathbf{D} \cdot \mathrm{d}\mathbf{a} = \int_{\mathscr{V}} \rho \, \mathrm{d}V.$$

$$\int_{\mathscr{A}} \operatorname{curl} \mathbf{H} \cdot \mathrm{d}\mathbf{a} = \int_{\mathscr{A}} \left(\mathbf{J} + \frac{\partial}{\partial t} \mathbf{D} \right) \cdot \mathrm{d}\mathbf{a},$$

$$\int_{\mathscr{A}} \operatorname{curl} \mathbf{E} \cdot \mathrm{d}\mathbf{a} = -\int_{\mathscr{A}} \frac{\partial}{\partial t} \mathbf{B} \cdot \mathrm{d}\mathbf{a},$$

$$\int_{\mathscr{V}} \operatorname{div} \mathbf{B} \, \mathrm{d}V = 0,$$

$$\int_{\mathscr{V}} \operatorname{div} \mathbf{D} \, \mathrm{d}V = \int_{\mathscr{V}} \rho \, \mathrm{d}V.$$

$$\operatorname{curl} \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t} \mathbf{D},$$

$$\operatorname{curl} \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B},$$

$$\operatorname{div} \mathbf{B} = 0,$$

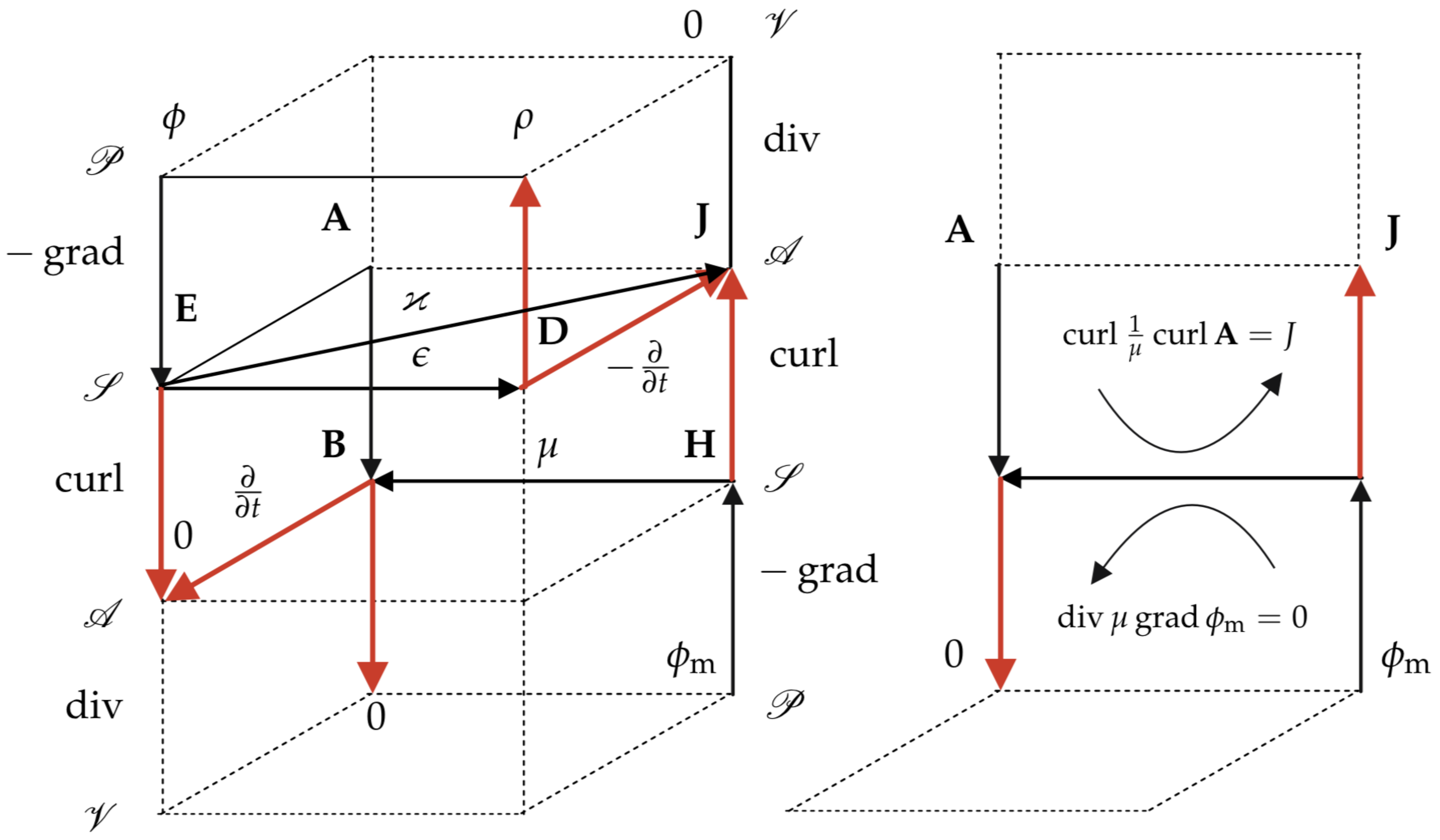
$$\operatorname{div} \mathbf{D} = \rho.$$

Maxwell's House

Faraday complex
Inner oriented

Ampere-Maxwell complex
Outer oriented

curl **B** is ugly !



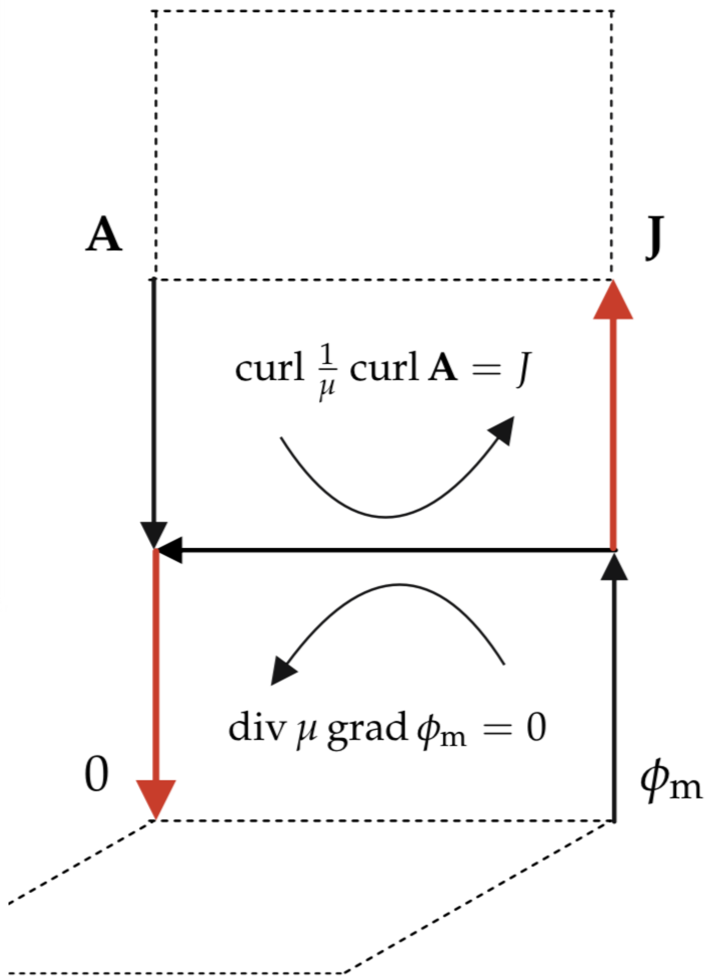
Maxwell’s Facade

$$\operatorname{curl}\frac{1}{\mu}\operatorname{curl}\mathbf{A} = \mathbf{J}$$

$$\frac{1}{\mu_0}\operatorname{curl}\operatorname{curl}\mathbf{A} = \mathbf{J}$$

$$\nabla^2\mathbf{A} - \operatorname{grad}\operatorname{div}\mathbf{A} = 0$$

$$\nabla^2 A_z = 0$$

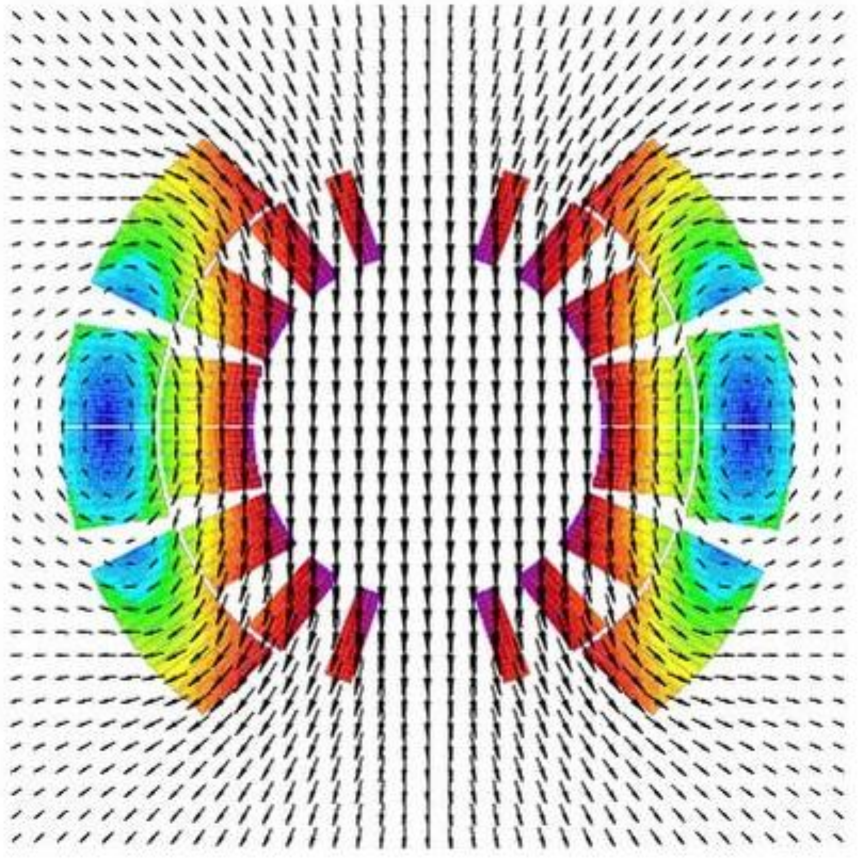


$$\operatorname{div}\mu\operatorname{grad}\phi_m = 0$$

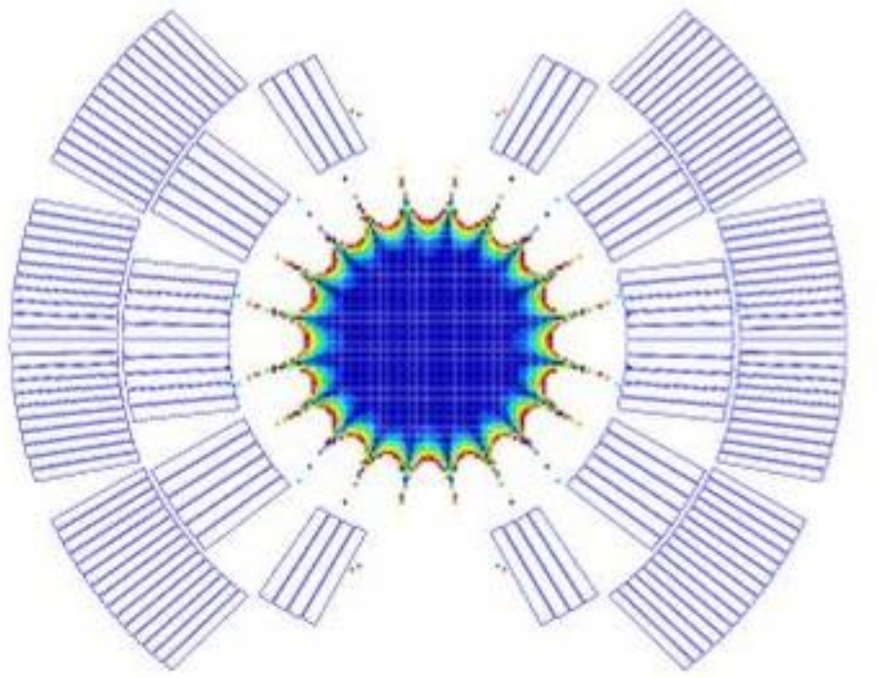
$$\mu_0\operatorname{div}\operatorname{grad}\phi_m = 0$$

$$\nabla^2\phi_m = 0$$

Field Quality



Field map

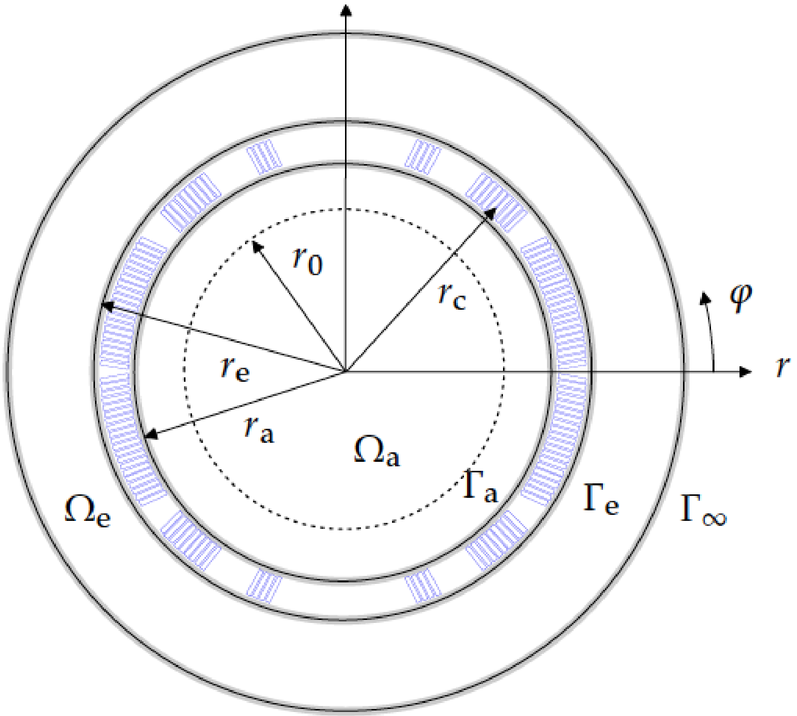


Good field region

1. Governing equation in the air domain

$$\nabla^2 A_z = 0,$$

2. Chose a suitable coordinate system



$$r^2 \frac{\partial^2 A_z}{\partial r^2} + r \frac{\partial A_z}{\partial r} + \frac{\partial^2 A_z}{\partial \varphi^2} = 0,$$

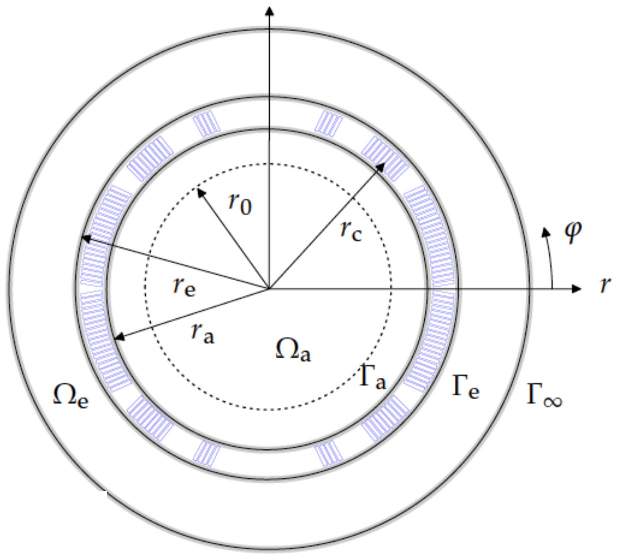
3. Make a guess, look it up in a book, use the method of separation: That is: find eigenfunctions. Coefficients are not know yet

$$A_z(r,\varphi) = \sum_{n=1}^{\infty} (\mathcal{E}_n r^n + \mathcal{F}_n r^{-n})(\mathcal{G}_n \sin n\varphi + \mathcal{H}_n \cos n\varphi).$$

Solving of Boundary Value Problems

4. Incorporate a bit of knowledge and rename

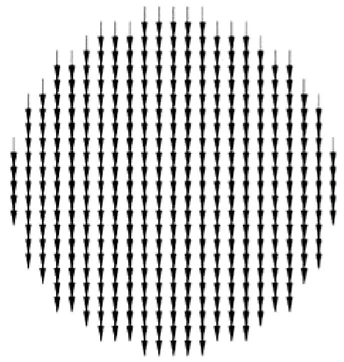
$$A_z(r,\varphi) = \sum_{n=1}^{\infty} r^n (\mathcal{A}_n \sin n\varphi + \mathcal{B}_n \cos n\varphi).$$

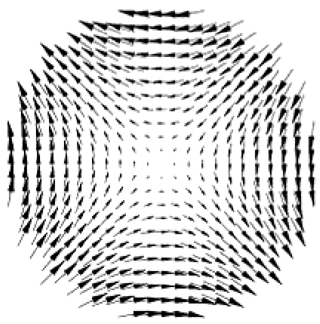


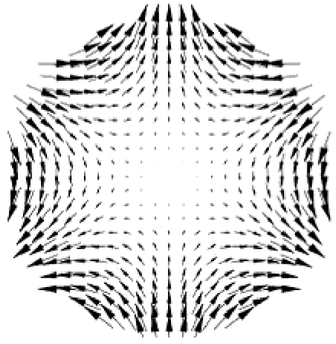
5. Calculate a field component

$$B_r(r,\varphi) = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1} (\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi),$$

$$B_\varphi(r,\varphi) = -\frac{\partial A_z}{\partial r} = -\sum_{n=1}^{\infty} n r^{n-1} (\mathcal{A}_n \sin n\varphi + \mathcal{B}_n \cos n\varphi),$$





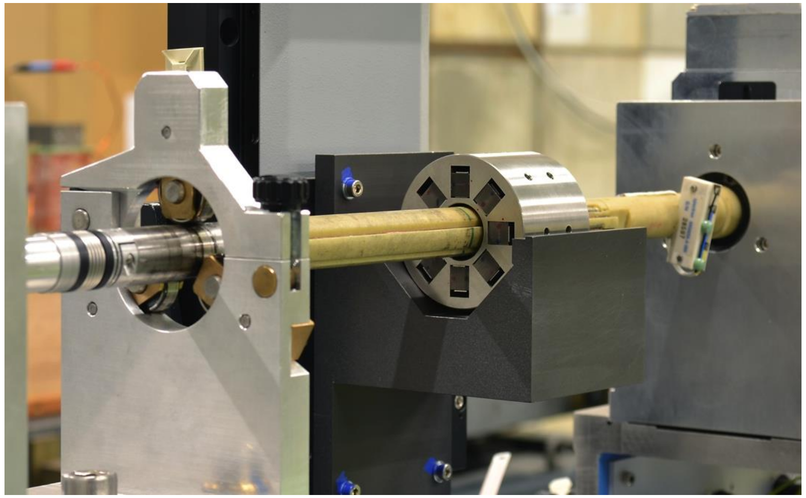


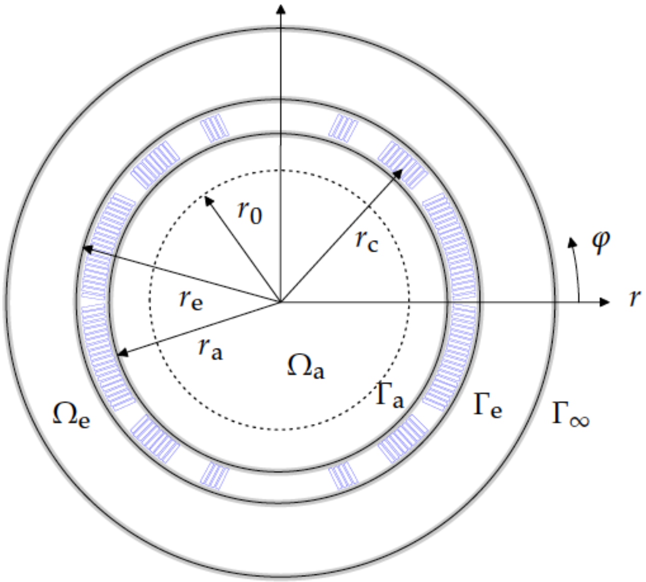
Solving of Boundary Value Problems

$$B_r(r,\varphi) = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1}(\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi),$$

6. Measure or calculate the field on a reference radius and perform Fourier analysis (develop into the eigenfunctions). Coefficients known here.

$$B_r(r_0,\varphi) = \sum_{n=1}^{\infty} (B_n(r_0)\sin n\varphi + A_n(r_0)\cos n\varphi),$$





Solving the Boundary Value Problem

7: Compare the known and unknown coefficients

$$B_r(r,\varphi) = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1} (\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi),$$

$$B_r(r_0,\varphi) = \sum_{n=1}^{\infty} (B_n(r_0) \sin n\varphi + A_n(r_0) \cos n\varphi),$$

$$\mathcal{A}_n = \frac{1}{n\, r_0^{n-1}} A_n(r_0), \qquad \mathcal{B}_n = \frac{-1}{n\, r_0^{n-1}} B_n(r_0).$$

8. Put this into the original solution for the entire air domain

$$A_z(r,\varphi) = -\sum_{n=1}^{\infty} \frac{r_0}{n} \left(\frac{r}{r_0}\right)^n (B_n(r_0) \cos n\varphi - A_n(r_0) \sin n\varphi).$$

9: Calculate fields and potential in the entire air domain

$$A_z(r,\varphi) = -\sum_{n=1}^{\infty} \frac{r_0}{n}\left(\frac{r}{r_0}\right)^n (B_n(r_0)\cos n\varphi - A_n(r_0)\sin n\varphi).$$

$$B_r(r,\varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} (B_n(r_0)\sin n\varphi + A_n(r_0)\cos n\varphi)$$

$$B_\varphi(r,\varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} (B_n(r_0)\cos n\varphi - A_n(r_0)\sin n\varphi)$$

$$B_x(r,\varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} (B_n(r_0)\sin(n-1)\varphi + A_n(r_0)\cos(n-1)\varphi)$$

$$B_y(r,\varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} (B_n(r_0)\cos(n-1)\varphi - A_n(r_0)\sin(n-1)\varphi)$$

Solving Boundary Value Problems

Take any 2π periodic function and develop according to

$$\frac{C_0}{2} + \sum_{n=1}^{\infty} (C_n(r_0) \sin n\varphi + D_n(r_0) \cos n\varphi) \, .$$

	B_r	B_φ	B_x	B_y	A_z	ϕ_{m}
$B_n =$	C_n	D_n	C_{n-1}	D_{n-1}	$\frac{-nD_n}{r_0}$	$\frac{-n\mu_0 C_n}{r_0}$
$A_n =$	D_n	$-C_n$	D_{n-1}	$-C_{n-1}$	$\frac{nC_n}{r_0}$	$\frac{-n\mu_0 D_n}{r_0}$

We can use fields, potentials, fluxes, or wire-oscillation amplitudes as “raw data”. The linear differential operators grad and rot transform into simple algebra in the L2 space of Fourier coefficients.

This is also the foundation for the method of superposition

Complex Potentials

$$\mathbf{H} = -\operatorname{grad}\phi = -\frac{\partial \phi}{\partial x}\mathbf{e}_x - \frac{\partial \phi}{\partial y}\mathbf{e}_y,$$

$$\mathbf{B} = \operatorname{curl}(\mathbf{e}_z A_z) = \frac{\partial A_z}{\partial y}\mathbf{e}_x - \frac{\partial A_z}{\partial x}\mathbf{e}_y\,.$$

This implies

$$\frac{\partial A_z}{\partial y} = -\mu_0 \frac{\partial \phi}{\partial x} \quad \text{and} \quad \frac{\partial A_z}{\partial x} = \mu_0 \frac{\partial \phi}{\partial y},$$

Which are the Cauchy Riemann equations of

$$w(z) := u(x,y) + iv(x,y) = A_z(x,y) + i\mu_0\phi(x,y)\,.$$

$$-\frac{\mathrm{d}w}{\mathrm{d}z} = -\frac{\partial A_z}{\partial x} - i\mu_0 \frac{\partial \phi}{\partial x} = i\frac{\partial A_z}{\partial y} - \mu_0 \frac{\partial \phi}{\partial y} = B_y(x,y) + iB_x(x,y) =: B(z).$$

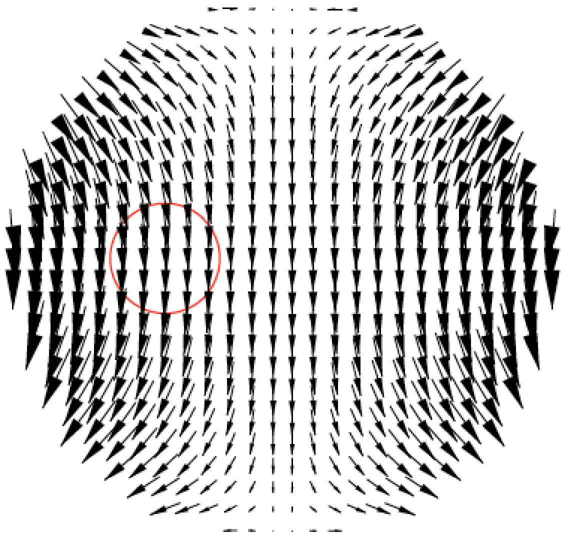
Complex Representation of the Field in Accelerator Magnets

$$B_x = B_r \cos\varphi - B_\varphi \sin\varphi, \qquad B_y = B_r \sin\varphi + B_\varphi \cos\varphi,$$

$$B_y + iB_x = (B_\varphi + iB_r)e^{-i\varphi}.$$

$$\begin{aligned} B_y + iB_x &= \sum_{n=1}^{\infty} (B_n(r_0) + i\,A_n(r_0)) \left(\frac{r}{r_0}\right)^{n-1} e^{i(n-1)\varphi} \\ &= \sum_{n=1}^{\infty} (B_n(r_0) + i\,A_n(r_0)) \left(\frac{z}{r_0}\right)^{n-1} \\ &= B_N \sum_{n=1}^{\infty} (b_n(r_0) + i\,a_n(r_0)) \left(\frac{z}{r_0}\right)^{n-1}, \end{aligned}$$

Feed-Down: Proof



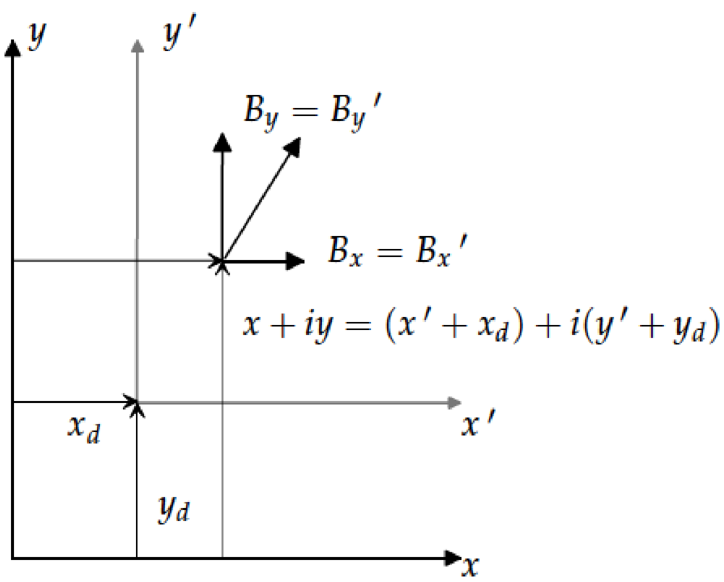
$$\sum_{n=1}^{\infty} C_n \left(\frac{z}{r_0}\right)^{n-1} = \sum_{n=1}^{\infty} \frac{C_n}{{r_0}^{n-1}} (z' + z_{\mathrm{d}})^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{C_n}{{r_0}^{n-1}} \sum_{k=1}^{n} \binom{n-1}{k-1} (z')^{k-1} {z_{\mathrm{d}}}^{n-k}$$

$$= \sum_{k=1}^{\infty} \left[\sum_{n=1}^{n} \frac{C_n}{{r_0}^{n-1}} \binom{n-1}{k-1} {z_{\mathrm{d}}}^{n-k} \right] (z')^{k-1}$$

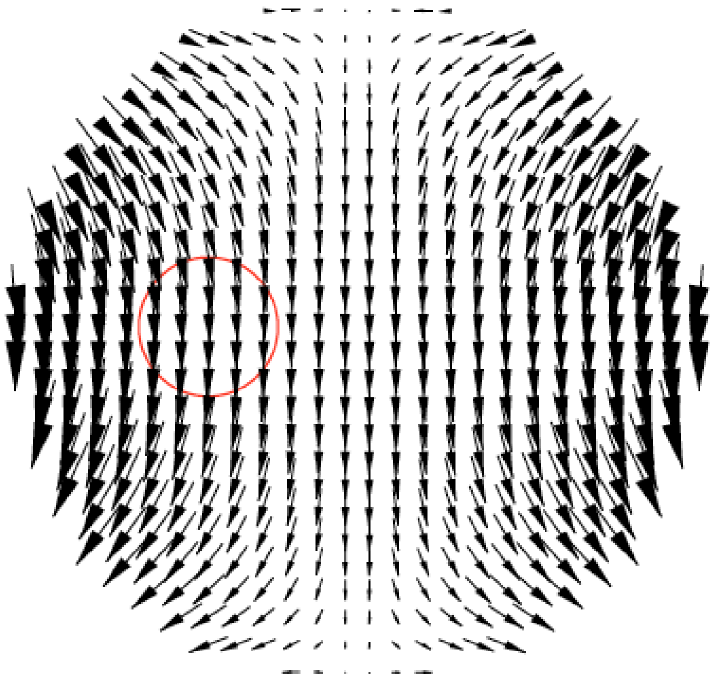
$$= \sum_{k=1}^{\infty} \left[\sum_{n=k}^{\infty} C_n \binom{n-1}{k-1} \left(\frac{z_{\mathrm{d}}}{r_0}\right)^{n-k} \right] \left(\frac{z'}{r_0}\right)^{k-1}$$

$$= \sum_{n=1}^{\infty} \left[\sum_{k=n}^{\infty} C_k \binom{k-1}{n-1} \left(\frac{z_{\mathrm{d}}}{r_0}\right)^{k-n} \right] \left(\frac{z'}{r_0}\right)^{n-1} .$$

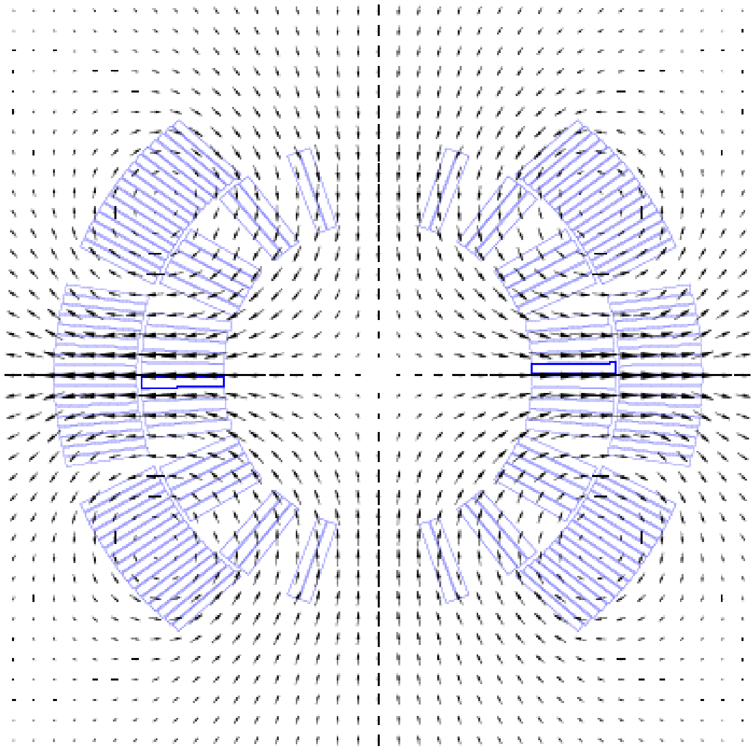


$$C_2' = C_2 + 2\, C_3 \left(\frac{z_{\mathrm{d}}}{r_0}\right) + 3\, C_4 \left(\frac{z_{\mathrm{d}}}{r_0}\right)^2 + \cdots ,$$

Feed-down: Enemy and Friend



- Measurement of magnetic axis in dipole by powering the coil as a quadrupole
- Feed-down can be used to center the measurement coil
  - Minimizing B_{10} which can only occur as feed-down from B_{11}



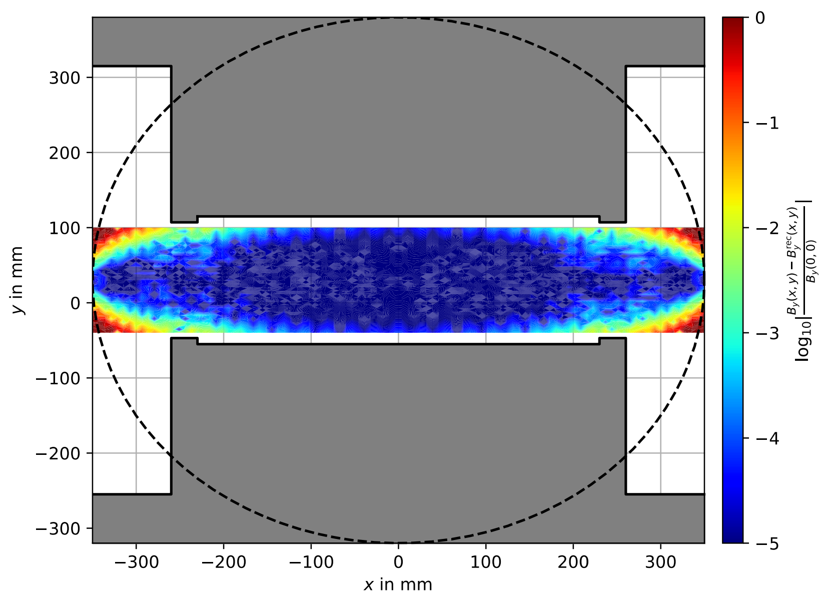
- Alignment tolerances of MCS and MCDO correctors w.r.t. MB
  - 0.3 mm radially
- Dipole magnetic axis has to be well aligned with respect to the closed orbit
  - ± 0.1 mm systematic, ± 0.5 mm random (r.m.s)

Maclaurin Series and the Analytical Continuation

$$B_y(x) = B_0 + \left.\frac{\mathrm{d}B_y}{\mathrm{d}x}\right|_{x=y=0} x + \cdots + \frac{1}{n!}\left.\frac{\mathrm{d}^n B_y}{\mathrm{d}x^n}\right|_{x=y=0} x^n + \cdots$$

$$B_y(r,\varphi) = \sum_{n=1}^{\infty}\left(\frac{r}{r_0}\right)^{n-1}\left(B_n(r_0)\cos(n-1)\varphi - A_n(r_0)\sin(n-1)\varphi\right)$$

$$b_n(r_0) = \frac{{r_0}^{n-1}}{B_N}\frac{1}{(n-1)!}\left.\frac{\mathrm{d}^{n-1}B_y}{\mathrm{d}x^{n-1}}\right|_{x=y=0}$$



Ideal Pole Shape of Conventional Magnets

Cauchy-Schwarz inequality $\quad | \langle \mathbf{a}, \mathbf{b} \rangle | \leq \| \mathbf{a} \| \| \mathbf{b} \|,$

For the directional derivative $\quad |\partial_{\mathbf{v}} \phi| \leq |\operatorname{grad} \phi| |\mathbf{v}|$

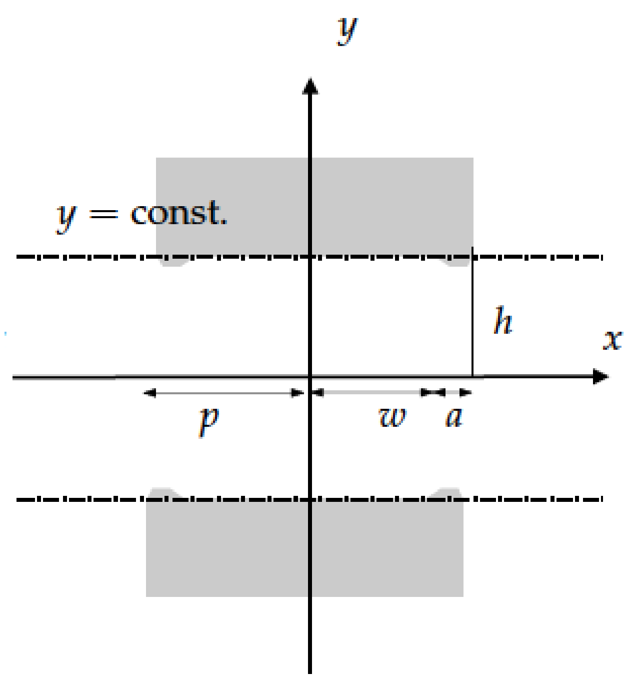
The directional derivative takes its maximum when **v** points in the direction of the gradient. Therefore, the gradient points in the direction of the steepest ascent of Φ and is thus normal to the surface of equipotential.

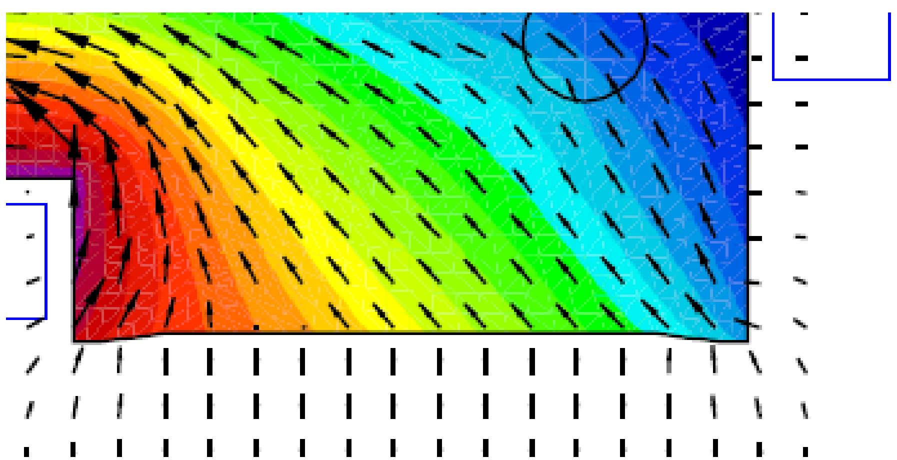
The flux density **B** exits a highly permeable surface in the normal direction. Therefore, the pole shape of normal conducting magnets can be seen as an equipotential of the magnetic scalar potential.

Ideal Pole Shape of Conventional Magnets

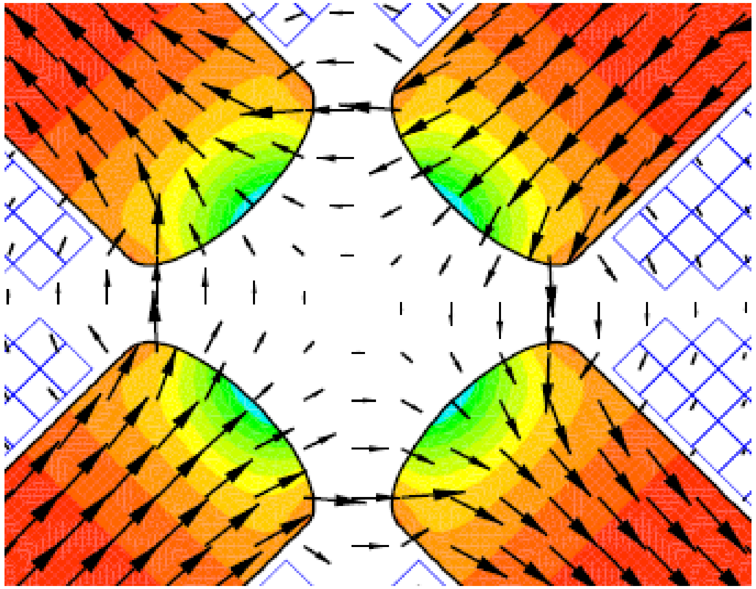
$$\phi_{\mathrm{m}}(r,\varphi) = -\sum_{n=1}^{\infty} \frac{r_0}{n\mu_0}\left(\frac{r}{r_0}\right)^n (A_n(r_0)\cos n\varphi + B_n(r_0)\sin n\varphi).$$

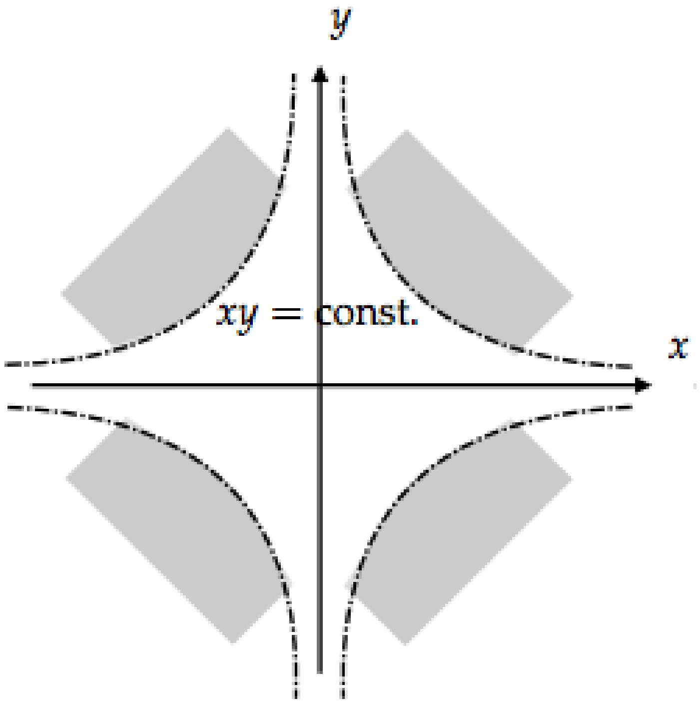
$$\phi_{\mathrm{m}}(x,y) = -\frac{1}{\mu_0}(B_1 y + A_1 x).$$





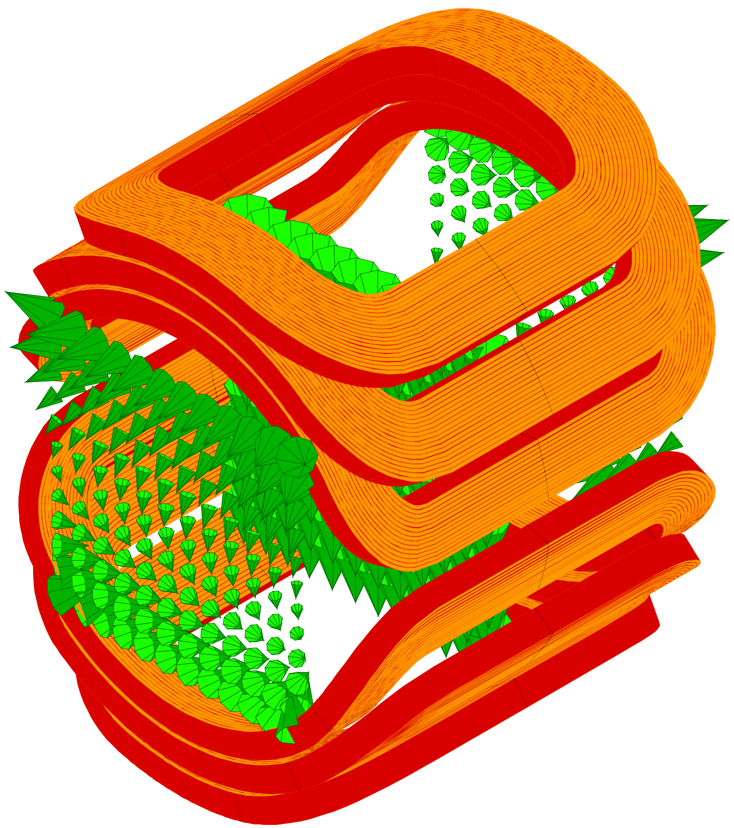
Ideal Pole Shape of Conventional Magnets

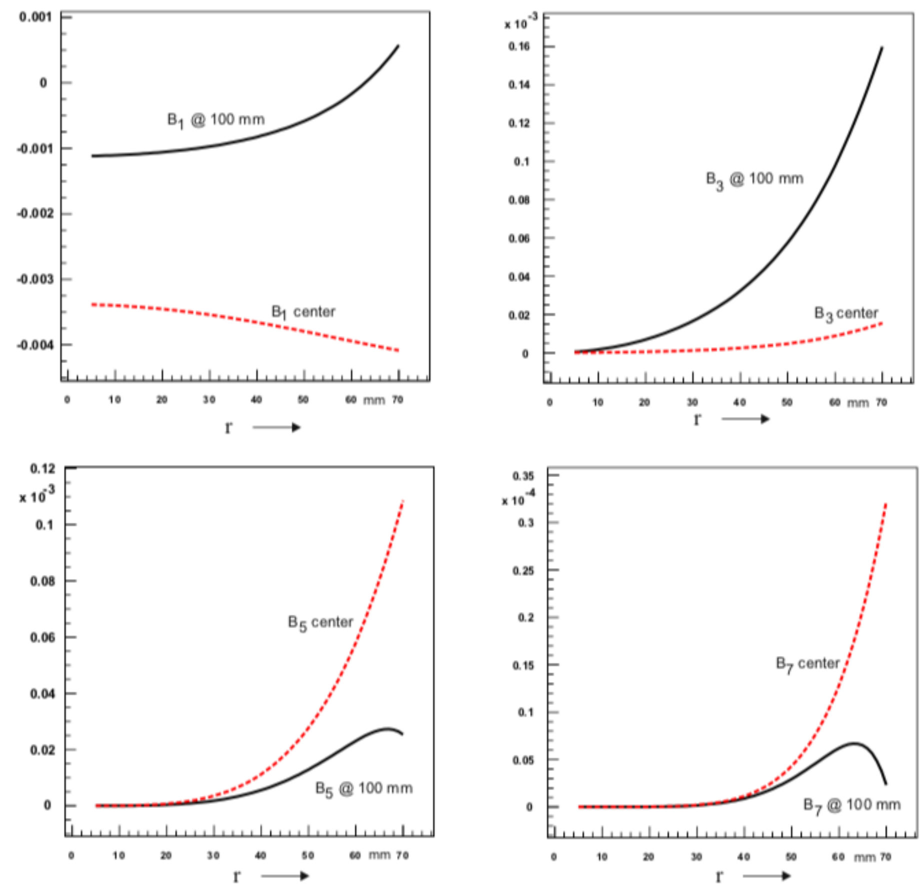




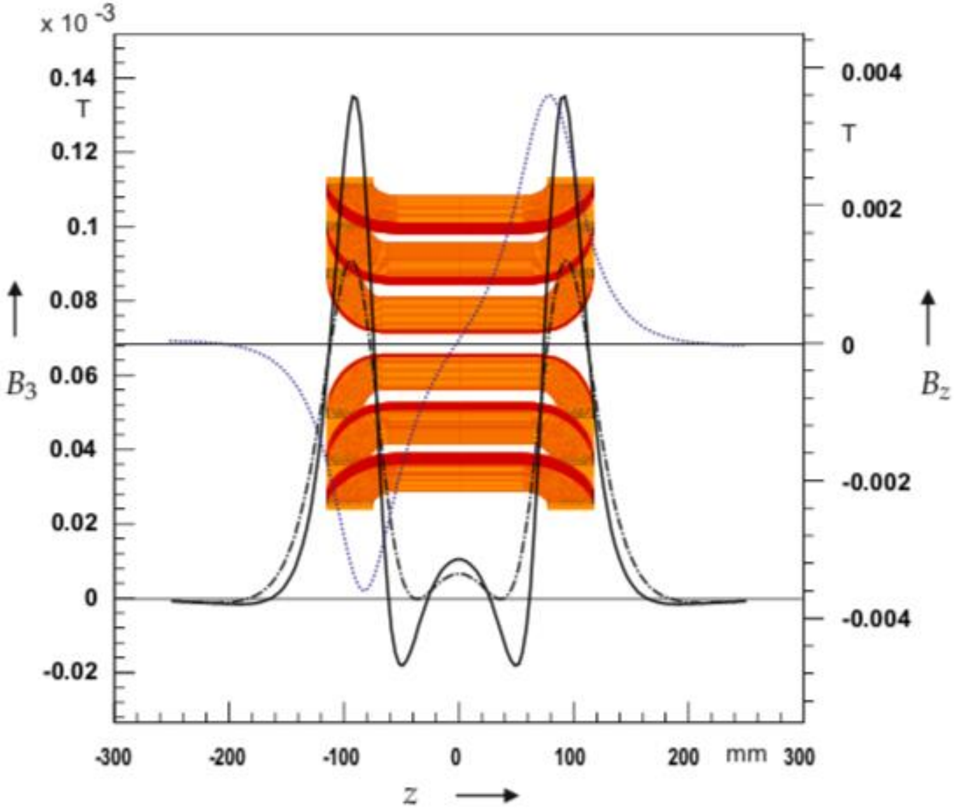
$$\phi_{\mathrm{m}}(r,\varphi) = \frac{1}{2\mu_0 r_0}\left(B_2(r_0)\,2xy + A_2(r_0)\,(x^2 - y^2)\right).$$

3D Field Harmonics





Integrated Harmonics



Local transverse harmonics calculated at different reference radii and scaled with the 2D laws

$$b_n(r_1) = \left(\frac{r_1}{r_0}\right)^{n-N} b_n(r_0),$$

wrong

Integrated Harmonics

$$\nabla^2 \phi_{\mathrm{m}}(x,y,z) = \frac{\partial^2 \phi_{\mathrm{m}}(x,y,z)}{\partial x^2} + \frac{\partial^2 \phi_{\mathrm{m}}(x,y,z)}{\partial y^2} + \frac{\partial^2 \phi_{\mathrm{m}}(x,y,z)}{\partial z^2} = 0$$

$$\overline{\phi}_{\mathrm{m}}(x,y) := \int_{-z_0}^{z_0} \phi_{\mathrm{m}}(x,y,z)\mathrm{d}z,$$

$$\begin{aligned}
\frac{\partial^2 \overline{\phi}_{\mathrm{m}}(x,y)}{\partial x^2} + \frac{\partial^2 \overline{\phi}_{\mathrm{m}}(x,y)}{\partial y^2} &= \int_{-z_0}^{z_0} \left(\frac{\partial^2 \phi_{\mathrm{m}}}{\partial x^2} + \frac{\partial^2 \phi_{\mathrm{m}}}{\partial y^2} \right) \mathrm{d}z \\
&= \int_{-z_0}^{z_0} \left(-\frac{\partial^2 \phi_{\mathrm{m}}}{\partial z^2} \right) \mathrm{d}z = - \left. \frac{\partial \phi_{\mathrm{m}}}{\partial z} \right|_{-z_0}^{z_0} \\
&= H_z(-z_0) - H_z(z_0) \stackrel{!}{=} 0 .
\end{aligned}$$

The 2D scaling laws hold for the integrated harmonics

Maxwell Equations

Integral Form

Local Form

Global Form

Laplace's Equation

The Curl-Curl Equation

Harmonic Fields

Green's Functions

Weak-Forms

Kirchhoff's Theorem

1D Calculation of NC Magnets

Field Quality in Accelerator Magnets

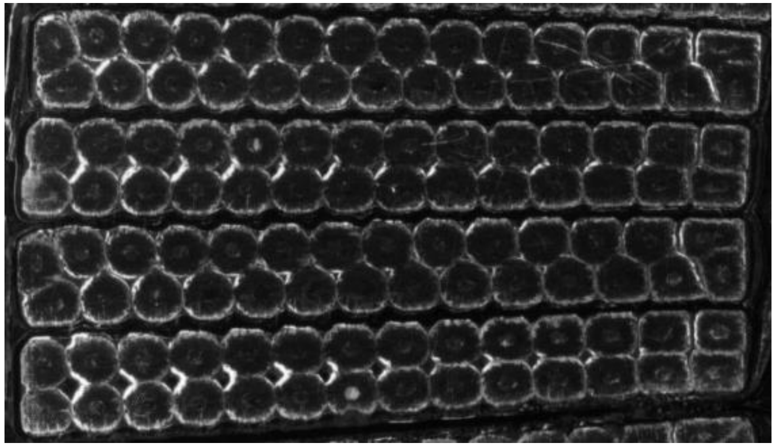
The Field of Line-currents Coil-Dominated Magnets

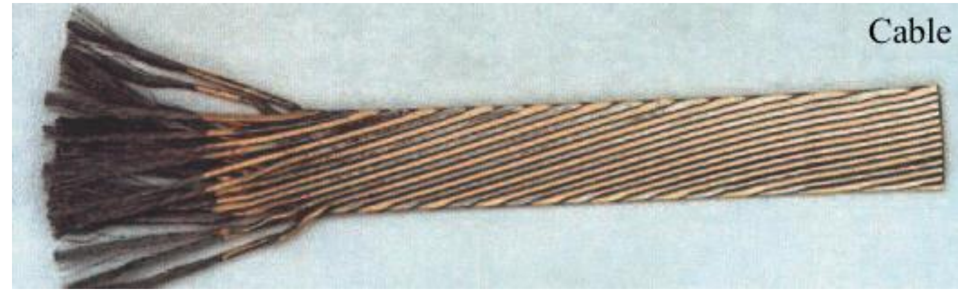
FEM

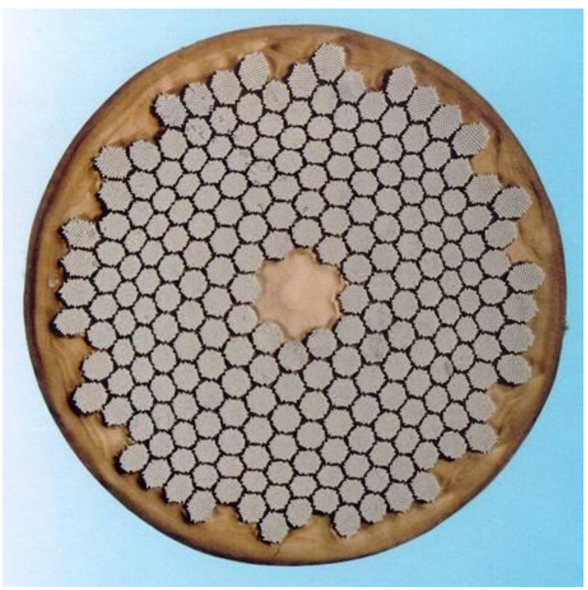
BEM

DEM

Rutherford (Roebel) Kabel, Strand, Nb-Ti Filament

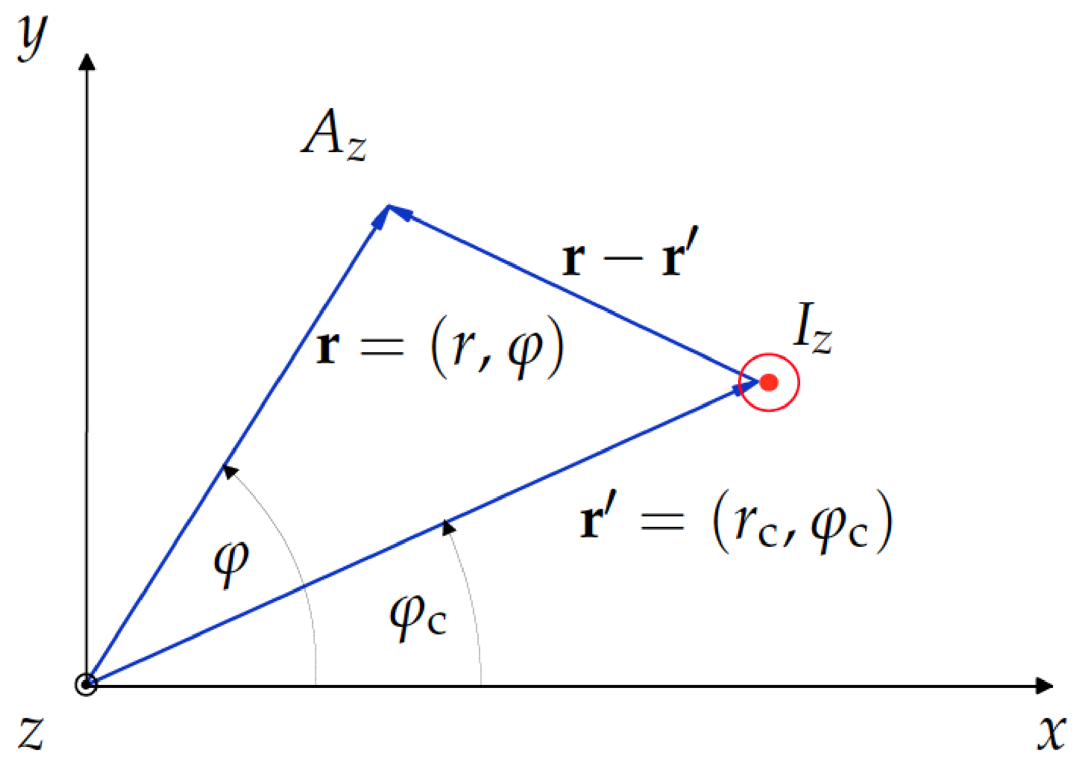






$$\mathbf{r} \mapsto \phi(|\mathbf{r} - \mathbf{r}'|)$$
$$\mathbf{r}' \mapsto \phi(|\mathbf{r} - \mathbf{r}'|)$$

$$\operatorname{grad} \phi(|\mathbf{r} - \mathbf{r}'|) = -\operatorname{grad}_{\mathbf{r}'} \phi(|\mathbf{r} - \mathbf{r}'|),$$
$$\operatorname{div} \mathbf{a}(|\mathbf{r} - \mathbf{r}'|) = -\operatorname{div}_{\mathbf{r}'} \mathbf{a}(|\mathbf{r} - \mathbf{r}'|),$$
$$\operatorname{curl} \mathbf{a}(|\mathbf{r} - \mathbf{r}'|) = -\operatorname{curl}_{\mathbf{r}'} \mathbf{a}(|\mathbf{r} - \mathbf{r}'|),$$
$$\nabla^2 \phi(|\mathbf{r} - \mathbf{r}'|) = \nabla^2_{\mathbf{r}'} \phi(|\mathbf{r} - \mathbf{r}'|).$$



Why bother?
Reciprocity; except for sign it does not matter if we exchange the source and field points

Greens Functions of Free Space

$$\mathcal{L}_{\mathbf{r}'}\phi(\mathbf{r}') = -f(\mathbf{r}')$$

$$\mathcal{L}_{\mathbf{r}'}G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'),$$

$$\int_{\mathcal{V}} \mathcal{L}_{\mathbf{r}'}G(\mathbf{r},\mathbf{r}')\,f(\mathbf{r})\mathrm{d}V = -\int_{\mathcal{V}} \delta(\mathbf{r}-\mathbf{r}')f(\mathbf{r})\mathrm{d}V = -f(\mathbf{r}').$$

$$\mathcal{L}_{\mathbf{r}'}\phi(\mathbf{r}') = \int_{\mathcal{V}} \mathcal{L}_{\mathbf{r}'}G(\mathbf{r},\mathbf{r}')f(\mathbf{r})\mathrm{d}V = \mathcal{L}_{\mathbf{r}'}\int_{\mathcal{V}} G(\mathbf{r},\mathbf{r}')f(\mathbf{r})\mathrm{d}V,$$

$$\phi(\mathbf{r}') = \int_{\mathcal{V}} G(\mathbf{r},\mathbf{r}')f(\mathbf{r})\mathrm{d}V.$$

$$G_2(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi}\ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{r_{\mathrm{ref}}}\right), \qquad G_3(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Biot-Savart's Law

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}\,, \qquad G_3(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathscr{V}} \frac{J_i(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}V'\,,$$

$$\mathbf{A}(\mathbf{r}) = A_x \mathbf{e_x} + A_y \mathbf{e_y} + A_z \mathbf{e_z} = \frac{\mu_0}{4\pi} \int_{\mathscr{V}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}V'.$$

This works only in Cartesian Coordinates

$$\mathbf{B}(\mathbf{r}) = \operatorname{curl} \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathscr{V}} \operatorname{curl} \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) \mathrm{d}V'$$

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathscr{V}} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \sum_{k=1}^{3} J_k(\mathbf{r}')(\mathbf{e}_i(\mathbf{r}) \cdot \mathbf{e}_k(\mathbf{r}'))\, \mathrm{d}V'. \quad \mathrm{d}V'$$

$$= \frac{\mu_0}{4\pi} \int_{\mathscr{V}} \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \mathrm{d}V'.$$

But wait a minute: Are we finished? Are we sure that the divergence of the vector potential is zero as it was required for the Laplace equation?

$$
\begin{aligned}
\operatorname{div}\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int_{\mathscr{V}} \operatorname{div}\left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)\mathrm{d}V' \\
&= \frac{\mu_0}{4\pi}\int_{\mathscr{V}} \left(\mathbf{J}(\mathbf{r}')\cdot\operatorname{grad}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) + \frac{1}{|\mathbf{r}-\mathbf{r}'|}\operatorname{div}\mathbf{J}(\mathbf{r}')\right)\mathrm{d}V' \\
&= \frac{\mu_0}{4\pi}\int_{\mathscr{V}} \mathbf{J}(\mathbf{r}')\cdot\operatorname{grad}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\mathrm{d}V' \\
&= -\frac{\mu_0}{4\pi}\int_{\mathscr{V}} \mathbf{J}(\mathbf{r}')\cdot\operatorname{grad}_{\mathbf{r}'}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\mathrm{d}V' \\
&= -\frac{\mu_0}{4\pi}\int_{\mathscr{V}} \left(\operatorname{div}_{\mathbf{r}'}\left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\operatorname{div}_{\mathbf{r}'}\mathbf{J}(\mathbf{r}')\right)\mathrm{d}V' \\
&= -\frac{\mu_0}{4\pi}\int_{\mathscr{V}} \operatorname{div}_{\mathbf{r}'}\left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)\mathrm{d}V' = -\frac{\mu_0}{4\pi}\int_{\partial\mathscr{V}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\cdot\mathrm{d}a'\,.
\end{aligned}
$$

Current loops must always be closed and must not leave the problem domain

Biot-Savart's Law for Line Currents

$$\mathbf{A}(\mathbf{r}) = A_x\mathbf{e_x} + A_y\mathbf{e_y} + A_z\mathbf{e_z} = \frac{\mu_0}{4\pi}\int_{\mathscr{V}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V'.$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_{\mathscr{C}} \frac{\mathrm{d}\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

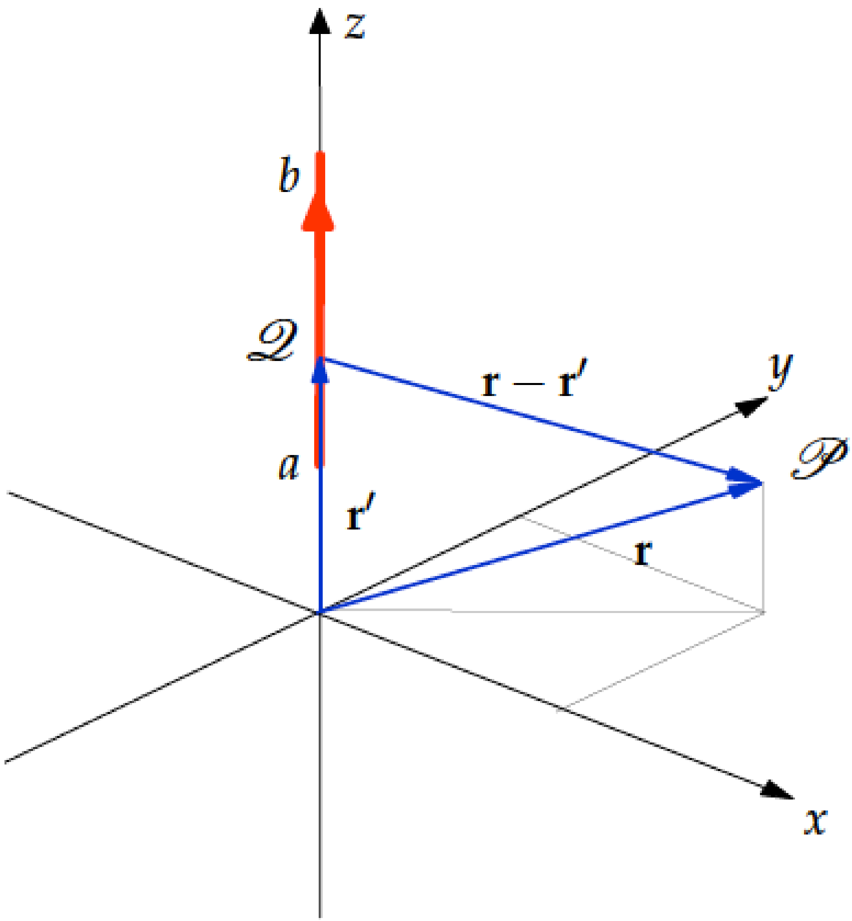
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_{\mathscr{C}} \frac{\mathrm{d}\mathbf{r}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3},$$

Vector Potential of a Line Current

$$A_z(x,y,z) = \frac{\mu_0 I}{4\pi}\int_a^b \frac{\mathrm{d}z_c}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0 I}{4\pi}\int_a^b \frac{\mathrm{d}z_c}{\sqrt{x^2+y^2+(z-z_c)^2}}$$

$$\frac{-\mu_0 I}{4\pi}\ln\left((z-z_c)+\sqrt{x^2+y^2+(z-z_c)^2}\right)\Big|_a^b$$

$$\frac{\mu_0 I}{4\pi}\ln\frac{z-a+\sqrt{x^2+y^2+(z-a)^2}}{z-b+\sqrt{x^2+y^2+(z-b)^2}}\,.$$



Field of a Line Current (Infinitely Long)

$$\lim_{a,b\to\pm\infty} \ln \frac{z-a+\sqrt{x^2+y^2+(z-a)^2}}{z-b+\sqrt{x^2+y^2+(z-b)^2}} = \lim_{a,b\to\pm\infty} \ln \frac{-a+|a|\sqrt{1+\frac{x^2+y^2}{a^2}}}{-b+|b|\sqrt{1+\frac{x^2+y^2}{b^2}}}$$

$$= \lim_{a,b\to\pm\infty} \ln \frac{-a-a(1+\frac{x^2+y^2}{2a^2}+\cdots)}{-b+b(1+\frac{x^2+y^2}{2b^2}+\cdots)} = \lim_{a,b\to\pm\infty} \ln \frac{-2a}{-b+b+\frac{x^2+y^2}{2b}}$$

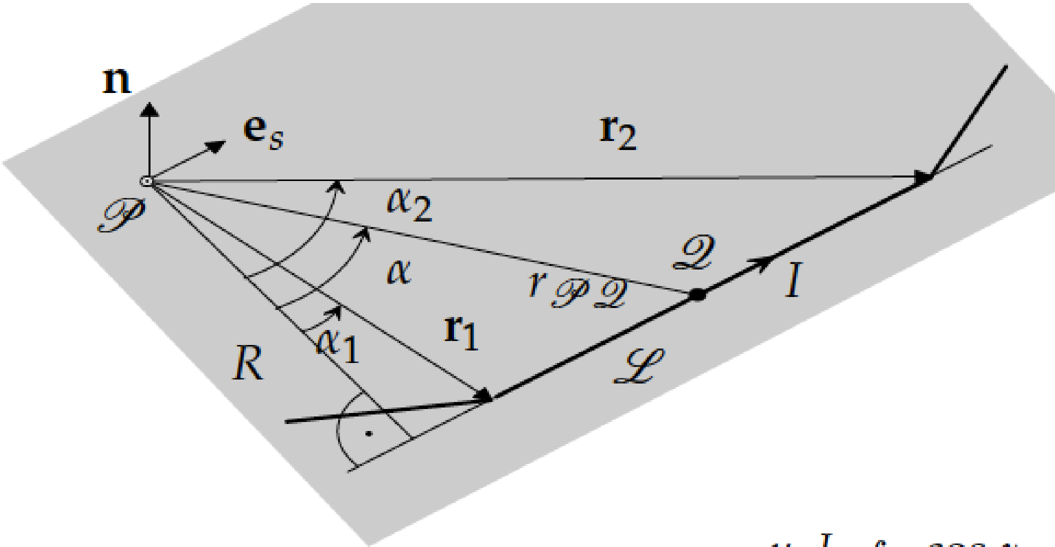
$$= \lim_{a,b\to\pm\infty} \ln \frac{-4ab}{x^2+y^2} \,.$$

$$A_z(x,y) = \lim_{a,b\to\pm\infty} \frac{\mu_0 I}{4\pi} \ln\left(\frac{-4ab}{x_0^2+y_0^2}\right) - \frac{\mu_0 I}{4\pi} \ln\left(\frac{x^2+y^2}{x_0^2+y_0^2}\right) .$$

Arbitrarily large but constant

$$\mathbf{A}(x,y) = -\frac{\mu_0 I}{4\pi} \ln\left(\frac{x^2+y^2}{x_0^2+y_0^2}\right) \mathbf{e}_z = -\frac{\mu_0 I}{2\pi} \ln\left(\frac{r}{r_{\text{ref}}}\right) \mathbf{e}_z \,,$$

Field of a Line Current Segment



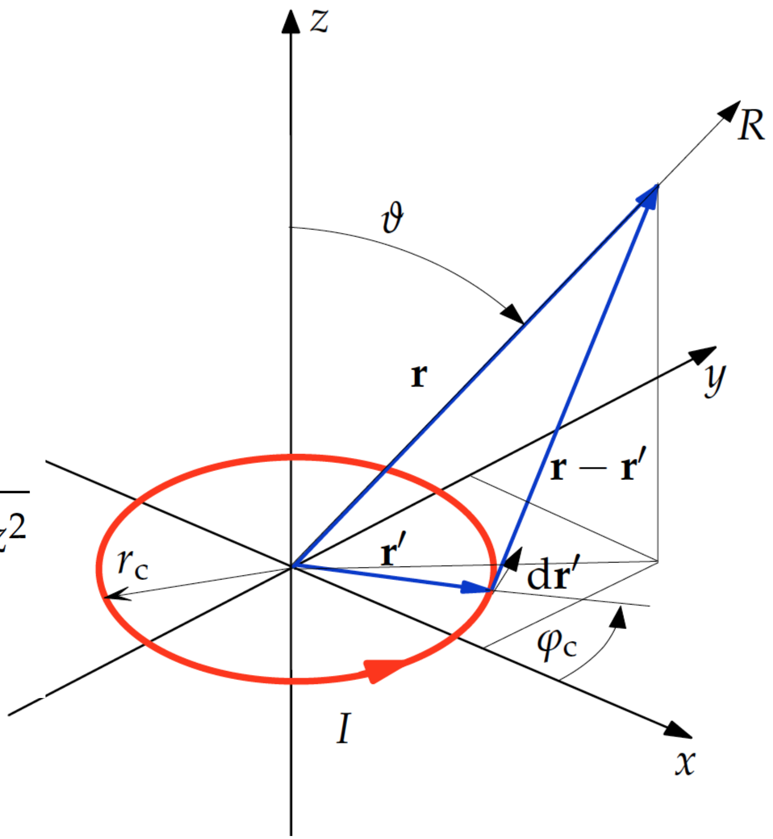
$$
\begin{aligned}
\mathbf{B}(\mathscr{P}) &= \frac{\mu_0 I}{4\pi}\int_{\mathscr{L}}\frac{\cos\alpha}{r^2_{\mathscr{P}\mathscr{Q}}}\mathrm{d}\mathbf{r}' = \frac{\mu_0 I}{4\pi R}\mathbf{n}\int_{\alpha_1}^{\alpha_2}\cos\alpha\,\mathrm{d}\alpha = \frac{\mu_0 I}{4\pi R}\left(\sin\alpha_2 - \sin\alpha_1\right)\mathbf{n} \\
&= \frac{\mu_0 I}{4\pi}\,\frac{\cos\alpha_2+\cos\alpha_1}{R}\,\frac{\sin\alpha_2-\sin\alpha_1}{\cos\alpha_2+\cos\alpha_1}\,\mathbf{n} \\
&= \frac{\mu_0 I}{4\pi}\left(\frac{1}{|\mathbf{r}_1|}+\frac{1}{|\mathbf{r}_2|}\right)\frac{\sin(\alpha_2-\alpha_1)}{1+\cos(\alpha_2-\alpha_1)}\,\mathbf{n} \\
&= \frac{\mu_0 I}{4\pi}\left(\frac{1}{|\mathbf{r}_1|}+\frac{1}{|\mathbf{r}_2|}\right)\frac{\sin(\alpha_2-\alpha_1)}{1+\frac{\mathbf{r}_1\cdot\mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|}}\,\frac{\mathbf{r}_1\times\mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|\sin(\alpha_2-\alpha_1)} \\
&= \frac{\mu_0 I}{4\pi}\,\frac{|\mathbf{r}_1|+|\mathbf{r}_2|}{|\mathbf{r}_1|\,|\mathbf{r}_2|+\mathbf{r}_1\cdot\mathbf{r}_2}\,\frac{\mathbf{r}_1\times\mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|}\,,
\end{aligned}
$$

Field of a Ring Current

$$\mathbf{r}' = \cos\varphi_c r_c\,\mathbf{e}_x + \sin\varphi_c r_c\,\mathbf{e}_y$$

$$\mathrm{d}\mathbf{r}' = -\sin\varphi_c r_c\,\mathrm{d}\varphi_c\mathbf{e}_x + \cos\varphi_c r_c\,\mathrm{d}\varphi_c\mathbf{e}_y$$

$$\begin{aligned}|\mathbf{r}-\mathbf{r}'| &= \sqrt{(x-x_c)^2+(y-y_c)^2+z^2}\\ &= \sqrt{(r\cos\varphi - r_c\cos\varphi_c)^2 + (r\sin\varphi - r_c\sin\varphi_c)^2 + z^2}\\ &= \sqrt{r^2+r_c^2+z^2-2rr_c\cos\varphi_c}\,,\end{aligned}$$



$$A_y(r,z) = \frac{\mu_0 I r_c}{2\pi}\int_0^{\pi}\frac{\cos\varphi_c\mathrm{d}\varphi_c}{\sqrt{r^2+r_c^2+z^2-2rr_c\cos\varphi_c}}$$

Field of a Ring Current

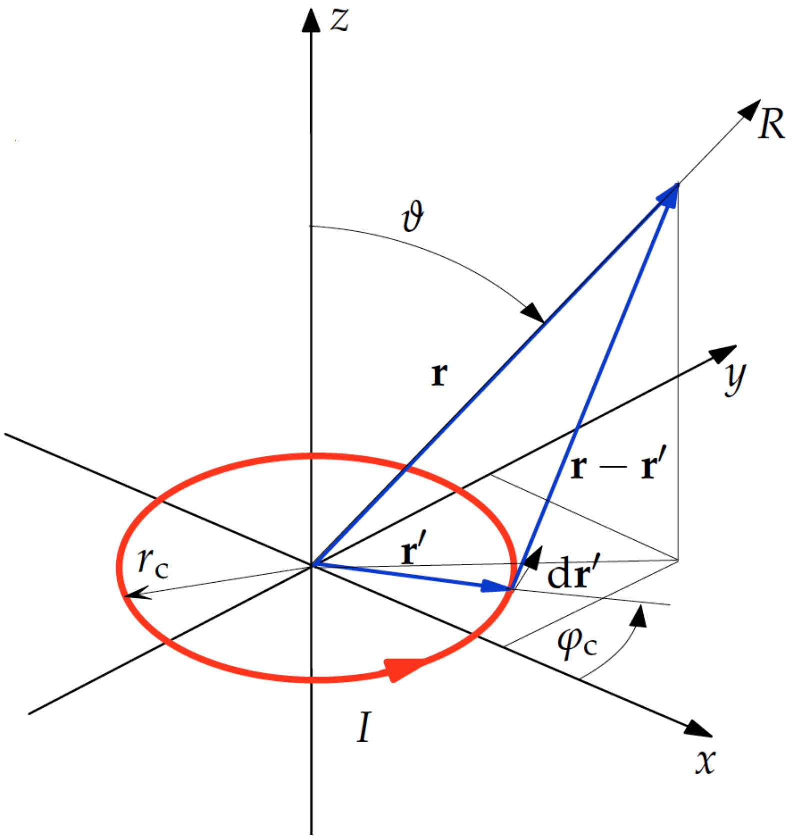
$$A_y(r,z) = \frac{\mu_0 I r_c}{2\pi} \int_0^{\pi} \frac{\cos\varphi_c \, d\varphi_c}{\sqrt{r^2 + r_c^2 + z^2 - 2 r r_c \cos\varphi_c}}$$

$$\psi := (\pi + \varphi_c)/2, \qquad k^2 := \frac{4 r r_c}{(r + r_c)^2 + z^2}$$

$$A_\varphi(r,z) = \frac{\mu_0 I r_c}{\pi \sqrt{(r + r_c)^2 + z^2}} \int_0^{\pi/2} \frac{2\sin^2\psi - 1}{\sqrt{1 - k^2 \sin^2\psi}} \, d\psi$$

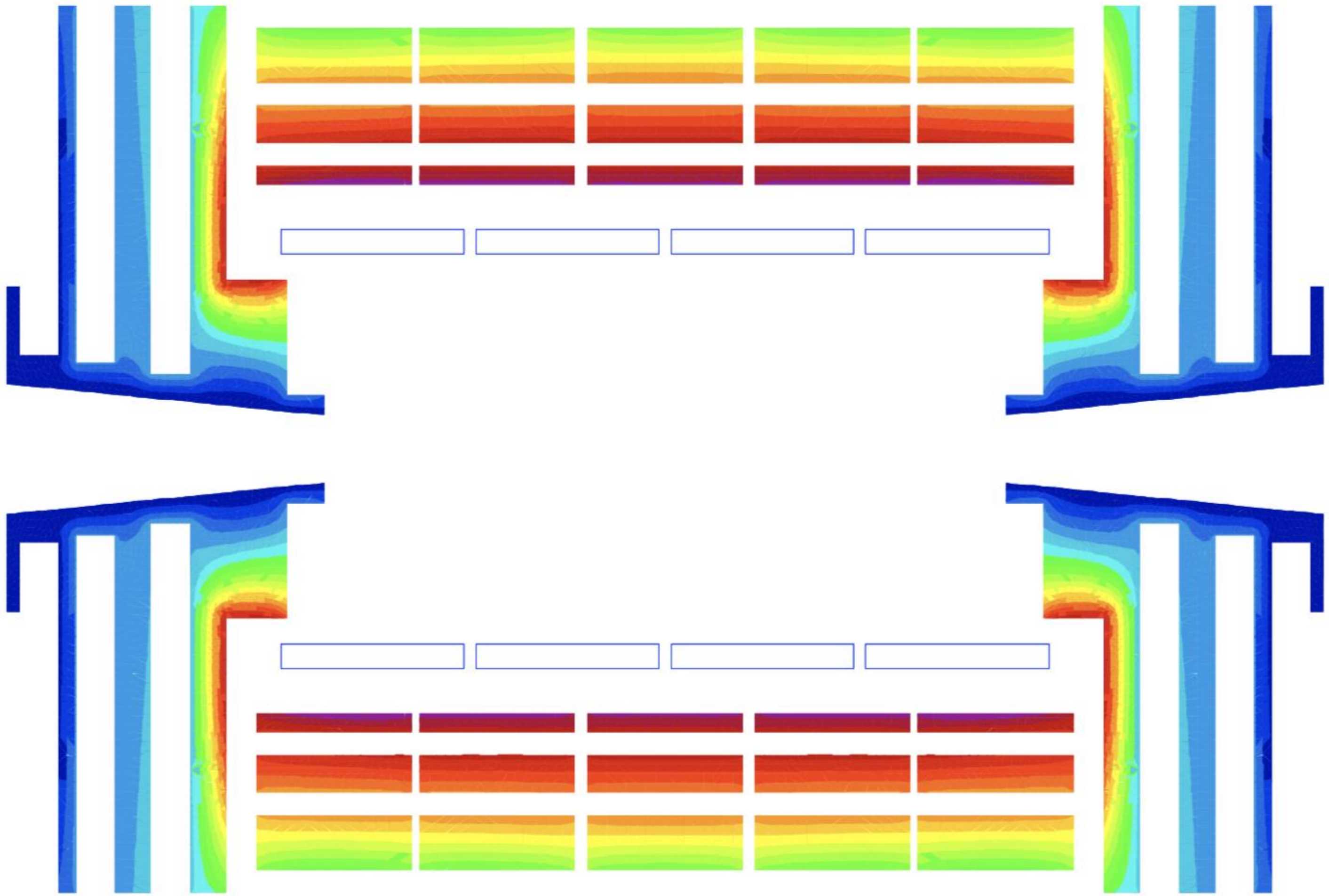
$$K\left(\frac{\pi}{2}, k\right) := \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - k^2 \sin^2\psi}},$$

$$E\left(\frac{\pi}{2}, k\right) := \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\psi} \, d\psi,$$



$$A_\varphi(r,z) = \frac{\mu_0 I}{2\pi r} \sqrt{(r + r_c)^2 + z^2} \left[\left(1 - \frac{k^2}{2}\right) K\left(\frac{\pi}{2}, k\right) - E\left(\frac{\pi}{2}, k\right) \right]$$

Field in the Return Yoke of CMS

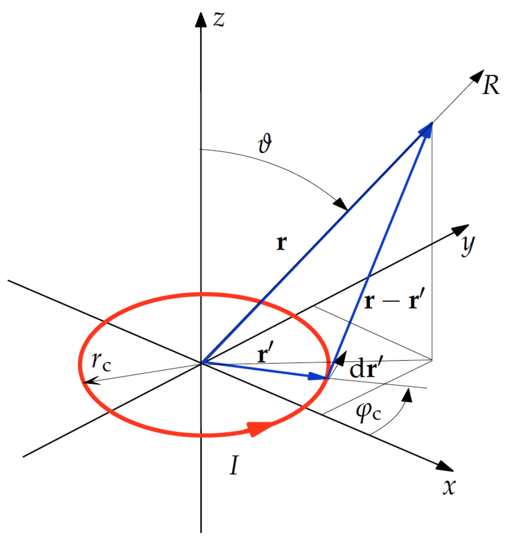


Field of a Ring Current (Helmholtz and Maxwell Coils)

On axis: $: k \ll 1$

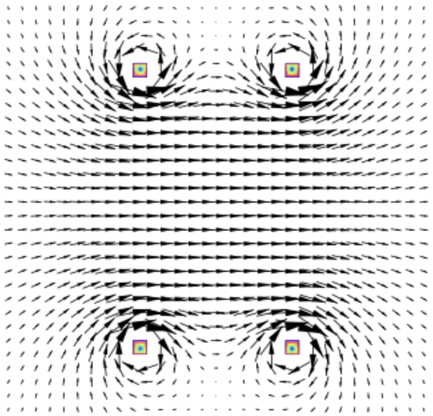
$$A_\varphi(r,z) = \frac{\mu_0 I r_c^2}{4} \frac{r}{(r_c^2 + z^2)^{\frac{3}{2}}},$$

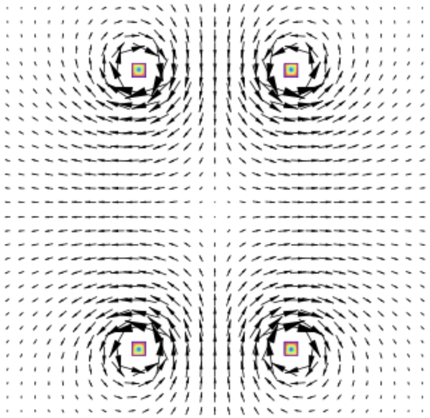
$$B_z(z) = \frac{\mu_0 I}{2} \frac{r_c^2}{(r_c^2 + z^2)^{\frac{3}{2}}}.$$



In the center:

$$B_z(z=0) = \frac{\mu_0 I}{2 r_c}.$$





$$B_z = \frac{\mu_0 I}{2} \left(\frac{r_c^2}{\sqrt{r_c^2 + (z+z_c)^2}^3} + \frac{r_c^2}{\sqrt{r_c^2 + (z-z_c)^2}^3} \right)$$

$$\frac{\mathrm{d}B_z}{\mathrm{d}z} = \frac{-3\mu_0 I r_c^2}{2} \left(\frac{z+z_c}{\sqrt{r_c^2 + (z+z_c)^2}^5} + \frac{z-z_c}{\sqrt{r_c^2 + (z-z_c)^2}^5} \right)$$

$$\frac{\mathrm{d}^2 B_z}{\mathrm{d}z^2} = \frac{-3\mu_0 I r_c^2}{2} \left(\frac{r_c^2 - 4(z+z_c)^2}{\sqrt{r_c^2 + (z+z_c)^2}^7} + \frac{r_c^2 - 4(z-z_c)^2}{\sqrt{r_c^2 + (z-z_c)^2}^7} \right)$$

Magnetic Dipole Moment

Far field approximation

$$m := I r_\mathrm{c}^2 \pi$$

$$A_\varphi(R,\vartheta) \approx \frac{\mu_0 I r_\mathrm{c}^2 \pi}{4\pi} \frac{\sin\vartheta}{R^2} = \frac{\mu_0 m}{4\pi} \frac{\sin\vartheta}{R^2}\,, \qquad [m] = 1\,\mathrm{A\,m^2}.$$

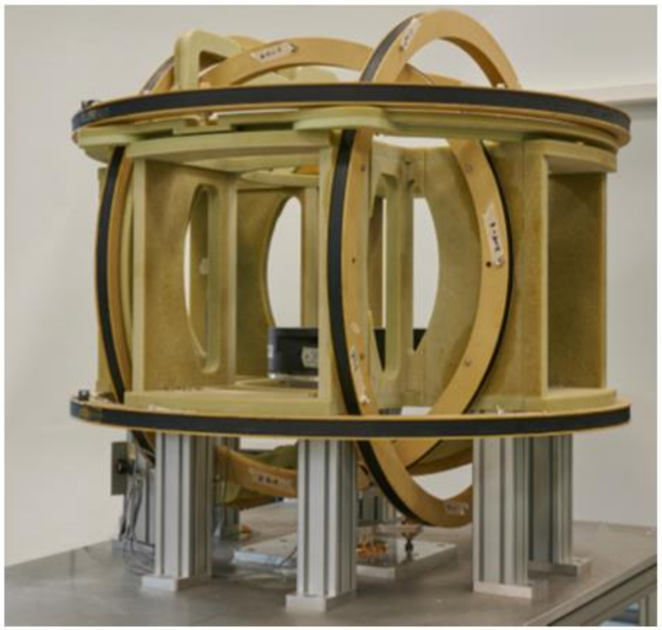
$$R = \sqrt{r^2 + z^2} \text{ and } \sin\vartheta = r/R,$$

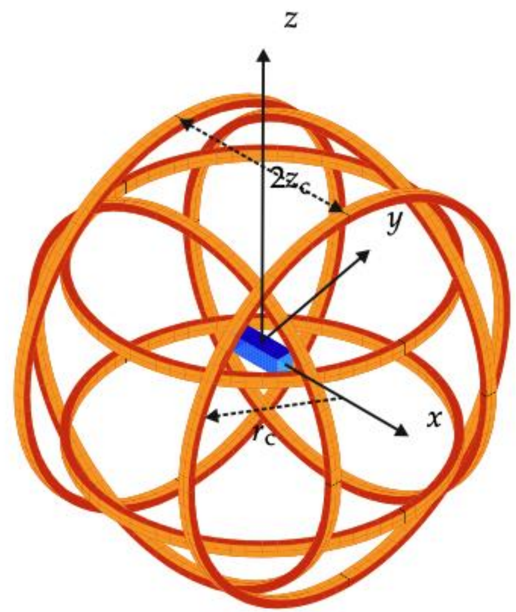
$$\mathbf{m} = I\mathbf{a}\,,$$

$$\mathbf{m} = \frac{I}{2}\int_{\mathscr{C}} \mathbf{r} \times \mathrm{d}\mathbf{r}\,,$$

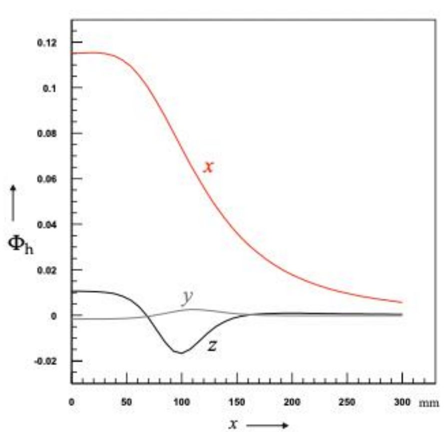
$$\mathbf{M}(\mathbf{r}) := \frac{\mathrm{d}\mathbf{m}}{\mathrm{d}V} = \frac{1}{2}\mathbf{r} \times \mathbf{J}(\mathbf{r}),$$

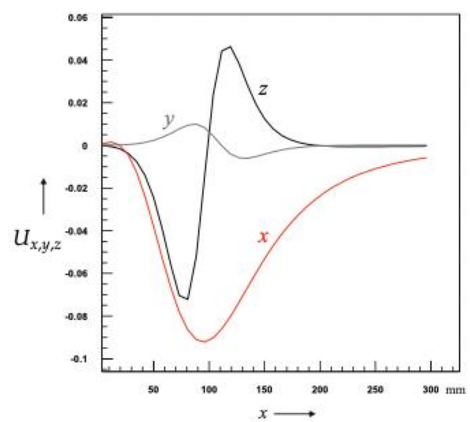
Measuring the Magnetization of PM Block in a Helmholtz Coil





$$\mathbf{m} = I_{\mathrm{p}}\,\mathbf{a}.$$





$$L_{\mathrm{ph}} = \frac{\Phi_{\mathrm{p}}(I_{\mathrm{h}})}{I_{\mathrm{h}}} = \frac{\Phi_{\mathrm{h}}(I_{\mathrm{p}})}{I_{\mathrm{p}}} = L_{\mathrm{hp}}$$

$$\Phi_{\mathrm{h}} = \frac{I_{\mathrm{p}}}{I_{\mathrm{h}}}\Phi_{\mathrm{p}}(I_{\mathrm{h}}) = \frac{I_{\mathrm{p}}}{I_{\mathrm{h}}}\mathbf{B}(I_{\mathrm{h}}) \cdot \mathbf{a} = \frac{\mathbf{m} \cdot \mathbf{B}(I_{\mathrm{h}})}{I_{\mathrm{h}}}$$

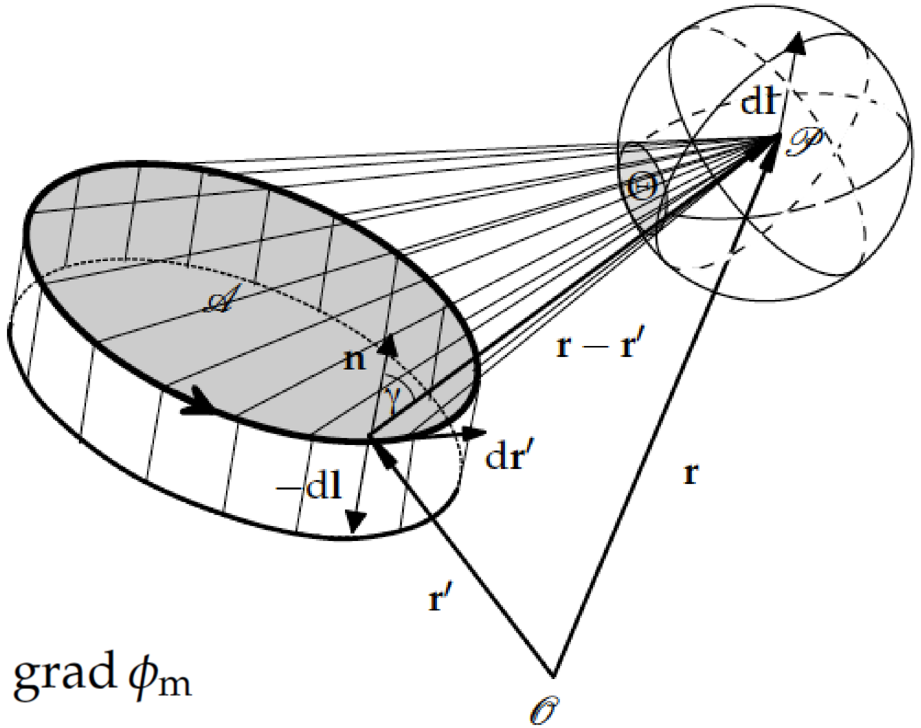
The Solid Angle and Magnetic Scalar Potential

$$\mathrm{d}\Theta = -\int_{\partial\mathscr{A}} \frac{1}{|\mathbf{r}-\mathbf{r}'|^2}(\mathrm{d}\mathbf{l}\times\mathrm{d}\mathbf{r}')\cdot\mathbf{e}_R = -\int_{\partial\mathscr{A}} \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\cdot(\mathrm{d}\mathbf{l}\times\mathrm{d}\mathbf{r}')$$

$$= -\mathrm{d}\mathbf{l}\int_{\partial\mathscr{A}} \frac{\mathrm{d}\mathbf{r}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}.$$

Expressing $\mathrm{d}\Theta$ as $\operatorname{grad}\Theta\cdot\mathrm{d}\mathbf{l}$

$$\operatorname{grad}\Theta = -\int_{\partial\mathscr{A}} \frac{\mathrm{d}\mathbf{r}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

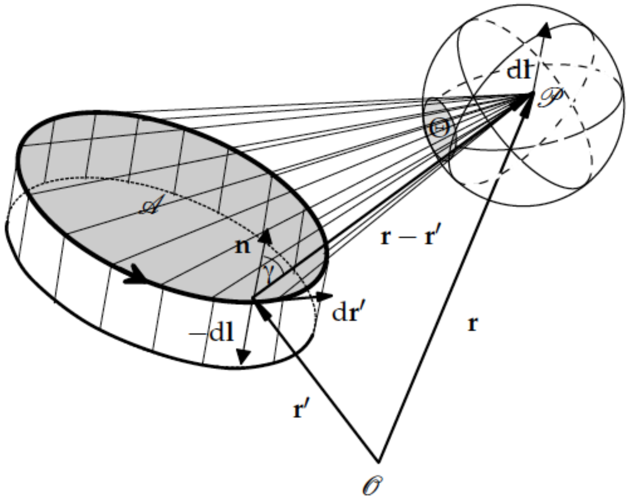


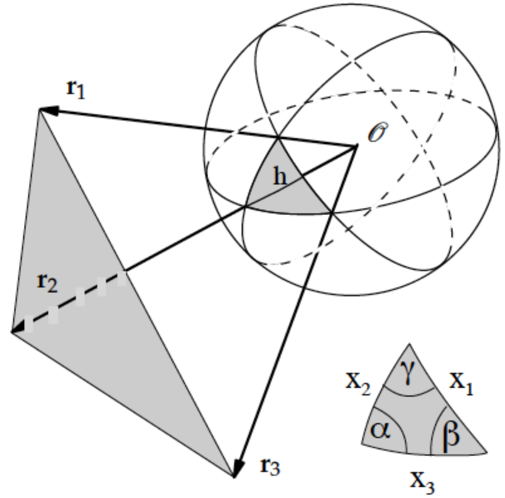
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_{\partial\mathscr{A}_c} \frac{\mathrm{d}\mathbf{r}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \mu_0\mathbf{H} = -\mu_0\operatorname{grad}\phi_m$$

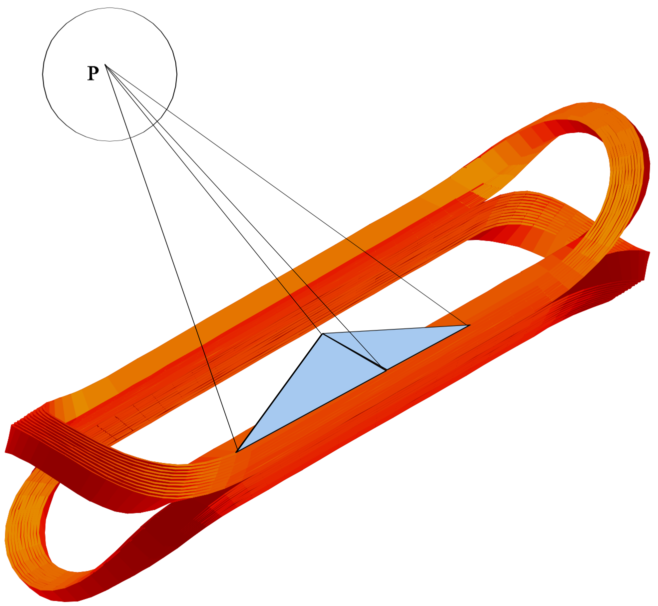
$$\phi_m(\mathbf{r}) = \frac{I}{4\pi}\Theta$$

Solid angle (easy to compute) yields the magnetic scalar potential of a current loop

The Solid Angle and Magnetic Scalar Potential





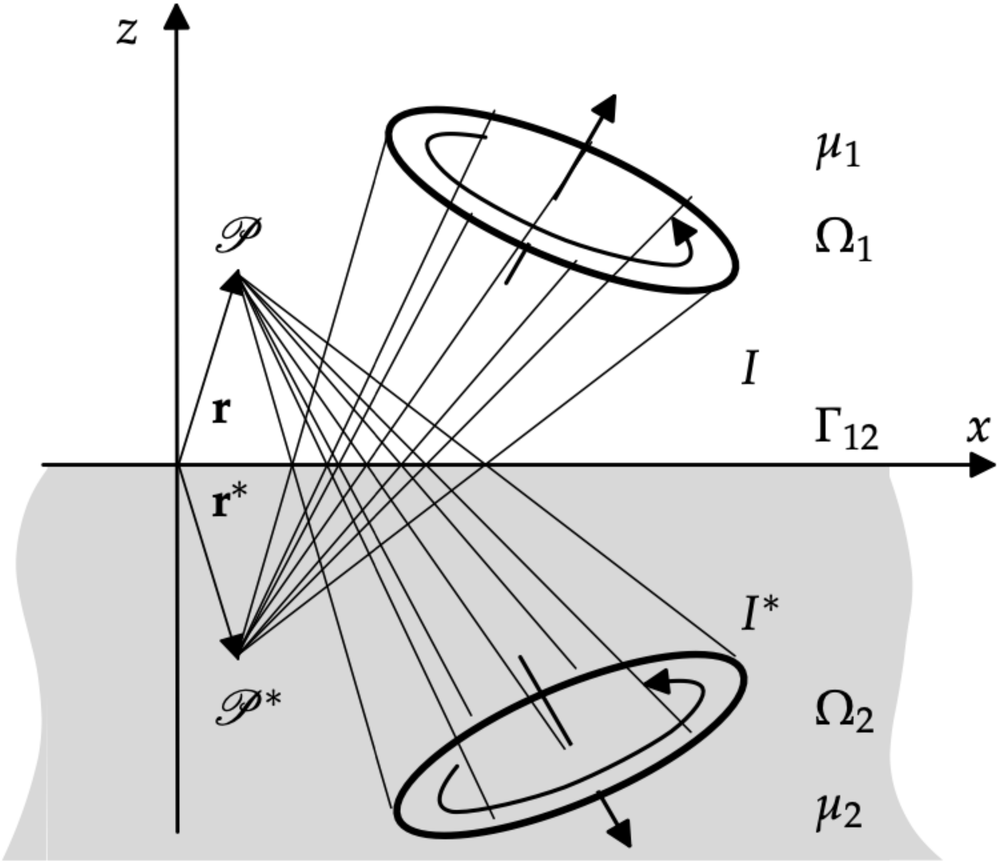


Euler Theorem
(spherical excess)

$$\Theta = \int_{\mathscr{A}} \frac{\cos\gamma}{R^2} \mathrm{d}a = \int_{\mathscr{A}} \frac{(\mathbf{r}-\mathbf{r}') \cdot \mathbf{n}}{|\mathbf{r}-\mathbf{r}'|^3} \mathrm{d}a\,,$$

$$\tan\left(\frac{\Theta}{2}\right) = \frac{\mathbf{r}_1 \cdot (\mathbf{r}_2 \times \mathbf{r}_3)}{r_1 r_2 r_3 + (\mathbf{r}_1 \cdot \mathbf{r}_2) r_3 + (\mathbf{r}_1 \cdot \mathbf{r}_3) r_2 + (\mathbf{r}_2 \cdot \mathbf{r}_3) r_1}\,.$$

Elegant Proof of the Imaging Current Method



Magnetic Energy

$$W = \frac{1}{2}\int_V \mathbf{H}\cdot\mathbf{B}\,\mathrm{d}V$$

Linear !

$$\mathrm{div}\,(\mathbf{A}\times\mathbf{H}) = \mathbf{H}\cdot\,\mathrm{curl}\,\mathbf{A} - \mathbf{A}\cdot(\,\mathrm{curl}\,\mathbf{H})$$

$$\begin{aligned} W &= \frac{1}{2}\int_V \mathbf{H}\cdot\,\mathrm{curl}\,\mathbf{A}\,\mathrm{d}V \\ &= \frac{1}{2}\int_V \mathrm{div}\,(\mathbf{A}\times\mathbf{H})\,\mathrm{d}V + \frac{1}{2}\int_V \mathbf{A}\cdot\,\mathrm{curl}\,\mathbf{H}\,\mathrm{d}V \\ &= \frac{1}{2}\int_{\partial V}(\mathbf{A}\times\mathbf{H})\cdot\mathrm{d}\mathbf{a} + \frac{1}{2}\int_V \mathbf{A}\cdot\,\mathrm{curl}\,\mathbf{H}\,\mathrm{d}V \end{aligned}$$

$$W = \frac{1}{2}\int_V \mathbf{A}\cdot\,\mathrm{curl}\,\mathbf{H}\,\mathrm{d}V = \frac{1}{2}\int_V \mathbf{A}\cdot\mathbf{J}\,\mathrm{d}V$$

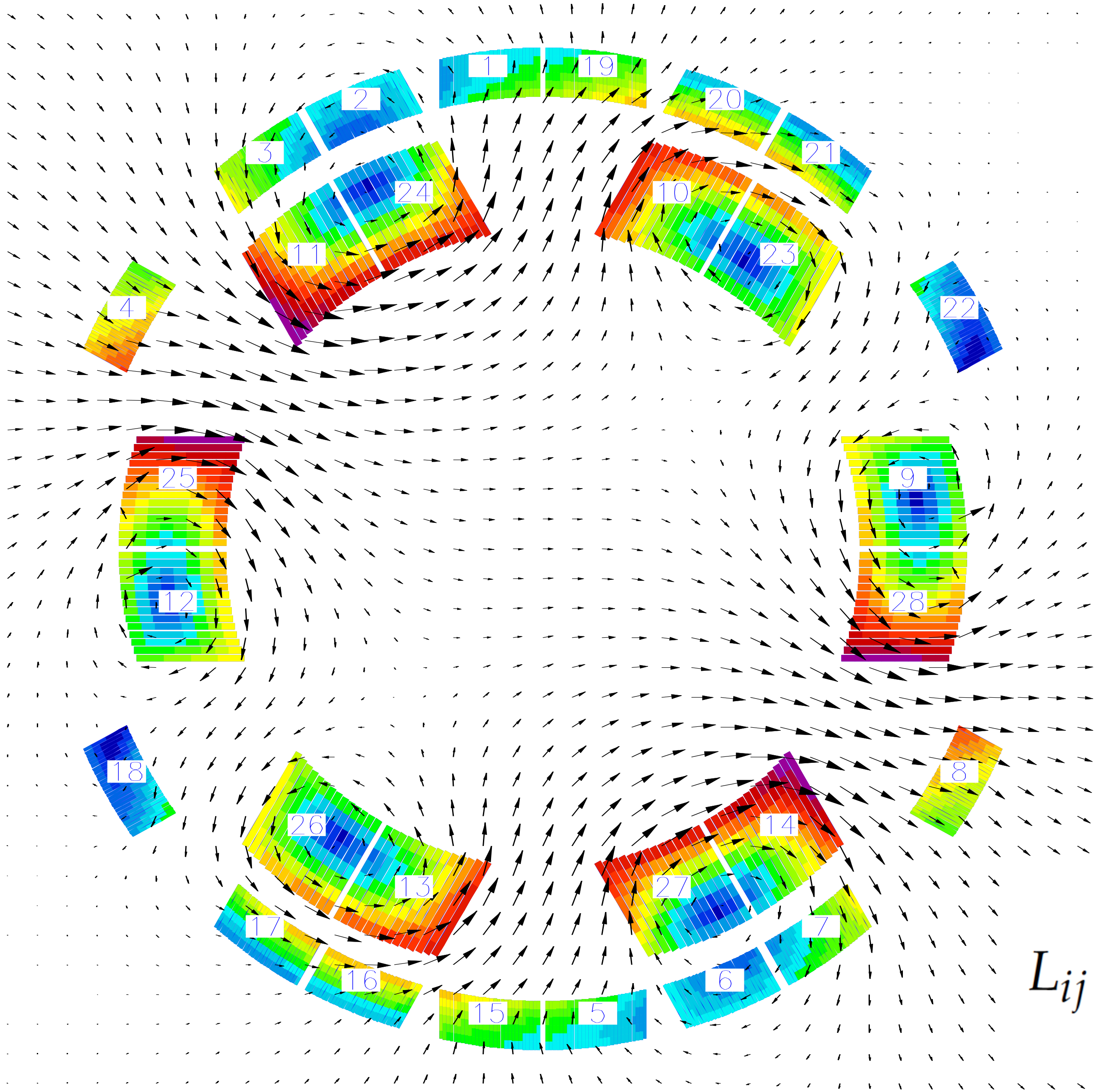
Inductance

$$W = \frac{\mu_0}{8\pi} \int_V \int_{V'} \frac{\mathbf{J}(\mathbf{r}) \cdot \mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}V' \mathrm{d}V$$

$$\begin{aligned} W = \sum_{i=1}^{n} \sum_{j=1}^{n} W_{ij} &= \frac{\mu_0}{8\pi} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_V \int_{V'} \frac{\mathbf{J}_i(\mathbf{r}) \cdot \mathbf{J}_j(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}V' \mathrm{d}V \\ &= \frac{\mu_0}{8\pi} \sum_{i=1}^{n} \sum_{j=1}^{n} I_i I_j \int_V \int_{V'} \frac{\mathbf{J}_i(\mathbf{r}) \cdot \mathbf{J}_j(\mathbf{r}')}{I_i I_j |\mathbf{r} - \mathbf{r}'|} \mathrm{d}V' \mathrm{d}V \end{aligned}$$

$$L_{ij} = \frac{\mu_0}{4\pi I_i I_j} \int_V \int_{V'} \frac{\mathbf{J}_i(\mathbf{r}) \cdot \mathbf{J}_j(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}V' \mathrm{d}V$$

Combined Dipole Sextupole Corrector



$$L_{ii} = \frac{2W_{ii}}{I^2}$$

$$L_{ij} = \frac{1}{2}\left(\frac{2W_{ij}}{I^2} - L_{ii} - L_{jj}\right)$$

Mutual Inductance Matrix

Coil	1	2	3	4	5	6	7	8
1	12.601	6.517	-0.245	0.252	0.478	-0.478	-0.252	0.245
2	6.517	12.601	-0.478	-0.252	0.245	-0.245	0.252	0.478
3	-0.245	-0.478	0.136	0.027	-0.010	0.009	-0.010	0.027
4	0.252	-0.252	0.027	0.136	0.027	-0.010	0.009	-0.010
5	0.478	0.245	-0.010	0.027	0.136	0.027	-0.010	0.009
6	-0.478	-0.245	0.009	-0.010	0.027	0.136	0.027	-0.010
7	-0.252	0.252	-0.010	0.009	-0.010	0.027	0.136	0.027
8	0.245	0.478	0.027	-0.010	0.009	-0.010	0.027	0.136

A coil of multipole order N does not couple into one of order K

Nonlinear Circuits (Differential Inductance)

$$U(t) = \frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\mathrm{d}(L\,I)}{\mathrm{d}t} = L\frac{\mathrm{d}I}{\mathrm{d}t} + I\frac{\mathrm{d}L}{\mathrm{d}t}$$

$$U = L^{\mathrm{d}}\,\frac{\mathrm{d}I}{\mathrm{d}t}$$

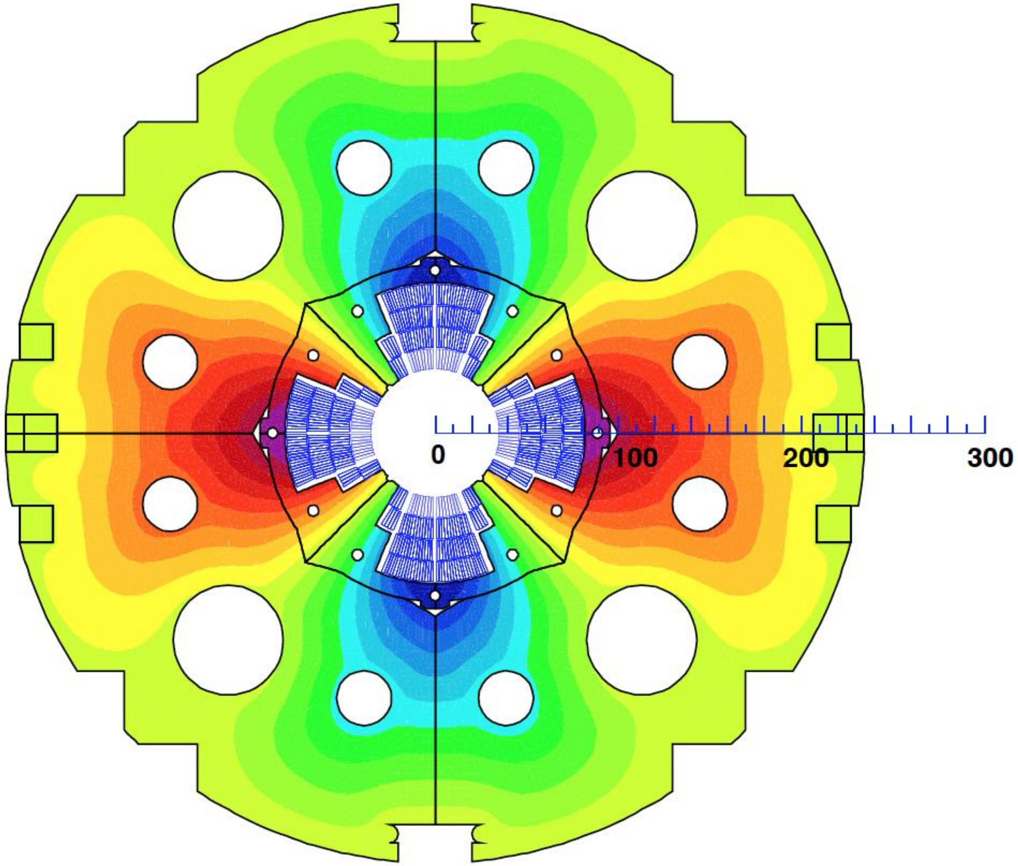
$$\mathrm{d}L = \frac{\partial L}{\partial I}\mathrm{d}I + \frac{\partial L}{\partial t}\mathrm{d}t$$

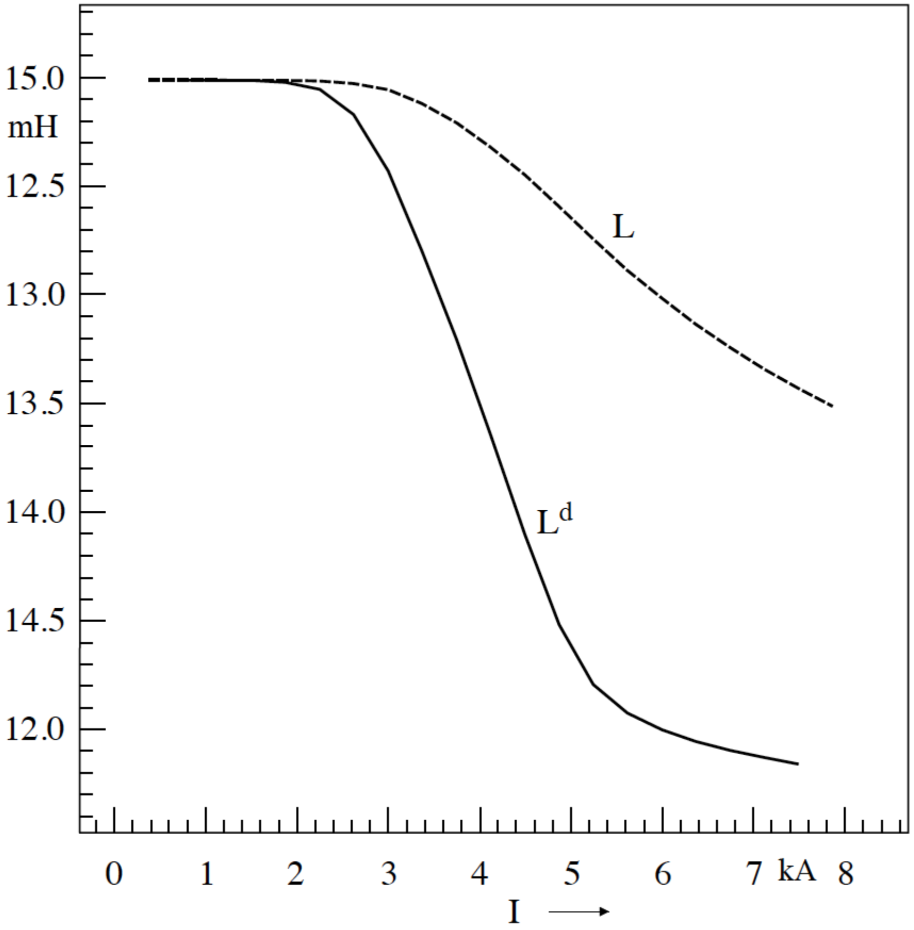
$$U(t) = \left(\frac{\partial L}{\partial I}\,I + L\right)\frac{\mathrm{d}I}{\mathrm{d}t} + I\frac{\partial L}{\partial t}$$

$$L^{\mathrm{d}} = L + I\frac{\partial L}{\partial I} = \frac{\mathrm{d}\Phi}{\mathrm{d}I}$$

For example,
machine rotor motion

Differential Inductance for the MQXY





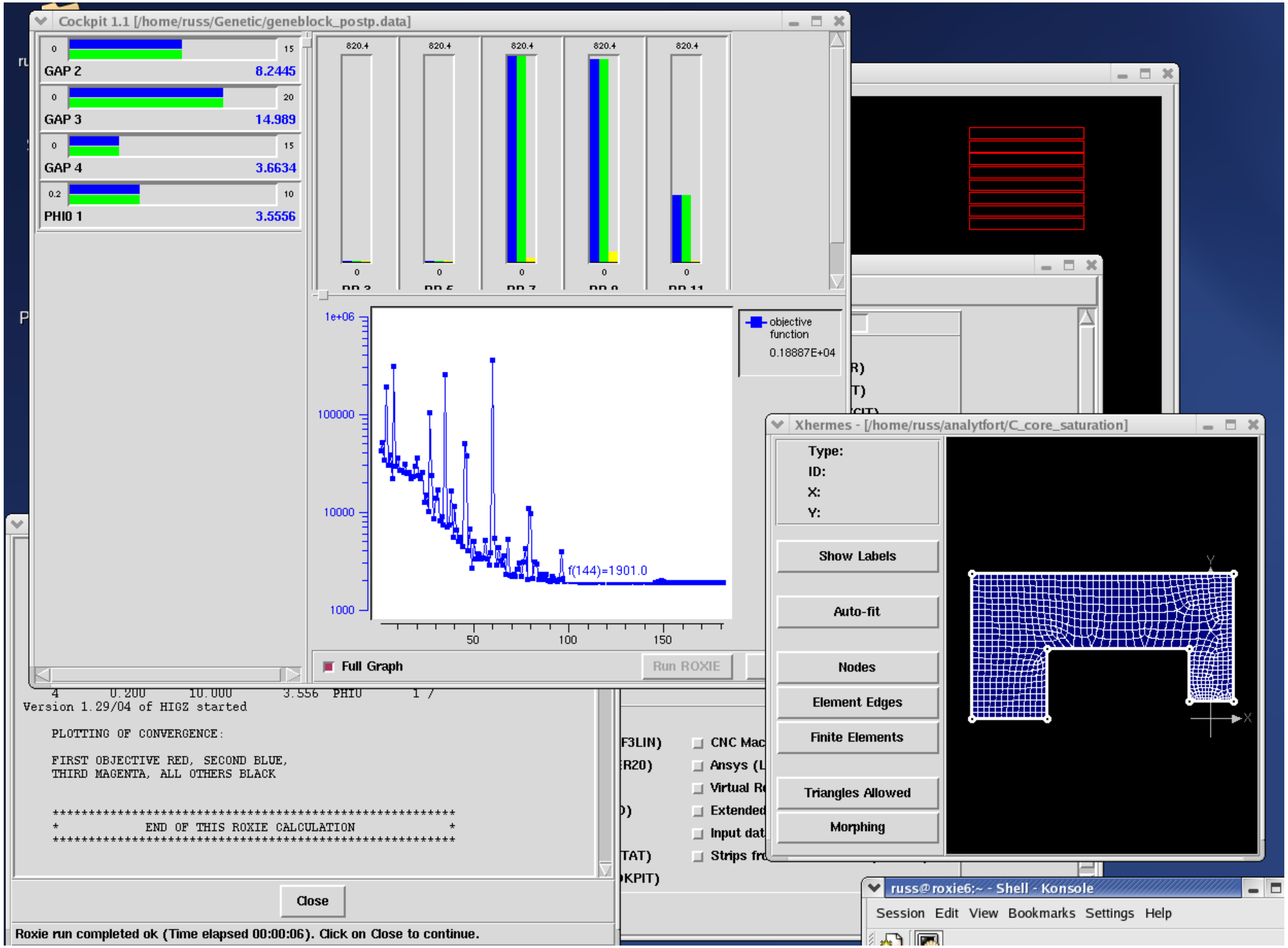
Nonlinear Circuits

$$W = \int_0^t U\, I\, \mathrm{d}\tau = \int_0^t \frac{\mathrm{d}\Phi(I(\tau))}{\mathrm{d}\tau} I(\tau)\, \mathrm{d}\tau = \int_0^t \frac{\mathrm{d}\Phi(I(\tau))}{\mathrm{d}I} \frac{\mathrm{d}I(\tau)}{\mathrm{d}\tau} I(\tau)\, \mathrm{d}\tau$$
$$= \int_0^{I(t)} \frac{\mathrm{d}\Phi}{\mathrm{d}I}\, I\, \mathrm{d}I = \int_0^{I(t)} L^{\mathrm{d}}\, I\, \mathrm{d}I\,,$$

$$W = \frac{1}{2} L^{\mathrm{W}} I^2 \qquad L^{\mathrm{W}} = \frac{2}{I(t)^2} \int_0^{I(t)} L^{\mathrm{d}}\, I\, \mathrm{d}I$$

No hyseresis !

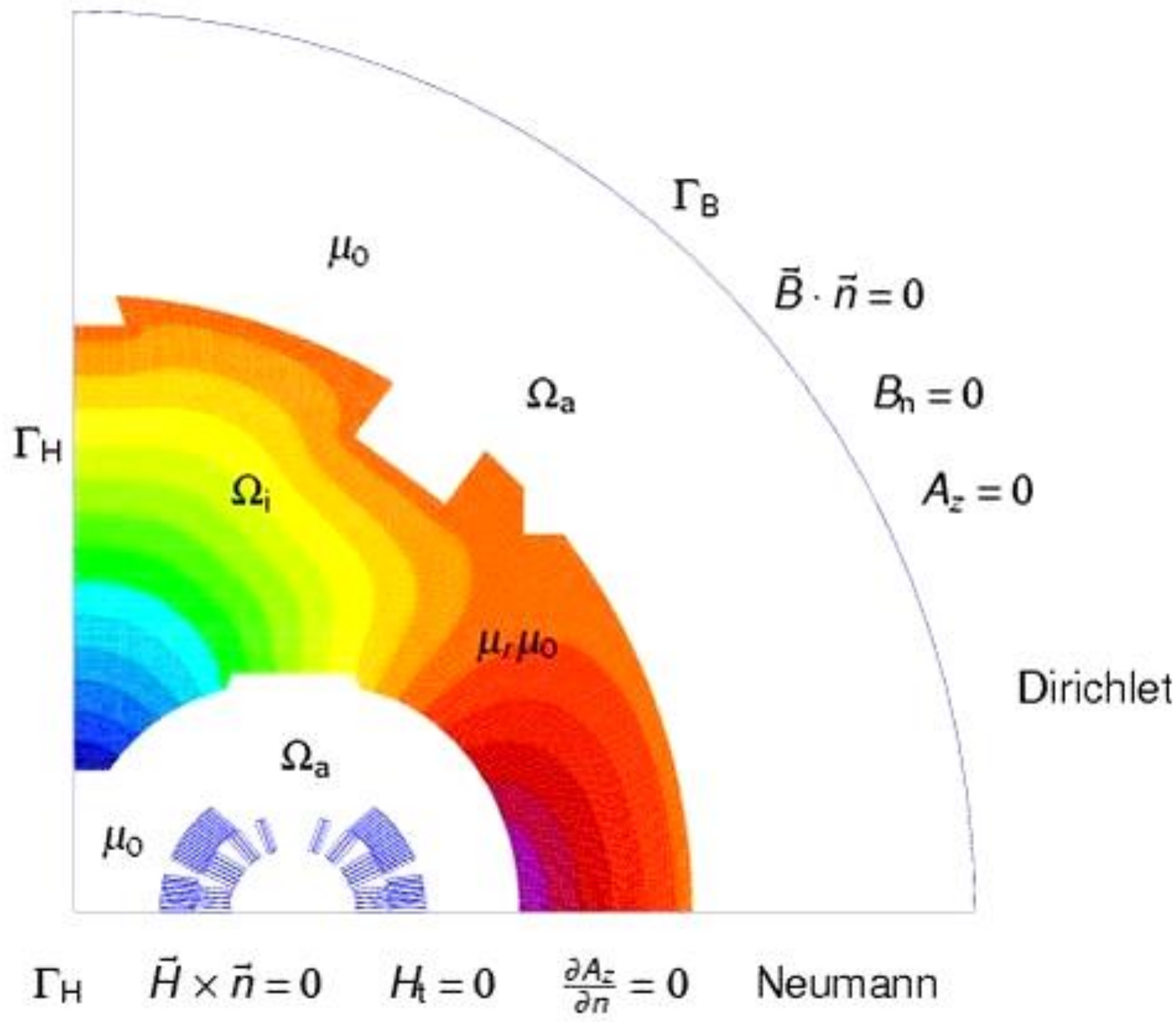
The CERN Field Computation Program ROXIE



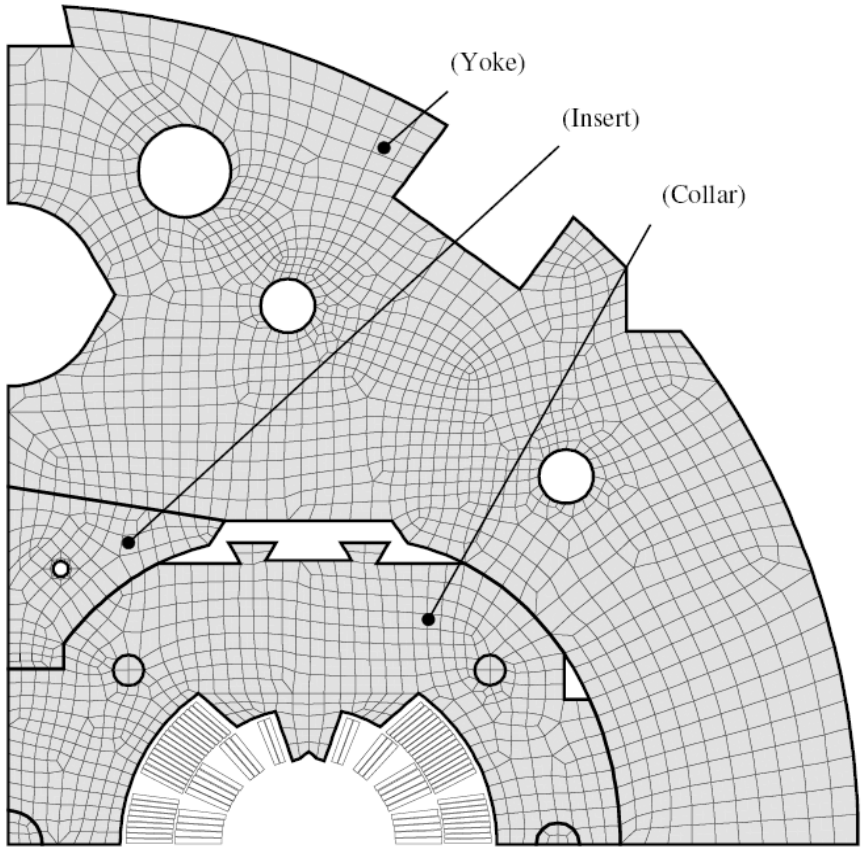
Objectives for the ROXIE Development

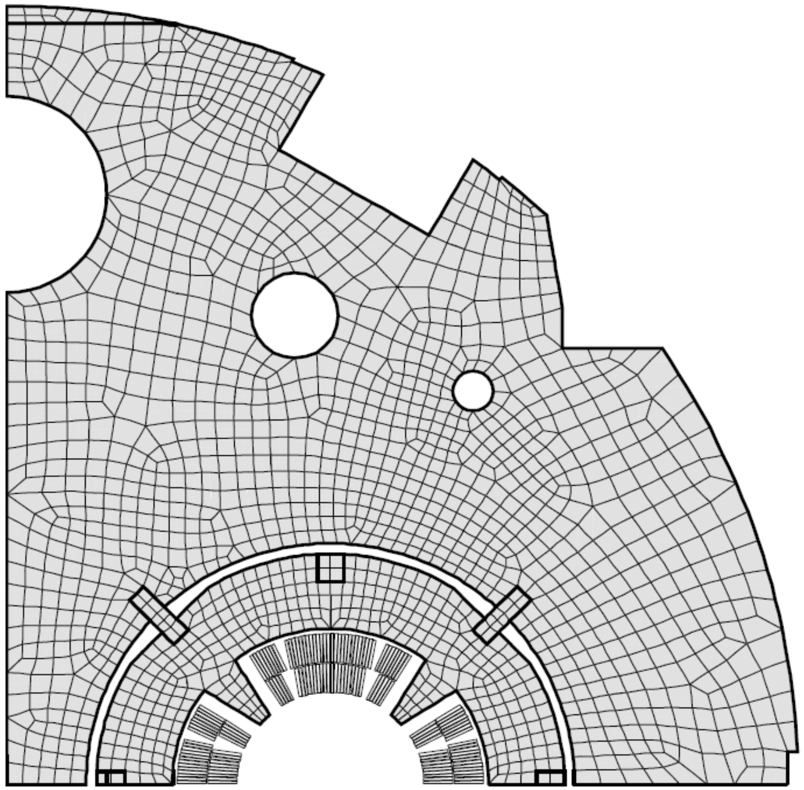
- ➔ Automatic generation of coil and yoke geometries
  - Features: Layers, coil-blocks, conductors, strands, holes, keys
- ➔ Field computation specially suited for magnet design (Ar, BEM-FEM)
  - No meshing of the coil
  - No artificial boundary conditions
  - Higher order quadrilateral meshes, Parametric mesh generator
  - Modeling of superconductor magnetization
- ➔ Mathematical optimization techniques
  - Genetic optimization, Pareto optimization, Search algorithms
- ➔ CAD/CAM interfaces
  - Drawings, End-spacer design and manufacture

The Problem Domain



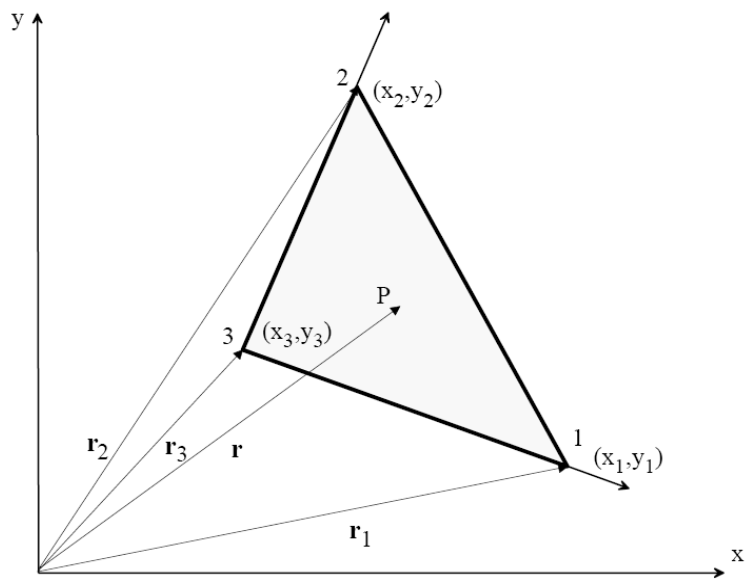
Examples for FEM Meshes





Shape Functions

$$A_j(\mathbf{x}) = \alpha_1 + \alpha_2 x + \alpha_3 y, \qquad \mathbf{x} \in \Omega_j \qquad A_j(\mathbf{x}) = A_{z_j}(x, y)$$



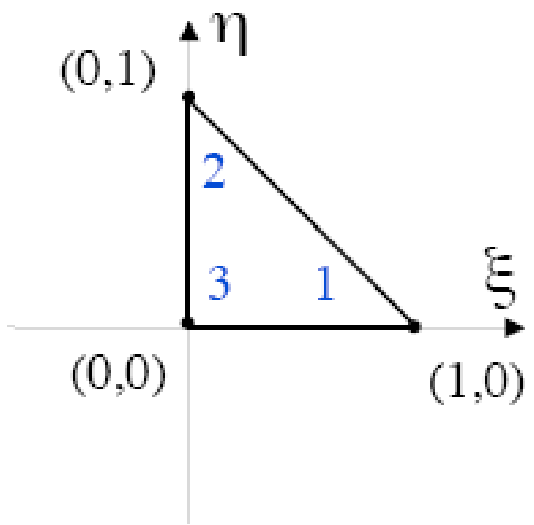
$$A^{(1)} = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$
$$A^{(2)} = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$
$$A^{(3)} = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$

$$\begin{pmatrix} A^{(1)} \\ A^{(2)} \\ A^{(3)} \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

$$\{\alpha\} = [C]^{-1}\{A\} \quad \longleftarrow \quad \{A\} = [C]\{\alpha\}$$

Transformation of Differential Operators

$$\frac{\partial N_k}{\partial x} = \frac{\partial N_k}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_k}{\partial \eta}\frac{\partial \eta}{\partial x}$$



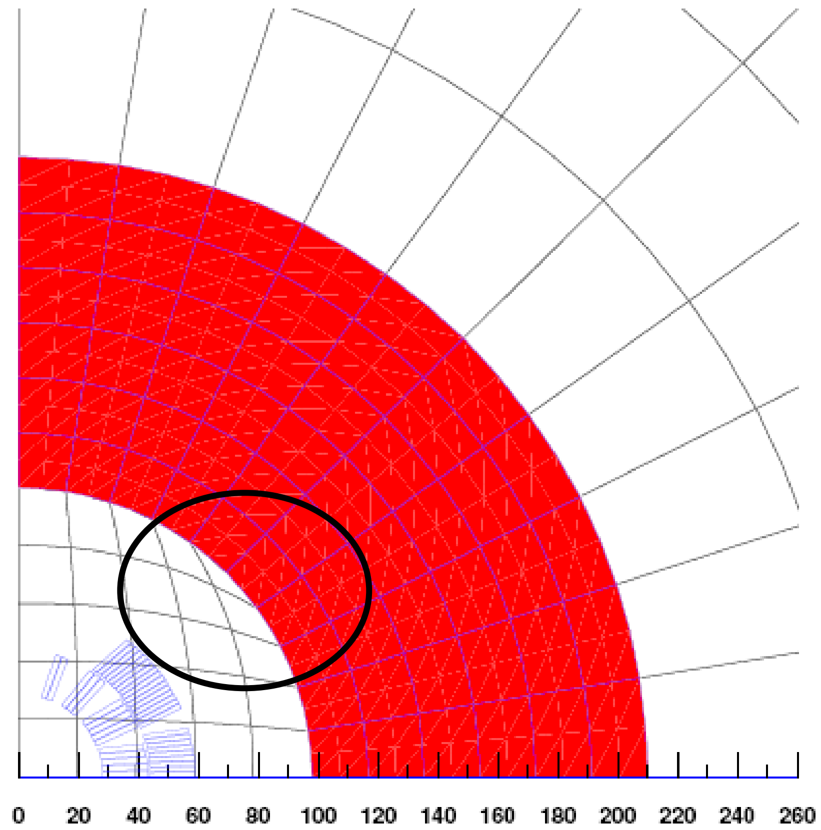
Complicated **Easy**

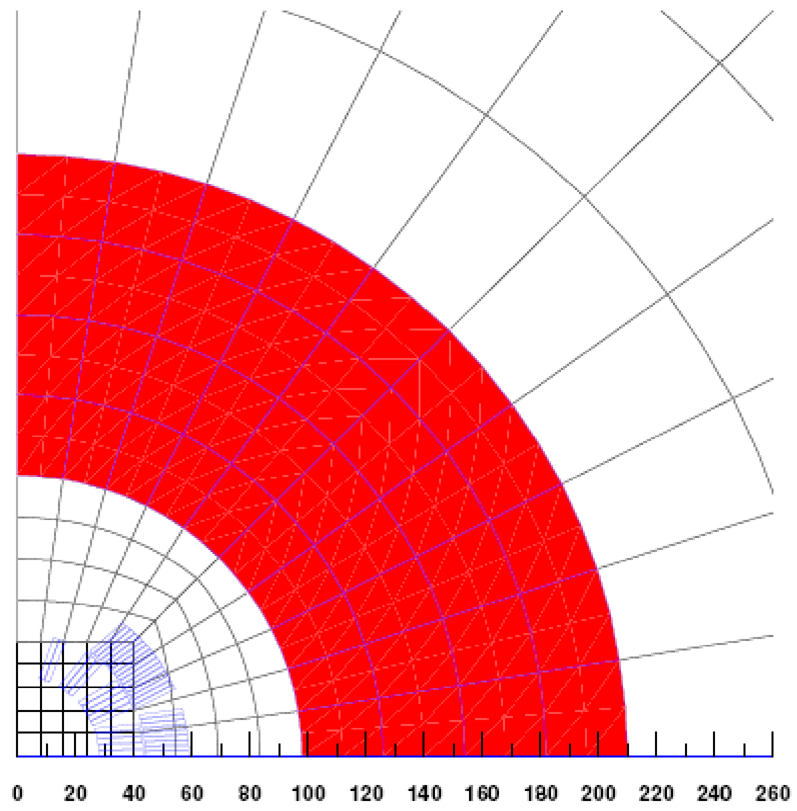
$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} N_k = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix} N_k = [J]_{T^{-1}} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix} N_k$$

$$[J]_{T^{-1}} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix}^{-1} = [J]_T^{-1}$$

$$[J]_T = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{K} \frac{\partial N_k}{\partial \xi} x^{(k)} & \sum_{k=1}^{K} \frac{\partial N_k}{\partial \xi} y^{(k)} \\ \sum_{k=1}^{K} \frac{\partial N_k}{\partial \eta} x^{(k)} & \sum_{k=1}^{K} \frac{\partial N_k}{\partial \eta} y^{(k)} \end{pmatrix} = \begin{pmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \cdots & \frac{\partial N_K}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \cdots & \frac{\partial N_K}{\partial \eta} \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_k & y_k \end{pmatrix}$$

Collinear Sides yield Singular Jacobi Matrices





Note: Bad meshing is not a trivial offence

Curl-Curl Equation

$$\mathbf{B} = \text{curl}\,\mathbf{A} \qquad \text{in}\,\Omega$$

$$\text{curl}\,\frac{1}{\mu}\,\text{curl}\,\mathbf{A} = \mathbf{J} \qquad \text{in}\,\Omega$$

$$\mathbf{H}_\text{t} = \mathbf{0} \quad \rightarrow \quad \frac{1}{\mu}(\text{curl}\,\mathbf{A}) \times \mathbf{n} \;=\; \mathbf{0} \;\; \text{on}\,\Gamma_H$$

$$B_\text{n} = 0 \quad \rightarrow \quad \mathbf{B}\cdot\mathbf{n} = \text{curl}\,\mathbf{A}\cdot\mathbf{n} \;=\; 0 \;\; \text{on}\,\Gamma_B$$

$$\left[\frac{1}{\mu}\,(\text{curl}\,\mathbf{A})\times\,\mathbf{n}\right]_\text{ai} \;=\; \mathbf{0} \;\; \text{on}\,\Gamma_\text{ai}$$

$$[\mathbf{A}]_\text{ai} \;=\; \mathbf{0} \;\; \text{on}\,\Gamma_\text{ai}$$

Problem in 3-D: Gauging

$$\mathbf{A} \rightarrow \mathbf{A}' : \mathbf{A}' = \mathbf{A} + \text{grad}\,\psi$$

$$\text{div}\,\mathbf{A}' = q$$

$$q = \text{div}\,\mathbf{A} + \nabla^2\psi$$

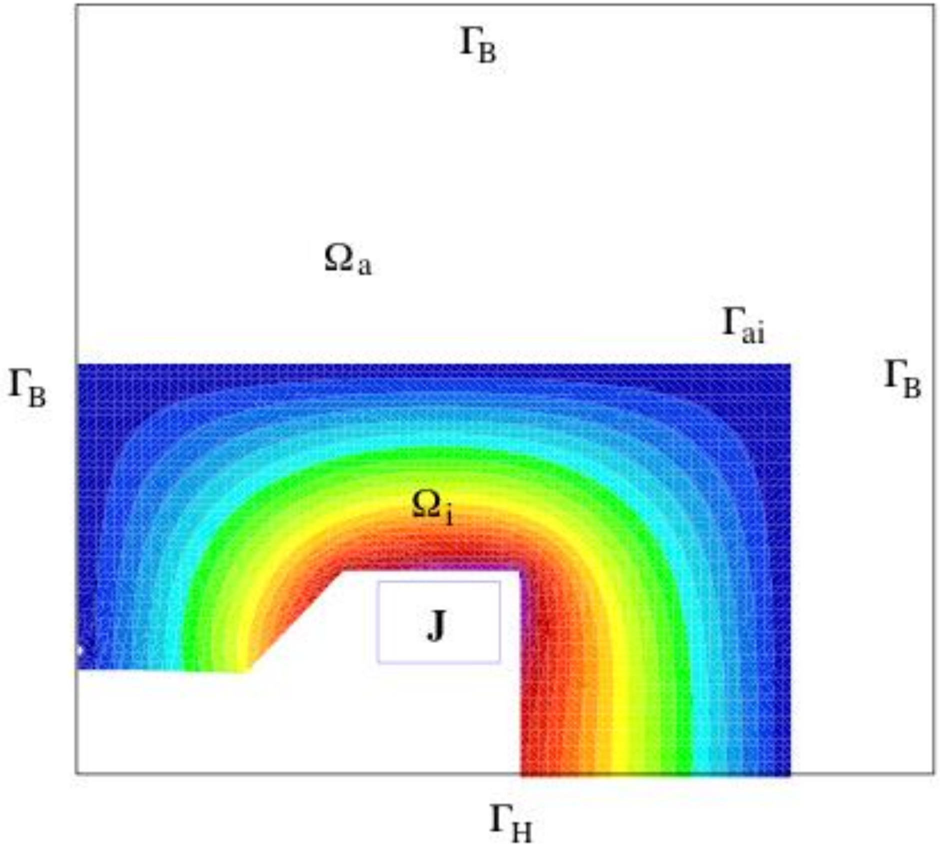
$$\frac{1}{\mu}\text{div}\,\mathbf{A} = 0 \;\; \text{in}\;\Omega$$

$$\mathbf{A}\cdot\mathbf{n} = 0 \;\; \text{on}\,\Gamma_H$$

$$\boxed{\text{curl}\,\frac{1}{\mu}\,\text{curl}\,\mathbf{A} - \;\text{grad}\,\frac{1}{\mu}\,\text{div}\;\mathbf{A} = \mathbf{J} \;\; \text{in}\,\Omega}$$

Weak Form in the FEM Problem

$$\mathrm{curl}\,\frac{1}{\mu}\,\mathrm{curl}\,\mathbf{A} - \mathrm{grad}\,\frac{1}{\mu}\,\mathrm{div}\,\mathbf{A} = \mathbf{J} \quad \mathrm{in}\,\Omega$$



$$\begin{aligned}
\mathbf{A}\cdot\mathbf{n} &= 0 \quad &&\text{on } \Gamma_H,\\
\frac{1}{\mu}\,\mathrm{div}\,\mathbf{A} &= 0 &&\text{on } \Gamma_B,\\
\mathbf{n}\times(\mathbf{A}\times\mathbf{n}) &= \mathbf{0} &&\text{on } \Gamma_B,\\
\mathbf{n}\times\left(\frac{1}{\mu}(\mathrm{curl}\,\mathbf{A})\times\mathbf{n}\right) &= \mathbf{0} &&\text{on } \Gamma_H,\\
\left[\frac{1}{\mu}\,\mathrm{div}\,\mathbf{A}\right]_{\mathrm{ai}} &= 0 &&\text{on } \Gamma_{\mathrm{ai}},\\
\left[\frac{1}{\mu}\,(\mathrm{curl}\,\mathbf{A})\times\mathbf{n}\right]_{\mathrm{ai}} &= \mathbf{0} &&\text{on } \Gamma_{\mathrm{ai}},\\
[\mathbf{A}]_{\mathrm{ai}} &= \mathbf{0} &&\text{on } \Gamma_{\mathrm{ai}}.
\end{aligned}$$

Weak Form in the FEM Problem (Green's First Theorem)

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A} - \operatorname{grad} \frac{1}{\mu} \operatorname{div} \mathbf{A} = \mathbf{J} \quad \text{in}\, \Omega$$

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A} - \operatorname{grad} \frac{1}{\mu} \operatorname{div} \mathbf{A} - \mathbf{J} = \mathbf{R}$$

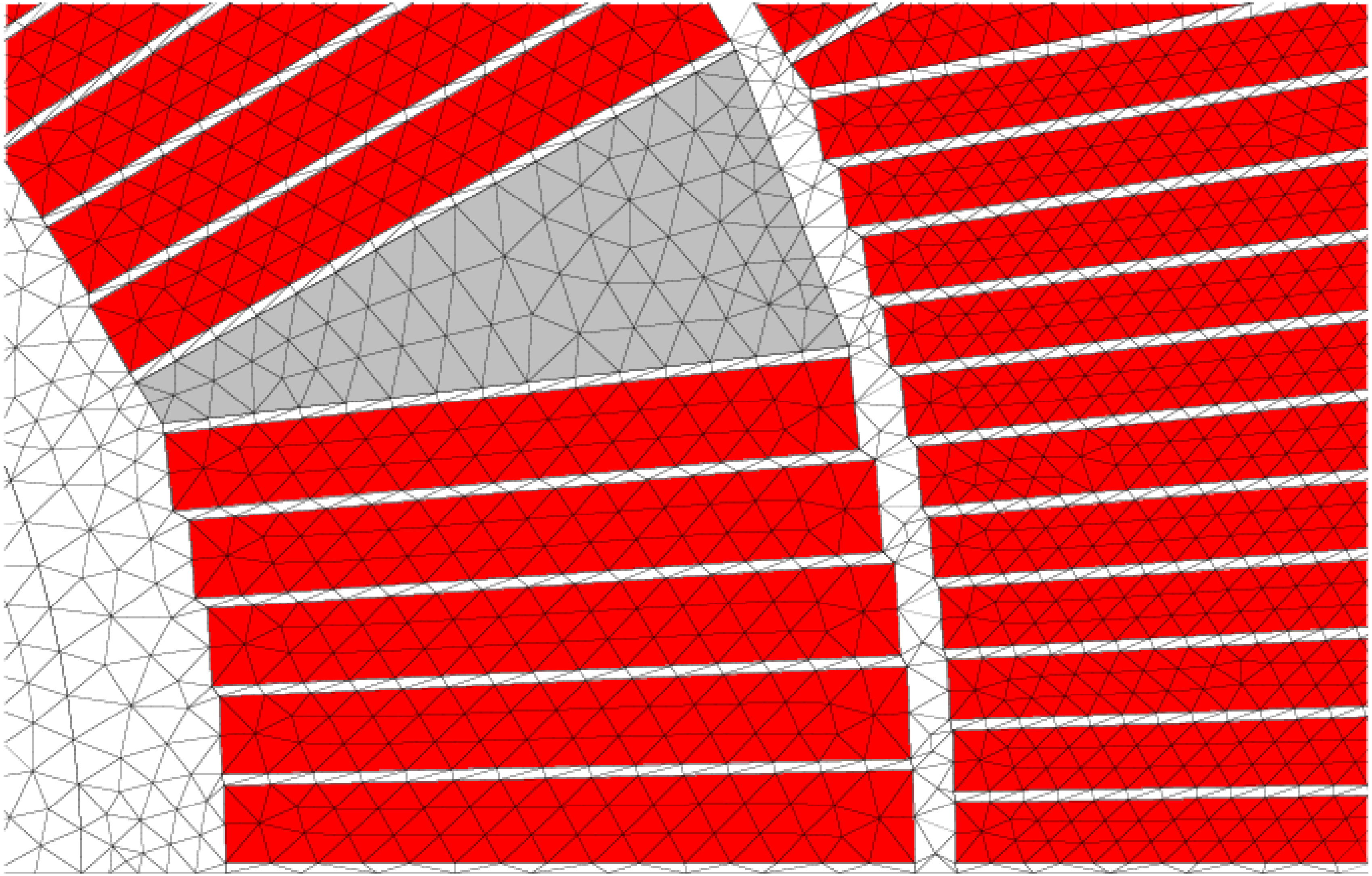
$$\int_\Omega \mathbf{w}_a \cdot \left(\operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A} - \operatorname{grad} \frac{1}{\mu} \operatorname{div} \mathbf{A} \right) \mathrm{d}\Omega = \int_\Omega \mathbf{w}_a \cdot \mathbf{J} \, \mathrm{d}\Omega \,, \qquad a = 1, 2, 3.$$

$$\int_\Omega \frac{1}{\mu} \operatorname{curl} \mathbf{A} \cdot \operatorname{curl} \mathbf{w}_a \mathrm{d}\Omega - \int_{\Gamma_\mathrm{H}} \frac{1}{\mu} (\operatorname{curl} \mathbf{A} \times \mathbf{n}) \cdot \mathbf{w}_a \mathrm{d}\Gamma_\mathrm{H} + \int_\Omega \frac{1}{\mu} \operatorname{div} \mathbf{A} \operatorname{div} \mathbf{w}_a \mathrm{d}\Omega -$$
$$\int_{\Gamma_\mathrm{B}} \frac{1}{\mu} \operatorname{div} \mathbf{A} (\mathbf{n} \cdot \mathbf{w}_a) \mathrm{d}\Gamma_\mathrm{B} - \int_{\Gamma_\mathrm{ai}} \left(\frac{1}{\mu} \operatorname{div} \mathbf{A}_\mathrm{i} (\mathbf{n}_\mathrm{i} \cdot \mathbf{w}_a) + \frac{1}{\mu_0} \operatorname{div} \mathbf{A}_\mathrm{a} (\mathbf{n}_\mathrm{a} \cdot \mathbf{w}_a) \right) \mathrm{d}\Gamma_\mathrm{ai} -$$
$$\int_{\Gamma_\mathrm{ai}} \left(\frac{1}{\mu} (\operatorname{curl} \mathbf{A}_\mathrm{i} \times \mathbf{n}_\mathrm{i}) + \frac{1}{\mu_0} (\operatorname{curl} \mathbf{A}_\mathrm{a} \times \mathbf{n}_\mathrm{a}) \right) \cdot \mathbf{w}_a \mathrm{d}\Gamma_\mathrm{ai} = \int_\Omega \mathbf{w}_a \cdot \mathbf{J} \, \mathrm{d}\Omega,$$

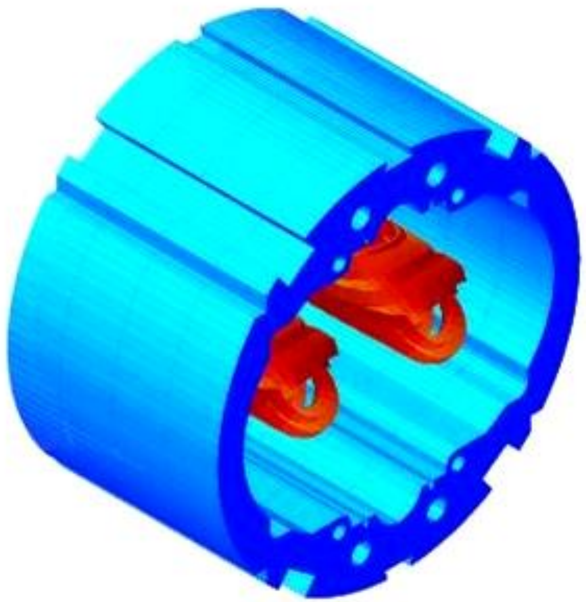
$$\boxed{\int_\Omega \frac{1}{\mu} \operatorname{curl} \mathbf{w}_a \cdot \operatorname{curl} \mathbf{A} \, \mathrm{d}\Omega + \int_\Omega \frac{1}{\mu} \operatorname{div} \mathbf{w}_a \operatorname{div} \mathbf{A} \, \mathrm{d}\Omega = \int_\Omega \mathbf{w}_a \cdot \mathbf{J} \, \mathrm{d}\Omega}$$

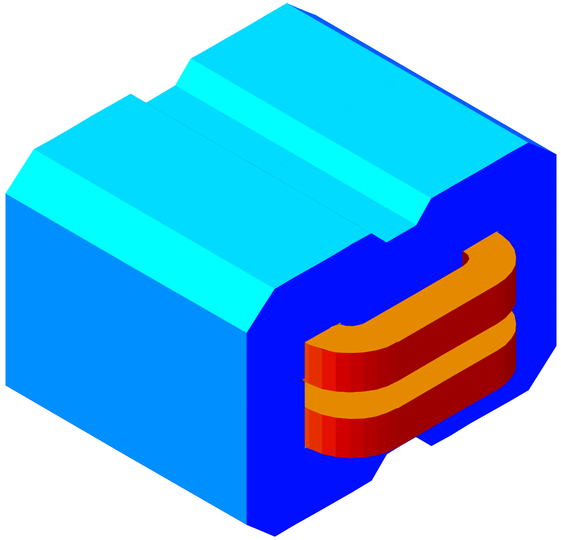
Removal of the second derivatives

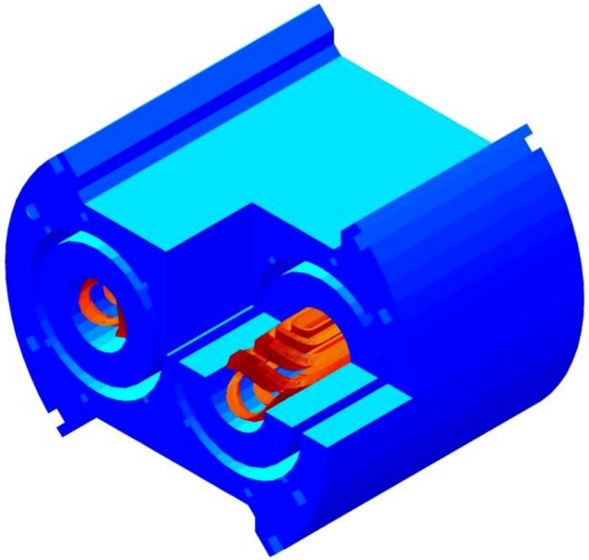
Meshing the Coil



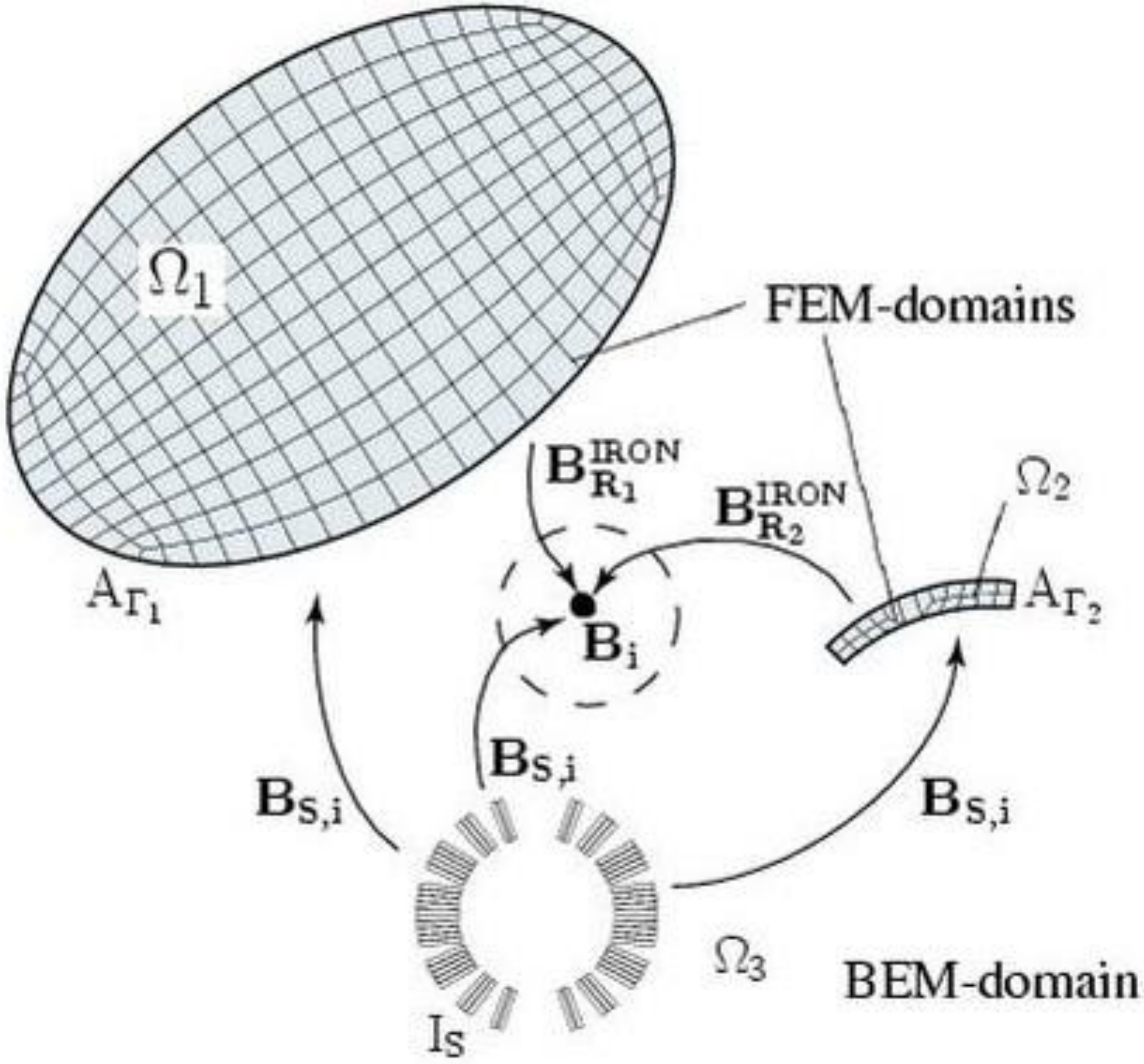
Magnet Extremities



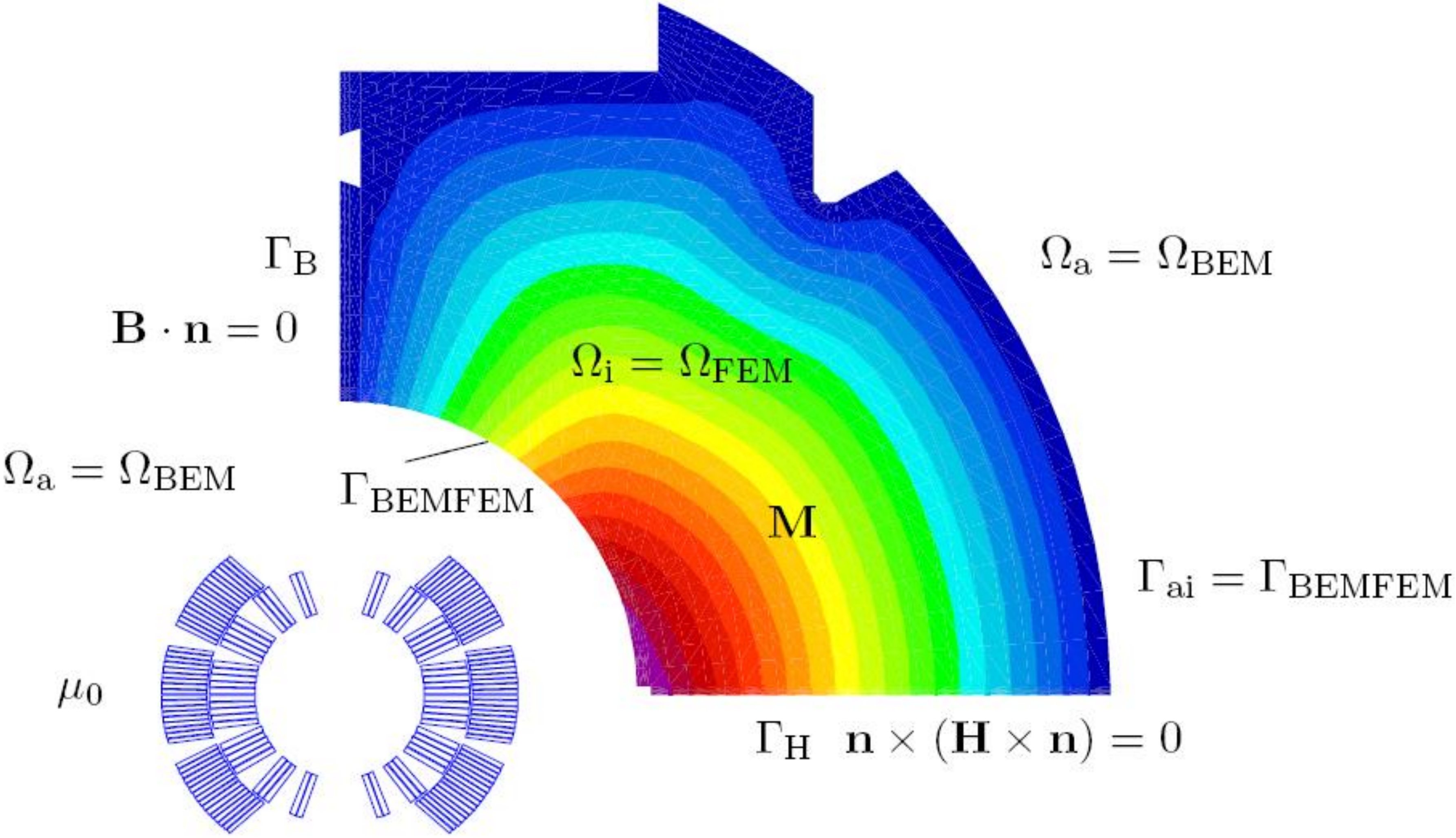




BEM-FEM Coupling (Elementary Model Problem)

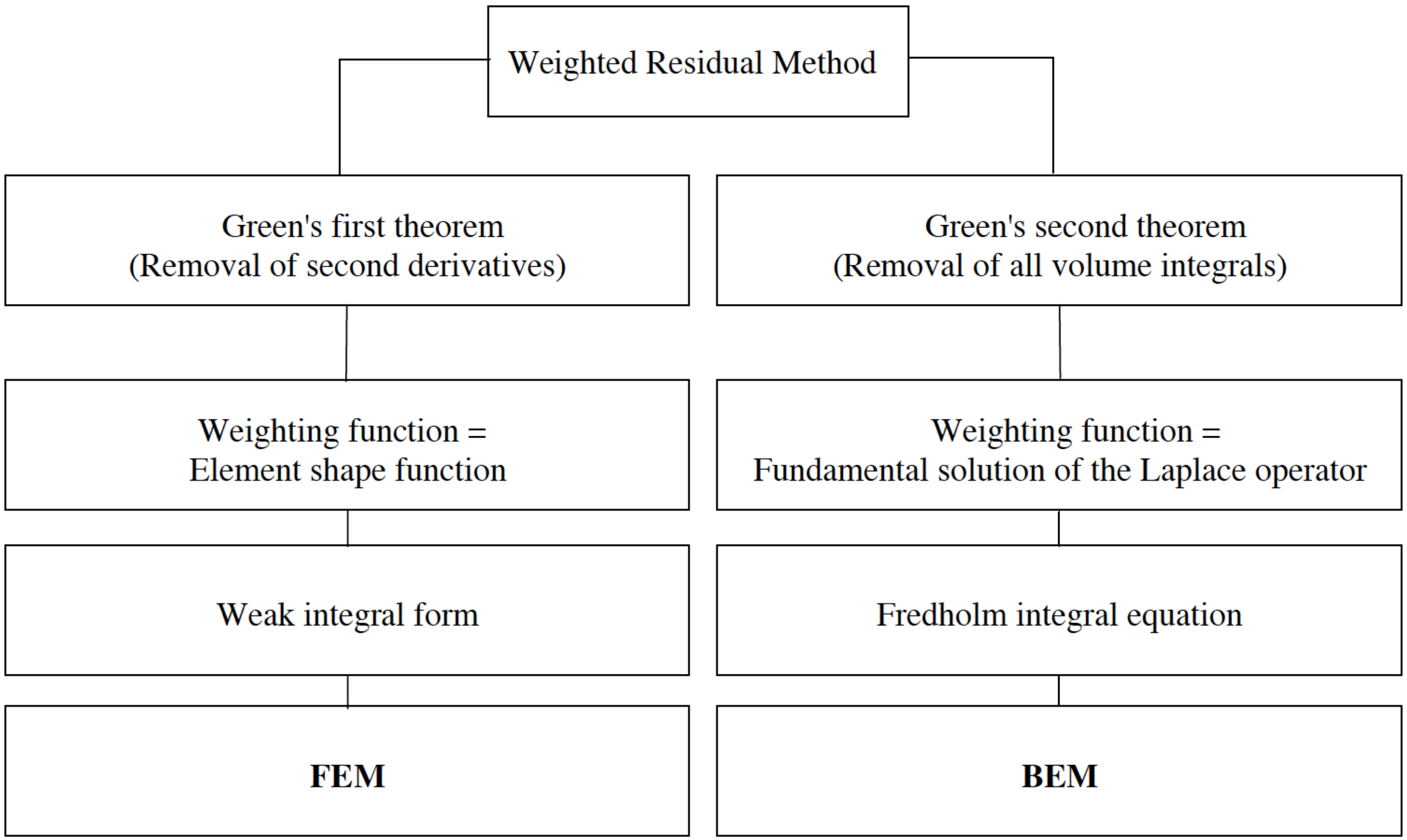


The Elementary Model Problem in Magnet Design



Green's First and Second Identities in FEM and BEM

$$\int_\Omega (\,\mathrm{grad}\,\phi \cdot \;\mathrm{grad}\,\psi + \phi\nabla^2\psi)\,\mathrm{d}V = \int_\Gamma \phi\;\mathrm{grad}\,\psi \cdot \mathbf{n}\,\mathrm{d}a\,,$$



$$\int_\Omega \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)\,\mathrm{d}V = \int_\Gamma \left(\phi\partial_{\mathbf{n}}\psi - \psi\partial_{\mathbf{n}}\phi\right)\,\mathrm{d}a\,,$$

The FEM Part (Iron-Domain)

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{P}_{\mathrm{m}}(\mathbf{H}) = \mu_0(\mathbf{H} + \mathbf{M}(\mathbf{H}))$$

Vector-Laplace

$$\boxed{-\frac{1}{\mu_0}\nabla^2 \mathbf{A} = \mathbf{J} + \operatorname{curl} \mathbf{M}} \qquad \text{in } \Omega_i,$$

$$\mathbf{A} \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma_H,$$

$$\frac{1}{\mu_0} \operatorname{div} \mathbf{A} = 0 \qquad \text{on } \Gamma_B,$$

$$\mathbf{n} \times (\mathbf{A} \times \mathbf{n}) = \mathbf{0} \qquad \text{on } \Gamma_B,$$

$$\frac{1}{\mu}(\operatorname{curl} \mathbf{A}) \times \mathbf{n} = \mathbf{0} \qquad \text{on } \Gamma_H,$$

$$\left[\frac{1}{\mu_0} \operatorname{div} \mathbf{A}_{\mathrm{a}}\right]_{\mathrm{ai}} = 0 \qquad \text{on } \Gamma_{\mathrm{ai}},$$

$$\boxed{\frac{1}{\mu_0}(\operatorname{curl} \mathbf{A}_{\mathrm{i}} - \mu_0 \mathbf{M}) \times \mathbf{n}_{\mathrm{i}} + \frac{1}{\mu_0}(\operatorname{curl} \mathbf{A}_{\mathrm{a}}) \times \mathbf{n}_{\mathrm{a}} = \mathbf{0}} \qquad \text{on } \Gamma_{\mathrm{ai}},$$

$$[\mathbf{A}]_{\mathrm{ai}} = \mathbf{0} \qquad \text{on } \Gamma_{\mathrm{ai}}.$$

FEM Part (Iron Magnetization but no Current Sources)

$$\frac{1}{\mu_0}\int_{\Omega_\mathrm{i}} \mathrm{grad}\,(\mathbf{A}\cdot\mathbf{e}_a)\cdot\mathrm{grad}\,w_a\,\mathrm{d}\Omega_\mathrm{i} - \frac{1}{\mu_0}\oint_{\Gamma_\mathrm{ai}}\left(\frac{\partial\mathbf{A}}{\partial n_\mathrm{i}} - (\mu_0\mathbf{M}\times\mathbf{n}_\mathrm{i})\right)\cdot\mathbf{w}_a\,\mathrm{d}\Gamma_\mathrm{ai} = \int_{\Omega_\mathrm{i}}\mathbf{M}\cdot\mathrm{curl}\,\mathbf{w}_a\,\mathrm{d}\Omega_\mathrm{i}$$

$$\frac{1}{\mu_0}\left(\mathrm{curl}\,\mathbf{A}_\mathrm{i}^\mathrm{FEM} - \mu_0\mathbf{M}\right)\times\mathbf{n}_\mathrm{i} + \frac{1}{\mu_0}\left(\mathrm{curl}\,\mathbf{A}_\mathrm{a}^\mathrm{BEM}\right)\times\mathbf{n}_\mathrm{a} = \mathbf{0} \quad \text{on } \Gamma_\mathrm{ai}$$

$$\rightarrow \quad \frac{\partial\mathbf{A}_\mathrm{i}^\mathrm{FEM}}{\partial n_\mathrm{i}} - (\mu_0\mathbf{M}\times\mathbf{n}_\mathrm{i}) + \frac{\partial\mathbf{A}_\mathrm{a}^\mathrm{BEM}}{\partial n_\mathrm{a}} = \mathbf{0} \quad \text{on } \Gamma_\mathrm{ai}$$

$$\mathbf{Q}_{\Gamma_\mathrm{ai}} := -\frac{\partial\mathbf{A}_{\Gamma_\mathrm{ai}}^\mathrm{BEM}}{\partial n_\mathrm{a}}$$

$$\frac{1}{\mu_0}\int_{\Omega_\mathrm{i}} \mathrm{grad}\,(\mathbf{A}\cdot\mathbf{e}_a)\cdot\mathrm{grad}\,w_a\,\mathrm{d}\Omega_\mathrm{i} - \frac{1}{\mu_0}\oint_{\Gamma_\mathrm{ai}}\mathbf{Q}_{\Gamma_\mathrm{ai}}\cdot\mathbf{w}_a\,\mathrm{d}\Gamma_\mathrm{ai} = \int_{\Omega_\mathrm{i}}\mathbf{M}\cdot\mathrm{curl}\,\mathbf{w}_a\,\mathrm{d}\Omega_\mathrm{i}$$

$$[K]\{A\} - [T]\{Q\} = \{F(\mathbf{M})\}$$

BEM Part (Air-Domain – No Iron, but Current Sources)

Vector Laplace

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}\,, \qquad \text{in } \Omega_\mathrm{a}$$

Weighted Residual

$$\int_{\Omega_\mathrm{a}} \nabla^2 A w \, \mathrm{d}\Omega_\mathrm{a} = -\int_{\Omega_\mathrm{a}} \mu_0 J w \, \mathrm{d}\Omega_\mathrm{a}$$

Apply Green's second theorem:

$$\int_{\Omega_\mathrm{a}} A \nabla^2 w \mathrm{d}\Omega_\mathrm{a} = -\int_{\Omega_\mathrm{a}} \mu_0 J w \, \mathrm{d}\Omega_\mathrm{a} + \int_{\Gamma_\mathrm{ai}} A \frac{\partial w}{\partial n_a} \mathrm{d}\Gamma_\mathrm{ai} - \int_{\Gamma_\mathrm{ai}} \frac{\partial A}{\partial n_a} w \mathrm{d}\Gamma_\mathrm{ai}$$

Biot-Savart

$$u^*(\mathbf{r}, \mathbf{r}') := w = \frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|}$$

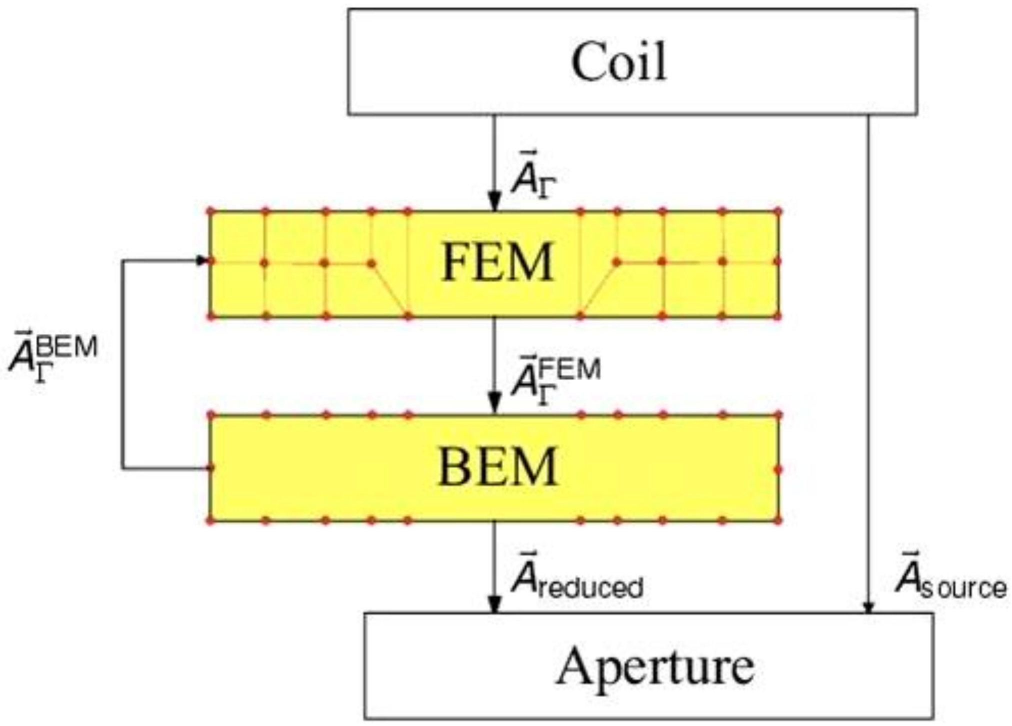
$$q^*(\mathbf{r}, \mathbf{r}') := \frac{\partial u^*}{\partial n_a} = \frac{\partial w}{\partial n_a} = -\frac{(\mathbf{r} - \mathbf{r}') \cdot \mathbf{n}_a}{4\pi |\mathbf{r} - \mathbf{r}'|^3}$$

$$\nabla^2 w = -\delta(|\mathbf{r} - \mathbf{r}'|)$$

$$\int_\Omega A(\mathbf{r}) \nabla^2 w \mathrm{d}\Omega = \int_\Omega A(\mathbf{r}) \delta(|\mathbf{r} - \mathbf{r}'|) \mathrm{d}\Omega = A(\mathbf{r}')$$

$$[G]\{Q\} + [H]\{A\} = \{A_\mathrm{s}\}$$

BEM-FEM Coupling



BEM

$$[G]\{Q\} + [H]\{A\} = \{A_s\}$$

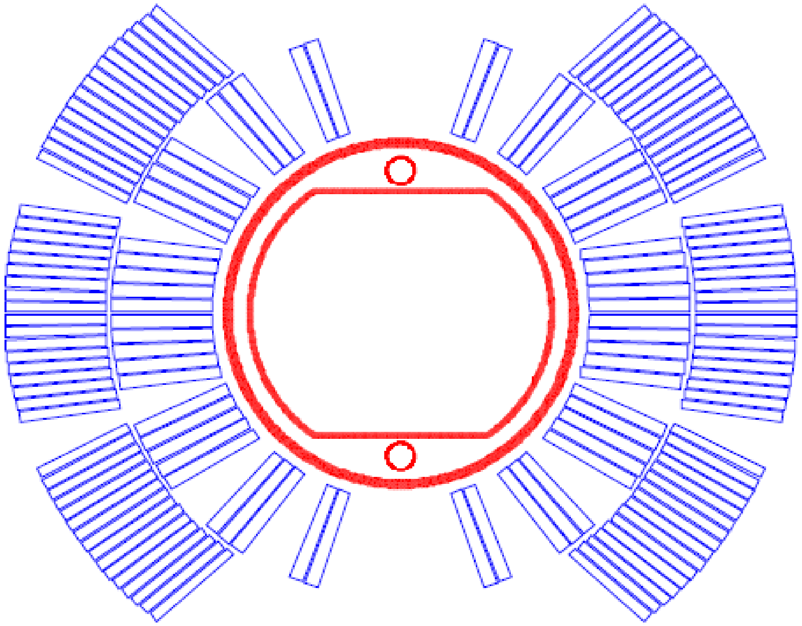
FEM

$$[K]\{A\} - [T]\{Q\} = \{F(\mathbf{M})\}$$

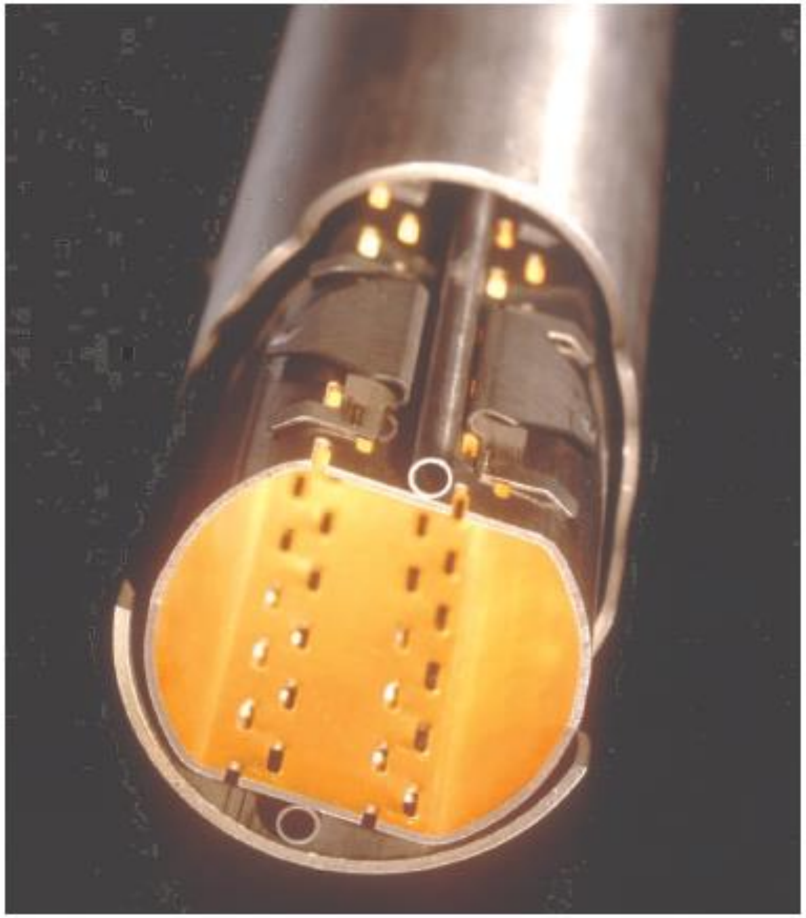
$$\{Q\} = -[G]^{-1}[H]\{A\} + [G]^{-1}\{A\}$$

$$\left([K] + [T][G]^{-1}[H]\right)\{A\} = \{F(\mathbf{M})\} + [T][G]^{-1}\{A_s\}$$

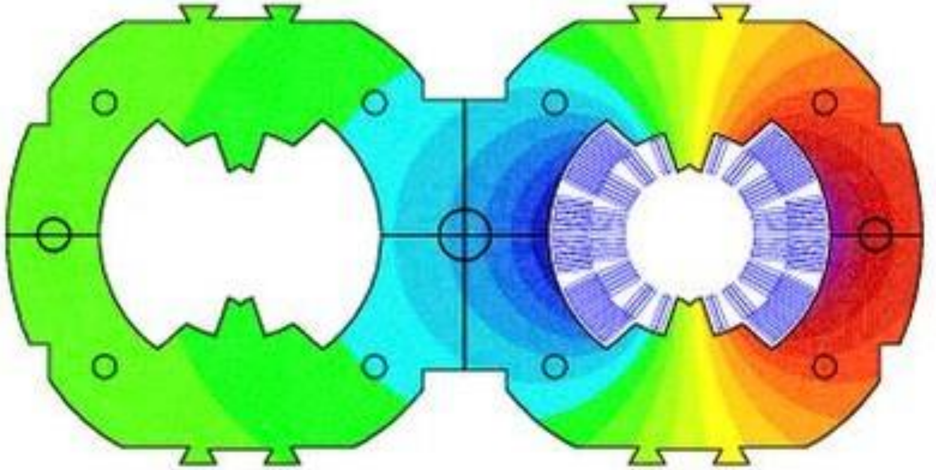
Open Boundary Problems (1)



LHC Beam Screen



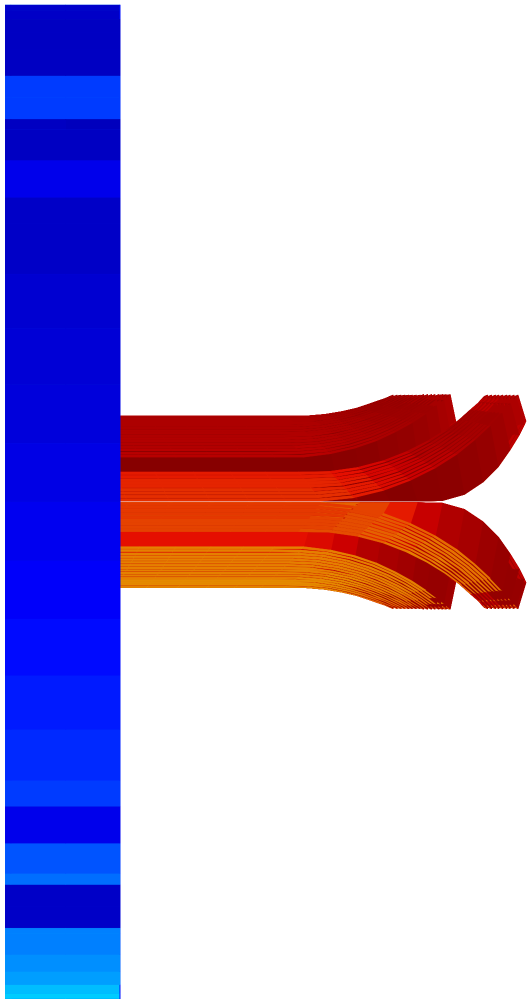
Open Boundary Problem (2)

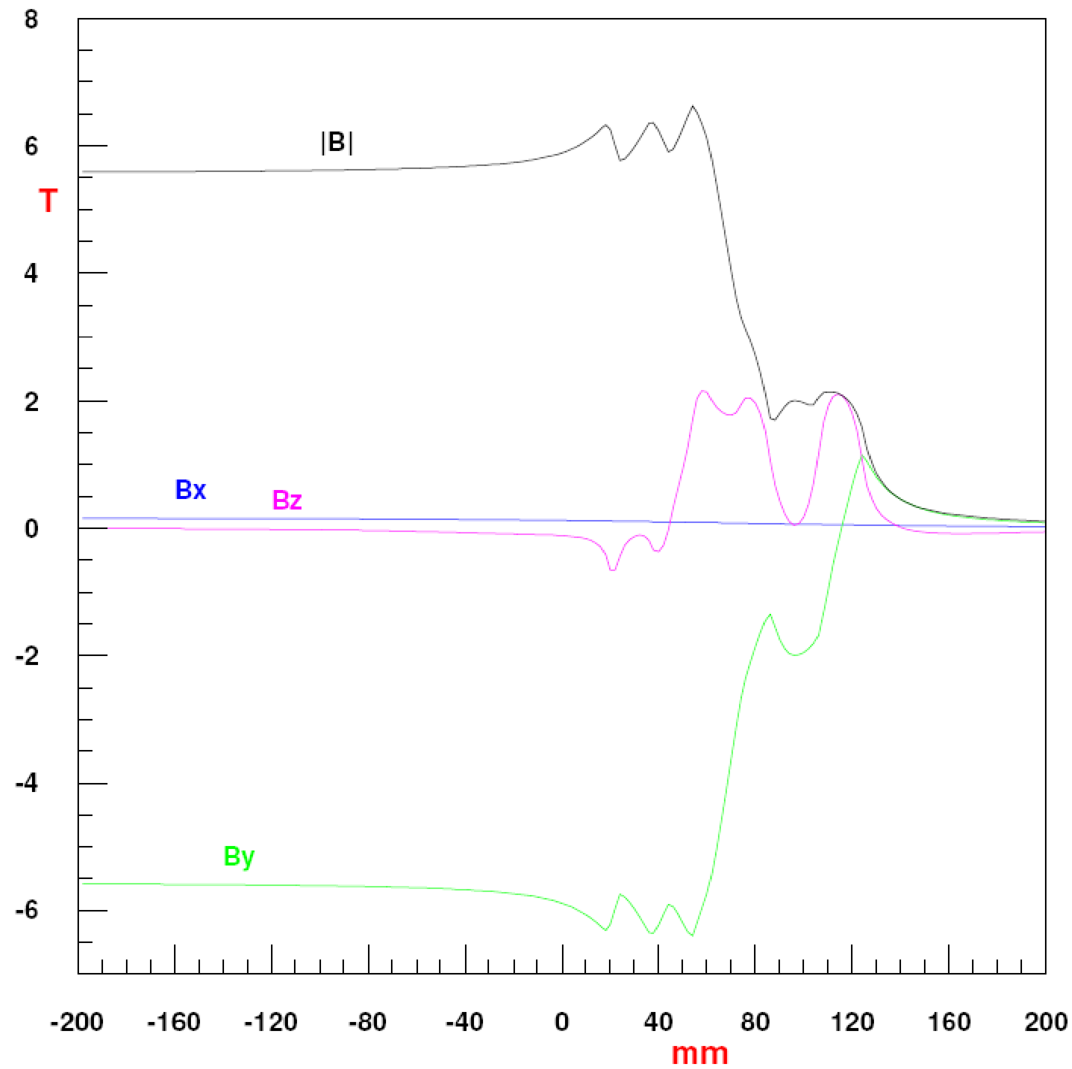


Collared Coil
Field Problem

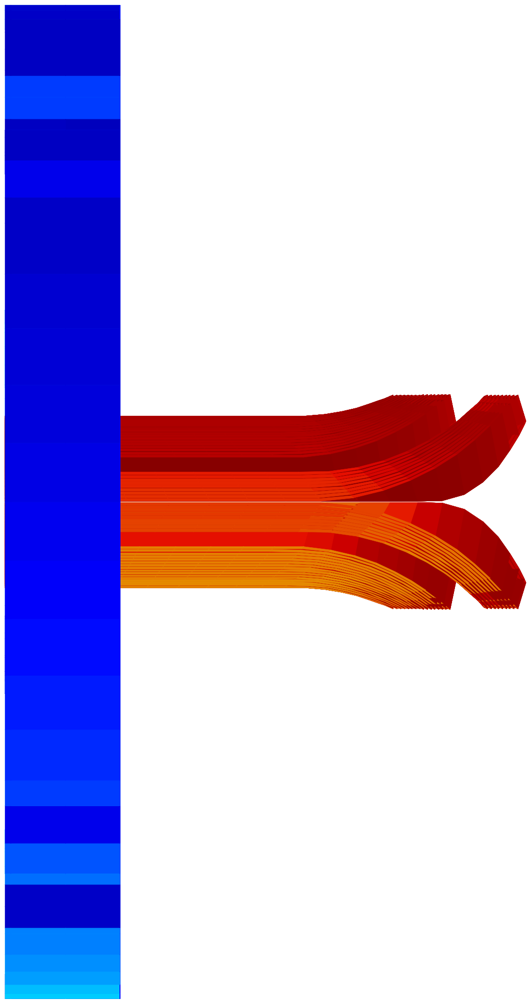
Collared Coil
Measurements in
Industry

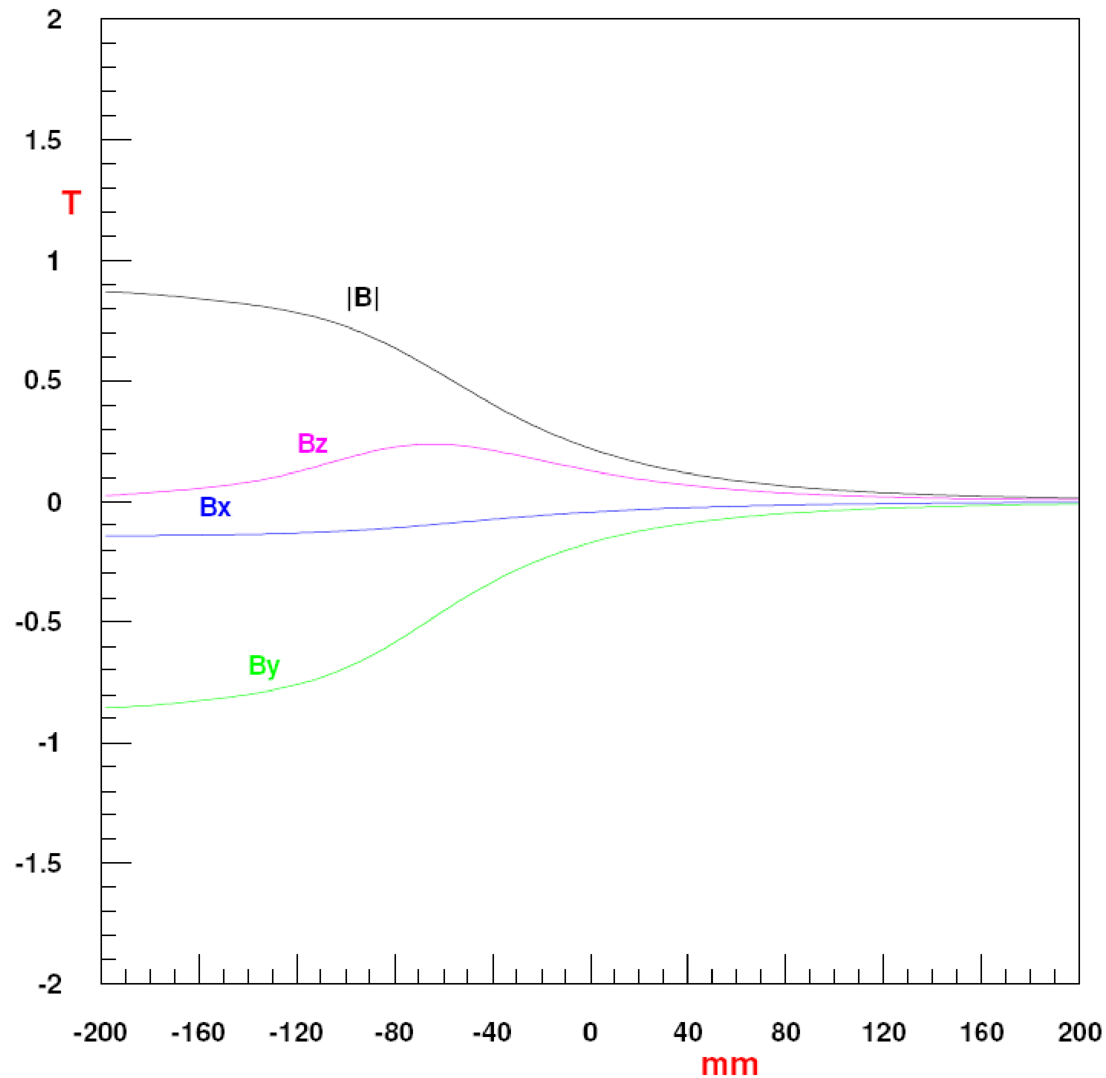
Source Field



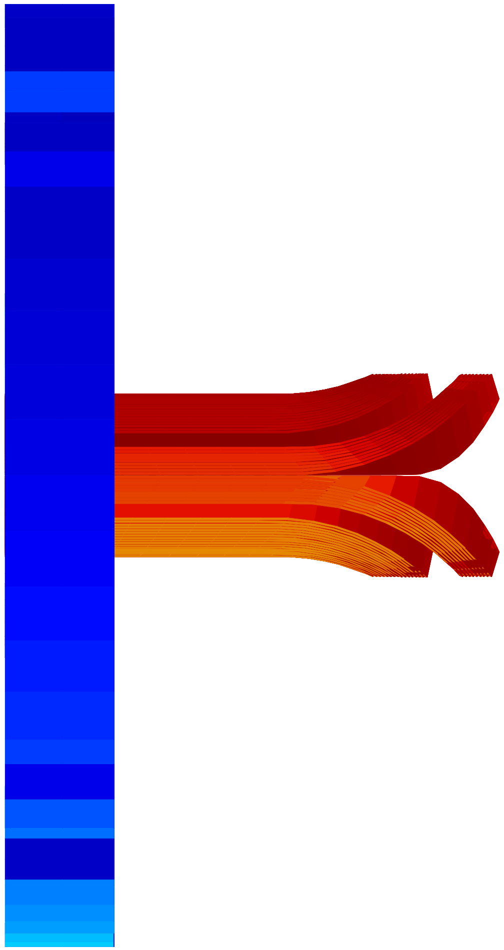


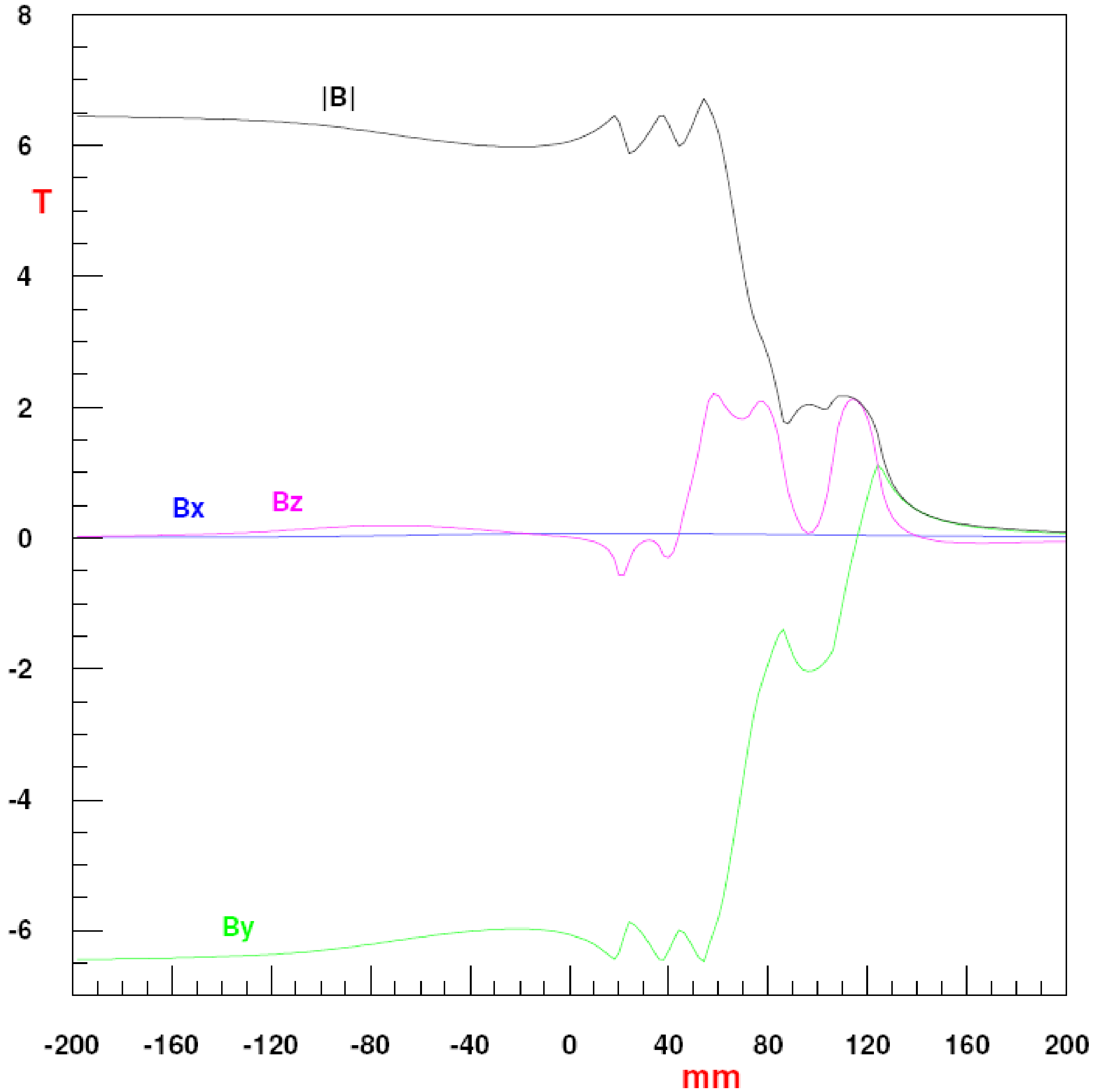
Reduced Field





Total Field





Forces (N) in the Connection Ends of the LHC Main Dipole

I	Fx	Fy	Fz
1	-39.7	-44.0	-45.4
2	-6.5	3.7	-41.7
3	-6.1	88.3	-38.2
4	1.25	3.9	-28.5
5	48.1	-46.7	-48.5
Sum	-2.95	5.2	-202.3