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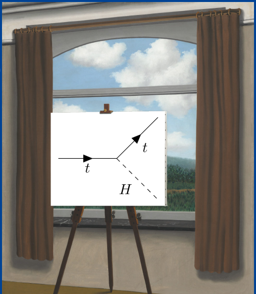
German Research Foundation

TTK Institut für Theoretische Teilchenphysik und Kosmologie
RWTH AACHEN UNIVERSITY

Higgs-boson production in top-quark fragmentation

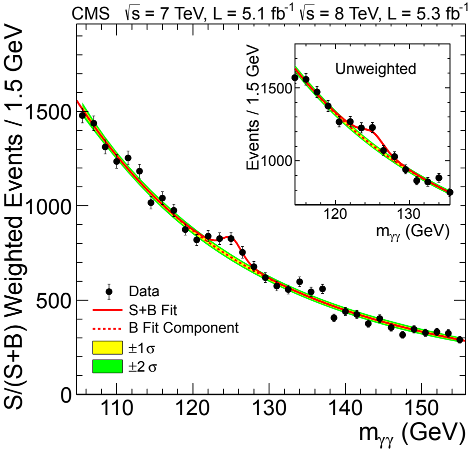
Colomba Brancaccio
Based on: JHEP 08 (2021) 145

The 19th Workshop of the LHC Higgs Working Group
November 28th, 2022



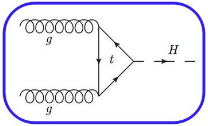
Introduction

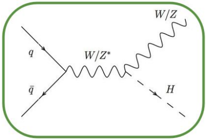
Happy Birthday Higgs Boson

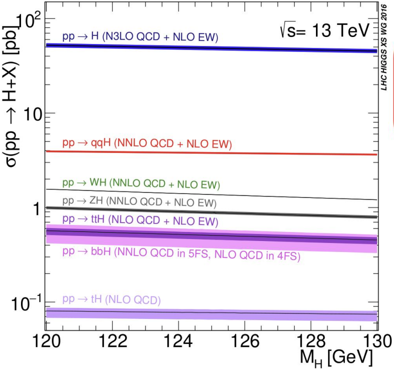


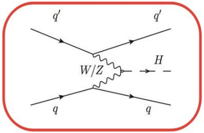
CMS collaboration, ’12

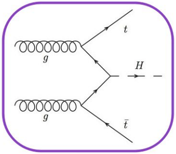
HIGGS BOSON PRODUCTION MODES





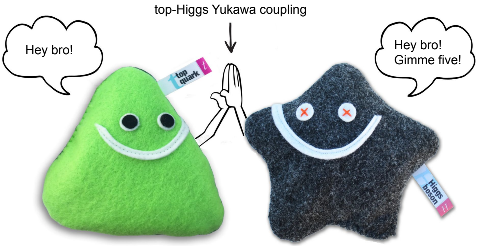


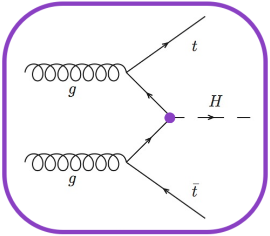




LHC Higgs Working Group, '17 ATLAS, '18 CMS, '18

PHENOMENOLOGICAL RELEVANCE OF $t\bar{t}H$

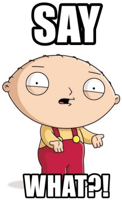




The **ttH production** is key **for assessing Higgs boson properties**:

- Provides direct access to the top-Higgs Yukawa coupling
  $\rightarrow$ Strongest coupling of the SM
- Allows to probe the CP structure of the Higgs boson
  $\rightarrow$ A CP-odd component would be an indication of new physics

CURRENT THEORETICAL DESCRIPTION

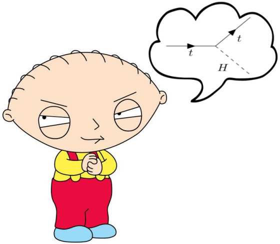


Goal

Computing the fagmentation functions to estimate higher order QCD corrections to $pp \to t\bar{t}H$ at high $p_{\mathrm{T},H}$.

$t \to tH$ fragmentation

- ✓ LO top-Higgs FF, [1]
- ✓ NLO top-Higgs FF in the limit $m_H^2 \ll m_t^2 \ll \hat{s}$ and based on soft-gluon approximation, [2]
- ✓ NLO top-Higgs FF. [3]



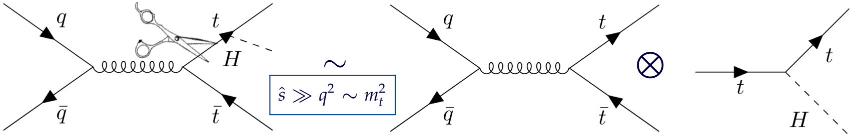
[1] Braaten, Zhang, ’16 [2] Dawson, Reina, ’98 [3] CB, Czakon, Generet, Krämer, ’21

For more details on state-of-art $t\bar{t}H$ higher order corrections stay tuned for Chiara’s talk!

The method

Final state factorisation

Hard scattering and collinear emission factorise in the collinear limit:

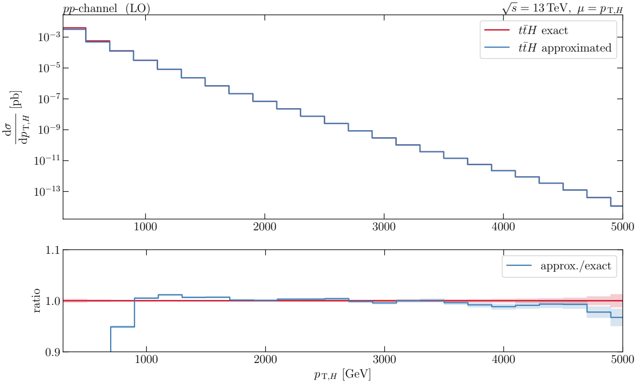


$$\mathrm{d}\hat{\sigma}_{q\bar{q}\to t\bar{t}H}(p_q, p_{\bar{q}}, p_H) = \int_0^1 \mathrm{d}z\, \mathrm{d}\tilde{\sigma}_{q\bar{q}\to t\bar{t}}(p_q, p_{\bar{q}}, p_t; \mu) \mathrm{D}_{t\to H}(z;\mu),$$

with $z = \dfrac{n \cdot p_H}{n \cdot p_t}$, $n^\mu = \dfrac{1}{\sqrt{2}}(1,0,0,1)$ light-cone vector in the Higgs direction.

- Analogous to the initial state factorisation (PDFs).
- $\mathrm{D}_{t\to H}(z;\mu)$ can be perturbatively computed.

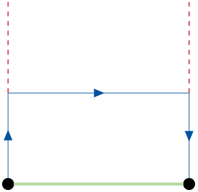
Fragmentation approach



- **Good approximation at large $p_{\text{T},H}$** $\to$ errors decrease to below 5% for $p_{\text{T},H} > 600\,\text{GeV}$.
- Enables to **resum logarithms** at high $p_{\text{T},H}$ $\to$ necessary for future colliders.

$t \to H$ fragmentation function

DEFINITION OF THE FRAGMENTATION FUNCTION



[LO example of the following general formula]

$$D_{q\to H}(z) = \frac{z^{d-3}}{4\pi}\int dx^- e^{-ip_H^+ x^-/z}\frac{1}{2N_c} Tr_{color} Tr_{Dirac}\Big[\not{n} \langle 0|\, \psi_q(0)$$

$$\bar{\mathrm{P}} \exp\left(ig\int_0^\infty dy^- n\cdot A_a(y^- n) T_a^T\right) a_H^\dagger(p_H) a_H(p_H)$$

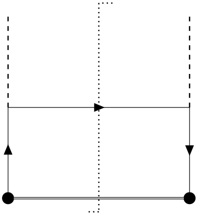
$$\mathrm{P} \exp\left(-ig\int_{x^-}^\infty dy^- n\cdot A_b(y^- n) T_b^T\right) \bar{\psi}_q(x^- n)\, |0\rangle \Big]$$

Wilson Lines

This definition is gauge invariant!

Collins, Soper, '82

Example: The LO fragmentation function

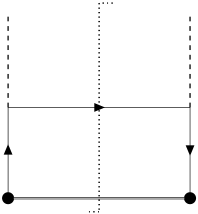


Applying the definition introduce in the previous slide, the LO fragmentation $D_{t\to H}$ reads:

$$D_{t\to H} = \frac{z^{d-3}}{4\pi} \int \frac{d^d p_t}{(2\pi)^d} (2\pi)\delta^+(p_t^2 - m_t^2)(2\pi)\delta^+(p_H^+/z - (p_t + p_H)^+) \, \frac{y_t^2 \tilde{\mu}^{2\epsilon}}{2N_c}$$

$$\times \sum_{spins,colors} Tr\left[\not{n} \frac{\not{p}_t + \not{p}_H + m_t}{(p_t + p_H)^2 - m_t^2} (\not{p}_t + m_t) \frac{\not{p}_t + \not{p}_H + m_t}{(p_t + p_H)^2 - m_t^2} \right].$$

Example: The LO fragmentation function



Using reverse unitarity, the phase-space becomes a loop integral

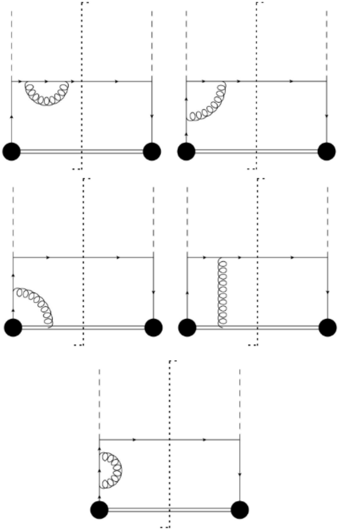
$$\delta(x) \to \frac{1}{2\pi i}\left(\frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon}\right)$$

$$D_{t\to H} = \frac{z^{d-3}}{4\pi}\int \frac{d^d p_t}{(2\pi)^d}(2\pi)\boxed{\delta^+(p_t^2 - m_t^2)}(2\pi)\boxed{\delta^+(p_H^+/z - (p_t+p_H)^+)}\ \frac{y_t^2\tilde{\mu}^{2\epsilon}}{2N_c}$$
$$\times \sum_{spins,colors} Tr\left[\not{n}\frac{\not{p}_t + \not{p}_H + m_t}{(p_t+p_H)^2 - m_t^2}(\not{p}_t + m_t)\frac{\not{p}_t + \not{p}_H + m_t}{(p_t+p_H)^2 - m_t^2}\right].$$

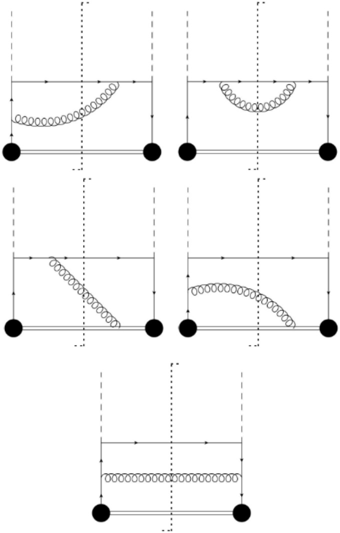
Anastasiou, Melnikov, '02

THE NLO FRAGMENTATION FUNCTION CONTRIBUTIONS

Virtual corrections



Real corrections



DIFFERENTIAL EQUATIONS METHOD

- **Reduction to MIs** performed by using the software *FIRE*[1]
- A **system of first order linear differential equations**[2] for the MIs can be derived:

$$\partial_{x_i} \vec{f}(\vec{x}, \epsilon) = A_{x_i}(\vec{x}, \epsilon)\, \vec{f}(\vec{x}, \epsilon).$$

- It is possible to choose a basis of MIs, the **Canonical basis**[3], such that:

$$\mathrm{d}\vec{f}(\vec{x}, \epsilon) = \epsilon\, \mathrm{d}A(\vec{x})\, \vec{f}(\vec{x}, \epsilon),$$

with

$$\mathrm{d}A(\vec{x})_{ij} = \sum_k c_{ijk}\, \mathrm{dlog}(\alpha_k(\vec{x})).$$

- The **solution of the differential equations system** is:

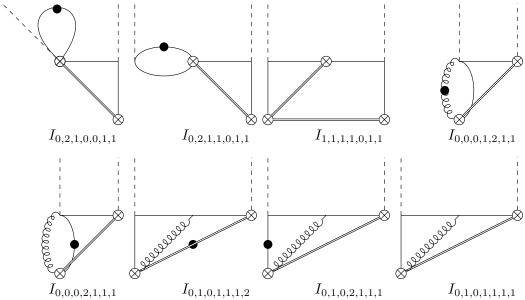
$$\vec{f}(\vec{x}, \epsilon) = \mathrm{P}\exp\left[\epsilon \int_\gamma \mathrm{d}\tilde{A}(\vec{x}')\right] \vec{f}(\vec{x}_0, \epsilon).$$

[1] Smirnov, Chukharev, '20 [2] Kotikov, '91 [3] Henn, '13

CANONICAL BASIS FOR VIRTUAL CORRECTIONS

- Generic canonical master: $f_i^{\text{virt}} = \epsilon^{n_i}\, B_i(m_t, m_h, z)\, T_i^{\text{virt}}$.
- Semi-algorithmic approach:
  - ✓ T_i^{virt} found by **maximizing symmetries**,
  - ✓ $B_i(m_t, m_h, z)$ found by applying **Magnus transformations**.

Pre-canonical T_i^{virt}:



Canonical basis for virtual corrections

- Generic canonical master: $f_i^{\text{virt}} = \epsilon^{n_i} B_i(m_t, m_h, z)\, T_i^{\text{virt}}$.
- Semi-algorithmic approach:
  - ✓ T_i^{virt} found by **maximizing symmetries**,
  - ✓ $B_i(m_t, m_h, z)$ found by applying **Magnus transformations**.

Canonical form for MIs of the virtual topology:

$$f_1^{virt} = \epsilon^2\, n \cdot p_h\, T_1^{virt},$$

$$f_2^{virt} = \epsilon^2 m_h \sqrt{4m_t^2 - m_h^2}\, n \cdot p_h\, T_2^{virt},$$

$$f_3^{virt} = \epsilon^3\, (n \cdot p_h)^2\, T_3^{virt},$$

$$f_4^{virt} = \epsilon^2\, n \cdot p_h\, T_4^{virt},$$

$$f_5^{virt} = \epsilon^2\, n \cdot p_h\, T_5^{virt},$$

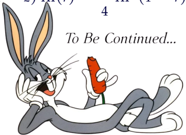
$$f_6^{virt} = \epsilon^2\, \frac{1-z}{z}\, (n \cdot p_h)^2\, T_6^{virt},$$

$$f_7^{virt} = \epsilon^2 m_h \sqrt{4m_t^2 - m_h^2}\, n \cdot p_h\, T_7^{virt},$$

$$f_8^{virt} = \epsilon^3\, n \cdot p_h\, T_8^{virt}.$$

Master Integrals Results

$$
\begin{aligned}
I_{0,0,1,1,1,1,1} = & -\frac{1}{4\epsilon^2}\ln(z) + \frac{1}{\epsilon}\Big\{ -\mathrm{Re}\Big[\mathrm{Li}_2\Big(\frac{z}{x^+}\Big)\Big] - \frac{1}{8}\arg^2\Big(\frac{x^+}{x^-}\Big) - \frac{1}{8}\ln^2(1-r) - \frac{1}{8}\ln^2(r) - \frac{1}{8}\ln^2(1-z) - \frac{1}{8}\ln^2(z \\
& - \frac{1}{4}\ln(r)\ln(1-z) + \frac{1}{4}\ln(1-r)\ln(r) + \frac{\pi^2}{12}\Big\} + \Big\{ -\mathrm{Re}\Big[\mathrm{Li}_3\Big(\frac{z-1}{z-x^+}\Big)\Big] - 4\,\mathrm{Re}\Big[\mathrm{Li}_3\Big(\frac{z}{z-x^+}\Big)\Big] - \mathrm{Re}\Big[\mathrm{Li}_3\Big(\frac{z(1-x^+)}{z-x^+}\Big) \\
& - \mathrm{Re}\Big[\mathrm{Li}_3\Big(1-\frac{z}{1-w^+}\Big)\Big] + \frac{1}{2}\mathrm{Re}\Big[\mathrm{Li}_3\Big(\frac{r+z(w^+-1)}{zw^+}\Big)\Big] + \frac{1}{2}\mathrm{Re}\Big[\mathrm{Li}_3\Big(\frac{zw^+}{r+z(w^+-1)}\Big)\Big] - \frac{1}{2}\mathrm{Re}\Big[\mathrm{Li}_3\Big(\frac{rw^+}{w^+z-z+r}\Big)\Big] \\
& - \frac{1}{2}\mathrm{Re}\Big[\mathrm{Li}_3\Big(\frac{w^+z-z+r}{rw^+}\Big)\Big] - \frac{1}{2}\mathrm{Re}\Big[\mathrm{Li}_3\Big(\frac{w^+}{w^-}\Big)\Big] + 2\arg\Big(\frac{x^+}{x^-}\Big)\mathrm{Im}\Big[\mathrm{Li}_2\Big(\frac{z}{x^+}\Big)\Big] - 2\,\mathrm{Im}\Big[\ln\Big(1-\frac{z-x^+}{z-x^-}\Big)\Big]\mathrm{Im}\Big[\mathrm{Li}_2\Big(\frac{z}{x^+} \\
& - \mathrm{Im}\Big[\ln\Big(1-\frac{z-x^+}{z-x^-}\Big)\Big]\mathrm{Im}\Big[\mathrm{Li}_2\Big(1-\frac{x^+}{x^-}\Big)\Big] + \frac{1}{2}\ln(z)\mathrm{Li}_2\Big(1-\frac{1-r}{1-z}\Big) - \frac{1}{2}\ln(r)\mathrm{Li}_2\Big(1-\frac{1-r}{1-z}\Big) - \frac{1}{2}\ln(z)\mathrm{Li}_2(r) + \frac{1}{2}\ln(\\
& + \frac{1}{2}\ln(r)\mathrm{Li}_2\Big(1-\frac{r}{z}\Big) + 2\ln\Big(\frac{m_t^2}{\mu^2}\Big)\mathrm{Re}\Big[\mathrm{Li}_2\Big(\frac{z}{x^+}\Big)\Big] - 2\ln(r)\,\mathrm{Re}\Big[\mathrm{Li}_2\Big(\frac{z}{x^+}\Big)\Big] - \frac{1}{48}\ln^3(1-z) + \frac{7}{24}\ln^3(z) + \frac{1}{48}\ln^3(1-r) - \\
& + \frac{1}{4}\ln^2(1-z)\ln\Big(\frac{m_t^2}{\mu^2}\Big) + \frac{1}{4}\ln^2(z)\ln\Big(\frac{m_t^2}{\mu^2}\Big) + \frac{1}{4}\ln^2(1-r)\ln\Big(\frac{m_t^2}{\mu^2}\Big) + \frac{1}{4}\ln^2(r)\ln\Big(\frac{m_t^2}{\mu^2}\Big) + \frac{1}{4}\arg^2\Big(\frac{x^+}{x^-}\Big)\ln\Big(\frac{m_t^2}{\mu^2}\Big) - \\
& - \frac{1}{16}\ln(1-z)\ln^2(1-r) - \frac{1}{16}\ln(z)\ln^2(1-r) - \frac{3}{4}\ln(1-z)\ln^2(r) + \frac{5}{8}\ln(z)\ln^2(r) + \frac{5}{8}\ln(1-r)\ln^2(r) - \frac{1}{16}\arg^2\Big(\frac{x^+}{x^-} \\
& + \frac{1}{16}\ln^2(1-z)\ln(1-r) + \frac{9}{16}\arg^2\Big(\frac{x^+}{x^-}\Big)\ln(1-r) - \frac{1}{4}\arg^2\Big(\frac{x^+}{x^-}\Big)\ln(r) - \frac{3}{4}\ln^2(1-z)\ln(r) - \frac{1}{4}\ln^2(1-r)\ln(r) + \frac{1}{4}
\end{aligned}
$$



Simplifying polylogarithmic expressions with symbols

Symbols ($\mathcal{S}$) were used as a systematic way of simplifying the polylogarithms appearing in the MIs analytic expressions.

Definition

- $\mathcal{S}(\mathrm{Li}_n(x)) = -\ln(1-x) \otimes \underbrace{\ln(x) \otimes \ldots \otimes \ln(x)}_{\text{n-1 times}}$,
- $\mathcal{S}(\ln(x)\ln(y)) = \ln(x) \otimes \ln(y) + \ln(y) \otimes \ln(x)$,
- $\ldots \otimes \ln(x \cdot y) \otimes \ldots = (\ldots \otimes \ln(x) \otimes \ldots) + (\ldots \otimes \ln(y) \otimes \ldots)$,
- $\mathcal{S}(\pi^n) = 0$ with $n \geq 2$,
- Symbols are unique up to $\sim \pi^n$.

Goncharov, '09 Duhr, Gangl, Rhodes,'12

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- $\mathcal{S}(\pi^n) = 0$ with $n \geq 2$,
- Symbols are unique up to $\sim \pi^n$.

Example

$$\mathcal{S}\left(-\mathrm{Li}_2(x) - \ln(1-x)\ln(x) + \frac{\pi^2}{6} \right)$$

$$= \ln(1-x) \otimes \ln(x) - (\ln(1-x) \otimes \ln(x) + \ln(x) \otimes \ln(1-x)) + 0$$

$$= -\ln(x) \otimes \ln(1-x)$$

$$= \mathcal{S}(\mathrm{Li}_2(1-x))$$

$$\downarrow$$

$$\mathrm{Li}_2(1-x) = -\mathrm{Li}_2(x) - \ln(1-x)\ln(x) + A\pi^2$$

$A = \frac{1}{6} \to$ Euler's reflection formula

Goncharov, '09

Duhr, Gangl, Rhodes,'12

Collinear renormalization

The bare $t \to h$ fragmentation function:

$$D_{h\to h} = \delta(1-z) + \mathcal{O}(y_t^2)$$

$$D^B_{t\to h}(z) = (Z_{th} \otimes D_{h\to h})(z) + (Z_{tt} \otimes D_{t\to h})(z) + \mathcal{O}(y_t^2\alpha_s^2, y_t^4).$$

The renormalization constants in terms of splitting functions are:

$$\begin{aligned} Z_{th}(z) =& \frac{y_t^2}{16\pi^2}\frac{1}{\epsilon}P^{(0)}_{th}(z) \\ &+ \frac{y_t^2}{16\pi^2}\frac{\alpha_s}{2\pi}\left(\underbrace{\frac{1}{2\epsilon}P^{(1)}_{th}(z)}_{?} + \frac{1}{2\epsilon^2}(P^{(0)}_{qq}\otimes P^{(0)}_{th})(z) - \frac{\beta^{(0)}_{th}}{4\epsilon^2}P^{(0)}_{th}(z)\right) \\ &+ \mathcal{O}(y_t^2\alpha_s^2, y_t^4), \\ Z_{tt}(z) =& \delta(1-z) + \frac{\alpha_s}{2\pi}\frac{1}{\epsilon}P^{(0)}_{qq} + \mathcal{O}(\alpha_s^2, y_t^2). \end{aligned}$$

$P^{(0)}_{qq}$, $P^{(0)}_{th}$ known $\to$ $P^{(1)}_{th}$ derived as a by-product of our computation.

Splitting Function Results

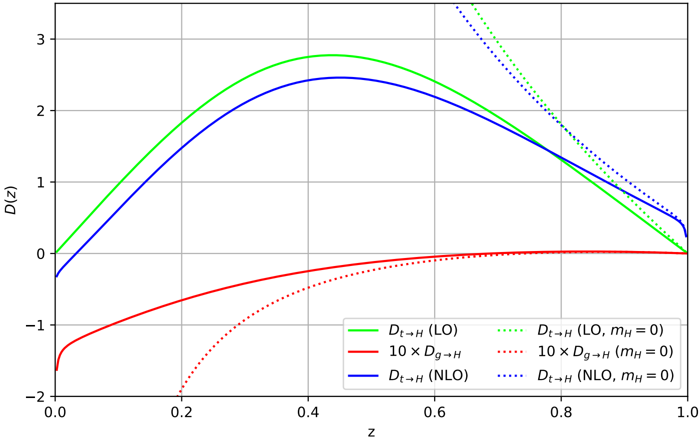
$$P_{th}^{(0)T}(z) = z$$

$$\boxed{P_{th}^{(1)T}(z)} = C_F\Bigg[-8z\, Li_2(z) + z\, ln^2(1-z) - \frac{1}{2} z\, ln^2(z) + 3z\, ln(1-z)$$

$$-4z\, ln(z)\, ln(1-z) + \left(-1 + \frac{1}{2}z\right) ln(z) + \left(-\frac{13}{2} + 15z\right)\Bigg]$$

$$\boxed{P_{gh}^{(1)T}(z)} = 2T_F\left[2(-3+2z+z^2) - (1+5z)\, ln(z) + z\, ln^2(z)\right]$$

$\mathrm{D}_{t\to H}$ AND $\mathrm{D}_{g\to H}$ FRAGMENTATION



Conclusions

CONCLUSIONS

SUMMARY

- Analytic computation of $\mathrm{D}_{t\to H}(z)$ fragmentation at $\mathcal{O}(y_t^2\alpha_s)$,
- Analytic computation of $D_{g\to H}(z)$ fragmentation at $\mathcal{O}(y_t^2\alpha_s)$,
- LO $pp \to t\bar{t}H$ approximated with errors $< 5\%$ for $p_{\mathrm{T},H} > 600\,\mathrm{GeV}$.

OUTLOOK

- Improving the NLO approximation of the $t\bar{t}H$ production,
- Use the formalism to resum large logs appearing in $t\bar{t}H$ production.

Theory status of $t\bar{t}H$ production

Next-to-leading order:

- NLO QCD corrections [Beenakker, Dittmaier, Kramer, Plumper, Spira, Zerwas, '01]
  [Dawson, Orr, Reina, Wackeroth, '01]
- NLO EW and QCD corrections [Frederix, Frixione, Hirschi, Pagani, Shao, Zaro, '14]
  [Zhang, Ma, Zhang, Chen, Guo, '15]

Next-to-leading order + top-quark decays:

- NLO+PS [Frederix, Frixione, Hirschi, Maltoni, Pittau, Torrielli, '11]
  [Garzelli, Kardos, Papadopoulos, Trocsanyi, '11]
  [Hartanto, Jager, Reina, Wackeroth, '15]
  [Maltoni, Pagani, Tsinikos, '16]
- NWA [Zhang, Ma, Zhang, Chen, Guo, '14]
- full off-shell effects in the di-lepton decay channel
  [Denner, Feger, Lang, Pellen, Uccirati, '15-'17]
  + Higgs boson decays in the NWA
  [Stremmer, Worek, '22]

THEORY STATUS OF $t\bar{t}H$ PRODUCTION

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- NLO EW and QCD corrections [Frederix, Frixione, Hirschi, Pagani, Shao, Zaro, '14]
  [Zhang, Ma, Zhang, Chen, Guo, '15]

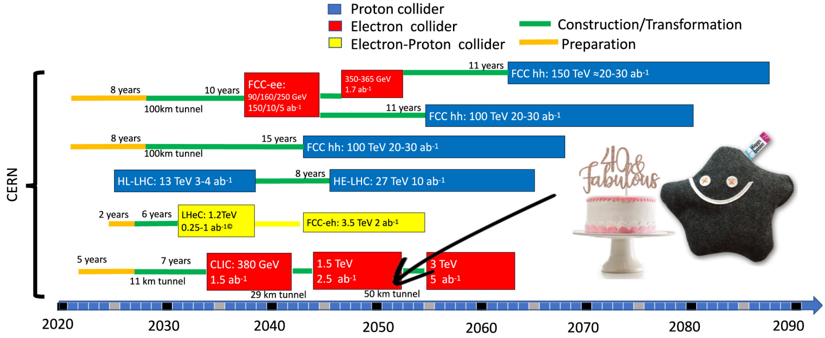
Beyond next-to-leading order:

- NLO+NNLL (soft gluons) [Kulesza, Motyka, Schwartländer, Stebel, Theeuwes, '16]
  [Broggio, Ferroglia, Frederix, Pecjak, Signer, Yang, Tsinikos, '16]
- NNLO in soft Higgs boson approximation
  [Catani, Devoto, Grazzini, Kallweit, Mazzitelli, Savoini, '22]

THE HIGH-$p_{\mathrm{T,H}}$ REGIME

	s [TeV]	σ [fb]	$\sigma_{p_{T,H}>600\,\mathrm{GeV}}$ [fb]	$\mathcal{L}$ [fb^{-1}]	N	$N_{p_{T,H}>600\,\mathrm{GeV}}$
LHC	13	580	0.9	79.8	$4.6 \cdot 10^4$	72
HL-LHC	14	690	1.2	$3 \cdot 10^3$	$2 \cdot 10^6$	$3.6 \cdot 10^3$
HE-LHC	27	$2.8 \cdot 10^3$	12	$10 \cdot 10^3$	$2.8 \cdot 10^{10}$	$1.2 \cdot 10^5$
FCC hh	100	$2.8 \cdot 10^4$	390	$30 \cdot 10^3$	$8.4 \cdot 10^{11}$	$1.2 \cdot 10^7$

Possible scenarios of future colliders



Leading power factorisation formula

The LO $pp \to t\bar{t}H$ cross section is decomposed as:

$$\mathrm{d}\hat{\sigma}_{pp\to t\bar{t}H}(P) \approx \underbrace{\mathrm{d}\tilde{\sigma}_{pp\to t\bar{t}H}(P,\mu)}_{\substack{\text{Direct Contribution} \\ q^2 \sim \hat{s}}} + \underbrace{2\int_0^1 \mathrm{d}z\, \mathrm{d}\tilde{\sigma}_{pp\to t\bar{t}}(p = P/z) \mathrm{D}_{t\to H}(z,\mu)}_{\substack{\text{Fragmentation Contribution} \\ q^2 \ll \hat{s}}}.$$

The direct (infrared-safe) contribution is computed as:

$$\mathrm{d}\tilde{\sigma}_{pp\to t\bar{t}H}(P,\mu) \approx \lim_{m_t\to 0}\Bigg(\lim_{m_H\to 0} \mathrm{d}\hat{\sigma}_{pp\to t\bar{t}H}(P) - 2\int_0^1 \mathrm{d}z\, \mathrm{d}\tilde{\sigma}_{pp\to t\bar{t}}(p = P/z) \lim_{m_H\to 0} \mathrm{D}_{t\to H}(z,\mu)\Bigg).$$

Reverse Unitarity

The Dirac δ distribution can be replaced by the imaginary part of an effective propagator:

$$\begin{aligned}
\mathrm{Disc}_z\left(\frac{1}{z}\right) &= \lim_{\epsilon\to 0}\left(\frac{1}{z-i\epsilon}-\frac{1}{z+i\epsilon}\right)\\
&= \lim_{\epsilon\to 0}\left(\frac{z+i\epsilon}{(z-i\epsilon)(z+i\epsilon)}-\frac{z-i\epsilon}{(z-i\epsilon)(z+i\epsilon)}\right)\\
&= \lim_{\epsilon\to 0}\left(\frac{2i\epsilon}{(z-i\epsilon)(z+i\epsilon)}\right)\\
&= 2i\lim_{\epsilon\to 0}\left(\frac{\epsilon}{z^2-\epsilon^2}\right)\\
&= 2\pi i\,\delta(z)
\end{aligned}$$

$$\boxed{\delta(z) = \frac{1}{2\pi i}\lim_{\epsilon\to 0}\left(\frac{1}{z-i\epsilon}-\frac{1}{z+i\epsilon}\right)}$$

Differential equations method

- A generic MI is a loop integral which can be represented as a function of the kinematic invariants $\vec{x}$ and the dimensional regulator ϵ: $\boldsymbol{f}(\vec{\boldsymbol{x}}, \boldsymbol{\epsilon})$.
- MI derivatives with respect to each kinematic invariant x_i can be computed by introducing the differential operators:

$$O_{jk} = p_j^{\mu} \sum_{i=1}^{n} \frac{\partial x_i}{\partial p_k^{\mu}} \frac{\partial f(\vec{x}, \epsilon)}{\partial x_i} = \sum_{i=1}^{n} a_{i,jk}(x_i) \frac{\partial f(\vec{x}, \epsilon)}{\partial x_i}.$$

- A **system of first order linear differential equations** for the MIs can be derived:

$$\partial_{x_i} \vec{f}(\vec{x}, \epsilon) = A_{x_i}(\vec{x}, \epsilon) \vec{f}(\vec{x}, \epsilon).$$

Kotikov, '91

CANONICAL BASIS APPROACH

- It is possible to choose a basis of MIs, the **Canonical basis**, such that:

$$d\vec{f}(\vec{x},\epsilon) = \epsilon \, dA(\vec{x}) \, \vec{f}(\vec{x},\epsilon),$$

with

$$dA(\vec{x})_{ij} = \sum_k c_{ijk} \, \mathrm{dlog}(\alpha_k(\vec{x})).$$

- The **solution of the differential equations system** is:

$$\vec{f}(\vec{x},\epsilon) = \mathrm{P}\exp\left[\epsilon \int_\gamma d\tilde{A}(\vec{x}')\right] \vec{f}(\vec{x}_0,\epsilon).$$

Canonical MIs can be expanded in Taylor series around $\epsilon = 0$:

$$\vec{f}(\vec{x},\epsilon) = \sum_{k=0}^{\infty} \epsilon^k \, \vec{f}^{(k)}(\vec{x}) \rightarrow \boxed{\vec{f}^{(k)}(\vec{x}) = \int_\gamma d\tilde{A}(\vec{x}') \, \vec{f}^{(k-1)}(\vec{x}') + \vec{f}^k(\vec{x}_0,\epsilon)}$$

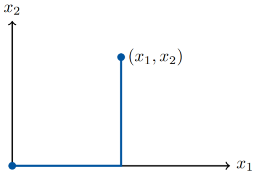
Henn, '13

Goncharov Polylogarithm (GPL)

A possible parametrization for the integration path γ is $\gamma = \cup\gamma_i$ with:

$$\gamma_i(\theta) = x'_k(\theta) = \begin{cases} x_k,\ k < i \\ x_k^0 + \theta(x_k - x_k^0),\ k = i \\ x_k^0,\ k > i \end{cases}$$

$\boxed{i = 2}$ $\rightarrow$



Setting the boundary condition to $\vec{x}_0 = (0, ..., 0)$

$$\vec{f}^{(k)}(\vec{x}) = \int_{(0,\dots,0)}^{(x_1,\dots,0)} A_1(\vec{x})\vec{f}^{(k-1)}(\vec{x})dx_1 + ... + \int_{(x_1,\dots,x_{n-1},0)}^{(x_1,\dots,x_n)} A_n(\vec{x})\vec{f}^{(k-1)}(\vec{x})dx_n + \vec{f}^{(k)}(\vec{0})$$

Since the $A(\vec{x})$ is in dlog-form:

$$\vec{f}^{(k)}(x) = \underbrace{\int^{x} \frac{dt_1}{t_1 - a_k} ... \int^{t_{k-2}} \frac{dt_{k-1}}{t_{k-1} - a_2} \int^{t_{k-1}} dt_k \frac{f^{(0)}}{t_k - a_1}}_{\text{GPL of weight } k \,\rightarrow\, G(a_1, ..., a_k; x)}$$

Goncharov, '01

Canonical matrix for virtual corrections

- **Rationalizing** roots: $m_t^2 \to \frac{m_h^2}{4}(-\tau^2 + 1)$.
- Canonical matrix in **dlog-form**:

$$
\begin{aligned}
\mathrm{d}A_\tau = & M_1 \,\mathrm{dlog}\,(\tau) + M_2 \,\mathrm{dlog}\,(1-\tau) + M_3 \,\mathrm{dlog}\,(1+\tau) \\
& + M_4 \,\mathrm{dlog}\,(2 - z\,(1-\tau)) + M_5 \,\mathrm{dlog}\,(-2 + z\,(1+\tau)) \\
& + M_6 \,\mathrm{dlog}\,\left(-4 + z\left(3 + \tau^2\right)\right)
\end{aligned}
$$

  where M_i are rational 8×8 matrices.

- Solution given in terms of **GPLs**.
- Integration constants matched in limit $m_t^2 \to \infty$.

Simplifying polylogarithms expressions with symbols

- Take the symbol of the initial form
- Use symbol properties to simplify the expression
- Write the simplified expression in terms of polylogaritmhs
  $\rightarrow$ Exploit that different types of contributions satisfy different **symmetry relations**

Example: At weight 2, there are 4 types of contributions:

$$\mathrm{Li}_2(a), \qquad \ln(a)\ln(b), \qquad \pi\ln(a), \qquad \pi^2,$$

with

$$\mathcal{S}(\mathrm{Li}_2(a)) = -\ln(1-a)\otimes\ln(a)$$
$$\mathcal{S}(\ln(a)\ln(b)) = \ln(a)\otimes\ln(b) + \ln(b)\otimes\ln(a).$$

Softwares - a summary

- Amplitude computation $\rightarrow$ *FORM*
- Reduction to MIs and differential equations $\rightarrow$ *FIRE*
- Integration in term of GPLs and simplification with symbols $\rightarrow$ *PolyLogTools*
- Computation of boundary conditions $\rightarrow$ *HypExp*
- Numerical checks of MIs $\rightarrow$ *FIESTA*

- Fragmentation convolution with massless cross section $\rightarrow$ *STRIPPER*
- Comparison with NLO $t\bar{t}H$ production $\rightarrow$ *MADGRAPH5*