HYDRODYNAMIZATION AND ASYMPTOTICS: THE EARLY TO LATE TIMES IN RELATIVISTIC HYDRODYNAMICS

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Nonperturbative Methods in QFT, 25 April 2022

EPSRC
Engineering and Physical Sciences Research Council

UNIVERSITY OF Southampton

$$\sum_{n=0}^{\infty} E_n\, g^n \quad e^{-A/g}$$

RELATIVISTIC HYDRODYNAMICS

Provides a reliable description of strongly coupled systems close to thermal equilibrium

Real life: strongly coupled quark-gluon plasma in particle accelerators;

To determine the kinetic parameters of hydrodynamic equations (e.g. shear viscosity): study the associated microscopic theory

- The associated microscopic theory can be a QFT, such as strongly coupled $\mathcal{N} = 4$ Super Yang-Mills (SYM), studied via holography

  [Heller,Janik,Witaszczyk'11,'13;IA,Meiring, Jankowski,Spalinski'18]

- Other microscopic models have been studied

  [Romatschke'17; Strickland,Noronha,Denicol'17]

STRONGLY COUPLED SYSTEMS

Kinematic regime: ***expanding plasma*** in the so-called central rapidity region, where one assumes ***longitudinal boost invariance*** (Bjorken flow)

[Bjorken '83]

In hydrodynamic theories the energy-momentum tensor is given by

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}(\mathcal{E})(\eta^{\mu\nu} + u^\mu u^\nu) + \Pi^{\mu\nu}$$

Energy density

Pressure, in 4d conformal theories given by:

$$\mathcal{P}(\mathcal{E}) = \mathcal{E}/3$$

flow velocity

Shear stress tensor: dissipative effects

Symmetries: conformal invariance, transversely homogeneous, invariance under longitudinal Lorentz boosts

STRONGLY COUPLED SYSTEMS

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[Bjorken '83]

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Strongly coupled boost invariant plasma:
all physics encoded in $\mathcal{E}(\tau)$.
Obtaining this function is in general too difficult:
perform a ***large proper time expansion*** $\tau \gg 1$.

LATE TIME BEHAVIOUR

Starting from highly non-equilibrium initial conditions, the microscopic theory will reveal the transition to hydrodynamic behaviour at late times

Conformal theories: late-time behaviour of energy density highly constrained

$$\mathcal{E}(\tau) = \frac{\Lambda}{(\Lambda\tau)^{1/3}} \left(1 + \sum_{k=1}^{+\infty} \frac{\epsilon_k}{(\Lambda\tau)^{2k/3}} \right), \quad \tau \gg 1$$

- Λ is a dimensionful parameter encoding initial non-eq. conditions
- Leading behaviour predicted by boost-invariant perfect fluid
- Subleading terms: dissipative hydrodynamic effects

Viscous hydrodynamics:	transition to equilibrium described by late time expansion

MIS CAUSAL HYDRODYNAMICS

Müller-Israel-Stewart (MIS) approach

- embed hydrodynamics in a framework compatible with relativistic causality;
- introduces non-hydrodynamic degrees of freedom;
- means of generating the hydrodynamic gradient expansion, studied as if it came from a microscopic theory

[Müller'67;Israel, Stewart'79]

MIS CAUSAL HYDRODYNAMICS

Solve evolution equations of the Energy momentum tensor

$$\nabla_\mu T^{\mu\nu} = 0$$

- Assume ***boost invariant flow, conformal invariance***
- ***Hydrodynamic gradient expansion***: approximate shear stress tensor by corrections to ideal fluid

Müller-Israel-Stuart (MIS) equations

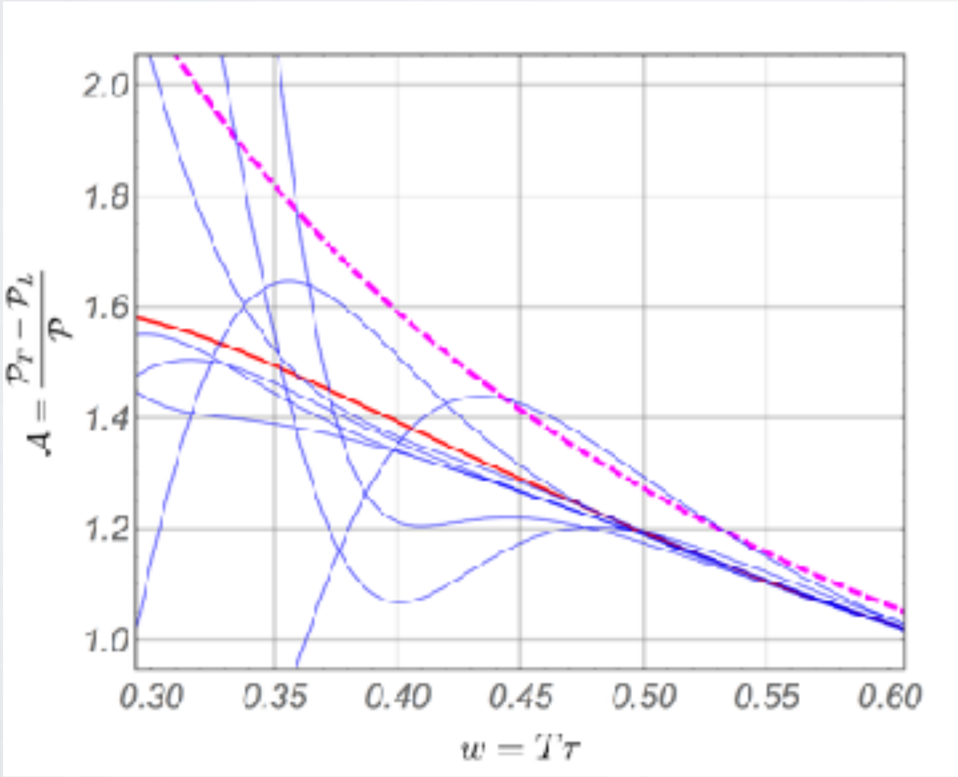
$$z\,C_{\tau\Pi} f\, f' + 4C_{\tau\Pi} f^2 + \left(z - \frac{16C_{\tau\Pi}}{3}\right) f - \frac{4C_\eta}{9} + \frac{16C_{\tau\Pi}}{9} - \frac{2z}{3} = 0$$

- Non-linear ODE describing the pressure anisotropy
- $C_{\tau\Pi},\ C_\eta$ are phenomenological parameters

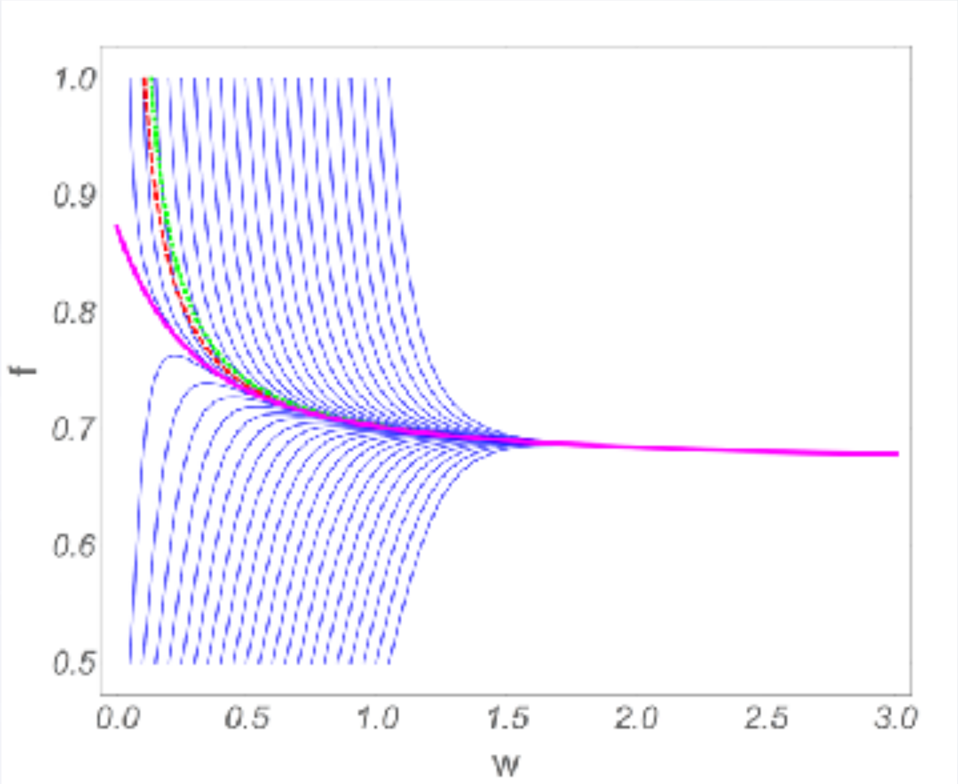
THE SUCCESS OF HYDRODYNAMICS

Hydrodynamic description accurate at earlier times than expected!

- Early time behaviour dictated by initial conditions
- Hydrodynamization: convergence to hydrodynamic description while the system is still very anisotropic and inhomogeneous
- Hydrodynamic description to equilibrium independent of initial conditions



$\mathcal{N} = 4SYM$ [Spalinski '17]



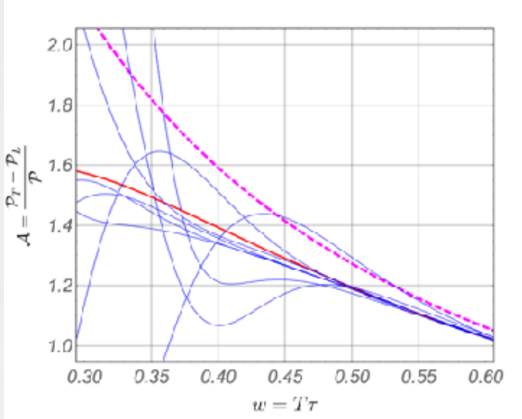
MIS [Heller,Spalinski '15]

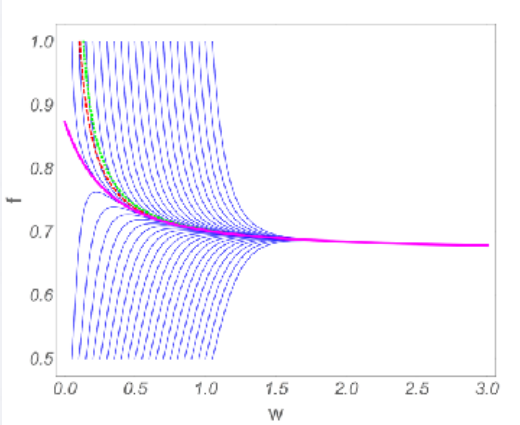
HYDRODYNAMIC ATTRACTORS

How to describe this early convergence towards hydrodynamics?

Hydrodynamic attractors

- The hydrodynamic model contains ***non-hydrodynamic degrees of freedom***, non-perturbative in nature
- These modes play a major role during the early times of the expanding plasma, very sensitive to initial conditions
- At ***hydrodynamisation scale*** still far from equilibrium, but the different initial solutions all become exponential close to to each other, and the
- Evolution of the system towards equilibrium effectively described by viscous hydrodynamics





LATE TIME ASYMPTOTICS

From asymptotics the phenomena of hydrodynamization is expected.

$$\mathcal{E}(\tau) = \frac{\Lambda}{(\Lambda\tau)^{1/3}}\left(1 + \sum_{k=1}^{+\infty} \frac{\epsilon_k}{(\Lambda\tau)^{2k/3}}\right), \quad \tau \gg 1$$

Series is asymptotic:
ϵ_k factorially divergent!

Late-time hydrodynamic attractor:

- described by a divergent, asymptotic perturbative series;
- resurgent properties encode all the information about the exponentially small non-hydrodynamic modes;
- initial conditions uniquely encoded in a set of parameters determining the strength of the non-hydrodynamic modes

RESURGENCE AND ASYMPTOTICS

$$\mathcal{E}(\tau) = \frac{\Lambda}{(\Lambda\tau)^{1/3}}\left(1 + \sum_{k=1}^{+\infty} \frac{\epsilon_k}{(\Lambda\tau)^{2k/3}}\right), \quad \tau \gg 1$$

Series is divergent, **resurgent**!

How can we **match the late-time behaviour to any given initial condition**?

Beyond a purely numerical analysis, can we **describe the system at all times**?

Can we hope to describe the **analytic behaviour** of our observable?

Resurgence and asymptotics

- asymptotic expansions "converge" quite quickly;
- established asymptotic ***summation methods with exponential accuracy***, effectively distinguishing between the exponentially close solutions at late-times;
- obtain ***global analytic properties*** of the asymptotic observables.

OUTLINE

0. ~~Motivation~~

1. Introduction to resurgent transseries
   - Resurgence and Borel transforms
   - Summations

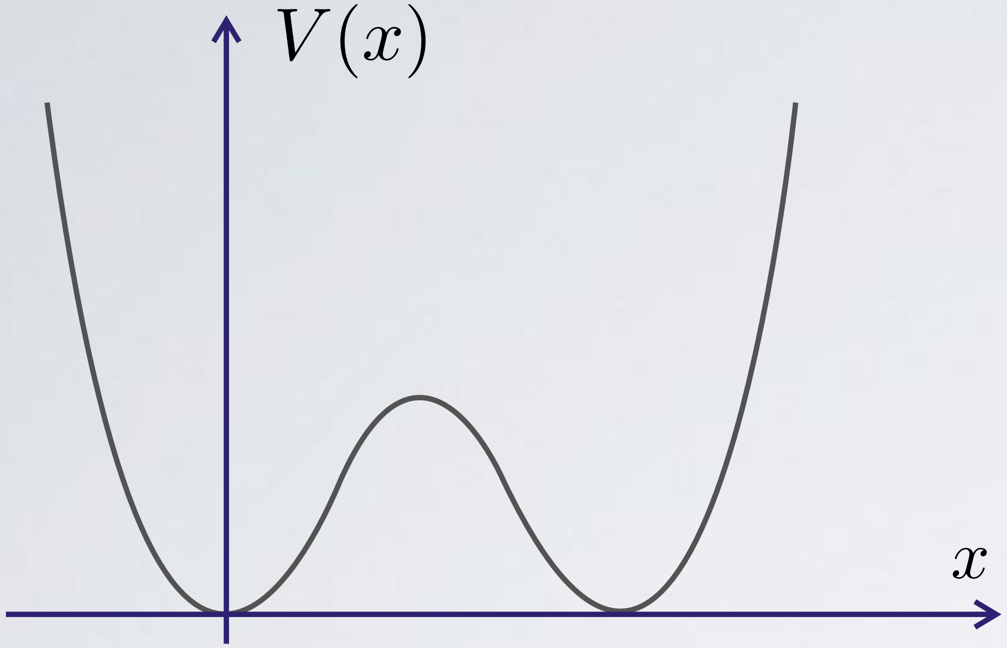
2. Müller-Israel-Stewart hydrodynamics
   - From late to early times: dependence on initial conditions
   - Branch points and global behaviour

3. Summary

1.
INTRODUCTION TO RESURGENT TRANSSERIES

[IA,Basar,Schiappa'18]

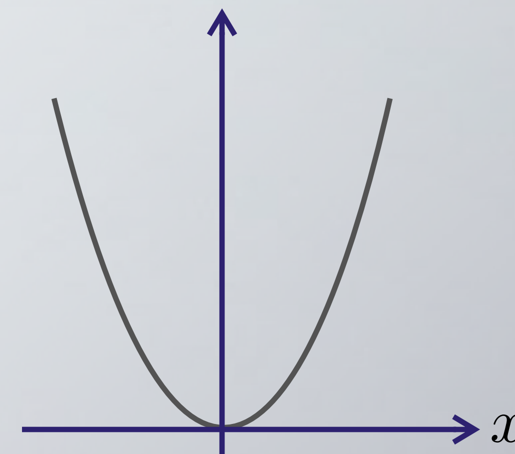
DOUBLE WELL IN QM



e.g. $V(x) = \frac{1}{2} x^2 \left(1 - \sqrt{g}\, x\right)^2$

Coupling ($\sim \hbar$)

$g = 0$ ➡ Harmonic oscillator



$$V_H(x) = \frac{1}{2} x^2$$

$$E_{g.s.} = \frac{1}{2}$$

$g > 0$ How can we solve it?

- Hamiltonian

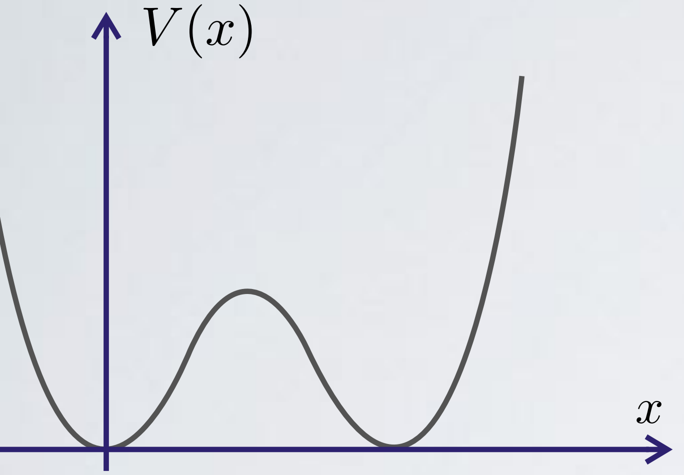
$$H = -\frac{1}{2}\left(\frac{d}{dx}\right)^2 + V(x)$$

- Schrödinger Eq:

$$H\,\psi(x,g) = E(g)\,\psi(x,g)$$

PERTURBATION THEORY IN QM

g very small → Perturbation theory → $E_{g.s.}(g) \simeq \sum_{n=0}^{\infty} E_n\, g^n$



$$V(x) = \frac{1}{2}x^2\,(1-\sqrt{g}\,x)^2$$

- ***Series is asymptotic***: For large enough n

$$E_n \sim n!\,A^{-n}$$

Why asymptotic? Existence of instantons

Corrections to $E_{g.s.} \sim \mathrm{e}^{-A/g} \sum_{n=0}^{\infty} E_n^{(1)}\, g^n$ Suppressed!

BEYOND PERTURBATION THEORY

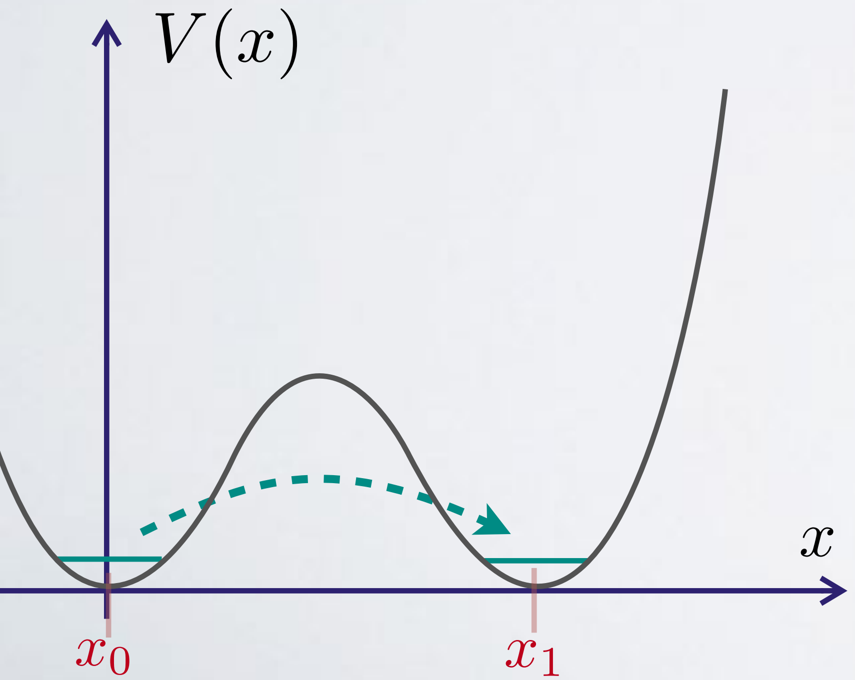
g very small → $E_{g.s.}(g) \simeq \sum_{n=0}^{\infty} E_n^{(0)} g^n$

$+$

Instanton corrections to $E_{g.s.}$ → $\sim \mathrm{e}^{-A/g} \sum_{n=0}^{\infty} E_n^{(1)} g^n$

$+$

Higher instanton corrections → $\mathcal{O}\left(\mathrm{e}^{-2A/g}\right)$



[Vanstein'64;Bender,Wu'73;Bogomolny,Zinn-Justin'80]

TRANSSERIES SOLUTION

$$E_{g.s.}(g,\sigma) \simeq \sum_{k=0}^{\infty} \sigma^k e^{-kA/g} E^{(k)}(g) \qquad E^{(k)}(g) \simeq \sum_{n=0}^{\infty} E_n^{(k)} g^n$$

k-instanton contribution,
each is asymptotic

$$E_n^{(k)} \sim n!\,(kA)^{-n}$$

Formal expansion in transmonomials

- the small parameter g
- non-perturbative term $e^{-A/g}$
- σ encodes boundary/initial conditions

$E_{g.s.}(g,\sigma)$ requires all instantons to be well defined

[Edgar'08]

ADDING ALL CONTRIBUTIONS

$$E_{g.s.}(g) \simeq \sum_{k=0}^{\infty} \mathrm{e}^{-k\,A/g}\, E^{(k)}(g)$$

ADDING ALL CONTRIBUTIONS

$$E_{g.s.}(g) \simeq E^{(0)}(g) + \mathrm{e}^{-A/g}\, E^{(1)}(g) + \mathrm{e}^{-2\,A/g}\, E^{(2)}(g) \,\cdots$$

Optimal error:

$$\left(E - E_N^{(0)}\right)(g) \sim \mathrm{e}^{-A/g}$$

Contributes at order

$$\mathrm{e}^{-A/g}$$

The one-instanton sector addresses the error from the perturbative series

ADDING ALL CONTRIBUTIONS

$$E_{g.s.}(g) \simeq E^{(0)}(g) + \mathrm{e}^{-A/g}\, E^{(1)}(g) + \mathrm{e}^{-2\,A/g}\, E^{(2)}(g) \cdots$$

Optimal error after summing

$$E^{(0)}(g) + \mathrm{e}^{-A/g}\, E^{(1)}(g) \;?$$

ADDING ALL CONTRIBUTIONS

$$E_{g.s.}(g) \simeq E^{(0)}(g) + \mathrm{e}^{-A/g} E^{(1)}(g) + \mathrm{e}^{-2A/g} E^{(2)}(g) \cdots$$

Optimal error after summing

$$E^{(0)}(g) + \mathrm{e}^{-A/g} E^{(1)}(g) \ ?$$

$$E^{(1)} \simeq \sum_{n=0}^{\infty} E_n^{(1)} g^n$$ asymptotic series!

Optimal error: $\left(E^{(1)} - E_N^{(1)}\right)(g) \sim \mathrm{e}^{-A/g}$

Error after summing perturbative and 1-instanton is

$$E_{g.s.} - \left(E^{(0)} + \mathrm{e}^{-A/g} E^{(1)}\right) \sim \mathrm{e}^{-2A/g}$$

ADDING ALL CONTRIBUTIONS

$$E_{g.s.}(g) \simeq E^{(0)}(g) + \mathrm{e}^{-A/g}\, E^{(1)}(g) + \mathrm{e}^{-2\,A/g}\, E^{(2)}(g)\, \cdots$$

Error after summing perturbative and 1-instanton is

$$E_{g.s.} - \left(E^{(0)} + \mathrm{e}^{-A/g}\, E^{(1)}\right) \sim \mathrm{e}^{-2\,A/g}$$

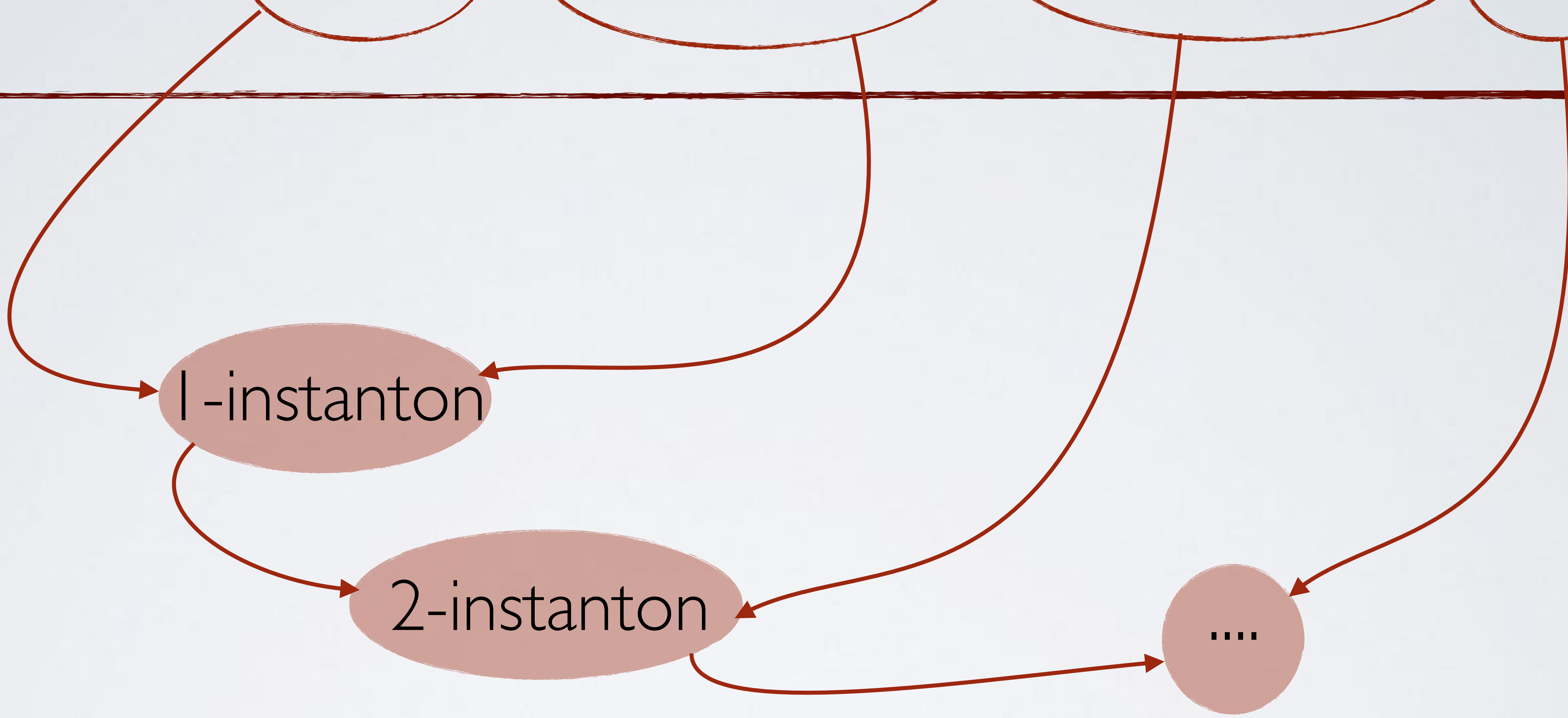
Again $E^{(2)} \simeq \sum_{n=0}^{\infty} E_n^{(2)}\, g^n$ asymptotic series! $\left(E^{(2)} - E_N^{(2)}\right)(g) \sim \mathrm{e}^{-A/g}$

Two-instanton sector addresses the new error

$$E_{g.s.} - \left(E^{(0)} + \mathrm{e}^{-A/g}\, E^{(1)} + \mathrm{e}^{-2A/g}\, E^{(2)}\right) \sim \mathrm{e}^{-3\,A/g}$$

ADDING ALL CONTRIBUTIONS

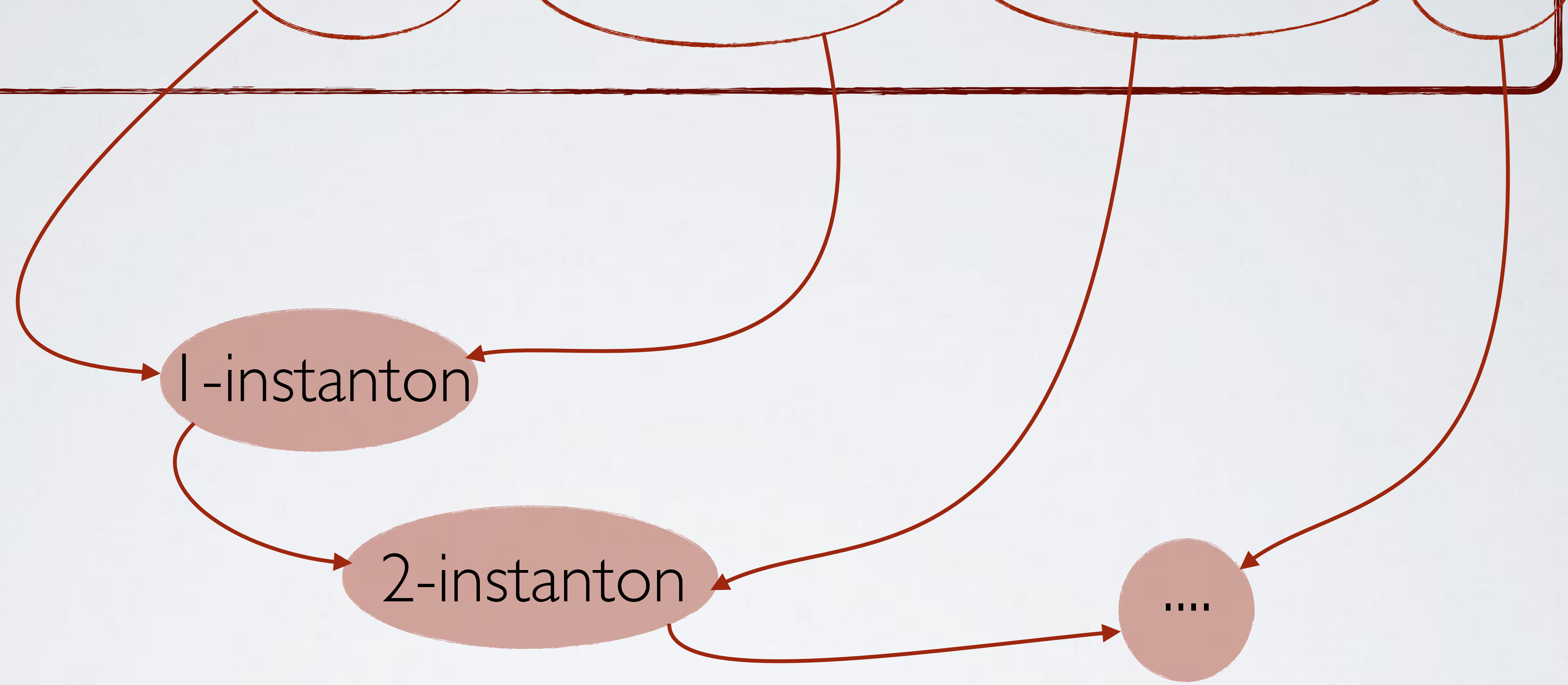
$$E_{g.s.}(g) \simeq E^{(0)}(g) + \mathrm{e}^{-A/g}\, E^{(1)}(g) + \mathrm{e}^{-2\,A/g}\, E^{(2)}(g) \cdots$$



All instanton contributions conspire to cancel the errors at higher and higher orders!

ADDING ALL CONTRIBUTIONS

$$E_{g.s.}(g) \simeq E^{(0)}(g) + \mathrm{e}^{-A/g}\, E^{(1)}(g) + \mathrm{e}^{-2\,A/g}\, E^{(2)}(g) \cdots$$



Resurgence

[Écalle'81]

A TRUE ANALYTIC SOLUTION?

Transseries $$E_{g.s.} \simeq \sum_{k=0}^{\infty} \sigma^k \mathrm{e}^{-kA/g} E^{(k)}(g) \qquad E^{(k)} \sim \sum_{n=0}^{\infty} E_n^{(k)} g^n$$

- Re-sum all asymptotic sectors $\mathcal{S}E^{(k)}(g)$
- Detemine σ from external data (boundary/initial conditions)
- This can be done for any value of g and encodes:

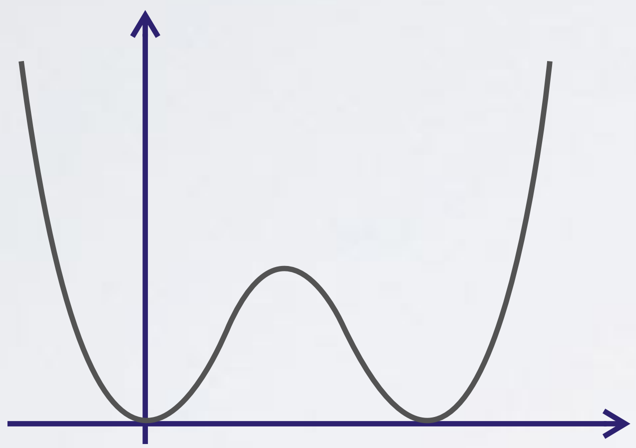
Analytic data (poles, zeros, branch cuts)

Phase transitions (Stokes phenomena)

Next: Borel transform and re-summation

BOREL TRANSFORMS

Determine NP phenomena from an asymptotic series



$$E_{g.s.}(g) \simeq \sum_{n=0}^{\infty} E_n^{(0)} g^n$$

$$E_n^{(0)} \sim \frac{n!}{A^n}$$ for large enough n

Remove the factorial growth to get a convergent series: inverse Laplace transform to each term

$$B_E(s) = \sum_{n=0}^{\infty} \frac{E_n^{(0)}}{n!} s^n$$

BOREL TRANSFORMS

Let's simplify our example!

Original Series:

$$E_{g.s.}(g) \simeq \sum_{n=0}^{\infty} E_n^{(0)}\, g^n$$

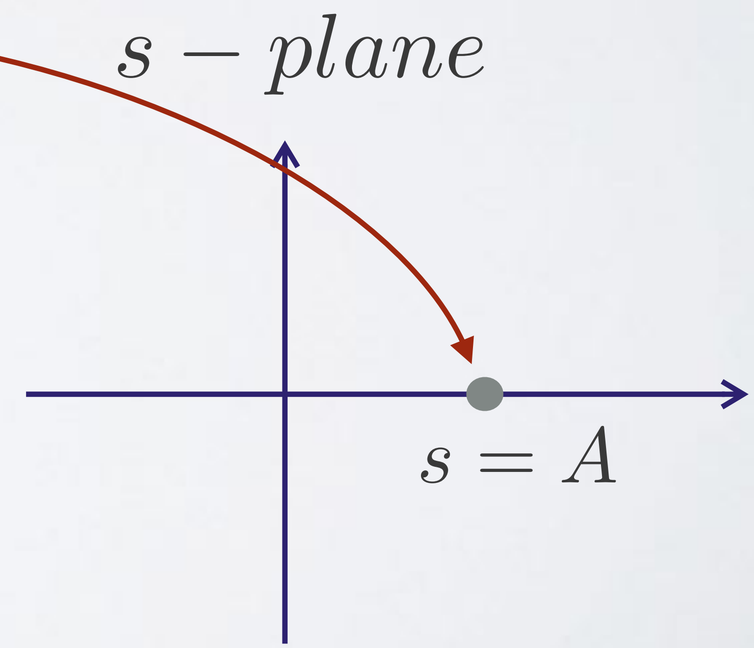
$$E_n^{(0)} = \frac{n!}{A^n}$$

Borel transform:

$$B_E(s) = \sum_{n=0}^{\infty} \frac{E_n^{(0)}}{n!} s^n = \sum_{n=0}^{\infty} \left(\frac{s}{A}\right)^n = \frac{A}{A-s}$$

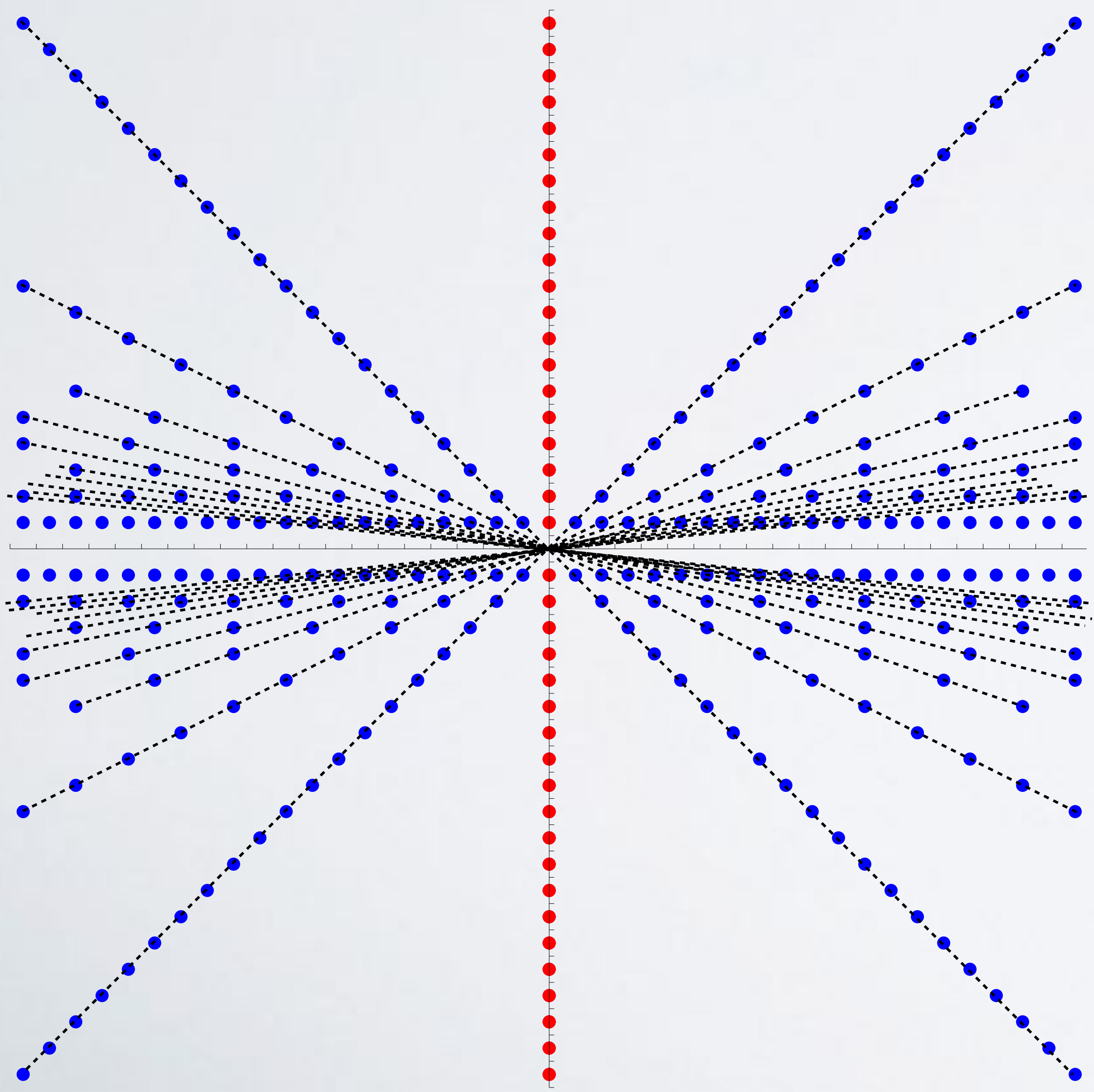
single pole at $s = A$

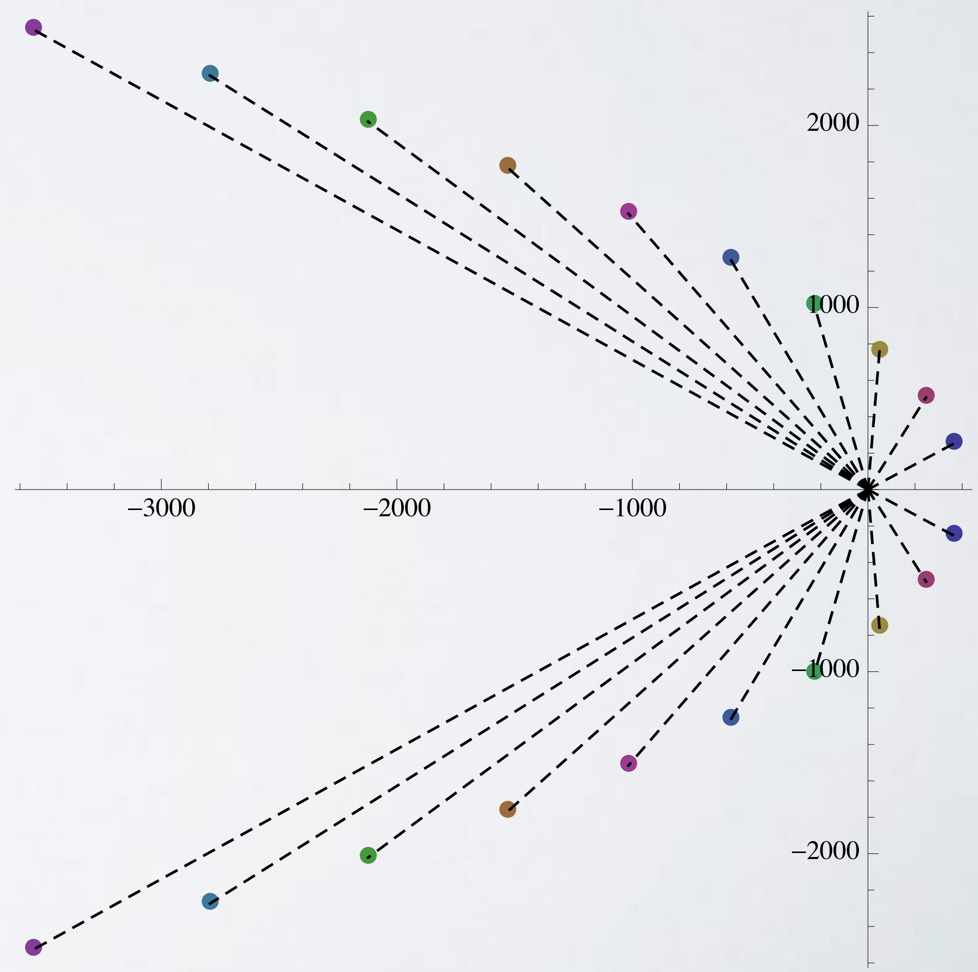
The position of the pole is controlled by instanton action



BOREL TRANSFORMS

- Non-perturbative phenomena: singularities in Borel plane
- Singularities usually will be branch cuts
- Singular directions: ***Stokes lines***
- Structure of singularities can be very complex





[IA,Russo,Schiappa'14]

TRANSSERIES SUMMATIONS

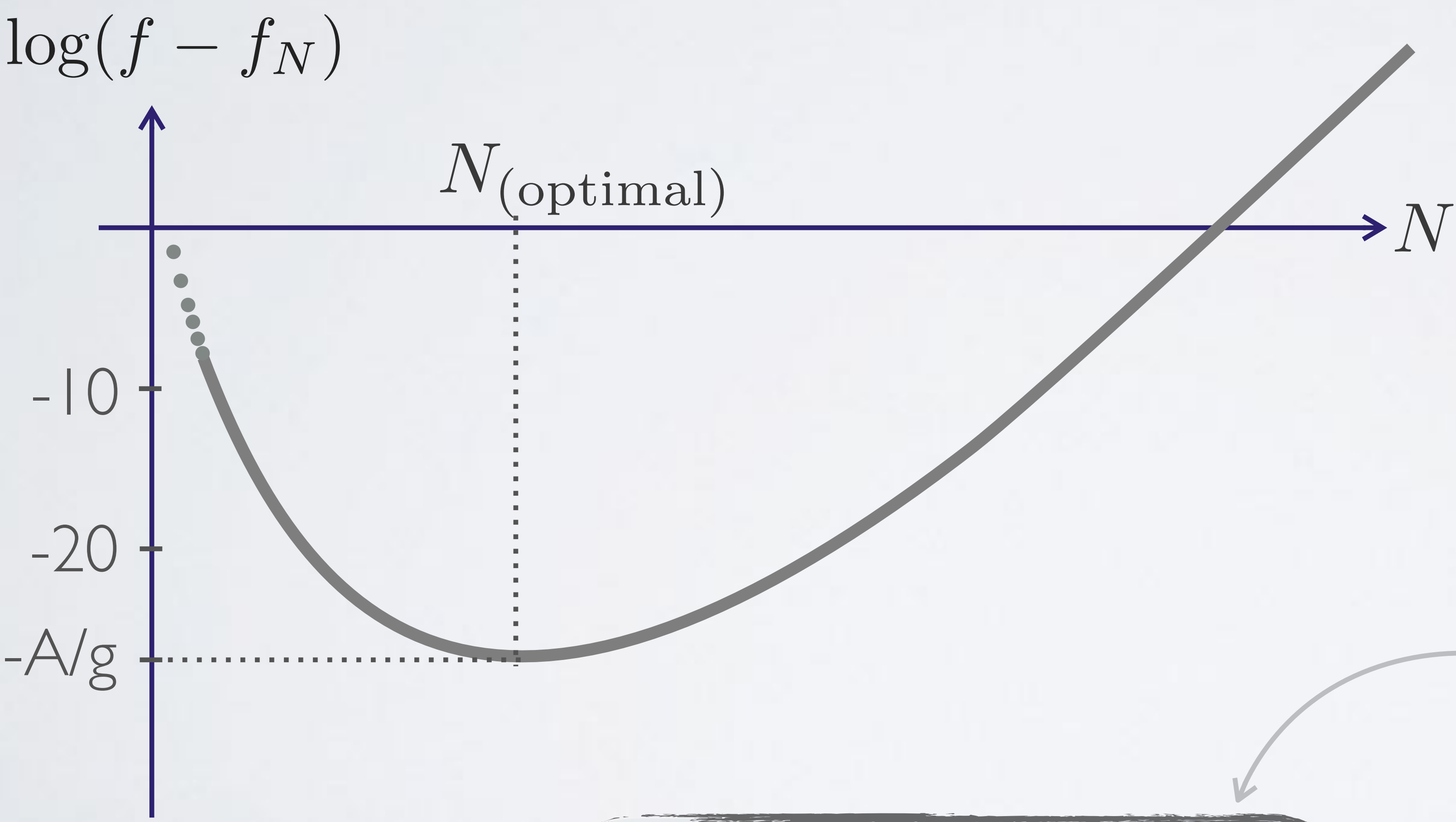
How to associate a function to the original asymptotic series?

- Optimal truncation and Hyperasymptotics
- Borel summation
- Transasymptotics

OPTIMAL TRUNCATION

$$f(g) \simeq \sum_{n=0}^{\infty} f_n \, g^n$$

- Assume g fixed and small
- Define $f_N(g) = \sum_{n=0}^{N} f_n \, g^n$



$$N_{(\text{optimal})} \approx A/g$$

Optimal error:

$$(f - f_N)(g) \sim \mathrm{e}^{-A/g}$$

for some value A

Non-perturbative effect!

LEVEL-1 HYPERASYMPTOTICS

Approximate the transseries including first exponential sector

$$E_{g.s.} \simeq E^{(0)}(g) + \sigma\, \mathrm{e}^{-A/g} E^{(1)}(g) + \cdots$$

$$E_{\mathrm{g.s.,Hyp}}(g) \simeq E_{\mathrm{Hyp},0}(g) + \sigma\, E_{\mathrm{Hyp},1}(g)$$ Error $\sim \mathrm{e}^{-2A/g}$

$$E_{\mathrm{Hyp},0}(g) = \sum_{m=0}^{N_{\mathrm{Hyp}}(g)-1} E_m^{(0)} g^m + g^{N_{\mathrm{Hyp}}(g)-1} \frac{S_1}{2\pi \mathrm{i}} \sum_{m=0}^{N_{\mathrm{Hyp}}(g)/2-1} E_m^{(1)} F^{(1)}\left(g; \begin{matrix} N_{\mathrm{Hyp}}(g) - m \\ -A \end{matrix} \right)$$

$$E_{\mathrm{Hyp},1}(g) = \mathrm{e}^{-A/g} \sum_{m=0}^{N_{\mathrm{Hyp}}(g)/2-1} E_m^{(1)} g^m$$

Truncation: $N_{\mathrm{Hyp}}(g) \sim 2A/g$

Hyperterminant: $F^{(1)}\left(g; \begin{matrix} M \\ a \end{matrix} \right)$

$$\mathrm{e}^{\frac{a}{g}+i\pi M} g^{1-M}\, \Gamma(M)\, \Gamma\left(1 - M, \frac{a}{g}\right)$$

[Berry,Howls'90,'91;Olde Daalhuis'95]

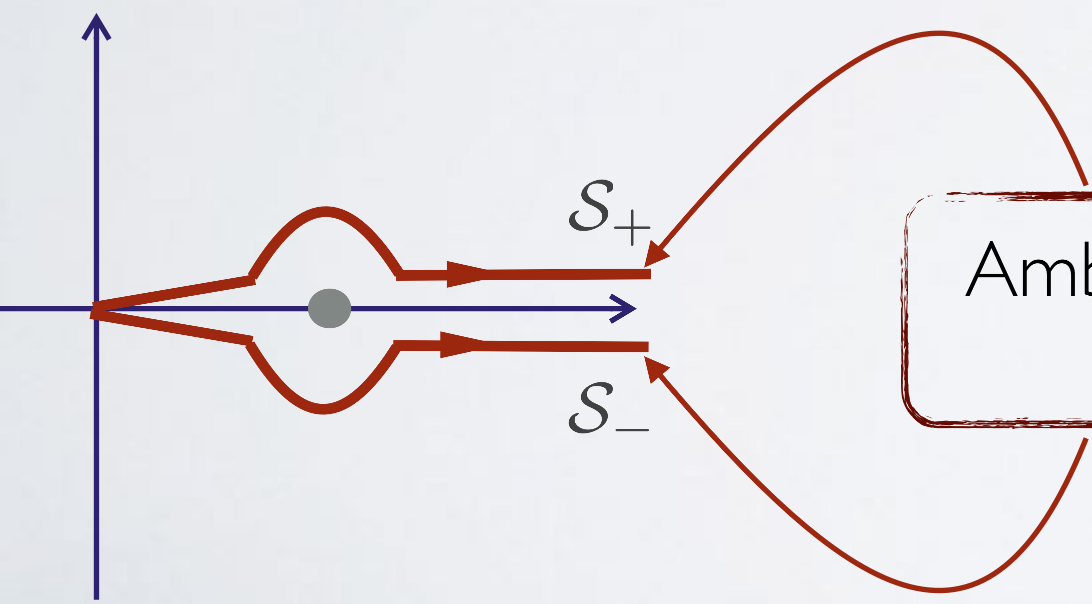
BOREL RESUMMATION

Borel resummation: Laplace transform

$$\mathcal{S}E_{g.s.}(g) = \int_0^\infty \mathrm{d}s B_E(s) \mathrm{e}^{-s/g}$$

$$B_E(s) = \frac{A}{A-s}$$

- Straightforward in the directions without singularities
- Re-summation along Stokes directions: ***Non-perturbative ambiguity***



Ambiguity in choice of contour

$$\mathcal{S}_+ - \mathcal{S}_- = \oint_{s=A} \mathrm{d}s \frac{A}{A-s} \mathrm{e}^{-s/g} \sim \mathrm{e}^{-A/g}$$

[IA,Basar,Schiappa'18]

ASIDE: NUMERICAL SUMMATION

What if we don't know the functional form of $B_E(s)$?

If we know finite number (N) of terms of Borel transform

- Approximation methods such as Padé approximants

$$\mathrm{BP}_E^{(N)}(s)$$

- Numerical re-summation for each value of g

$$\mathcal{S}_N E_{g.s.}(g) = \int_0^\infty \mathrm{d}s\, \mathrm{BP}_E^{(N)}(s) \mathrm{e}^{-s/g}$$

BOREL RESUMMATION

Approximate the transseries including first exponential sector

$$E_{g.s.} \simeq E^{(0)}(g) + \sigma\, \mathrm{e}^{-A/g} E^{(1)}(g) + \cdots$$

$$E_{\mathrm{g.s.,B}}(g) \simeq \mathcal{S}_{N_0} E^{0}(g) + \sigma\, \mathrm{e}^{-A/g}\, \mathcal{S}_{N_0} E^{(1)}(g)$$

Error $\sim \mathrm{e}^{-2A/g}$

$$\mathcal{S}_{N_0} E^{(j)} = \int_0^{+\infty} \mathrm{d}\xi\, \mathrm{e}^{-\xi/g}\, \mathrm{BP}_{N_0}[E^{(j)}](\xi)$$

Truncation: N_0 any, but at least $2A/g$ to minimise error

TRANSASYMPTOTIC SUMMATION

Transseries $E_{g.s.} \simeq \sum_{k=0}^{\infty} \sigma^k \mathrm{e}^{-kA/g} E^{(k)}(g)$ $\qquad E^{(k)} \sim \sum_{n=0}^{\infty} E_n^{(k)} g^n$

Change the order of summation: sum all exponentials for each order g^n

$$E_{\text{g.s.}} \simeq \sum_{k=0}^{+\infty} \sigma^k \mathrm{e}^{-kA/g} \sum_{n=0}^{+\infty} E_n^{(k)} g^n = \sum_{n=0}^{+\infty} g^n \sum_{k=0}^{+\infty} \Big(\underbrace{\sigma\, \mathrm{e}^{-A/g}}_{\tau}\Big)^k E_n^{(k)} = \sum_{n=0}^{+\infty} g^n F_n(\tau)$$

where $F_n(\tau) = \sum_{k=0}^{+\infty} \tau^k E_n^{(k)}$

- **Validity**: small g, exponentials can be order 1
- Study **analytic properties** of solution: e.g. branch points

[Costin'99,'01]

3.

MÜLLER-ISRAEL-STEWART HYDRODYNAMICS

MIS CAUSAL HYDRODYNAMICS

Look back at the Müller-Israel-Stewart (MIS) ODE:

$$z\, C_{\tau\Pi} f\, f' + 4C_{\tau\Pi} f^2 + \left(z - \frac{16C_{\tau\Pi}}{3}\right) f - \frac{4C_\eta}{9} + \frac{16C_{\tau\Pi}}{9} - \frac{2z}{3} = 0$$

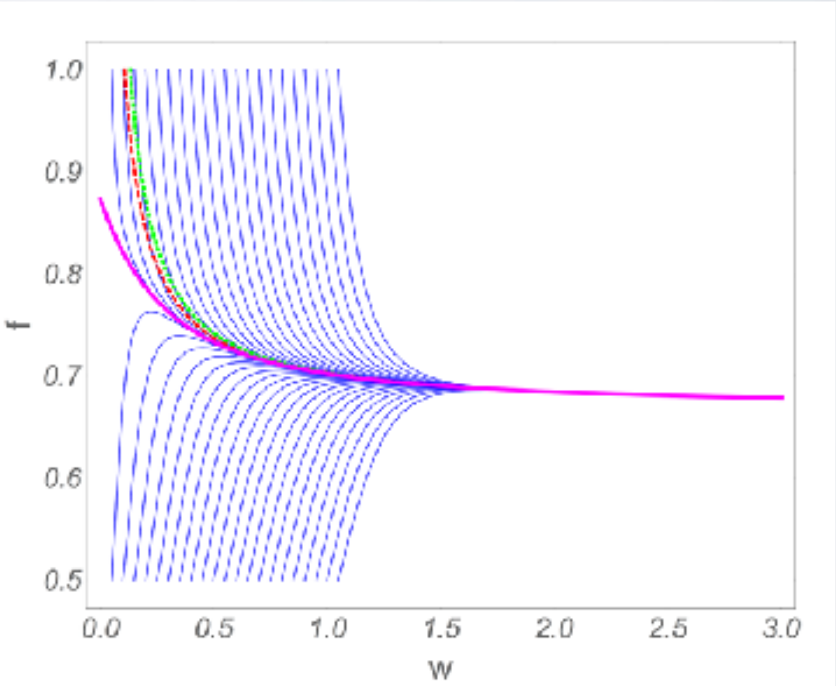
Write **solution at late times as a resurgent transseries**:

- single, purely decaying non-hydrodynamic mode
- describes decay to hydrodynamic attractor

$$\mathcal{F}\left(z,\sigma\right) = \sum_{n=0}^{+\infty} \sigma^n\, \mathrm{e}^{-nAz}\, \Phi_n\left(z\right)$$

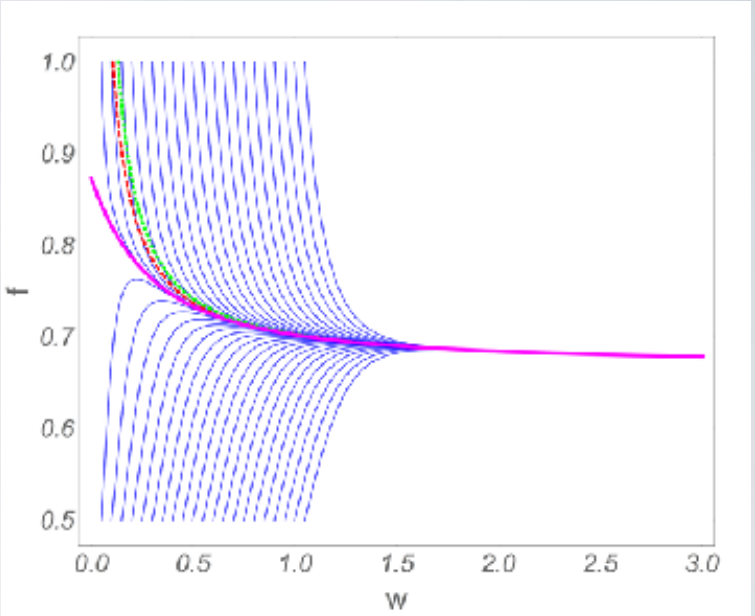
$$\Phi_n\left(z\right) = z^{-n\beta} \sum_{k=0}^{+\infty} a_k^{(n)} z^{-k}$$

$$\beta = -\frac{C_\eta}{C_{\tau\Pi}} \qquad A = \frac{3}{2C_{\tau\Pi}}$$



[Heller,Spalinski'15; Basar,Dunne'15; IA,Spalinski'15]

SOLUTION AT EARLY TIMES



[Heller,Spalinski '15]

Finite solution: Stable solution, converging to a finite value at early times

$$f_{\text{finite}}(z) = \frac{2}{3}\left(1 - \beta + \mathcal{O}(z)\right), \quad z \to 0$$

Generic solution: 1-parameter family of solutions divergent at early times

$$f_C(z) = \frac{2}{3}\left(\frac{C}{z^4} + 2 + \mathcal{O}(z)\right), \quad z \to 0$$

Can we relate transseries parameter σ and parameter C ?

FROM LATE TO EARLY TIMES

Approximation attime z_0 : hyperasymptotics

$$f_{\text{approx}}(z_0) \simeq f_{\text{Hyp},0}(z_0) + \sigma f_{\text{Hyp},1}(z_0)$$

Approximation at time z_0 : Borel summation

$$f_{\text{approx}}(z_0) \simeq \mathcal{S}_{N_0}\Phi^0(z_0) + \sigma\, \mathrm{e}^{-A z_0}\, \mathcal{S}_{N_0}\Phi^{(1)}(z_0)$$

Approximation at time z_0 : Taylor series method

Use analytic continuation via numerical Taylor series method, to bring early times solution to the finite value z_0 : $f_{\text{an}}(z_0)$

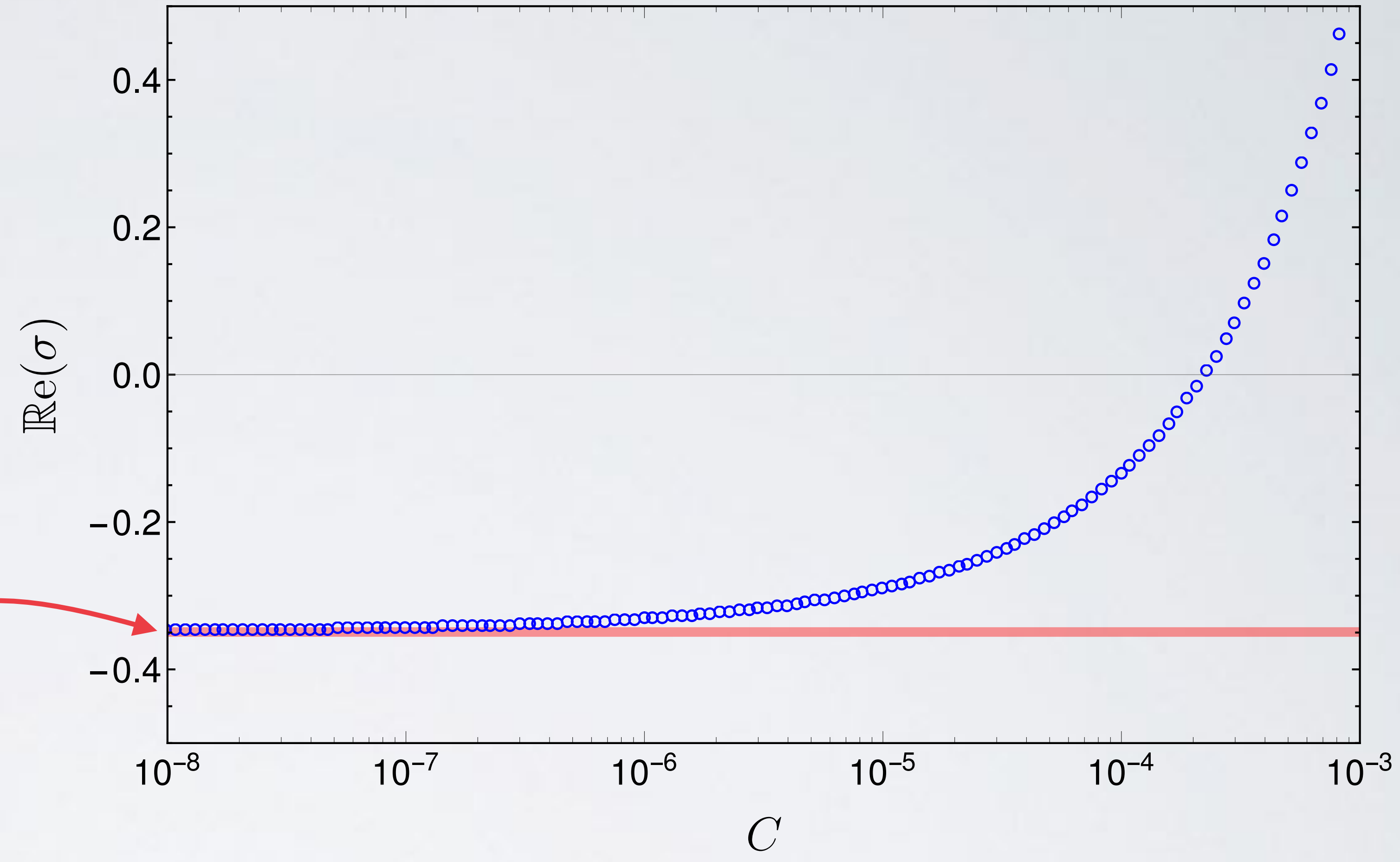
Solve $f_{\text{approx}}(z_0) = f_{\text{an}}(z_0)$ to obtain $\sigma(C)$

FROM LATE TO EARLY TIMES

- ***Ambiguity cancelation***:
  $\mathrm{Im}\sigma$ constrained
  by Stokes constant

$$\mathrm{i\,Im}\sigma = \frac{S_1}{2} \sim \mathrm{i}0.0027$$

$$\mathrm{Re}\left(\sigma_{f_+}\right) \sim -0.3493$$



- Strength of non-hydrodynamic mode $\boldsymbol{\sigma}$ highly sensitive to initial condition
- Choice of interpolation point z_0 changes accuracy:
  1. Larger z_0 increases accuracy
  2. Include extra exponential sectors to increase accuracy at smaller z_0

TRANSASYMPTOTICS

Rearrange order of transmonomials in the transseries:

$$\mathcal{F}(z,\sigma) = \sum_{k=0}^{+\infty} z^{-k} \sum_{n=0}^{+\infty} \left(\sigma\, z^{-\beta}\, \mathrm{e}^{-Az}\right)^n a_k^{(n)} = \sum_{k=0}^{+\infty} z^{-k} \sum_{n=0}^{+\infty} \tau^n\, a_k^{(n)}$$

Sum the transseries in a new regime $z^{-1} \ll \tau \ll 1$:

$$\mathcal{F}(z,\tau) = \sum_{k=0}^{+\infty} z^{-k}\, F_k(\tau) \qquad\qquad F_k(\tau) = \sum_{n=0}^{+\infty} \tau^n\, a_k^n$$

The sum over powers of τ can be done exactly!

Study analytic behaviour: poles, branch points…

[Costin et al'01-13; IA,Schiappa,Vonk 'to appear]

TRANSASYMPTOTICS

$$\mathcal{F}(z,\tau) = \sum_{k=0}^{+\infty} z^{-k}\, F_k(\tau) \qquad F_k(\tau) = \sum_{n=0}^{+\infty} \tau^n\, a_k^n$$

Recursive calculation:

$$F_0(\tau) = \frac{2}{3}\left(1 + W\left(\frac{3}{2}\tau\right)\right)$$

Lambert-W function

$$W(x)\,\mathrm{e}^{W(x)} = x$$

$$\vdots$$

$$F_k(\tau) = \frac{P_k\left(F_0(\tau)\right)}{Q_k\left(F_0(\tau)\right)}$$

Polynomials

- This summation can also be used for determining $\sigma(C)$
- $\mathscr{F}(z,\tau) = 0$ are square root branch points, related to the single square root branch point of $W(\tau)$ at $\tau = -\mathrm{e}^{-1}$.

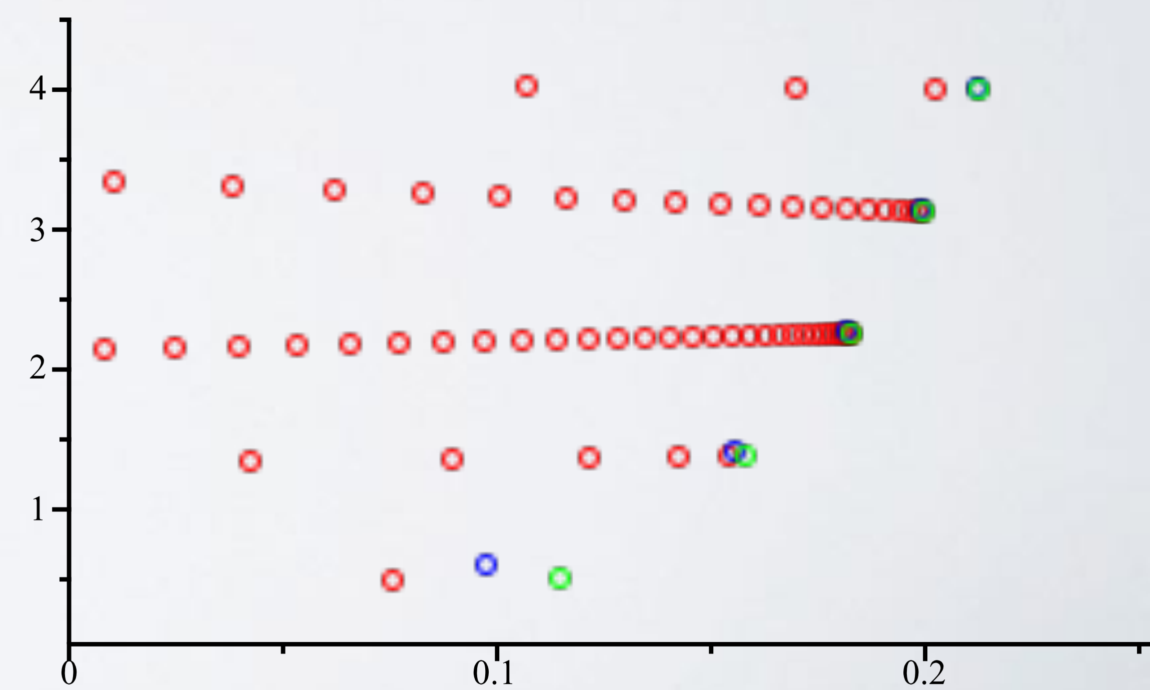
BRANCH POINTS

Asymptotic prediction:

$$w_{\rm bp}(t) \simeq \frac{t}{A} - \frac{\beta}{A}\log(t) + \frac{1}{A\,t}\left(\beta^2\log(t) + \beta^2 + 5\beta - \frac{3}{A}\right), \quad \text{as} \quad t \to \infty$$

$$\text{where} \quad t \equiv \log\left(\frac{3\sigma e}{2A^{-\beta}}\right) + \pi i(1+2n), \qquad n \in \mathbb{Z}$$

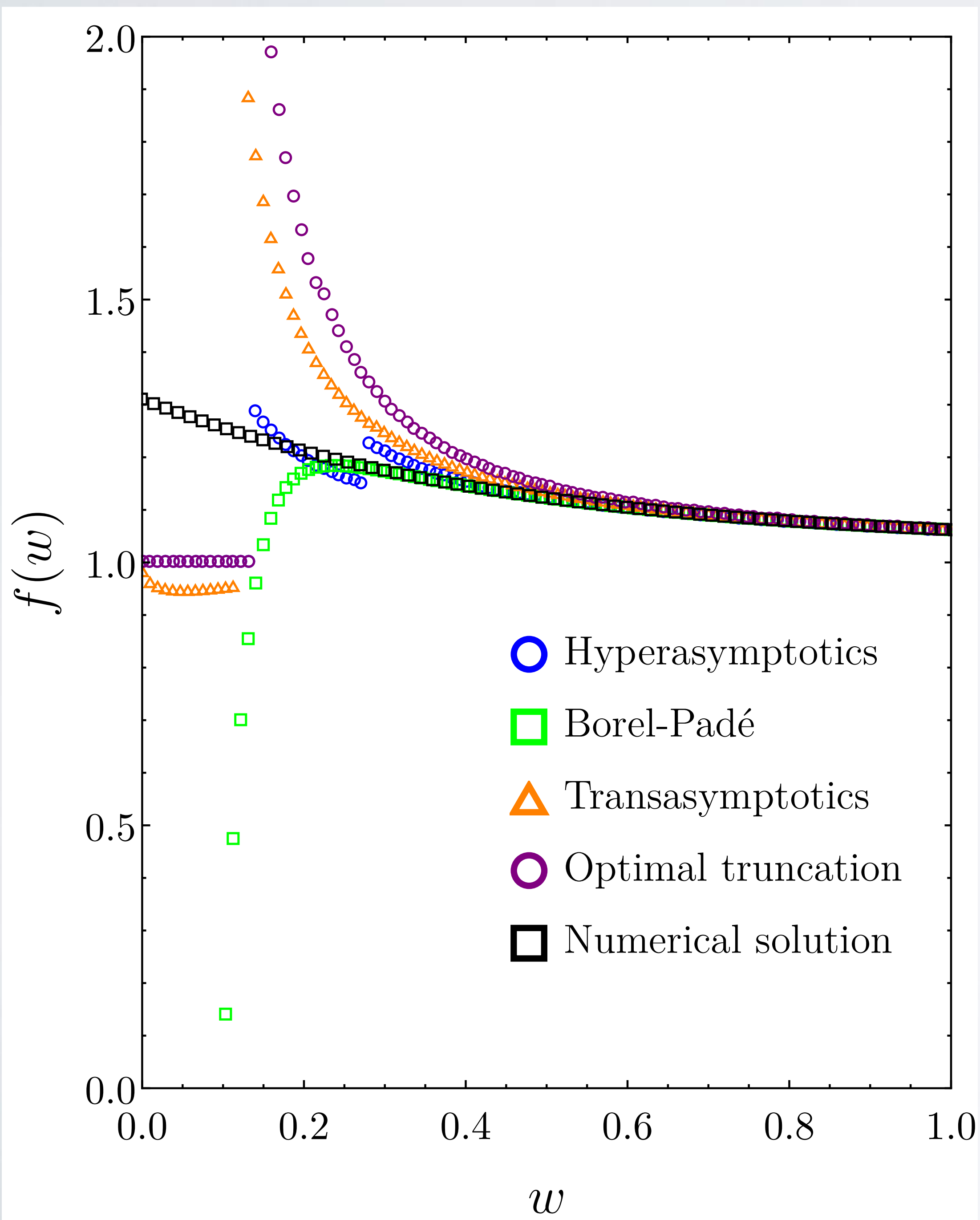
- numerical branch cut
- numerical branch point
- predicted branch point

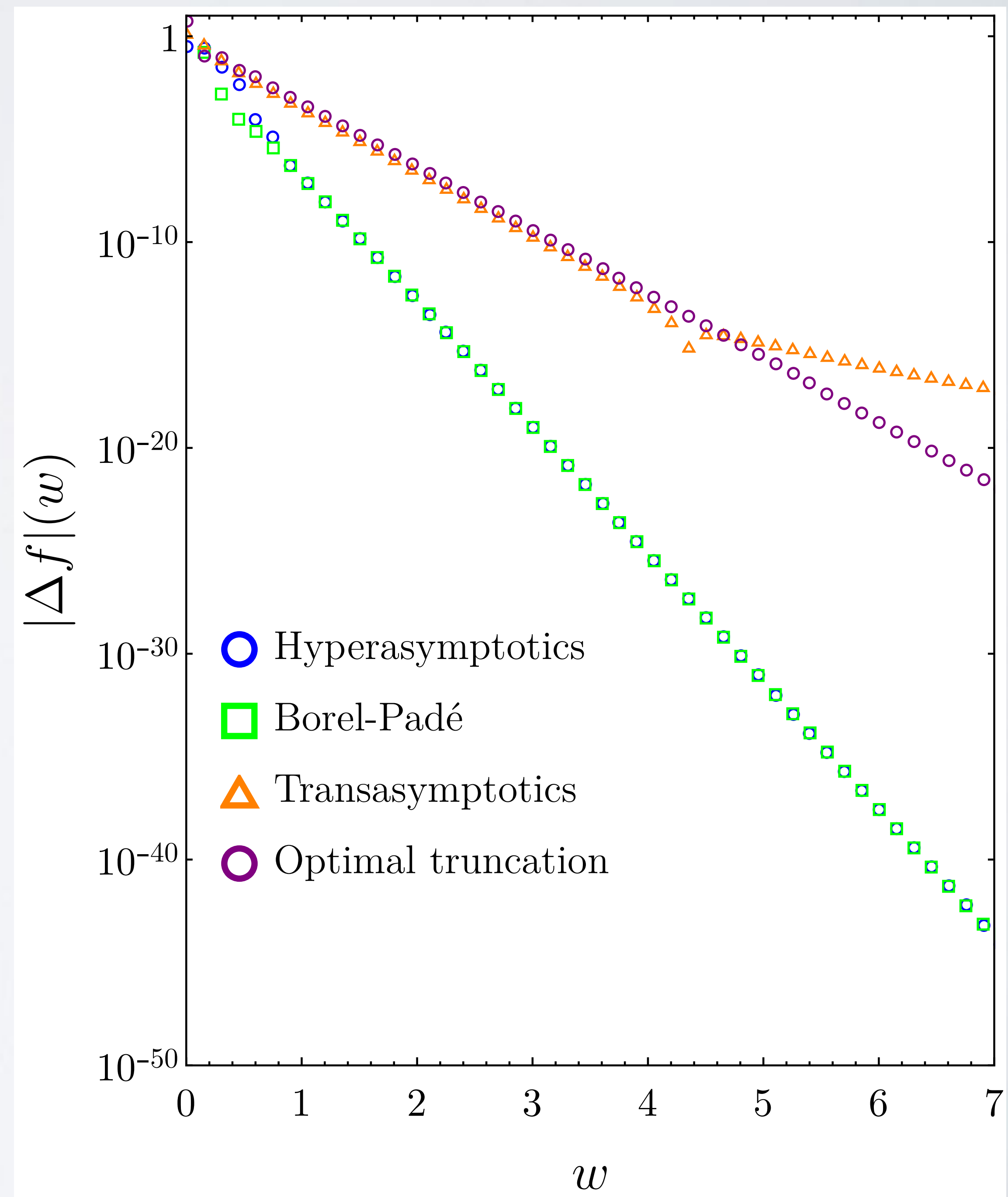


$\sigma = 2/3$

NUMERICS VS ASYMPTOTICS

For the finite solution at early times $f_+(z)$





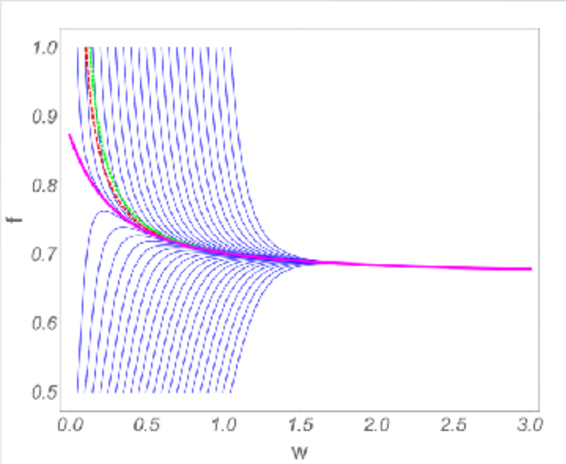
4.

SUMMARY/ FUTURE DIRECTIONS

SUMMARY

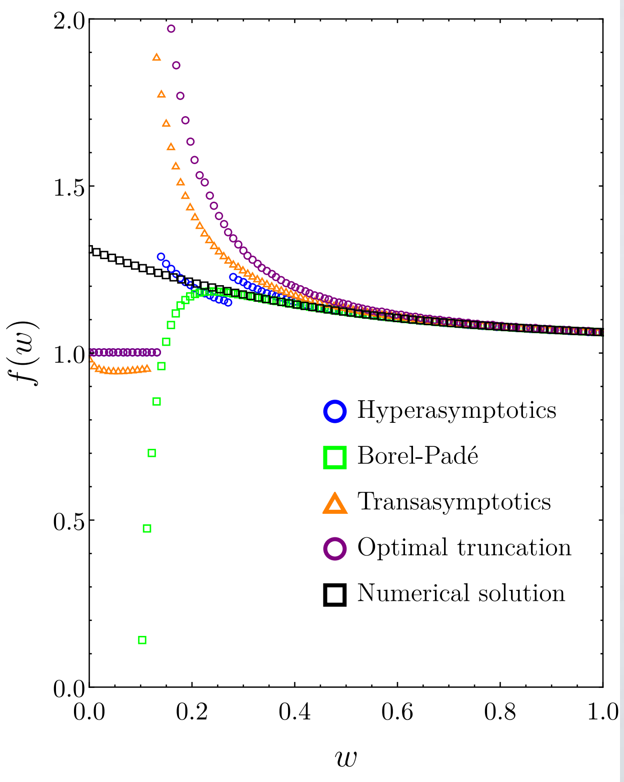
Hydrodynamic gradient expansion

- convergence towards attractor at late times
- description of the system after the decay of non-hydro modes,
- asymptotic properties encode the non-perturbative scales



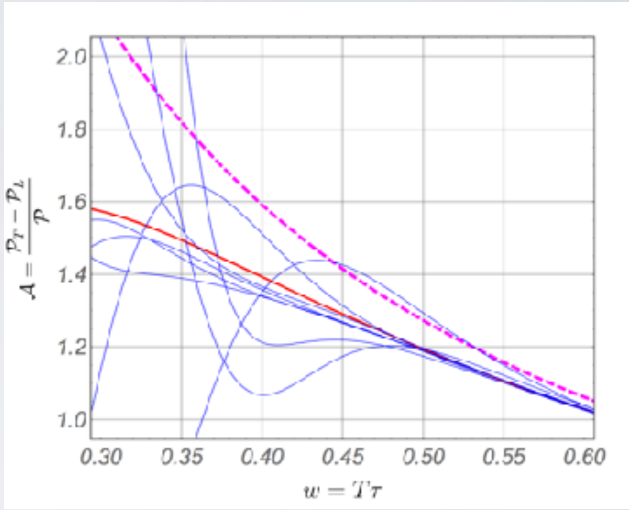
Hydrodynamic Transseries

- encodes multiple scales of early and late times
- exponentially accurate summations interpolate between initial conditions and hydrodynamic regime
- global analytic behaviour: branch points



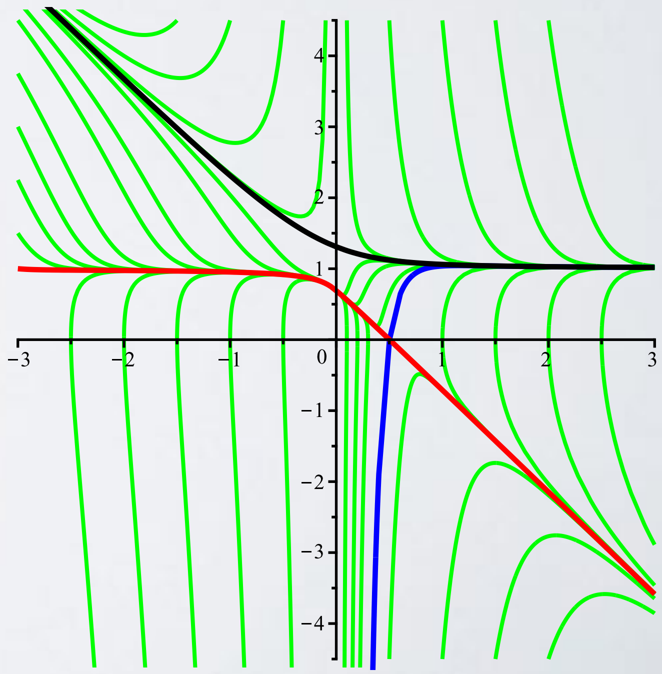
FUTURE DIRECTIONS

Applications to SYM plasma/other microscopic theories



- existence of a hydrodynamic attractor
- early time descriptions attainable
- multiple non-hydro modes

Analytic properties beyond hydrodynamics



- zeros of partition functions and phase transitions
- prediction of bifurcation phenomena in discrete systems
- connect distinct regimes of asymptotics

THANK YOU!

$$\sum_{n=0}^{\infty} E_n\, g^n \quad e^{-A/g}$$