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Institut d'Ingénierie du territoire

**CERN Academic Training**

# Introduction to Geodesy

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# Program

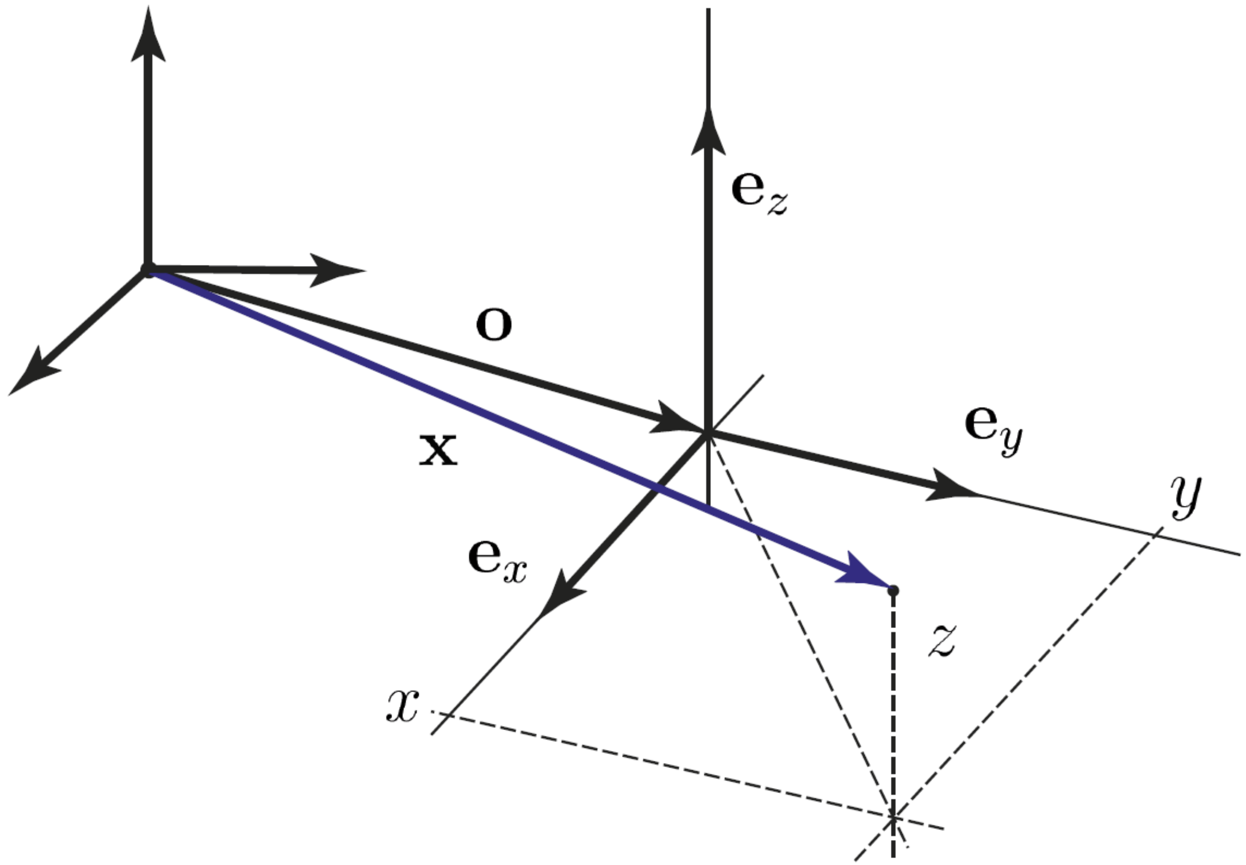
- Coordinate Systems
- Reference Systems and Frames
- Gravity Field
- Height Systems

# Coordinate Systems

## Cartesian Coordinates

# Cartesian Coordinates

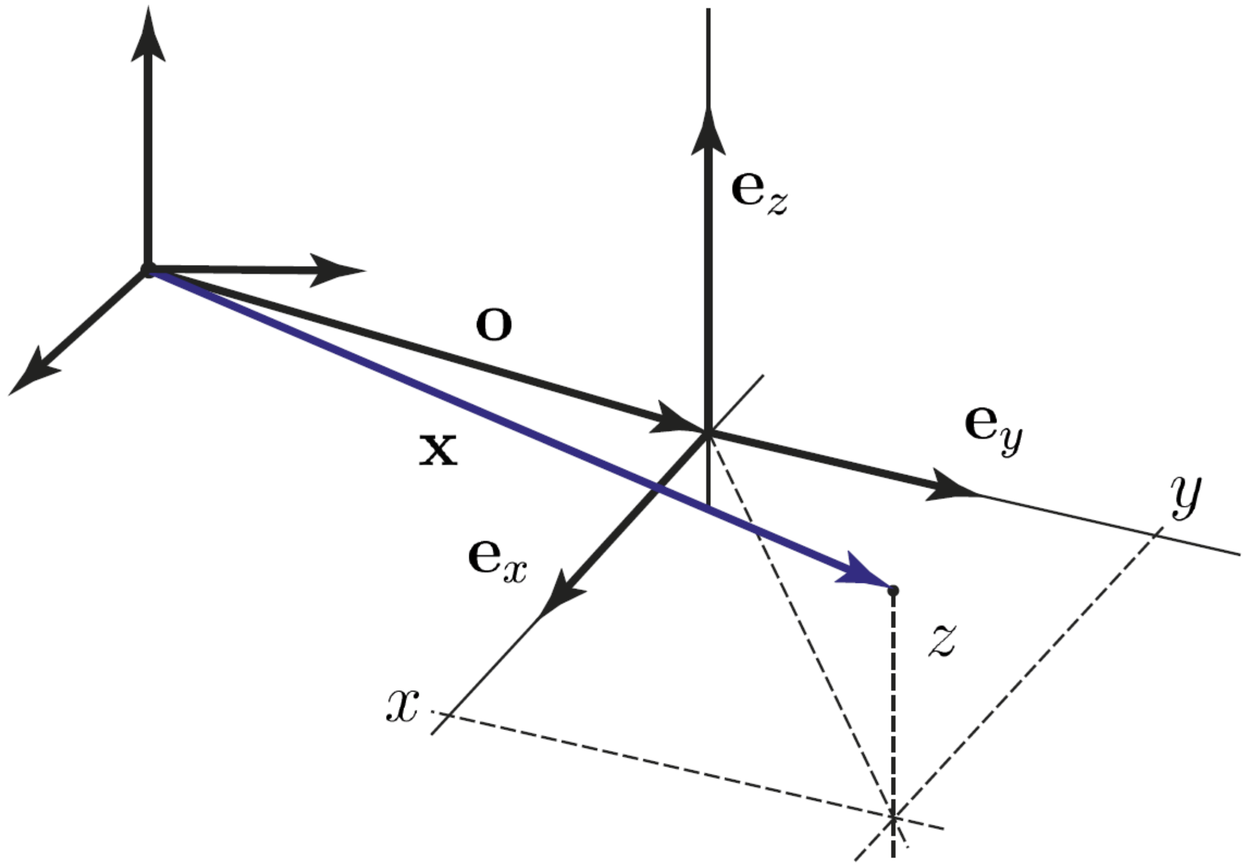
$$\mathbf{x} = \mathbf{o} + x \cdot \mathbf{e}_x + y \cdot \mathbf{e}_y + z \cdot \mathbf{e}_z$$



# Cartesian Coordinates

$$\mathbf{x} = \mathbf{o} + x \cdot \mathbf{e}_x + y \cdot \mathbf{e}_y + z \cdot \mathbf{e}_z$$

$$\begin{aligned}\mathbf{x} &= \mathbf{o} + \begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} o_x \\ o_y \\ o_z \end{pmatrix} + \begin{pmatrix} e_{x_x} & e_{y_x} & e_{z_x} \\ e_{x_y} & e_{y_y} & e_{z_y} \\ e_{x_z} & e_{y_z} & e_{z_z} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \mathbf{o} + \mathbf{T} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}\end{aligned}$$



# Cartesian Coordinates

## Example :

$$\mathbf{o} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{e}_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{e}_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \mathbf{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \Rightarrow \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\mathbf{x} = \mathbf{o} + x \cdot \mathbf{e}_x + y \cdot \mathbf{e}_y + z \cdot \mathbf{e}_z$$

$$\mathbf{x} = \mathbf{o} + \begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} o_x \\ o_y \\ o_z \end{pmatrix} + \begin{pmatrix} e_{x_x} & e_{y_x} & e_{z_x} \\ e_{x_y} & e_{y_y} & e_{z_y} \\ e_{x_z} & e_{y_z} & e_{z_z} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

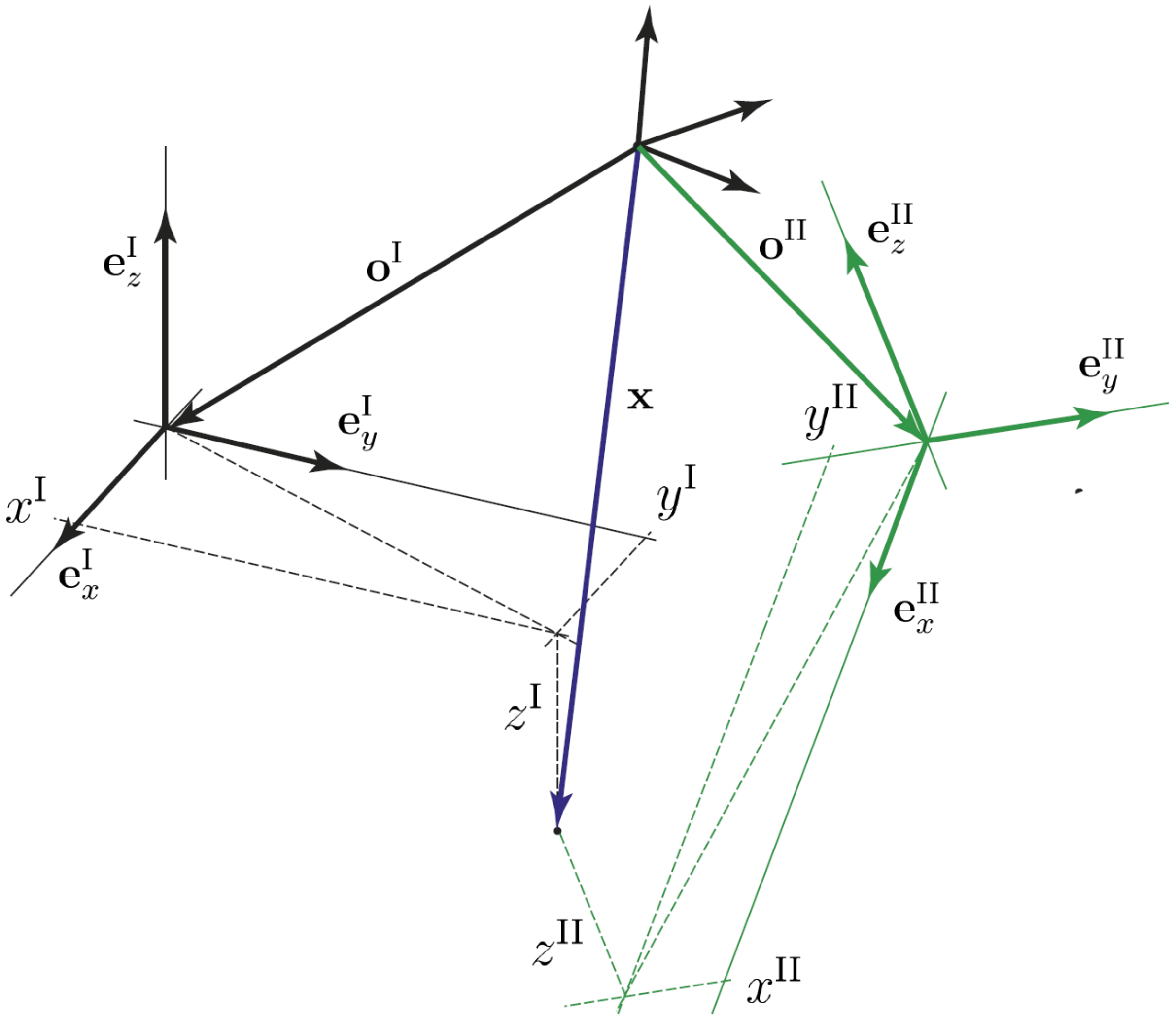
$$= \mathbf{o} + \mathbf{T} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

# Cartesian Coordinates

## 2 Systems :

$$\mathbf{x} = \mathbf{o}^{\mathrm{I}} + x^{\mathrm{I}} \cdot \mathbf{e}_x^{\mathrm{I}} + y^{\mathrm{I}} \cdot \mathbf{e}_y^{\mathrm{I}} + z^{\mathrm{I}} \cdot \mathbf{e}_z^{\mathrm{I}} = \mathbf{o}^{\mathrm{I}} + \mathbf{T}^{\mathrm{I}} \cdot \begin{pmatrix} x^{\mathrm{I}} \\ y^{\mathrm{I}} \\ z^{\mathrm{I}} \end{pmatrix}$$

$$\mathbf{x} = \mathbf{o}^{\mathrm{II}} + x^{\mathrm{II}} \cdot \mathbf{e}_x^{\mathrm{II}} + y^{\mathrm{II}} \cdot \mathbf{e}_y^{\mathrm{II}} + z^{\mathrm{II}} \cdot \mathbf{e}_z^{\mathrm{II}} = \mathbf{o}^{\mathrm{II}} + \mathbf{T}^{\mathrm{II}} \cdot \begin{pmatrix} x^{\mathrm{II}} \\ y^{\mathrm{II}} \\ z^{\mathrm{II}} \end{pmatrix}$$



# Cartesian Coordinates

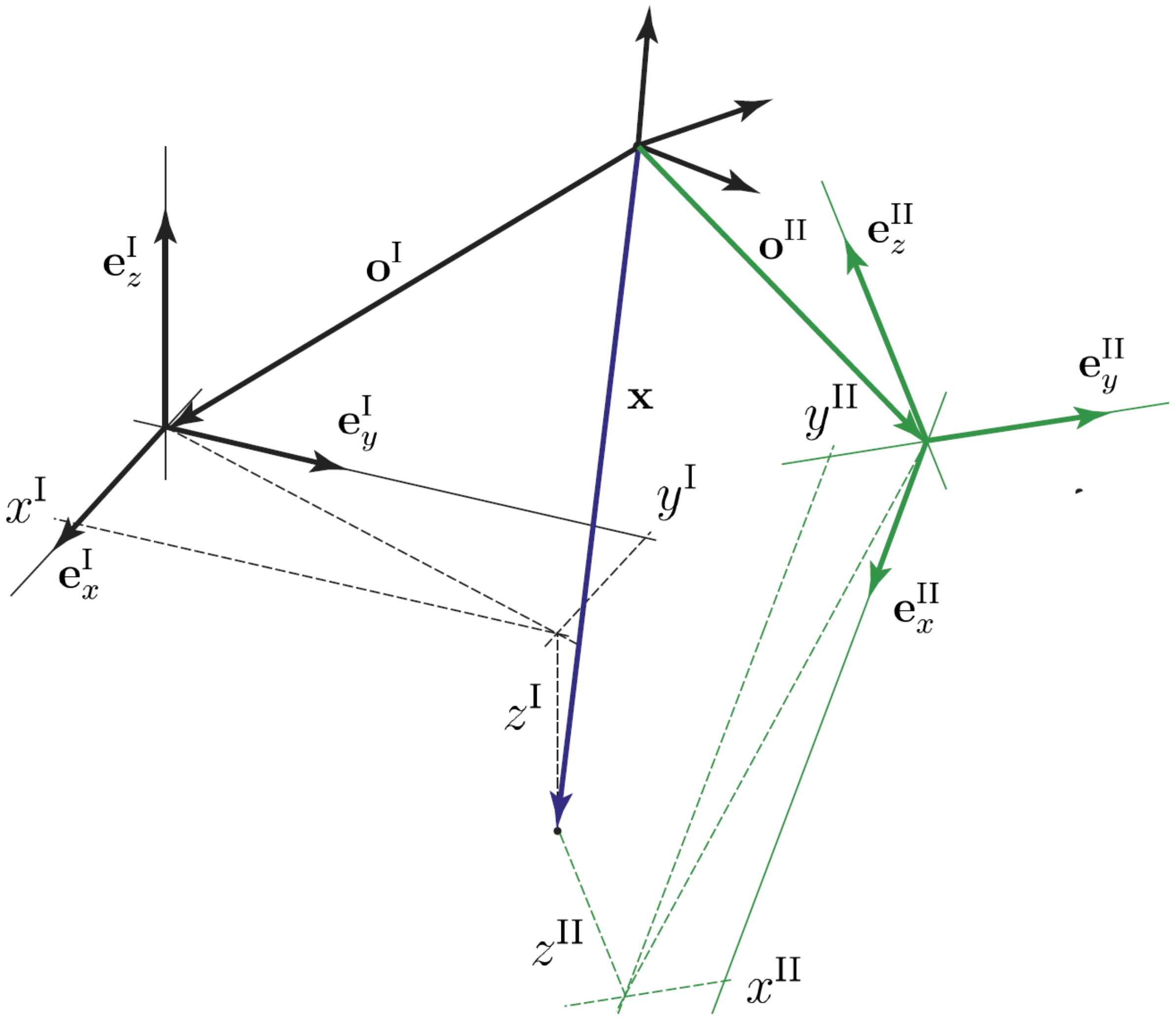
## 2 Systems :

$$\mathbf{x} = \mathbf{o}^{\mathrm{I}} + \mathbf{T}^{\mathrm{I}} \cdot \begin{pmatrix} x^{\mathrm{I}} \\ y^{\mathrm{I}} \\ z^{\mathrm{I}} \end{pmatrix} = \mathbf{o}^{\mathrm{II}} + \mathbf{T}^{\mathrm{II}} \cdot \begin{pmatrix} x^{\mathrm{II}} \\ y^{\mathrm{II}} \\ z^{\mathrm{II}} \end{pmatrix}$$

$$\begin{pmatrix} x^{\mathrm{II}} \\ y^{\mathrm{II}} \\ z^{\mathrm{II}} \end{pmatrix} = \left(\mathbf{T}^{\mathrm{II}}\right)^{-1} \cdot \left(\mathbf{o}^{\mathrm{I}} - \mathbf{o}^{\mathrm{II}}\right) + \left(\mathbf{T}^{\mathrm{II}}\right)^{-1} \cdot \mathbf{T}^{\mathrm{I}} \cdot \begin{pmatrix} x^{\mathrm{I}} \\ y^{\mathrm{I}} \\ z^{\mathrm{I}} \end{pmatrix}$$

If $\mathbf{e}_x^{\mathrm{I}}, \mathbf{e}_y^{\mathrm{I}}, \mathbf{e}_z^{\mathrm{I}}$ / $\mathbf{e}_x^{\mathrm{II}}, \mathbf{e}_y^{\mathrm{II}}, \mathbf{e}_z^{\mathrm{II}}$ are orthonormal

$$\begin{pmatrix} x^{\mathrm{II}} \\ y^{\mathrm{II}} \\ z^{\mathrm{II}} \end{pmatrix} = \left(\mathbf{T}^{\mathrm{II}}\right)^{T} \cdot \left(\mathbf{o}^{\mathrm{I}} - \mathbf{o}^{\mathrm{II}}\right) + \left(\mathbf{T}^{\mathrm{II}}\right)^{T} \cdot \mathbf{T}^{\mathrm{I}} \cdot \begin{pmatrix} x^{\mathrm{I}} \\ y^{\mathrm{I}} \\ z^{\mathrm{I}} \end{pmatrix}$$



# Cartesian Coordinates

## Rigid transformation :

$$\begin{pmatrix} x^{\mathrm{II}} \\ y^{\mathrm{II}} \\ z^{\mathrm{II}} \end{pmatrix} = \underbrace{\left(\mathbf{T}^{\mathrm{II}}\right)^T \cdot \left(\mathbf{o}^{\mathrm{I}} - \mathbf{o}^{\mathrm{II}}\right)}_{\mathbf{t}_{\mathrm{I}}^{\mathrm{II}}} + \underbrace{\left(\mathbf{T}^{\mathrm{II}}\right)^T \cdot \mathbf{T}^{\mathrm{I}}}_{\mathbf{R}_{\mathrm{I}}^{\mathrm{II}}} \cdot \begin{pmatrix} x^{\mathrm{I}} \\ y^{\mathrm{I}} \\ z^{\mathrm{I}} \end{pmatrix} \Rightarrow \begin{pmatrix} x^{\mathrm{II}} \\ y^{\mathrm{II}} \\ z^{\mathrm{II}} \end{pmatrix} = \mathbf{t}_{\mathrm{I}}^{\mathrm{II}} + \mathbf{R}_{\mathrm{I}}^{\mathrm{II}} \cdot \begin{pmatrix} x^{\mathrm{I}} \\ y^{\mathrm{I}} \\ z^{\mathrm{I}} \end{pmatrix}$$

## Similarity transformation :

adding a scaling factor :

$$\Rightarrow \begin{pmatrix} x^{\mathrm{II}} \\ y^{\mathrm{II}} \\ z^{\mathrm{II}} \end{pmatrix} = \mathbf{t}_{\mathrm{I}}^{\mathrm{II}} + m \cdot \mathbf{R}_{\mathrm{I}}^{\mathrm{II}} \cdot \begin{pmatrix} x^{\mathrm{I}} \\ y^{\mathrm{I}} \\ z^{\mathrm{I}} \end{pmatrix}$$

with : $\mathbf{t}_{\mathrm{I}}^{\mathrm{II}} = \left(\mathbf{T}^{\mathrm{II}}\right)^T \cdot \left(\mathbf{o}^{\mathrm{I}} - \mathbf{o}^{\mathrm{II}}\right) = \begin{pmatrix} t_{\mathrm{I}\,x}^{\mathrm{II}} \\ t_{\mathrm{I}\,y}^{\mathrm{II}} \\ t_{\mathrm{I}\,z}^{\mathrm{II}} \end{pmatrix}$ and : $\mathbf{R}_{\mathrm{I}}^{\mathrm{II}} = \left(\mathbf{T}^{\mathrm{II}}\right)^T \cdot \mathbf{T}^{\mathrm{I}} = \begin{pmatrix} r_{\mathrm{I}\,11}^{\mathrm{II}} & r_{\mathrm{I}\,12}^{\mathrm{II}} & r_{\mathrm{I}\,13}^{\mathrm{II}} \\ r_{\mathrm{I}\,21}^{\mathrm{II}} & r_{\mathrm{I}\,22}^{\mathrm{II}} & r_{\mathrm{I}\,23}^{\mathrm{II}} \\ r_{\mathrm{I}\,31}^{\mathrm{II}} & r_{\mathrm{I}\,32}^{\mathrm{II}} & r_{\mathrm{I}\,33}^{\mathrm{II}} \end{pmatrix}$

# Cartesian Coordinates

**Parametrizations of rotation matrix :**

- 3 angles
- 1 unit vector + 1 angle
- 1 unit quaternion

# Cartesian Coordinates

## 3 angles / Cardan Angles

$$\mathbf{R}_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \qquad \mathbf{R}_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} \qquad \mathbf{R}_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

$$\begin{aligned}
\mathbf{R}_{\mathrm{I}}^{\mathrm{II}}(\alpha,\beta,\gamma) &= \mathbf{R}_z(\gamma)\cdot\mathbf{R}_y(\beta)\cdot\mathbf{R}_x(\alpha) \\
&= \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \\
&= \begin{pmatrix} \cos\gamma\cos\beta & \cos\gamma\sin\beta\sin\alpha + \sin\gamma\cos\alpha & -\cos\gamma\sin\beta\cos\alpha + \sin\gamma\sin\alpha \\ -\sin\gamma\cos\beta & -\sin\gamma\sin\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\gamma\sin\beta\cos\alpha + \cos\gamma\sin\alpha \\ \sin\beta & -\cos\beta\sin\alpha & \cos\beta\cos\alpha \end{pmatrix}
\end{aligned}$$

# Cartesian Coordinates

## 3 angles / Cardan Angles

$$\alpha = \begin{cases} -\,\mathtt{atan2}\left(\dfrac{r_{32}}{\cos\beta}, \dfrac{r_{33}}{\cos\beta}\right) & \text{si}: r_{31} \neq -1, \text{ou}: r_{31} \neq +1 \\ +\,\mathtt{atan2}\left(r_{12}, r_{13}\right) & \text{si}: r_{31} = -1 \\ +\,\mathtt{atan2}\left(-r_{12}, -r_{13}\right) & \text{si}: r_{31} = +1 \end{cases}$$

$$\beta = \begin{cases} +\arcsin\left(r_{31}\right) & \text{si}: r_{31} \neq -1, \text{ou}: r_{31} \neq +1 \\ -90^\circ & \text{si}: r_{31} = -1 \\ +90^\circ & \text{si}: r_{31} = +1 \end{cases}$$

$$\gamma = \begin{cases} -\,\mathtt{atan2}\left(\dfrac{r_{21}}{\cos\beta}, \dfrac{r_{11}}{\cos\beta}\right) & \text{si}: r_{31} \neq -1, \text{ou}: r_{31} \neq +1 \\ +0^\circ & \text{si}: r_{31} = -1 \\ +0^\circ & \text{si}: r_{31} = +1 \end{cases}$$

# Cartesian Coordinates

## 1 unit quaternion

$$\mathbf{q} = \begin{pmatrix} q_w \\ q_x \\ q_y \\ q_z \end{pmatrix} = \begin{pmatrix} \cos\frac{\alpha}{2} \\ \sin\frac{\alpha}{2}\cdot\mathbf{n} \end{pmatrix} = \begin{pmatrix} \cos\frac{\alpha}{2} \\ n_x\cdot\sin\frac{\alpha}{2} \\ n_y\cdot\sin\frac{\alpha}{2} \\ n_z\cdot\sin\frac{\alpha}{2} \end{pmatrix}$$

with :

$$|\mathbf{q}| = \sqrt{q_w^2 + q_x^2 + q_y^2 + q_z^2} = 1$$

Rotation :

$$\begin{pmatrix} 0 \\ x^{\mathrm{II}} \\ y^{\mathrm{II}} \\ z^{\mathrm{II}} \end{pmatrix} = \overline{\mathbf{q}}_{\mathrm{I}}^{\mathrm{II}} \otimes \begin{pmatrix} 0 \\ x^{\mathrm{I}} \\ y^{\mathrm{I}} \\ z^{\mathrm{I}} \end{pmatrix} \otimes \mathbf{q}_{\mathrm{I}}^{\mathrm{II}}$$

Rotation matrix :

$$\mathbf{R}_{\mathrm{I}}^{\mathrm{II}}(\mathbf{q}) = \begin{pmatrix} q_w^2 + q_x^2 - q_y^2 - q_z^2 & 2q_xq_y + 2q_wq_z & 2q_xq_z - 2q_wq_y \\ 2q_xq_y - 2q_wq_z & q_w^2 - q_x^2 + q_y^2 - q_z^2 & 2q_yq_z + 2q_wq_x \\ 2q_xq_z + 2q_wq_y & 2q_yq_z - 2q_wq_x & q_w^2 - q_x^2 - q_y^2 + q_z^2 \end{pmatrix}$$

# Cartesian Coordinates

## Example

3 Cardan angles (x=>y=>z) :

$$\mathbf{R}_{\mathrm{I}}^{\mathrm{II}}(\alpha, \beta, \gamma) = \mathbf{R}_{\mathrm{I}}^{\mathrm{II}}(10^\circ, 20^\circ, 30^\circ)$$

1 unit quaternion :

$$\mathbf{R}_{\mathrm{I}}^{\mathrm{II}}(\mathbf{q}) = \mathbf{R}_{\mathrm{I}}^{\mathrm{II}}\left(\begin{pmatrix} +0.943714 \\ +0.127679 \\ +0.144878 \\ +0.268536 \end{pmatrix}\right)$$

1 unit vector + 1 angle :

$$\mathbf{R}_{\mathrm{I}}^{\mathrm{II}}(\mathbf{n}, \alpha) = \mathbf{R}_{\mathrm{I}}^{\mathrm{II}}\left(\begin{pmatrix} +0.38601 \\ +0.43801 \\ +0.81187 \end{pmatrix}, +38.6300^\circ\right)$$

Same rotation matrix

$$\mathbf{R}_{\mathrm{I}}^{\mathrm{II}} = \begin{pmatrix} +0.813798 & +0.543838 & -0.204874 \\ -0.469846 & +0.823173 & +0.318796 \\ +0.34202 & -0.163176 & +0.925417 \end{pmatrix}$$

# Coordinate Systems

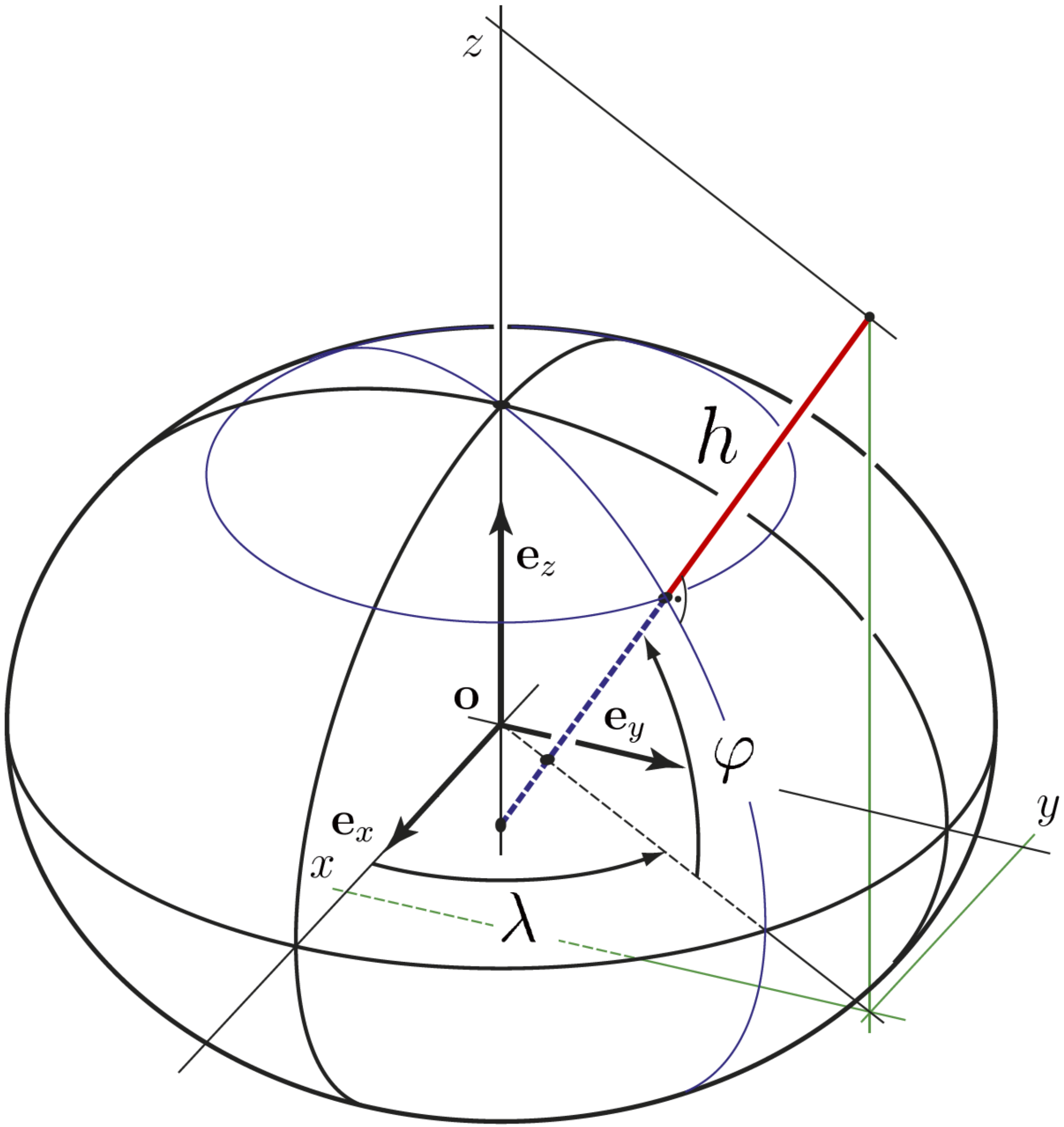
## Ellipsoidal Coordinates

# Ellipsoidal Coordinates

## Ellipsoid of revolution

Canonical form :

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$$



# Ellipsoidal Coordinates

## Ellipsoid of revolution

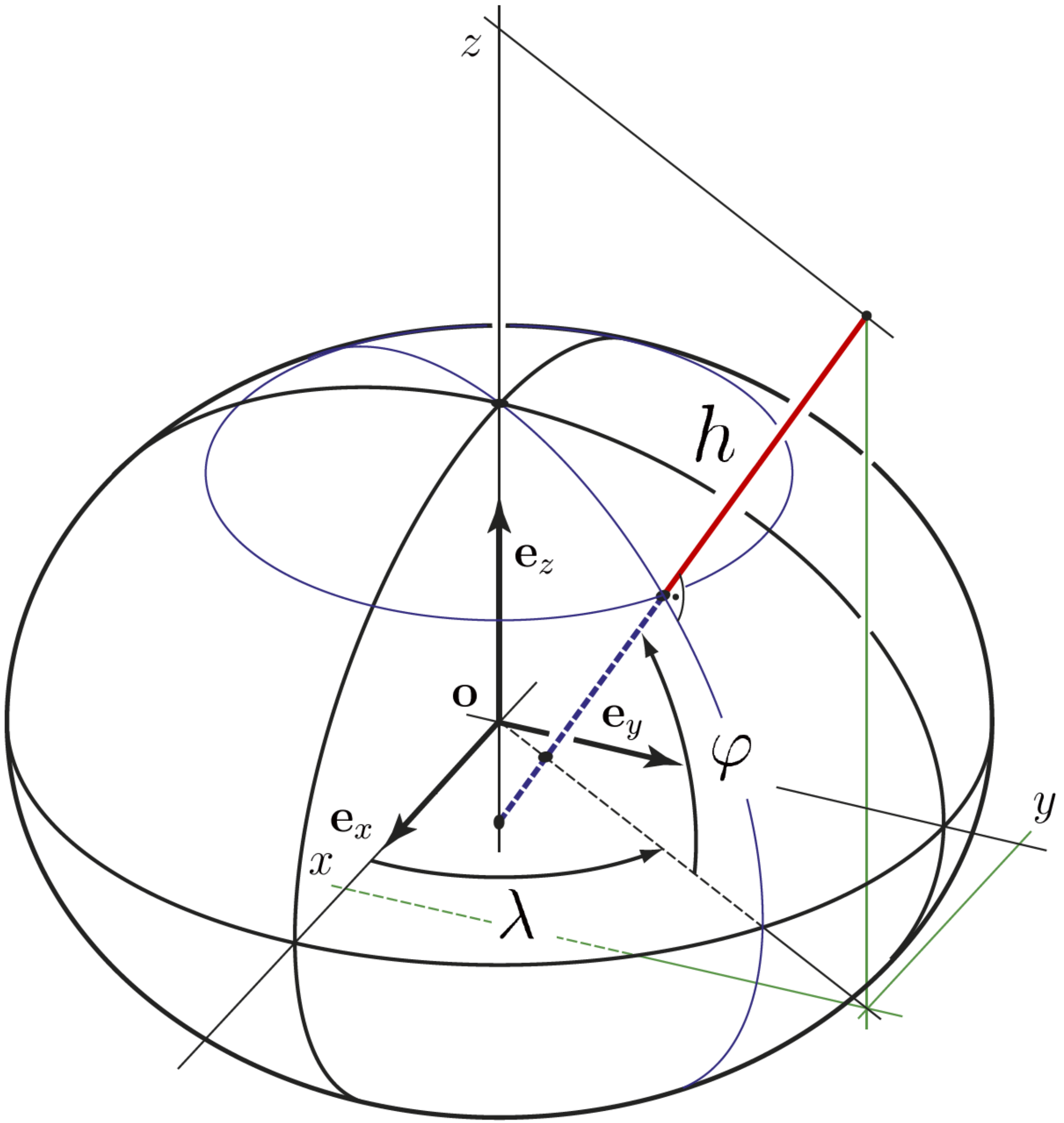
Canonical form :

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$$

Geodetic parametrization :

$$x(\lambda, \varphi, h) = (R_N + h) \cdot \cos\varphi \cdot \cos\lambda$$
$$y(\lambda, \varphi, h) = (R_N + h) \cdot \cos\varphi \cdot \sin\lambda$$
$$z(\lambda, \varphi, h) = \left[R_N \cdot (1 - e^2) + h\right] \cdot \sin\varphi$$

$$R_N(a, e, \varphi) = \frac{a}{\sqrt{1 - e^2 \cdot \sin^2\varphi}}$$



# Ellipsoidal Coordinates

## Ellipsoid of revolution

Canonical form :

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$$

Geodetic parametrization to cartesian :

$$x(\lambda, \varphi, h) = (R_N + h) \cdot \cos\varphi \cdot \cos\lambda$$
$$y(\lambda, \varphi, h) = (R_N + h) \cdot \cos\varphi \cdot \sin\lambda$$
$$z(\lambda, \varphi, h) = \left[R_N \cdot (1 - e^2) + h\right] \cdot \sin\varphi$$

$$R_N(a, e, \varphi) = \frac{a}{\sqrt{1 - e^2 \cdot \sin^2\varphi}}$$

Cartesian to geodetic parametrization :

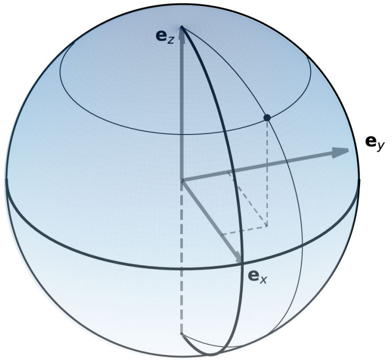
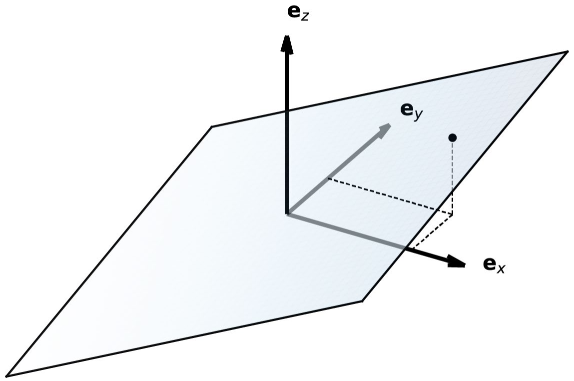
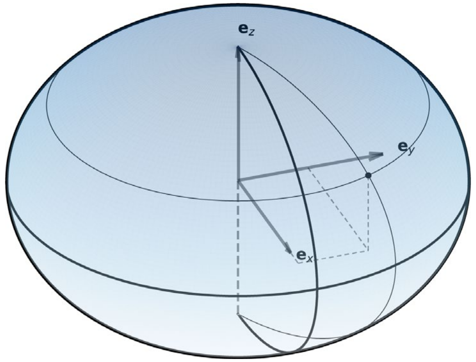
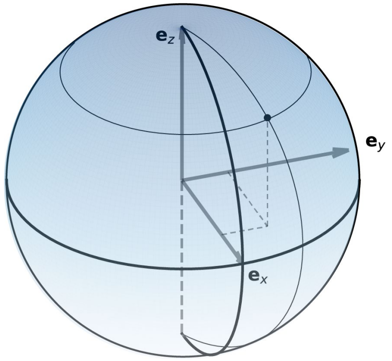
$$\lambda(x, y, z) = \arctan\left(\frac{y}{x}\right)$$
$$\varphi(x, y, z) = \arctan\left(\frac{z}{\sqrt{x^2 + y^2}} \cdot \frac{1}{1 - \frac{R_N}{R_N + h} \cdot e^2}\right)$$
$$h(x, y, z) = \frac{\sqrt{x^2 + y^2}}{\cos\varphi} - R_N$$

$$R_N(a, e, \varphi) = \frac{a}{\sqrt{1 - e^2 \cdot \sin^2\varphi}}$$

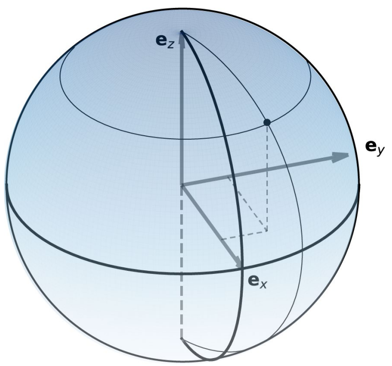
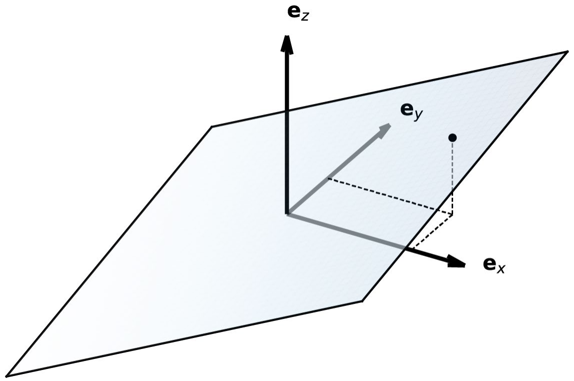
# Coordinate Systems

## Projective Coordinates

# Projective Coordinates

| Surface of REFERENCE | Surface of PROJECTION | Formulas of PROJECTION |
|---|---|---|
|   |   | $E(\lambda_{\text{sph}}, \phi_{\text{sph}})$ <br> $N(\lambda_{\text{sph}}, \phi_{\text{sph}})$ |
|   |   | $\lambda_{\text{sph}}(\lambda_{\text{ell}}, \varphi_{\text{ell}})$ <br> $\phi_{\text{sph}}(\lambda_{\text{ell}}, \varphi_{\text{ell}})$ |

# Projective Coordinates

| Surface of REFERENCE | Surface of PROJECTION | Formulas of PROJECTION |
|---|---|---|
|   |   | $E(\lambda_{\text{sph}}, \phi_{\text{sph}})$ <br> $N(\lambda_{\text{sph}}, \phi_{\text{sph}})$ |

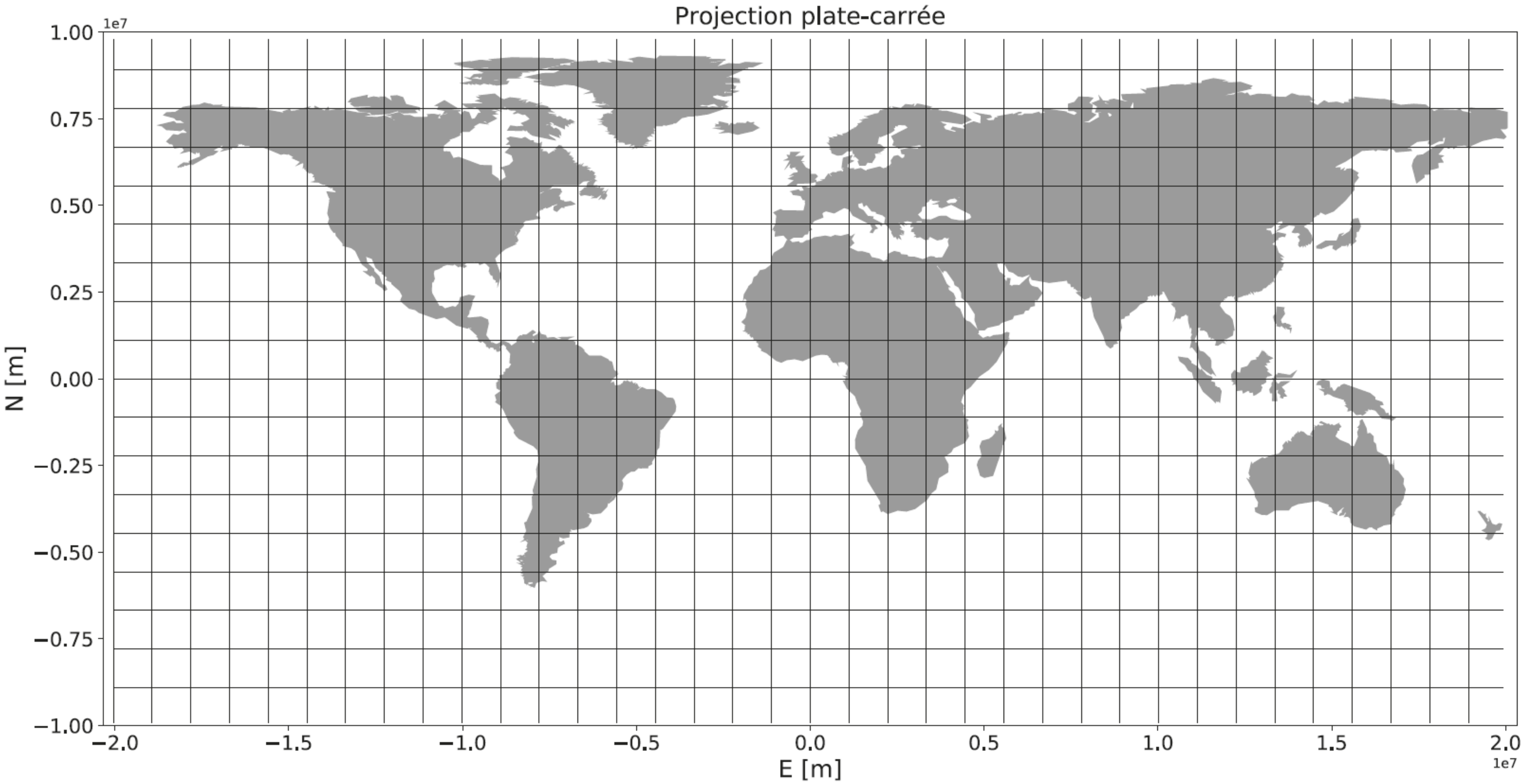
**Basic example :**

**plane chart projection**

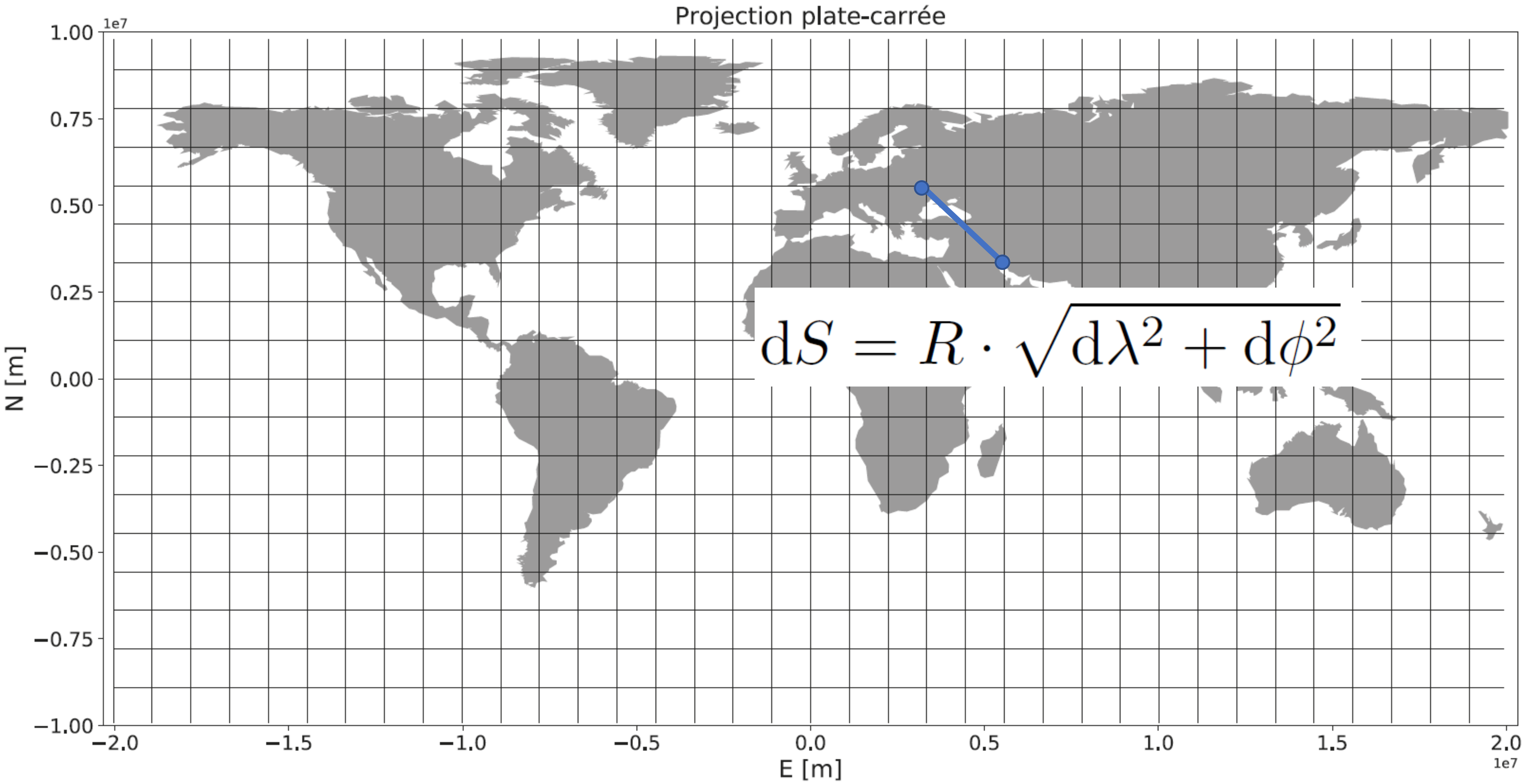
$$E(\lambda, \phi) = R \cdot \lambda$$
$$N(\lambda, \phi) = R \cdot \phi$$

# Projective Coordinates

# Projective Coordinates



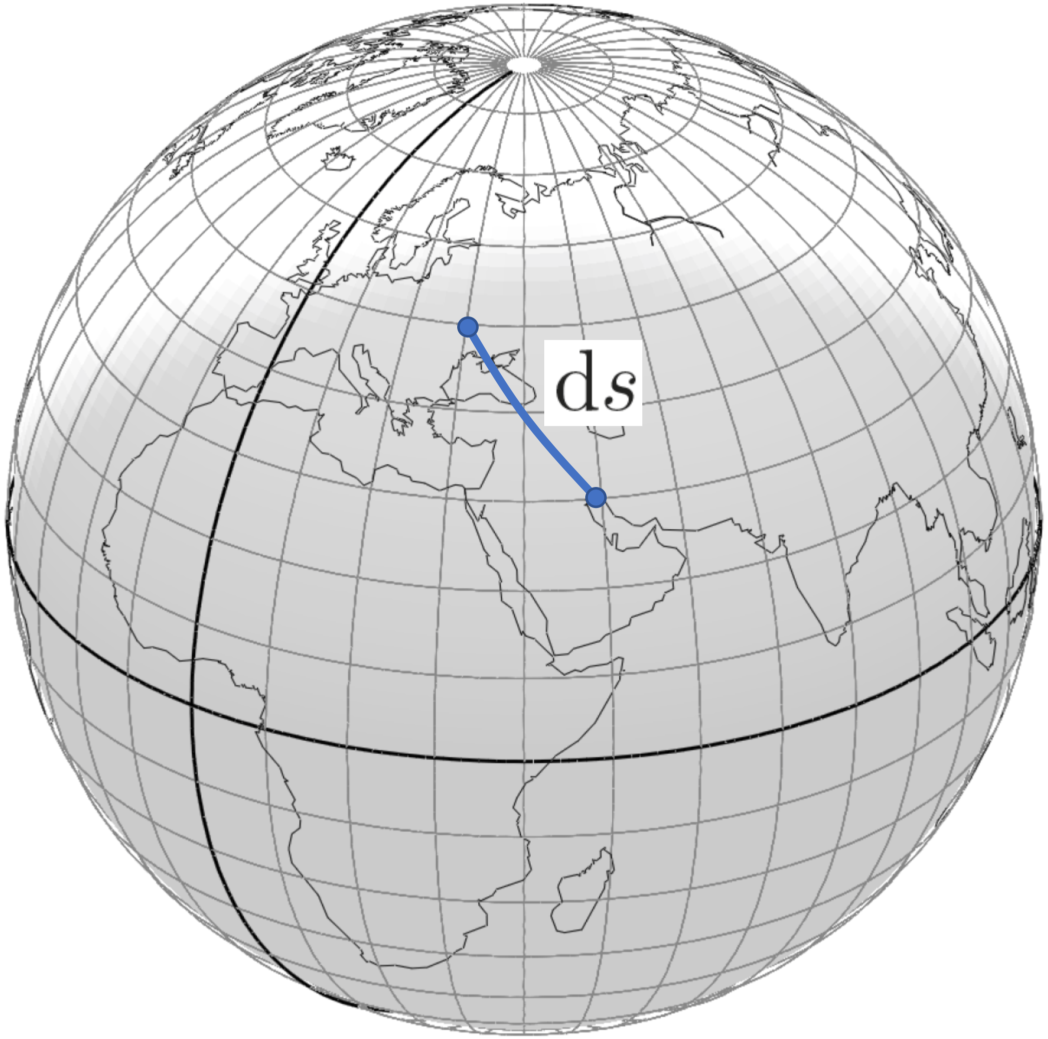
# Projective Coordinates




$$\mathrm{d}S = R \cdot \sqrt{\mathrm{d}\lambda^2 + \mathrm{d}\phi^2}$$

# Projective Coordinates

$$\mathrm{d}s = R \cdot \sqrt{\cos^2\phi \cdot \mathrm{d}\lambda^2 + \mathrm{d}\phi^2}$$

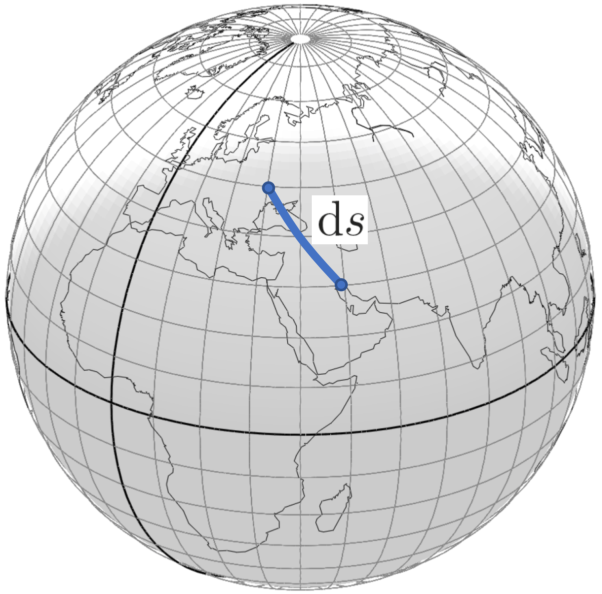


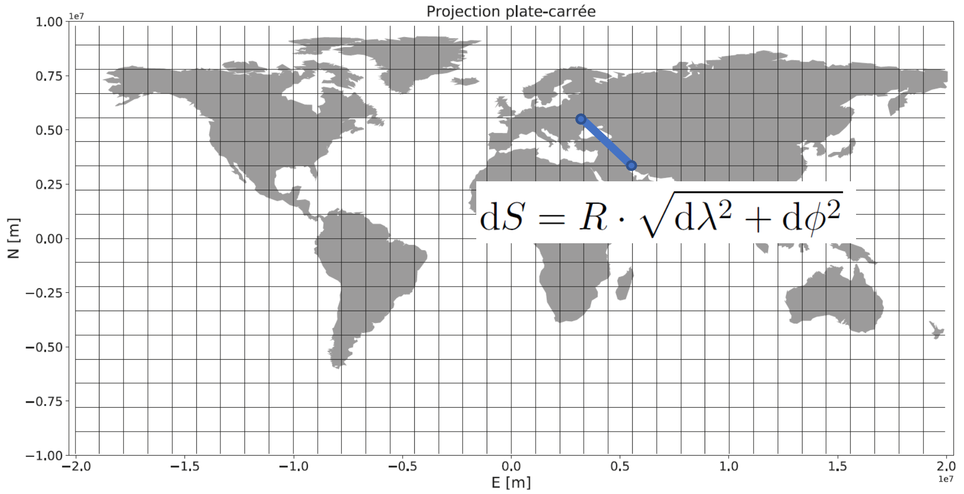
# Projective Coordinates

## Scale factor map/sphere (plane chart map)

$$\frac{\mathrm{d}S}{\mathrm{d}s} = \frac{R \cdot \sqrt{\mathrm{d}\lambda^2 + \mathrm{d}\phi^2}}{R \cdot \sqrt{\cos^2\phi \cdot \mathrm{d}\lambda^2 + \mathrm{d}\phi^2}}$$

- Changes with respect to  position
- Changes with respect to direction



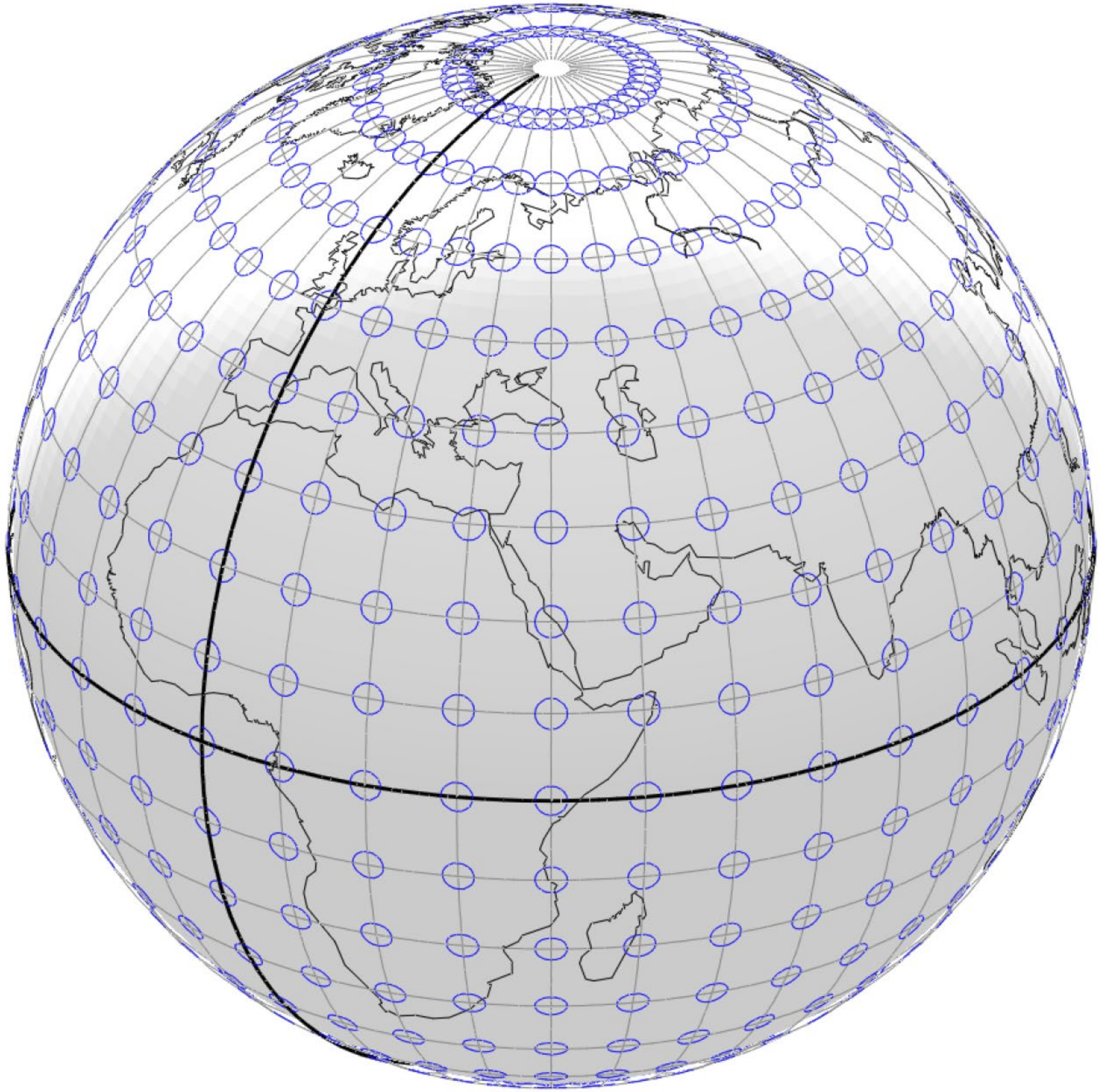



# Projective Coordinates

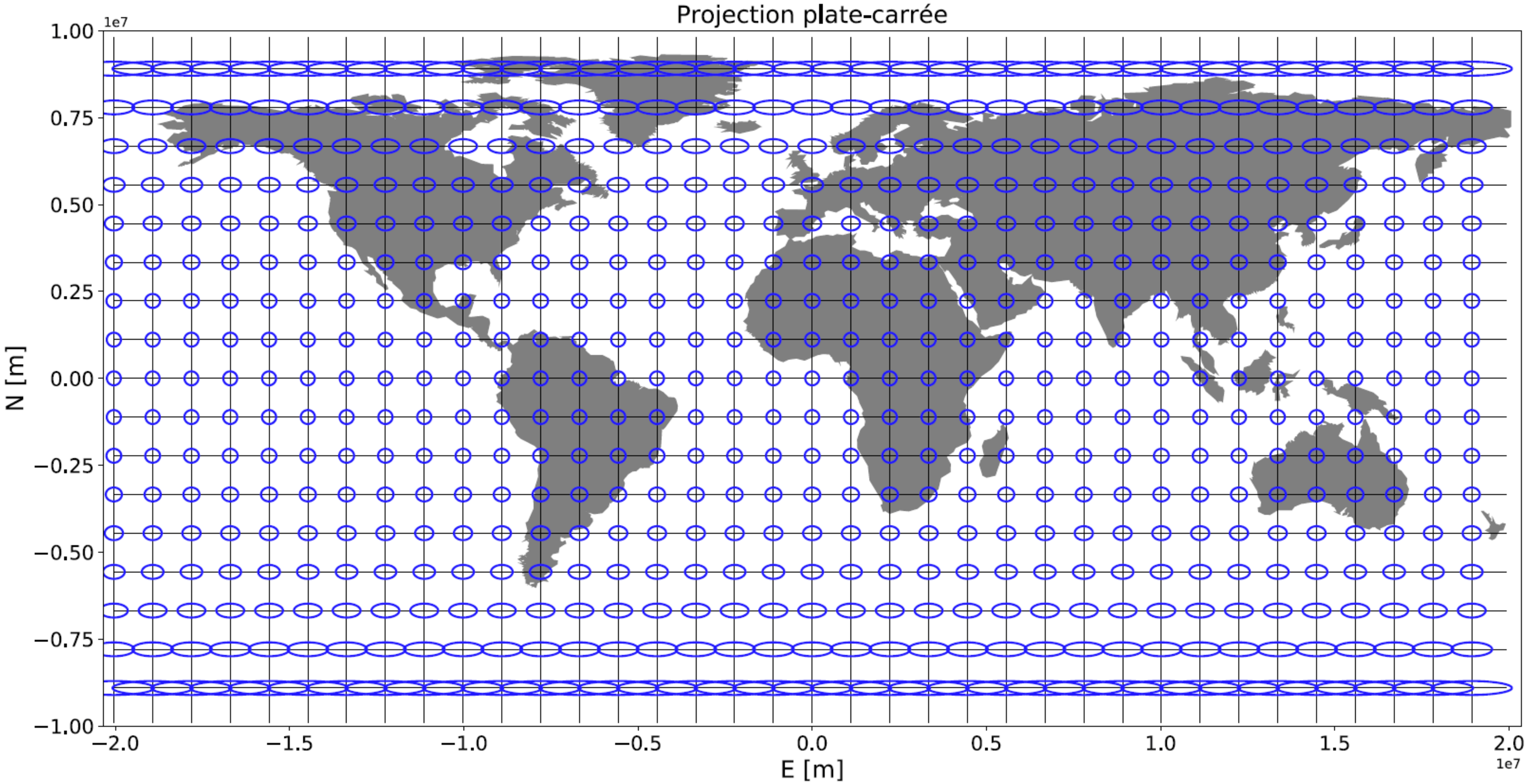
**Generalization of Analysis :**

Images of small circles on the reference surface

➡ **Tissot Indicatrix**



# Projective Coordinates



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# Projective Coordinates

## Computation of Tissot Indicatrix

Differential Distances on reference surfaces

$$\mathrm{d}s^2 = \mathrm{d}\mathbf{u}^T \cdot \mathbf{T} \cdot \mathrm{d}\mathbf{u}$$

General Parametrization :

$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

Tangent vectors :

$$\mathbf{t}_u(u,v) = \begin{pmatrix} \frac{\partial x(u,v)}{\partial u} \\ \frac{\partial y(u,v)}{\partial u} \\ \frac{\partial z(u,v)}{\partial u} \end{pmatrix} \qquad \mathbf{t}_v(u,v) = \begin{pmatrix} \frac{\partial x(u,v)}{\partial v} \\ \frac{\partial y(u,v)}{\partial v} \\ \frac{\partial z(u,v)}{\partial v} \end{pmatrix}$$

Metric Tensor :

$$\mathbf{T} = \begin{pmatrix} \mathbf{t}_u \cdot \mathbf{t}_u & \mathbf{t}_u \cdot \mathbf{t}_v \\ \mathbf{t}_v \cdot \mathbf{t}_u & \mathbf{t}_v \cdot \mathbf{t}_v \end{pmatrix} = \begin{pmatrix} t_{uu} & t_{uv} \\ t_{vu} & t_{vv} \end{pmatrix}$$

# Projective Coordinates

## Computation of Tissot Indicatrix

Differential Distances on reference surfaces

$$\mathrm{d}s^2 = \mathrm{d}\mathbf{u}^T \cdot \mathbf{T} \cdot \mathrm{d}\mathbf{u}$$

General Parametrization :

$$\mathbf{x}(\lambda, \phi) = \begin{pmatrix} x(\lambda, \phi) \\ y(\lambda, \phi) \\ z(\lambda, \phi) \end{pmatrix} = R \cdot \begin{pmatrix} \cos\phi \cdot \cos\lambda \\ \cos\phi \cdot \sin\lambda \\ \sin\phi \end{pmatrix}$$

Metric Tensor :

$$\mathbf{T}_{\mathrm{sph}} = \begin{pmatrix} \mathbf{t}_\lambda \cdot \mathbf{t}_\lambda & \mathbf{t}_\lambda \cdot \mathbf{t}_\phi \\ \mathbf{t}_\phi \cdot \mathbf{t}_\lambda & \mathbf{t}_\phi \cdot \mathbf{t}_\phi \end{pmatrix} = \begin{pmatrix} R^2 \cdot \cos^2\phi & 0 \\ 0 & R^2 \end{pmatrix}$$

Tangent vectors :

$$\mathbf{t}_\lambda = \begin{pmatrix} \frac{\partial x(\lambda,\phi)}{\partial\lambda} \\ \frac{\partial y(\lambda,\phi)}{\partial\lambda} \\ \frac{\partial z(\lambda,\phi)}{\partial\lambda} \end{pmatrix} = R \cdot \begin{pmatrix} -\cos\phi \cdot \sin\lambda \\ \cos\phi \cdot \cos\lambda \\ 0 \end{pmatrix}$$

$$\mathbf{t}_\phi = \begin{pmatrix} \frac{\partial x(\lambda,\phi)}{\partial\phi} \\ \frac{\partial y(\lambda,\phi)}{\partial\phi} \\ \frac{\partial z(\lambda,\phi)}{\partial\phi} \end{pmatrix} = R \cdot \begin{pmatrix} -\sin\phi \cdot \cos\lambda \\ -\sin\phi \cdot \sin\lambda \\ \cos\phi \end{pmatrix}$$

# Projective Coordinates

## Computation of Tissot Indicatrix

Differential Distances on projection surfaces

$$\mathrm{d}S^2 = \mathrm{d}\mathbf{u}^T \cdot \mathbf{P} \cdot \mathrm{d}\mathbf{u}$$

General Parametrization :

$$\mathbf{X}(u,v) = \begin{pmatrix} X(u,v) \\ Y(u,v) \end{pmatrix}$$

Tangent vectors :

$$\mathbf{p}_u(u,v) = \begin{pmatrix} \frac{\partial X(u,v)}{\partial u} \\ \frac{\partial Y(u,v)}{\partial u} \end{pmatrix} \qquad \mathbf{p}_v(u,v) = \begin{pmatrix} \frac{\partial X(u,v)}{\partial v} \\ \frac{\partial Y(u,v)}{\partial v} \end{pmatrix}$$

Metric Tensor :

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_u \cdot \mathbf{p}_u & \mathbf{p}_u \cdot \mathbf{p}_v \\ \mathbf{p}_v \cdot \mathbf{p}_u & \mathbf{p}_v \cdot \mathbf{p}_v \end{pmatrix} = \begin{pmatrix} p_{uu} & p_{uv} \\ p_{uv} & p_{vv} \end{pmatrix}$$

# Projective Coordinates

## Computation of Tissot Indicatrix

Differential Distances on projection surfaces

$$\mathrm{d}S^2 = \mathrm{d}\mathbf{u}^T \cdot \mathbf{P} \cdot \mathrm{d}\mathbf{u}$$

General Parametrization :

$$\mathbf{X}(\lambda,\phi) = \begin{pmatrix} X(\lambda,\phi) \\ Y(\lambda,\phi) \end{pmatrix} = \begin{pmatrix} E(\lambda,\phi) \\ N(\lambda,\phi) \end{pmatrix} = \begin{pmatrix} R\cdot\lambda \\ R\cdot\phi \end{pmatrix}$$

Metric Tensor :

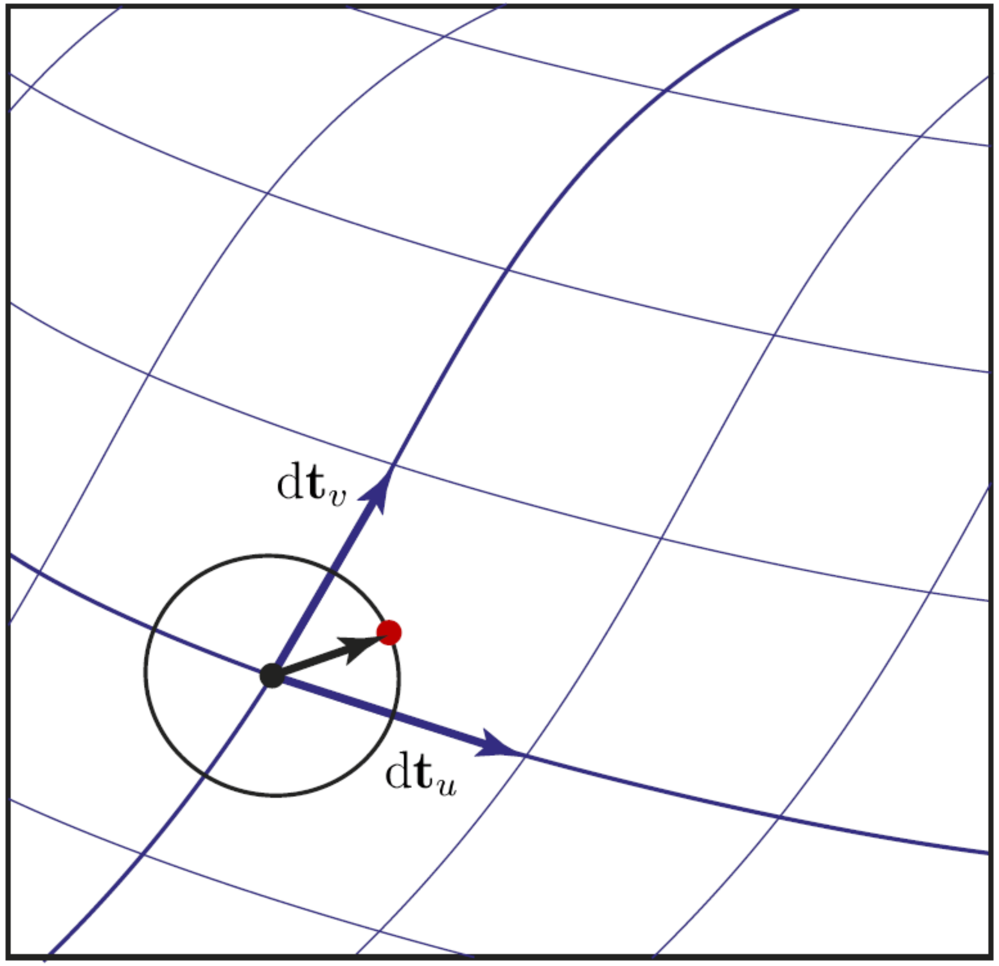
$$\mathbf{P}_{\text{plate-carrée/sph}} = \begin{pmatrix} \mathbf{p}_\lambda\cdot\mathbf{p}_\lambda & \mathbf{p}_\lambda\cdot\mathbf{p}_\phi \\ \mathbf{p}_\phi\cdot\mathbf{p}_\lambda & \mathbf{p}_\phi\cdot\mathbf{p}_\phi \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \end{pmatrix}$$

Tangent vectors :

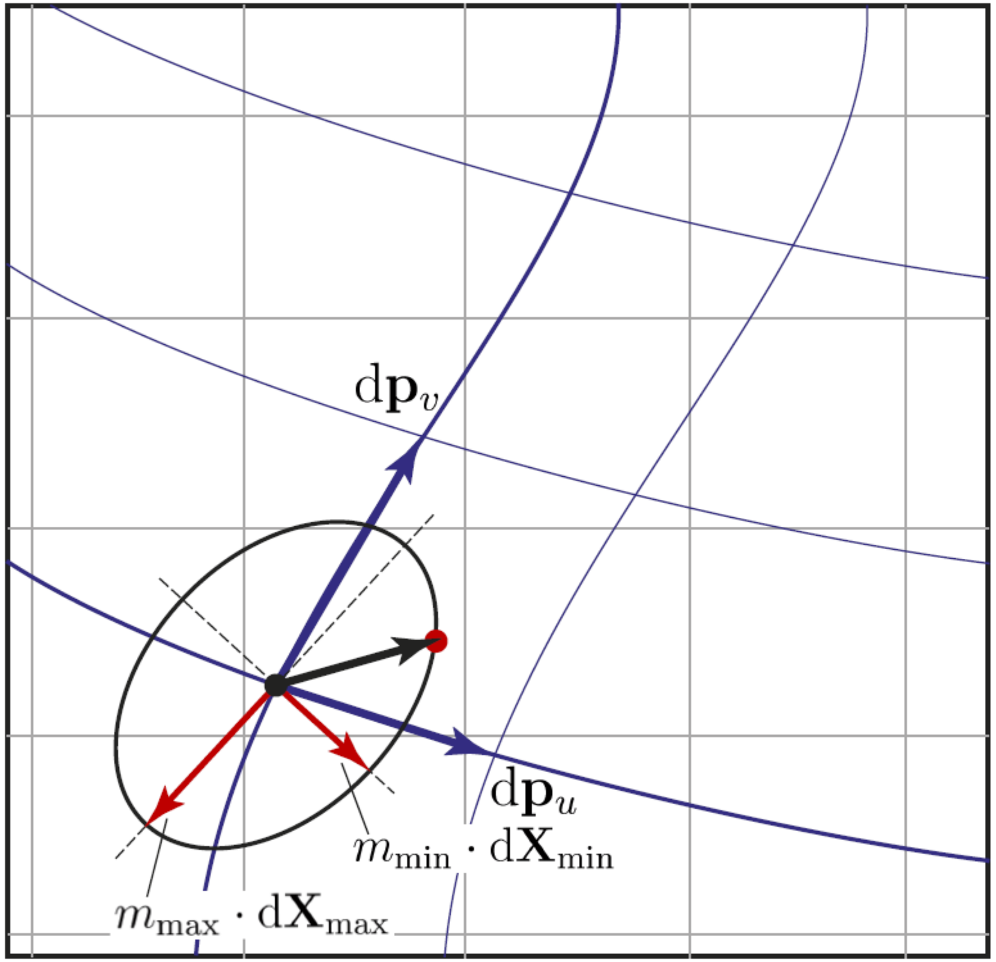
$$\mathbf{p}_\lambda = \begin{pmatrix} \frac{\partial E(\lambda,\phi)}{\partial\lambda} \\ \frac{\partial N(\lambda,\phi)}{\partial\lambda} \end{pmatrix} = \begin{pmatrix} R \\ 0 \end{pmatrix} \qquad \mathbf{p}_\phi = \begin{pmatrix} \frac{\partial E(\lambda,\phi)}{\partial\phi} \\ \frac{\partial N(\lambda,\phi)}{\partial\phi} \end{pmatrix} = \begin{pmatrix} 0 \\ R \end{pmatrix}$$

# Projective Coordinates

**Surface of REFERENCE**




**Surface of PROJECTION**



# Projective Coordinates

## Computation of Tissot Indicatrix

Scale between reference and projection surfaces

$$m(\mathrm{d}\mathbf{u}) = \frac{\mathrm{d}S}{\mathrm{d}s} = \frac{\sqrt{\mathrm{d}\mathbf{u} \cdot \mathbf{P} \cdot \mathrm{d}\mathbf{u}^T}}{\sqrt{\mathrm{d}\mathbf{u} \cdot \mathbf{T} \cdot \mathrm{d}\mathbf{u}^T}}$$

Deformation tensor :

$$\mathbf{D} = \mathbf{T}^{-1} \cdot \mathbf{P}$$

Principal axes with eigenvalue decomposition

$$\mathbf{D} = \mathbf{V} \cdot \mathbf{\Lambda} \cdot \mathbf{V}^{-1}$$

→

$$m_{\max} = \sqrt{\lambda_1} \qquad \mathrm{d}\mathbf{u}_{\max} = \mathbf{v}_1 = \begin{pmatrix} v_{1_u} \\ v_{1_v} \end{pmatrix}$$

$$m_{\min} = \sqrt{\lambda_2} \qquad \mathrm{d}\mathbf{u}_{\min} = \mathbf{v}_2 = \begin{pmatrix} v_{2_u} \\ v_{2_v} \end{pmatrix}$$

# Projective Coordinates

## Computation of Tissot Indicatrix

Example : spherical plan chart projection

Metric tensors :

$$\mathbf{T}_{\text{sph}} = \begin{pmatrix} \mathbf{t}_\lambda \cdot \mathbf{t}_\lambda & \mathbf{t}_\lambda \cdot \mathbf{t}_\phi \\ \mathbf{t}_\phi \cdot \mathbf{t}_\lambda & \mathbf{t}_\phi \cdot \mathbf{t}_\phi \end{pmatrix} = \begin{pmatrix} R^2 \cdot \cos^2\phi & 0 \\ 0 & R^2 \end{pmatrix}$$

$$\mathbf{P}_{\text{plate-carrée/sph}} = \begin{pmatrix} \mathbf{p}_\lambda \cdot \mathbf{p}_\lambda & \mathbf{p}_\lambda \cdot \mathbf{p}_\phi \\ \mathbf{p}_\phi \cdot \mathbf{p}_\lambda & \mathbf{p}_\phi \cdot \mathbf{p}_\phi \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \end{pmatrix}$$

Deformation matrix

$$\begin{aligned} \mathbf{D} &= \mathbf{T}_{\text{sph}}^{-1} \cdot \mathbf{P}_{\text{plate-carrée/sph}} \\ &= \begin{pmatrix} \frac{1}{R^2 \cdot \cos^2\phi} & 0 \\ 0 & \frac{1}{R^2} \end{pmatrix} \cdot \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{R^2}{R^2 \cdot \cos^2\phi} & 0 \\ 0 & \frac{R^2}{R^2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\cos^2\phi} & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

# Projective Coordinates

## Computation of Tissot Indicatrix

Example : spherical plan chart projection

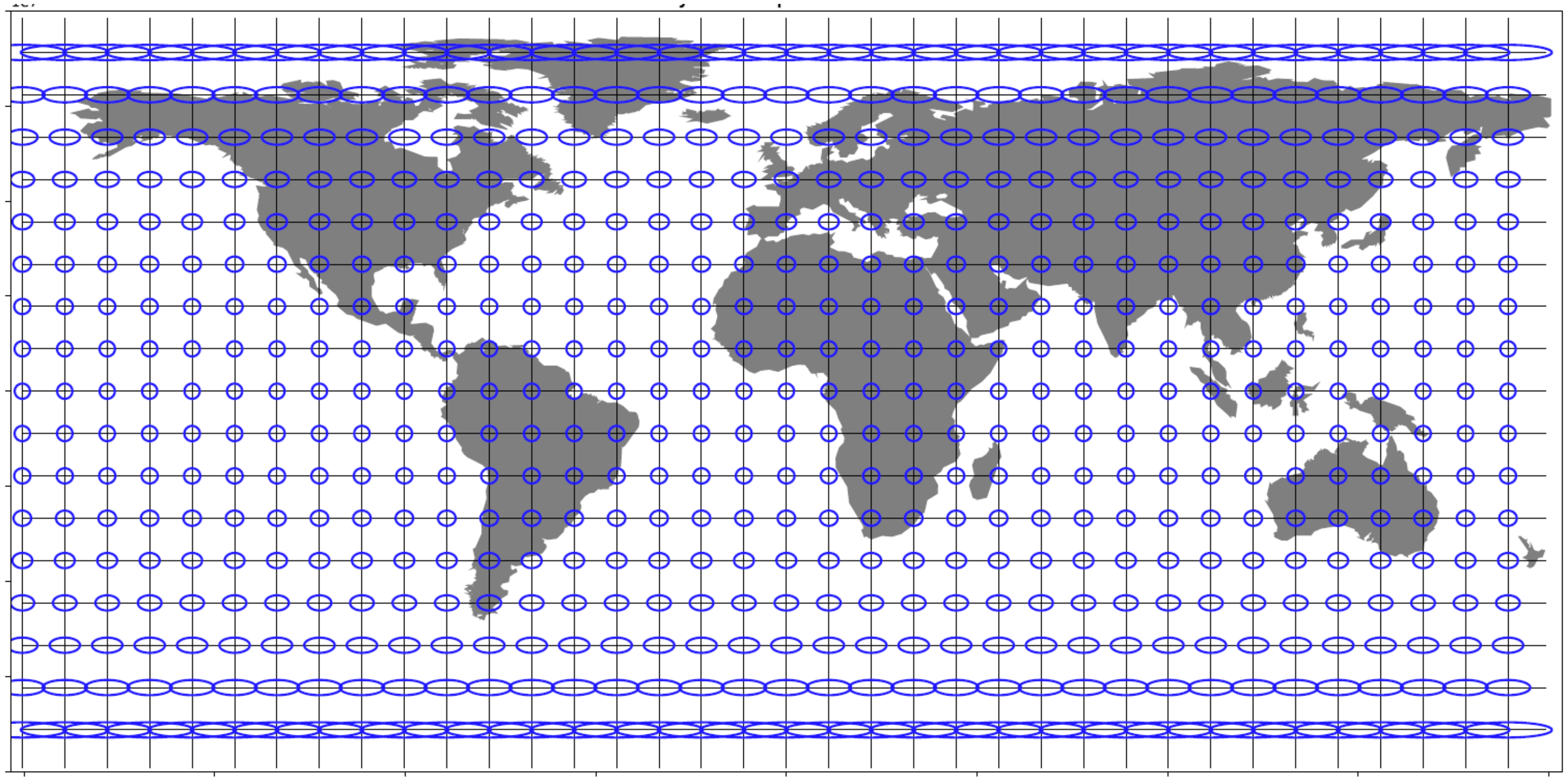
Tissot Indicatrix :

$$m_{\max} = \sqrt{\lambda_1} = \frac{1}{\cos\phi}$$

$$m_{\min} = \sqrt{\lambda_2} = 1$$

$$\mathrm{d}\mathbf{u}_{\max} = \mathbf{v}_1 = \begin{pmatrix} 1 \cdot \mathrm{d}\lambda \\ 0 \cdot \mathrm{d}\phi \end{pmatrix} = \begin{pmatrix} \mathrm{d}\lambda \\ 0 \end{pmatrix}$$

$$\mathrm{d}\mathbf{u}_{\min} = \mathbf{v}_2 = \begin{pmatrix} 0 \cdot \mathrm{d}\lambda \\ 1 \cdot \mathrm{d}\phi \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{d}\phi \end{pmatrix}$$



# Projective Coordinates

## Properties of projections

- Conformal : $m_{\max} = m_{\min} \quad \forall u, v$
- Equivalent : $m_{\max} \cdot m_{\min} = 1 \quad \forall u, v$

## Classes of projections

Cylindrical :

$$E(\lambda, \phi) = R \cdot \lambda$$
$$N(\lambda, \phi) = R \cdot f(\phi)$$

Conical :

$$E(\lambda, \phi) = \rho(\lambda, \phi) \cdot \cos\left[\alpha(\lambda, \phi)\right]$$
$$N(\lambda, \phi) = \rho(\lambda, \phi) \cdot \sin\left[\alpha(\lambda, \phi)\right]$$

$$\rho(\lambda, \phi) = R \cdot f(\phi)$$
$$\alpha(\lambda, \phi) = \sin \phi_0 \cdot \lambda + \text{cst}$$

Azimuthal :

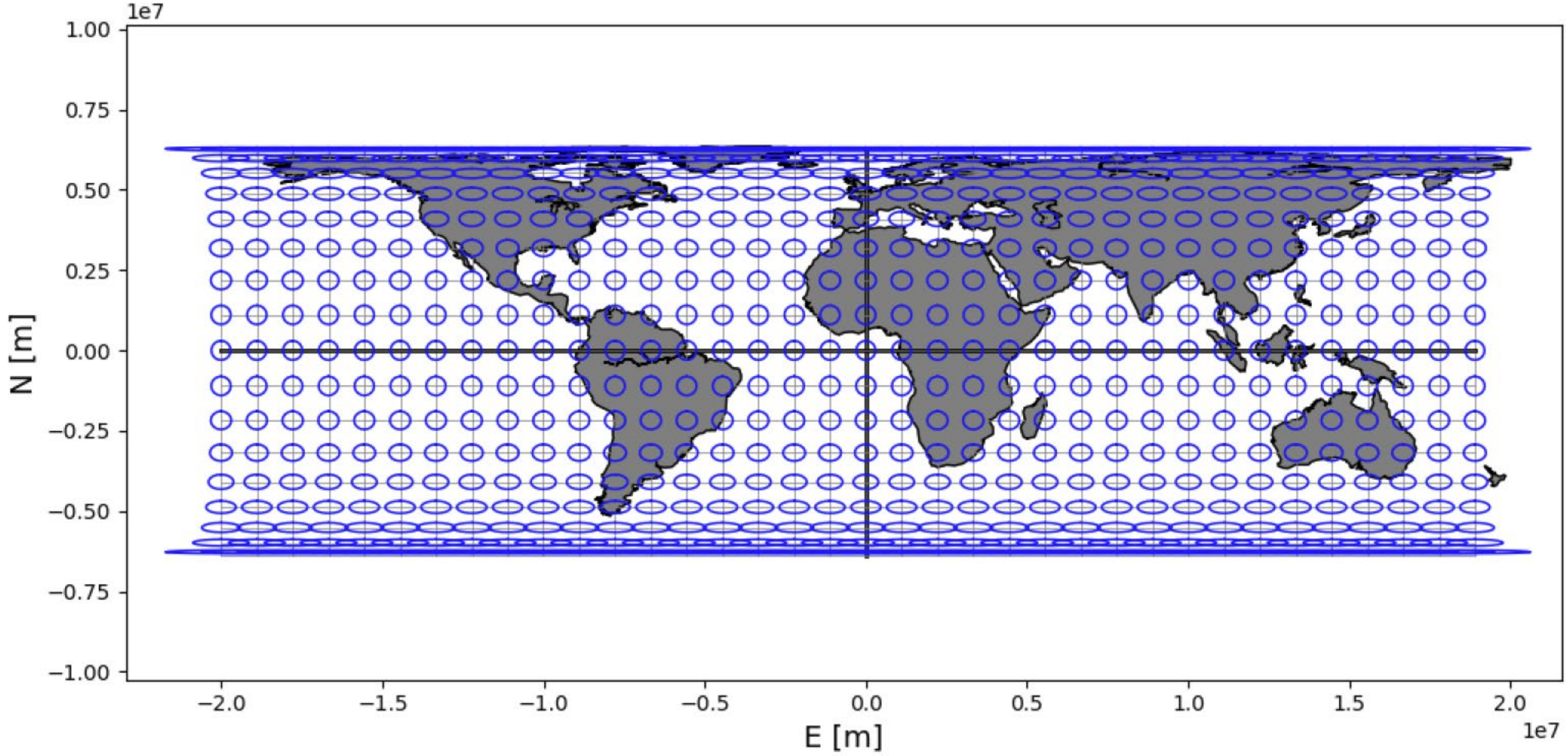
$$E(\lambda, \phi) = \rho(\lambda, \phi) \cdot \cos\left[\alpha(\lambda, \phi)\right]$$
$$N(\lambda, \phi) = \rho(\lambda, \phi) \cdot \sin\left[\alpha(\lambda, \phi)\right]$$

$$\rho(\lambda, \phi) = R \cdot f(\phi)$$
$$\alpha(\lambda, \phi) = \lambda + \text{cst}$$

# Projective Coordinates

## Cylindrical (Equivalent)

$$E(\lambda, \phi) = R \cdot \lambda$$
$$N(\lambda, \phi) = R \cdot \sin \phi$$

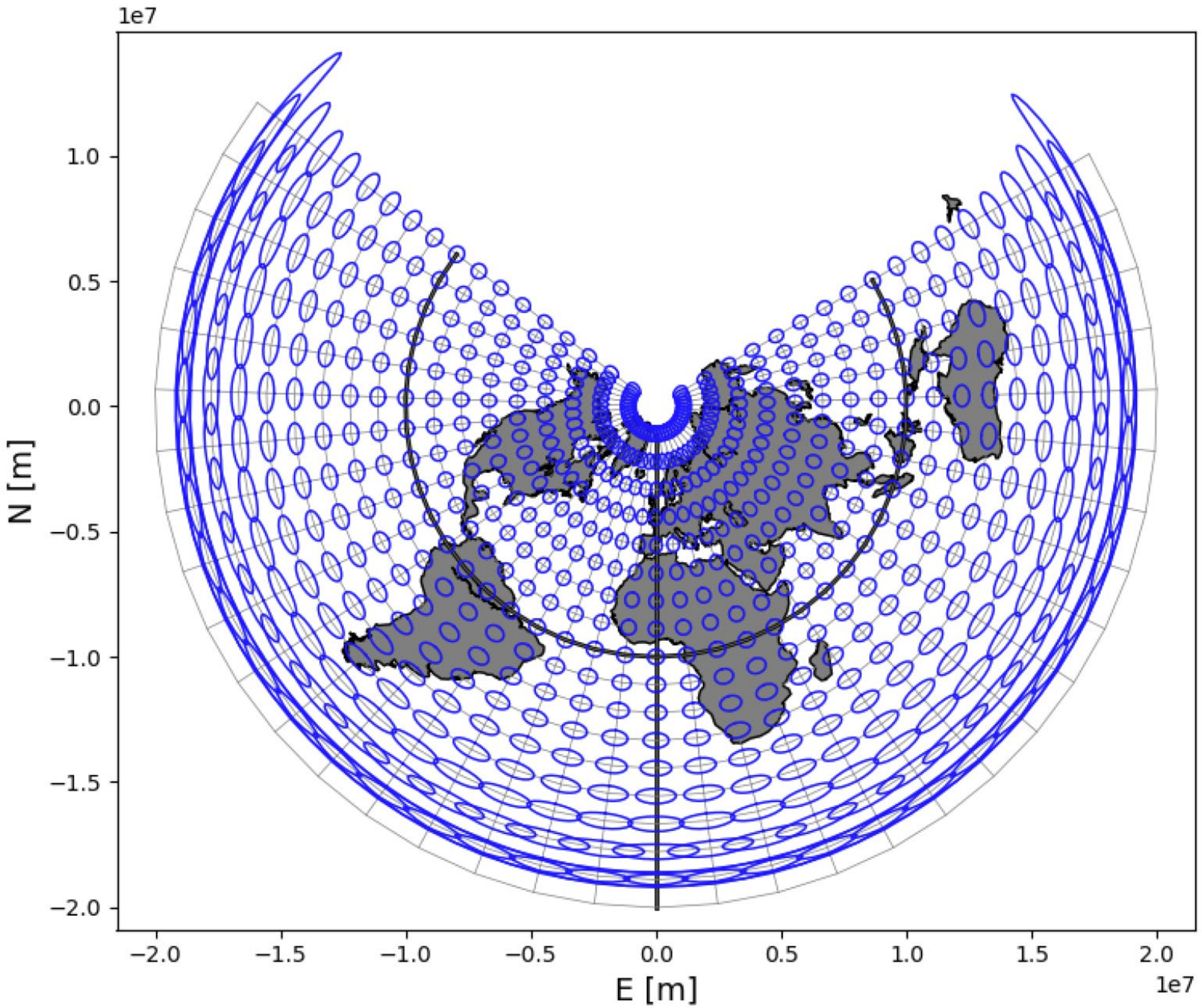


# Projective Coordinates

## Conical

$$E(\lambda,\phi) = \rho(\lambda,\phi)\cdot\cos\left[\alpha(\lambda,\phi)\right]$$
$$N(\lambda,\phi) = \rho(\lambda,\phi)\cdot\sin\left[\alpha(\lambda,\phi)\right]$$

$$\rho(\lambda,\phi) = R\cdot\left(\frac{\pi}{2}-\phi\right)$$
$$\alpha(\lambda,\phi) = \lambda\cdot\sin\frac{\pi}{4}-\frac{\pi}{2}$$



# Projective Coordinates

## Mercator projection (cylindrical conformal)

$$\mathbf{X}(\lambda,\phi) = \begin{pmatrix} E(\lambda,\phi) \\ N(\lambda,\phi) \end{pmatrix} = \begin{pmatrix} R\cdot\lambda \\ R\cdot\ln\left[\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right] \end{pmatrix}$$

$$\mathbf{P}_{\mathrm{Mercator/sph}} = \begin{pmatrix} R^2 & 0 \\ 0 & \frac{R^2}{\cos^2\phi} \end{pmatrix}$$

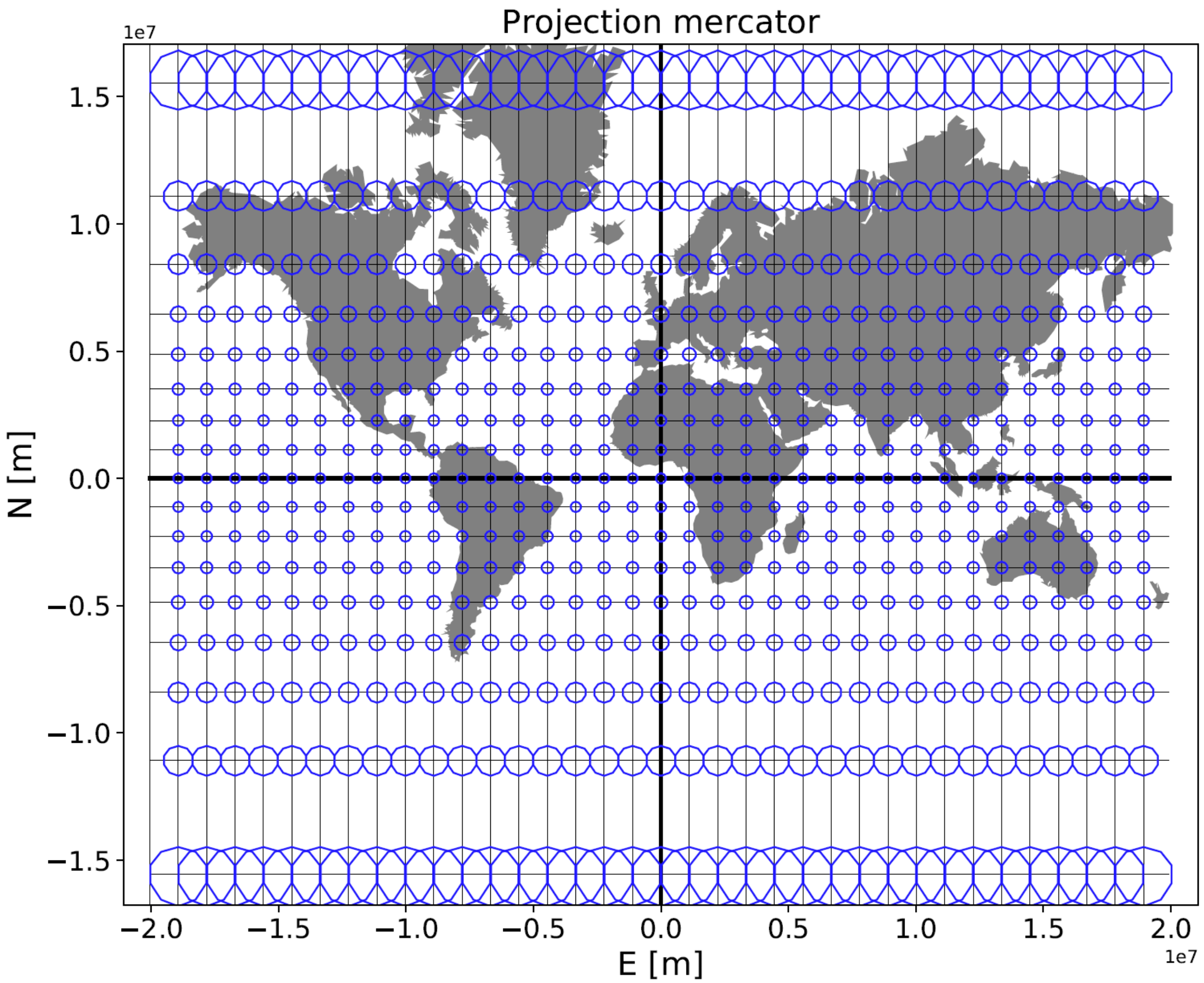
$$\mathbf{D}_{\mathrm{Mercator/sph}} = \begin{pmatrix} \frac{1}{\cos^2\phi} & 0 \\ 0 & \frac{1}{\cos^2\phi} \end{pmatrix}$$

Scale factor :

$$m_{\max} = m_{\min} = \frac{1}{\cos\phi}$$

Meridian convergence :

$$\mu(\lambda,\phi) = 0$$



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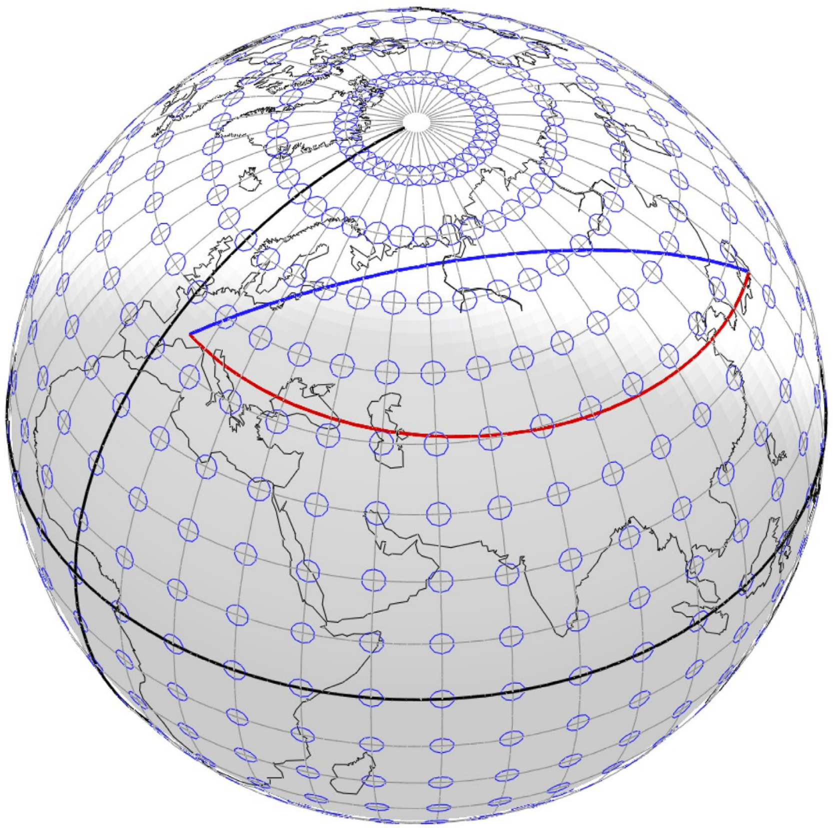
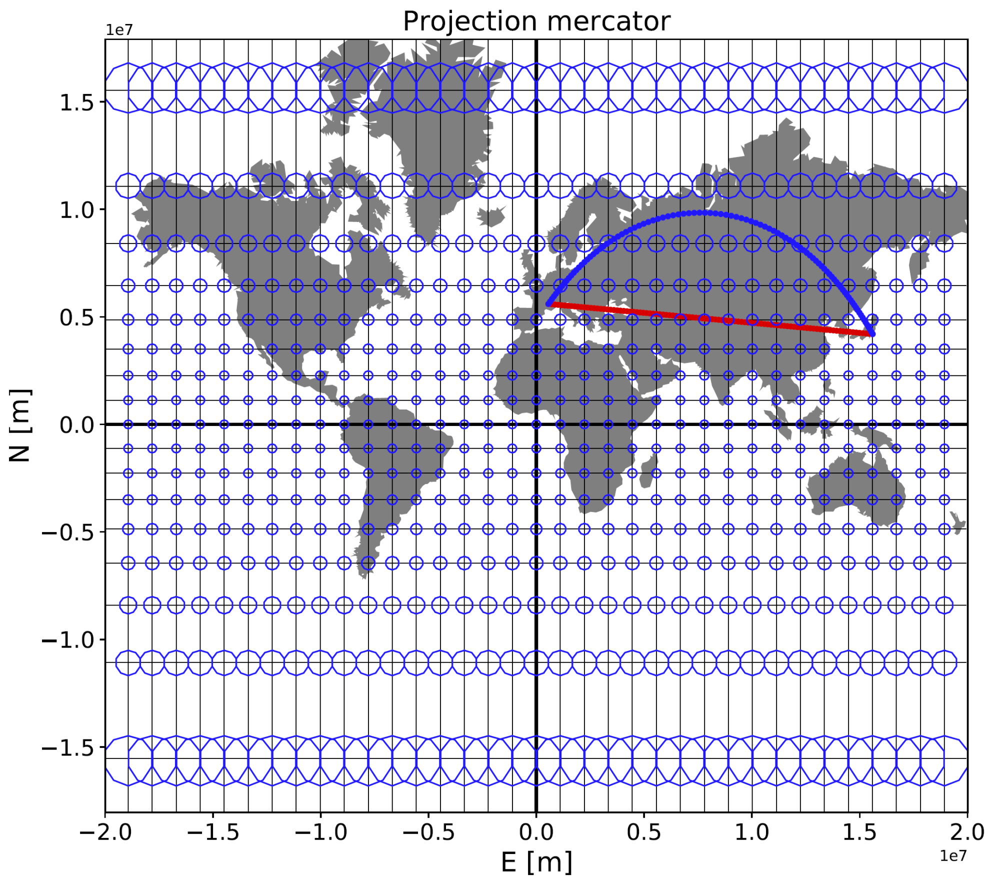
# Projective Coordinates

## Mercator projection (cylindrical conformal)

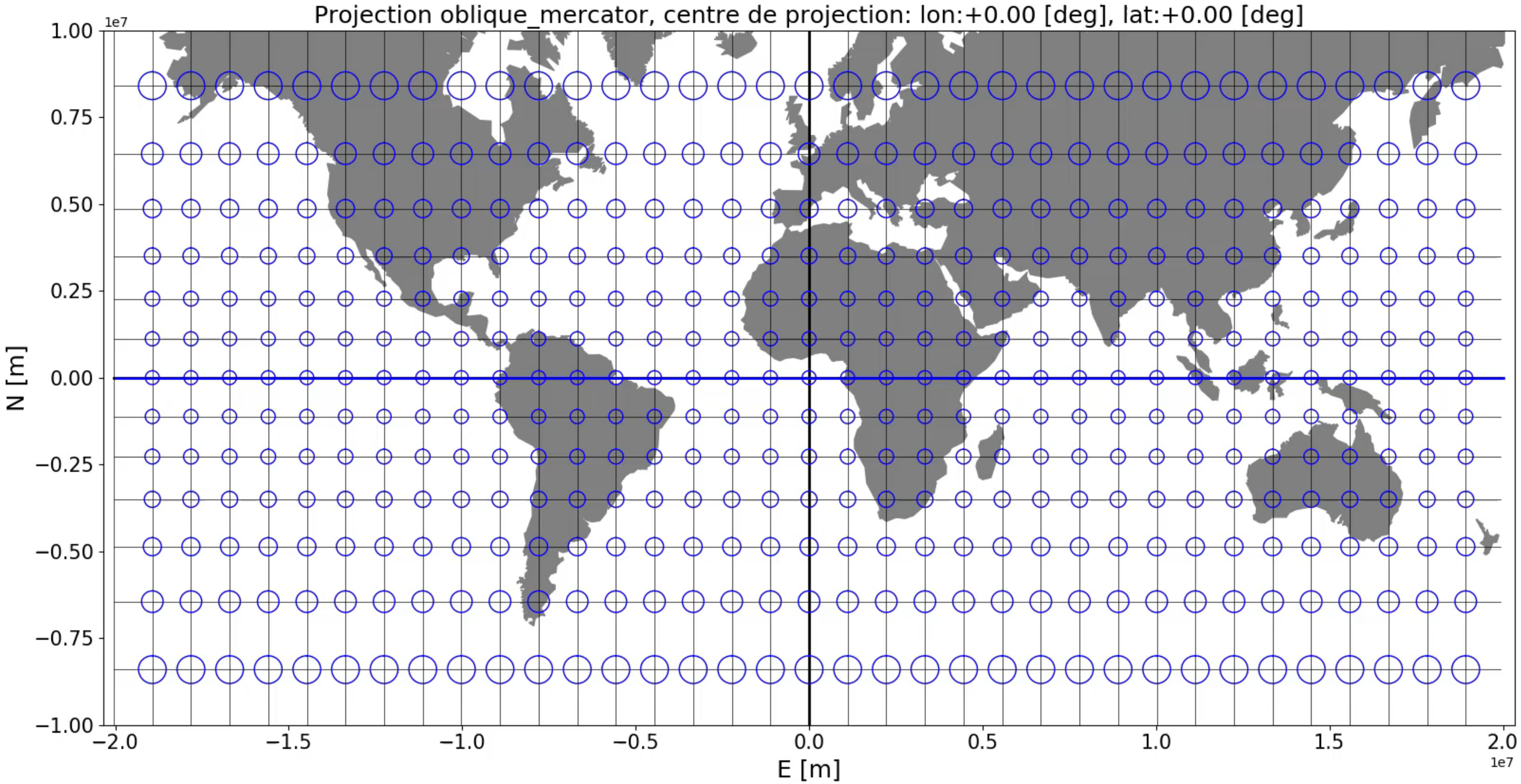


loxodrome orthodrome



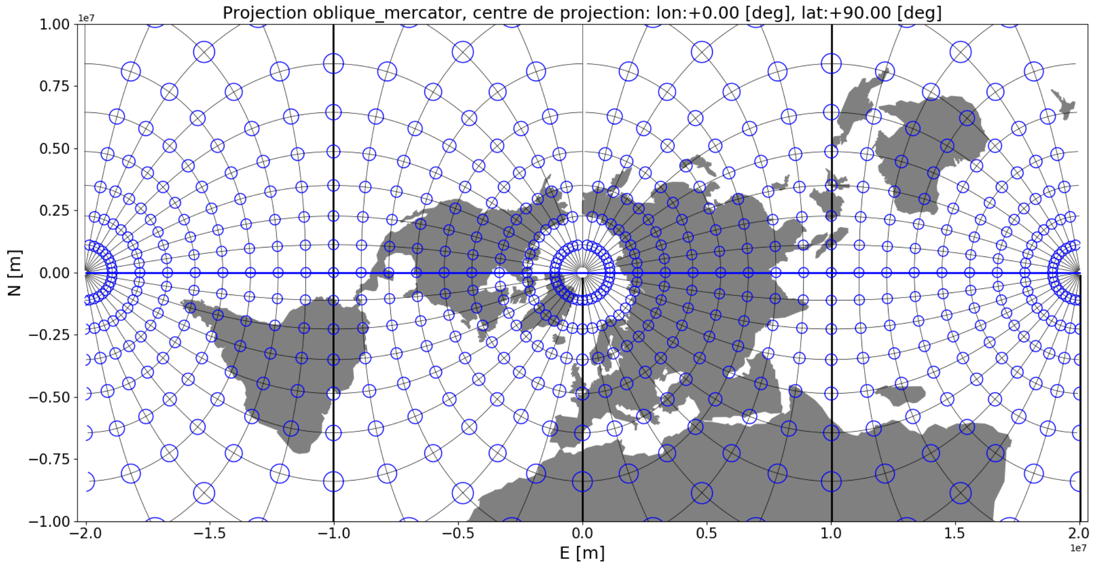
# Projective Coordinates

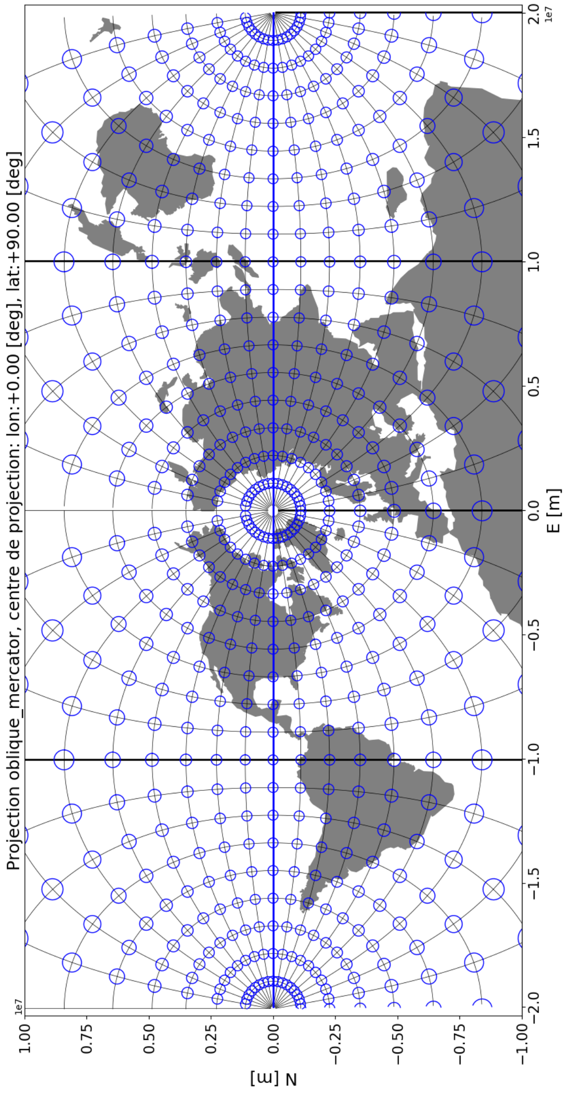
## Oblique Mercator projection (cylindrical conformal)



# Projective Coordinates

## Transverse Mercator projection (cylindrical conformal)



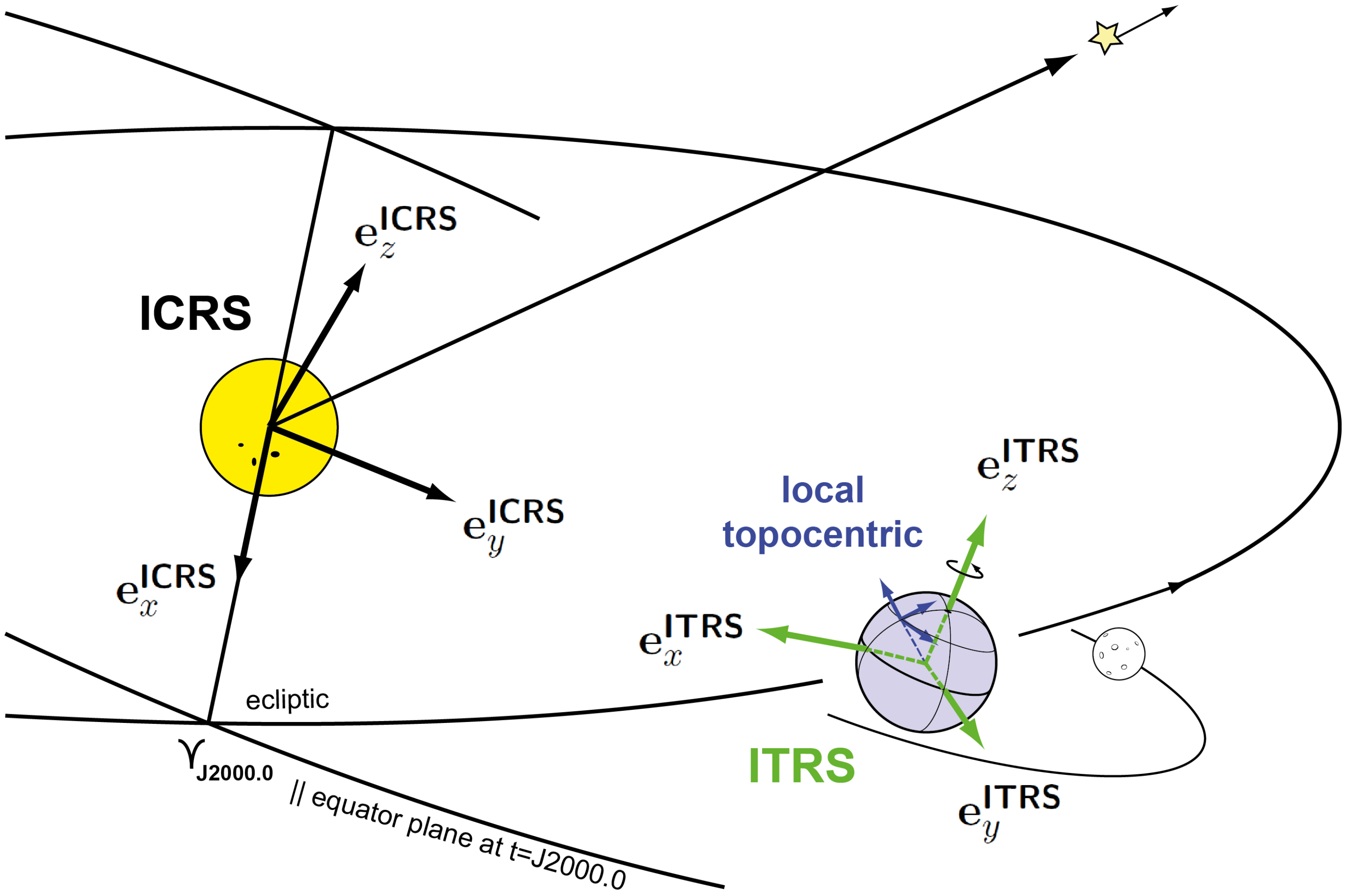
# Reference Systems and Frames

# Reference Systems and Frames

## ICRS
**International Celestial Reference System**

## ITRS
**International Terrestrial Reference System**

## local topocentric
**Natural system of an observer aligned with gravity vector**

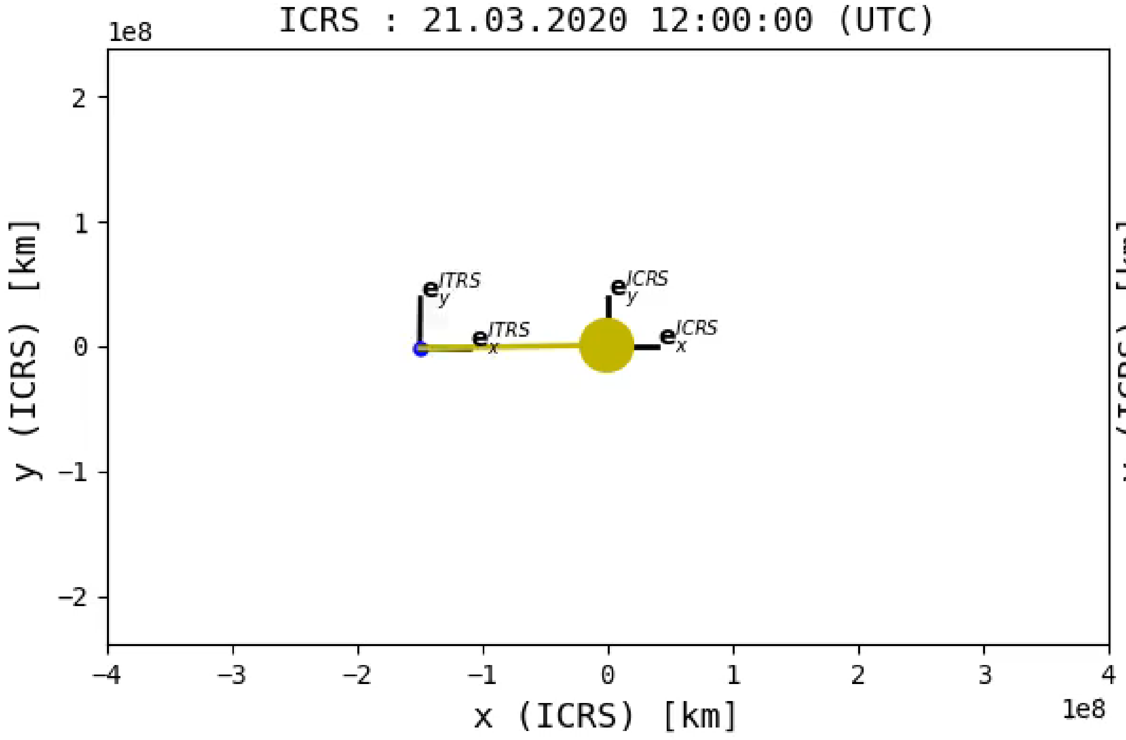


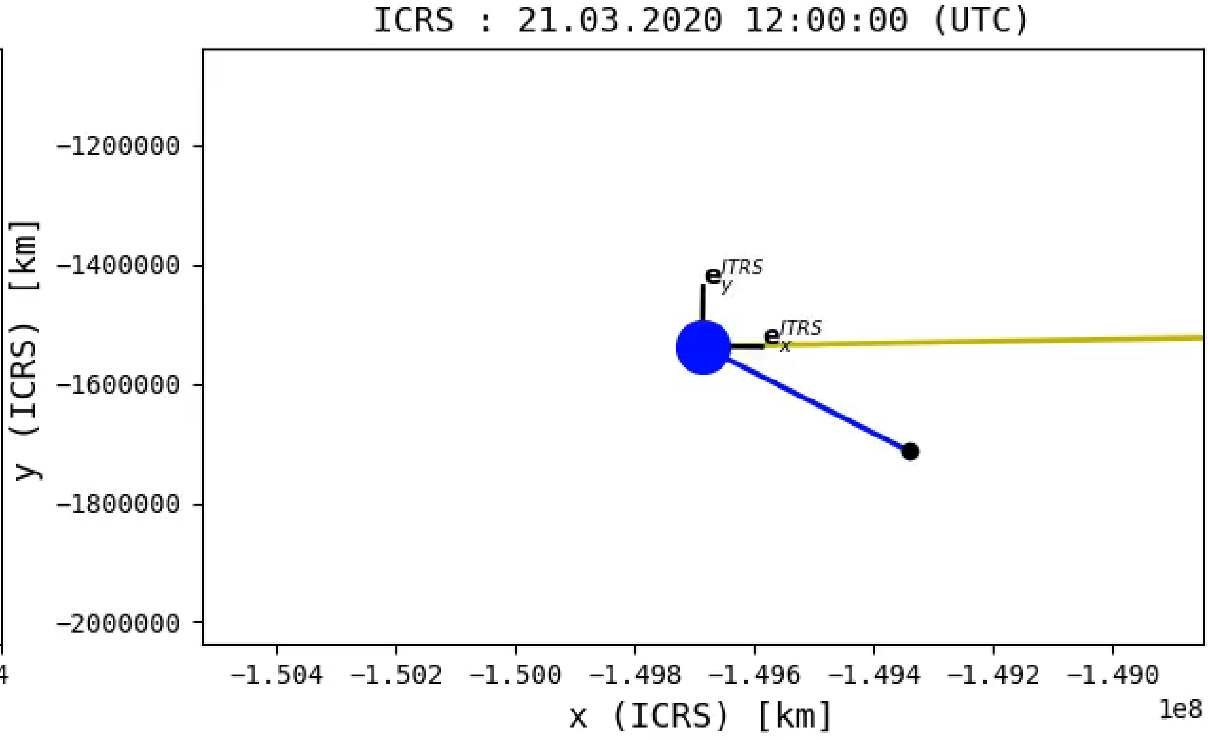
# Reference Systems and Frames

ICRS / ICRF – International Celestial Reference System/Frame

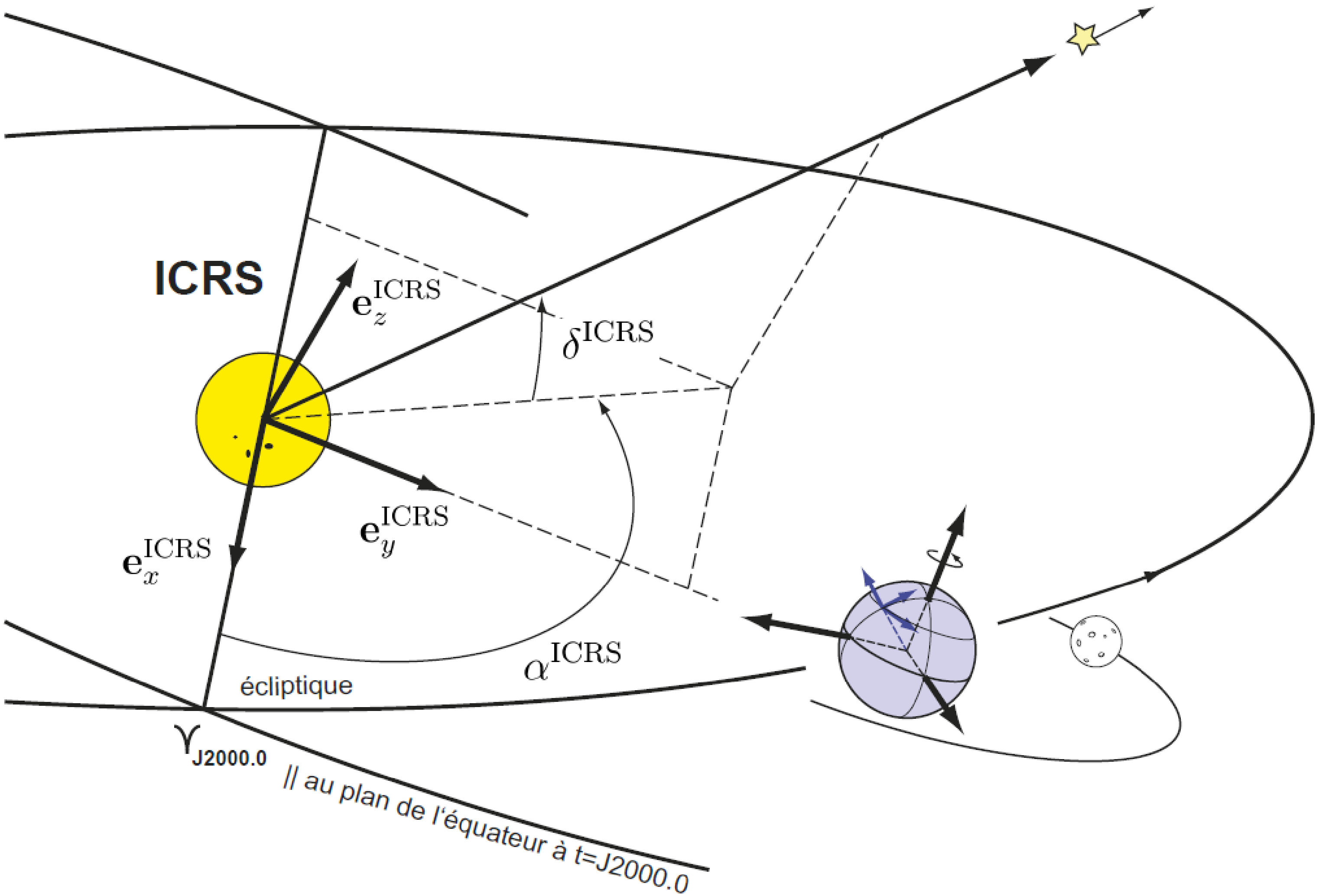
# ICRS / ICRF – International Celestial Reference System/Frame

$$m \cdot \mathbf{a} = m \cdot \frac{\mathrm{d}^2\mathbf{x}(t)}{\mathrm{d}t^2} = \sum \mathbf{F}_{\text{ext}} \quad ✔$$



# ICRS / ICRF – International Celestial Reference System/Frame



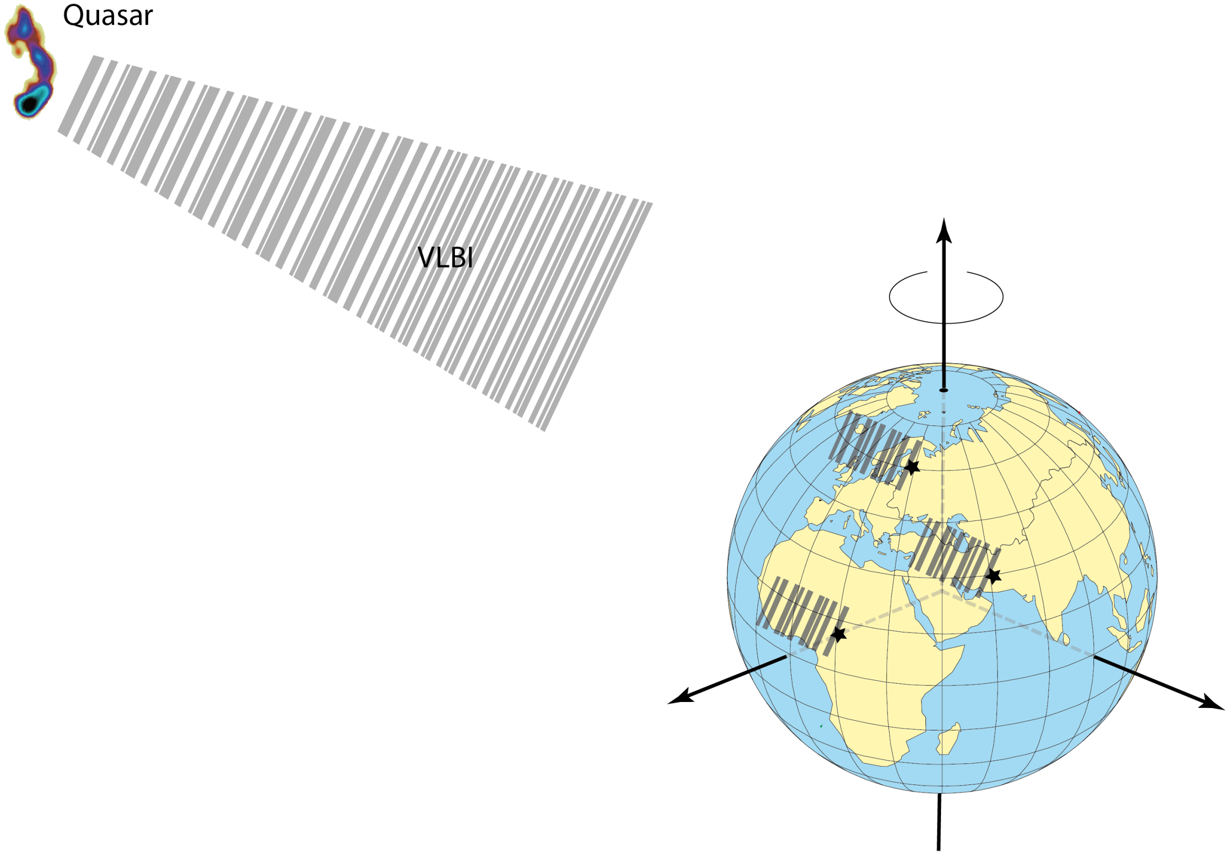
# ICRS / ICRF – International Celestial Reference System/Frame

$$\mathbf{x}^{\mathsf{ICRF}} = s^{\mathsf{ICRF}} \cdot \begin{pmatrix} \cos\delta^{\mathsf{ICRF}} \cdot \cos\alpha^{\mathsf{ICRF}} \\ \cos\delta^{\mathsf{ICRF}} \cdot \sin\alpha^{\mathsf{ICRF}} \\ \sin\delta^{\mathsf{ICRF}} \end{pmatrix}$$

| ICRF Designation (1) | IERS Des. (2) | Inf. (3) | Right Ascension J2000.0 h m s | Declination J2000.0 o ' " | Uncertainty R.A. s | Dec. " |
|---|---|---|---|---|---|---|
| ICRF J000020.3-322101 | 2357-326 | | 00 00 20.39997606 | -32 21 01.2337415 | 0.00000804 | 0.0002624 |
| ICRF J000027.0+030715 | 2357+028 | | 00 00 27.02251377 | 03 07 15.6463606 | 0.00005931 | 0.0003421 |
| ICRF J000053.0+405401 | 2358+406 | | 00 00 53.08106320 | 40 54 01.8096518 | 0.00001504 | 0.0002670 |
| ICRF J000105.3-155107 | 2358-161 | | 00 01 05.32873479 | -15 51 07.0752302 | 0.00000702 | 0.0002261 |
| ICRF J000107.0+605122 | 2358+605 | | 00 01 07.09981547 | 60 51 22.7980875 | 0.00003378 | 0.0001948 |
| ICRF J000108.6+191433 | 2358+189 | | 00 01 08.62156616 | 19 14 33.8017136 | 0.00000260 | 0.0000472 |
| ICRF J000211.9-215309 | 2359-221 | | 00 02 11.98142614 | -21 53 09.8655460 | 0.00001333 | 0.0004673 |
| ICRF J000315.9-194150 | 0000-199 | | 00 03 15.94940393 | -19 41 50.4018049 | 0.00000936 | 0.0002972 |

# Reference Systems and Frames

source: https://en.wikipedia.org/wiki/Geodetic_Observatory_Wettzell



# Reference Systems and Frames

TRS / TRF – Terrestrial Reference Systems/Frames

# TRS / TRF – Terrestrial Reference System/Frame

**TRS Terrestrial Reference Systems :**

Theoretical definitions and conventions

**ITRS** International Terrestrial Reference System

**ETRS89** European Terrestrial Reference System

**CHTRS89** Swiss Terrestrial Reference System

**?? RGS ??**

**TRF Terrestrial Reference Frames :**

Realization of a TRS

List of coordinates and velocities of benchmarks

**ITRF89, ITRF90, …, ITRF2008, ITRF2014**

**ETRF89, ETRF90, …, ETRF2014**

**CHTRF95**

**RGF93**

# TRS / TRF – Terrestrial Reference System/Frame

Definition of a **T**errestrial **R**eference **S**ystem

$$(\mathbf{o}^{\mathsf{TRS}}, \mathbf{e}_x^{\mathsf{TRS}}, \mathbf{e}_y^{\mathsf{TRS}}, \mathbf{e}_z^{\mathsf{TRS}})$$

Reference points in a **T**errestrial **R**eference **S**ystem

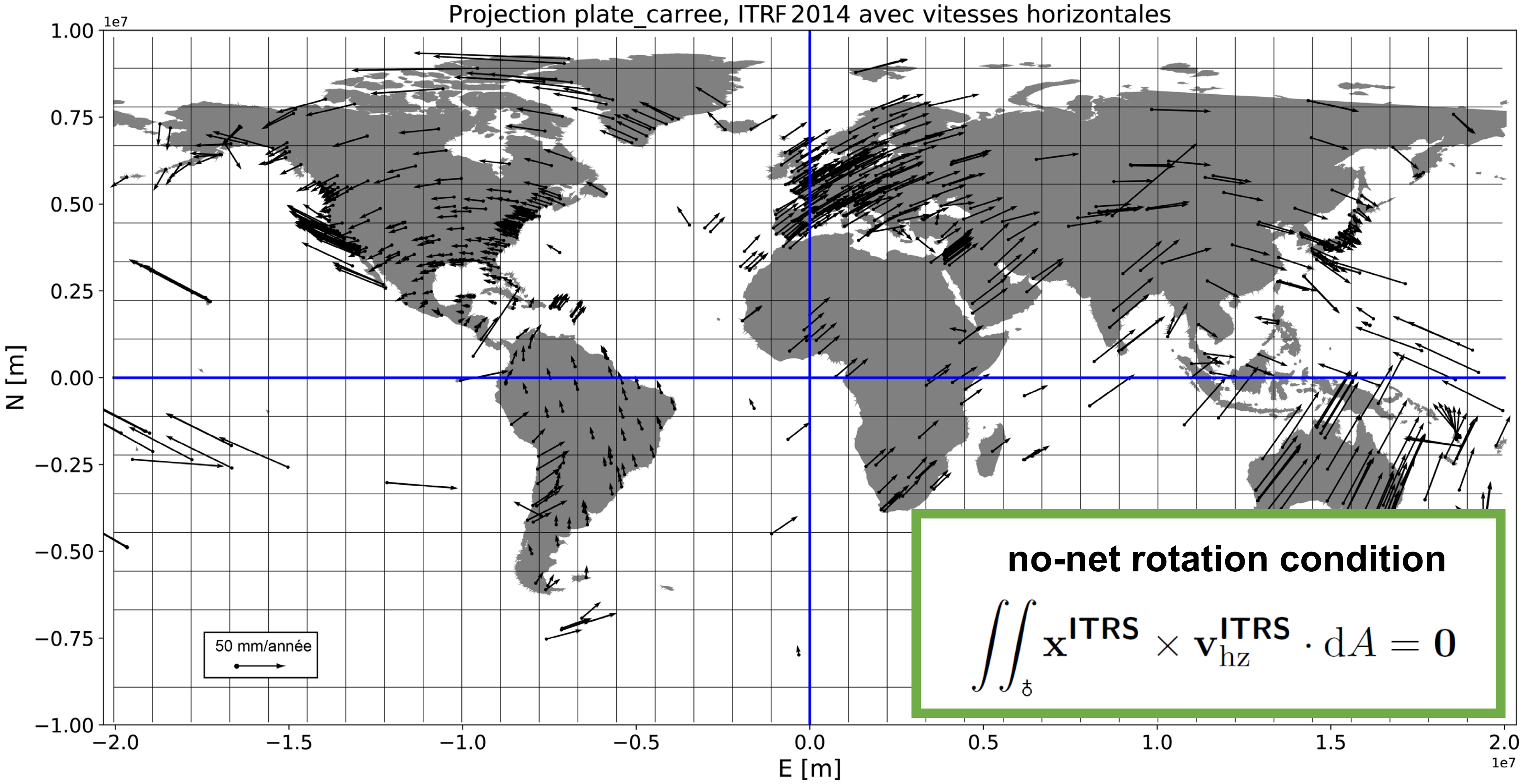
$$\mathbf{x}^{\mathsf{TRS}}(t) = \mathbf{x}^{\mathsf{TRS}}(t_0) + \dot{\mathbf{x}}^{\mathsf{TRS}}(t_0) \cdot (t - t_0)$$
$$\dot{\mathbf{x}}^{\mathsf{TRS}}(t) = \dot{\mathbf{x}}^{\mathsf{TRS}}(t_0)$$

# TRS / TRF – Terrestrial Reference System/Frame

ITRF2014 STATION POSITIONS AT EPOCH 2010.0 AND VELOCITIES
VLBI STATIONS

| DOMES NB. | SITE NAME | TECH. | ID. | X/Vx | Y/Vy | Z/Vz | Sigmas | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | ------------------------ | ----m/m/y---- | -------------------- | | | |
| 10003M003 | Toulouse | VLBI | 7608 | 4627949.8413 | 119843.9181 | 4372863.1808 | 0.0394 | 0.0353 | 0.0391 |
| 10003M003 | | | | -.01148 | 0.01934 | 0.01202 | .00198 | .00198 | .00198 |
| 14001S001 | Zimmerwald | SLR | 7810 | 4331283.3113 | 567549.9584 | 4633140.2352 | 0.0007 | 0.0008 | 0.0006 |
| 14001S001 | | | | -.01392 | 0.01806 | 0.01169 | .00003 | .00004 | .00003 |
| 14001M004 | Zimmerwald | GNSS | ZIMM | 4331296.9927 | 567555.9663 | 4633133.9907 | 0.0006 | 0.0006 | 0.0005 |
| 14001M004 | | | | -.01393 | 0.01809 | 0.01169 | .00003 | .00004 | .00003 |
| 14001M006 | Zimmerwald | GNSS | ZIMJ | 4331293.8539 | 567542.2203 | 4633135.7774 | 0.0006 | 0.0006 | 0.0006 |
| 14001M006 | | | | -.01393 | 0.01810 | 0.01168 | .00003 | .00004 | .00003 |
| 14001M008 | Zimmerwald | GNSS | ZIM2 | 4331299.8080 | 567537.4098 | 4633133.7766 | 0.0006 | 0.0006 | 0.0006 |
| 14001M008 | | | | -.01393 | 0.01810 | 0.01168 | .00003 | .00004 | .00003 |
| 14014M002 | Wabern | GNSS | WAB2 | 4327318.3250 | 566955.8288 | 4636425.8268 | 0.0006 | 0.0006 | 0.0005 |
| 14014M002 | | | | -.01414 | 0.01815 | 0.01112 | .00003 | .00004 | .00003 |
| 97301S006 | KOUROU | DORIS | KRWB | 3855260.4203 | -5049735.5530 | 563056.6395 | 0.0038 | 0.0035 | 0.0023 |
| 97301S006 | | | | -.00589 | -.00109 | 0.01443 | .00030 | .00027 | .00020 |

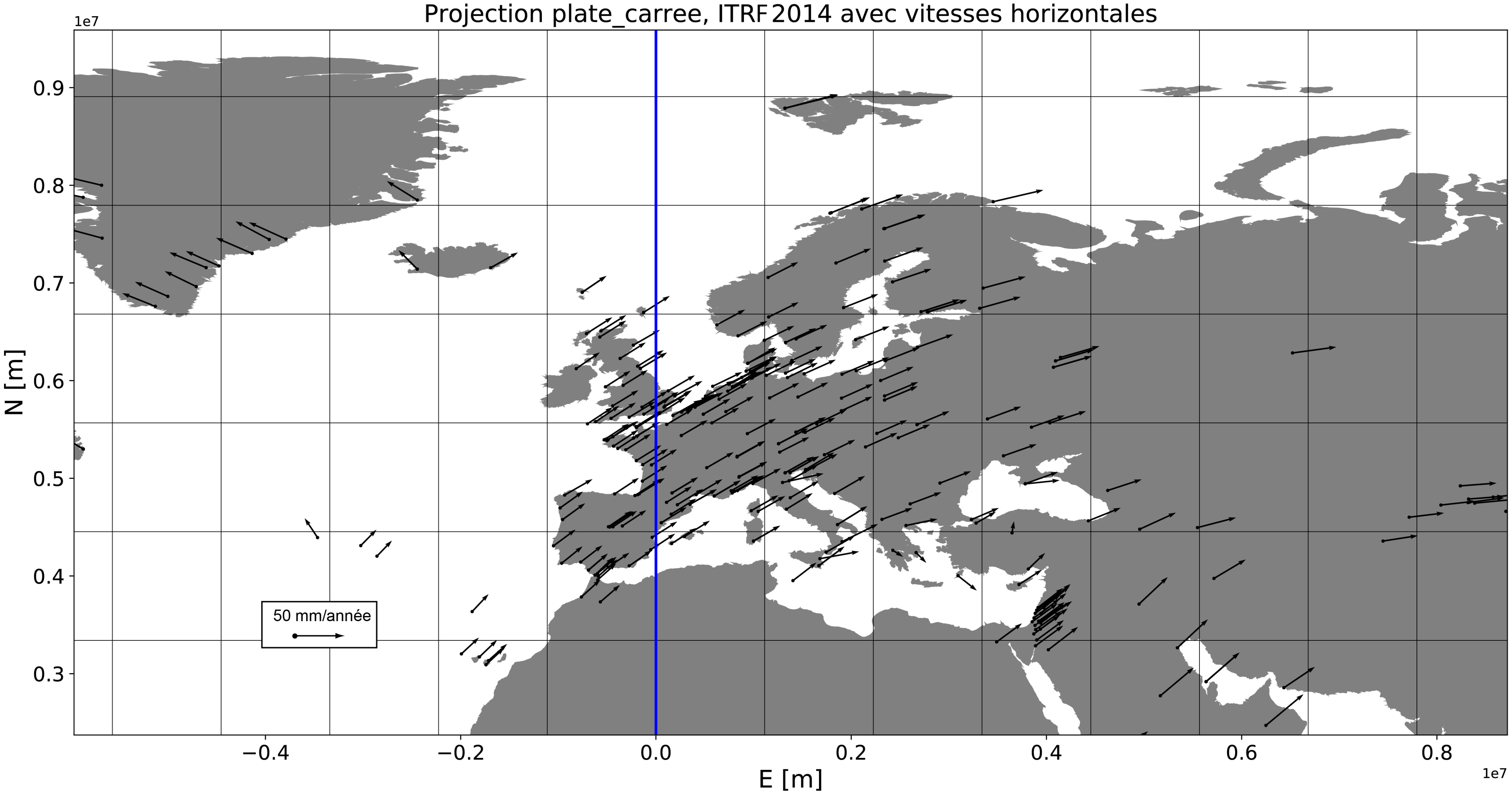
# TRS / TRF – Terrestrial Reference System/Frame




**no-net rotation condition**

$$\iint_{\text{♁}} \mathbf{x}^{\text{ITRS}} \times \mathbf{v}_{\text{hz}}^{\text{ITRS}} \cdot \mathrm{d}A = \mathbf{0}$$

# TRS / TRF – Terrestrial Reference System/Frame



# TRS / TRF – Terrestrial Reference System/Frame

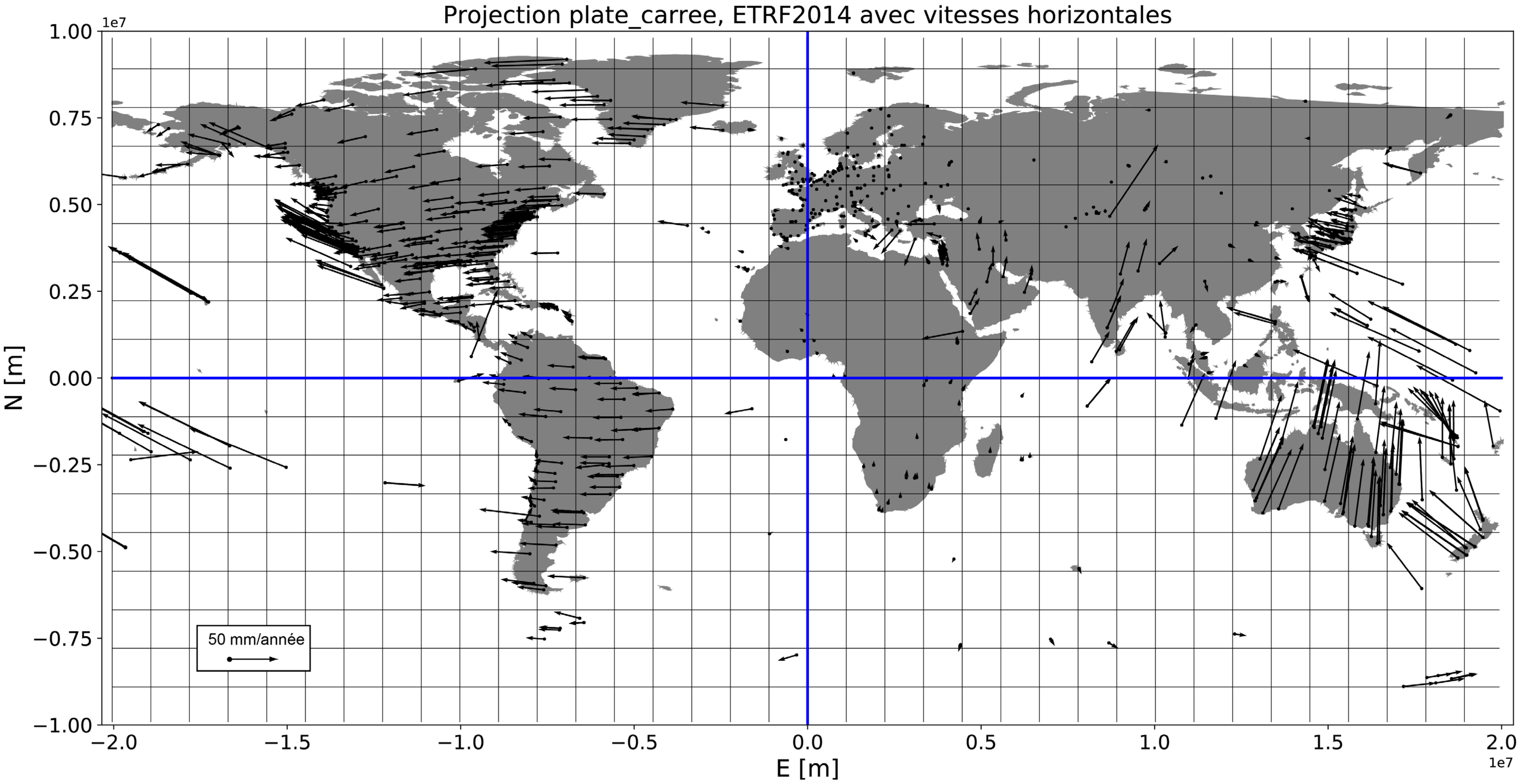
ETRF2005 STATION POSITIONS AT EPOCH 2000.0 AND VELOCITIES

| DOMES NB. | SITE NAME | TECH. | ID. | X/Vx | Y/Vy | Z/Vz | Sigmas | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | m/m/y | | | | | |
| 14001M004 | Zimmerwald | GNSS | ZIMM | 4331297.3481 | 567555.6414 | 4633133.7268 | 0.0010 | 0.0004 | 0.0010 |
| 14001M004 | | | | 0.00118 | 0.00052 | 0.00195 | .00014 | .00007 | .00015 |

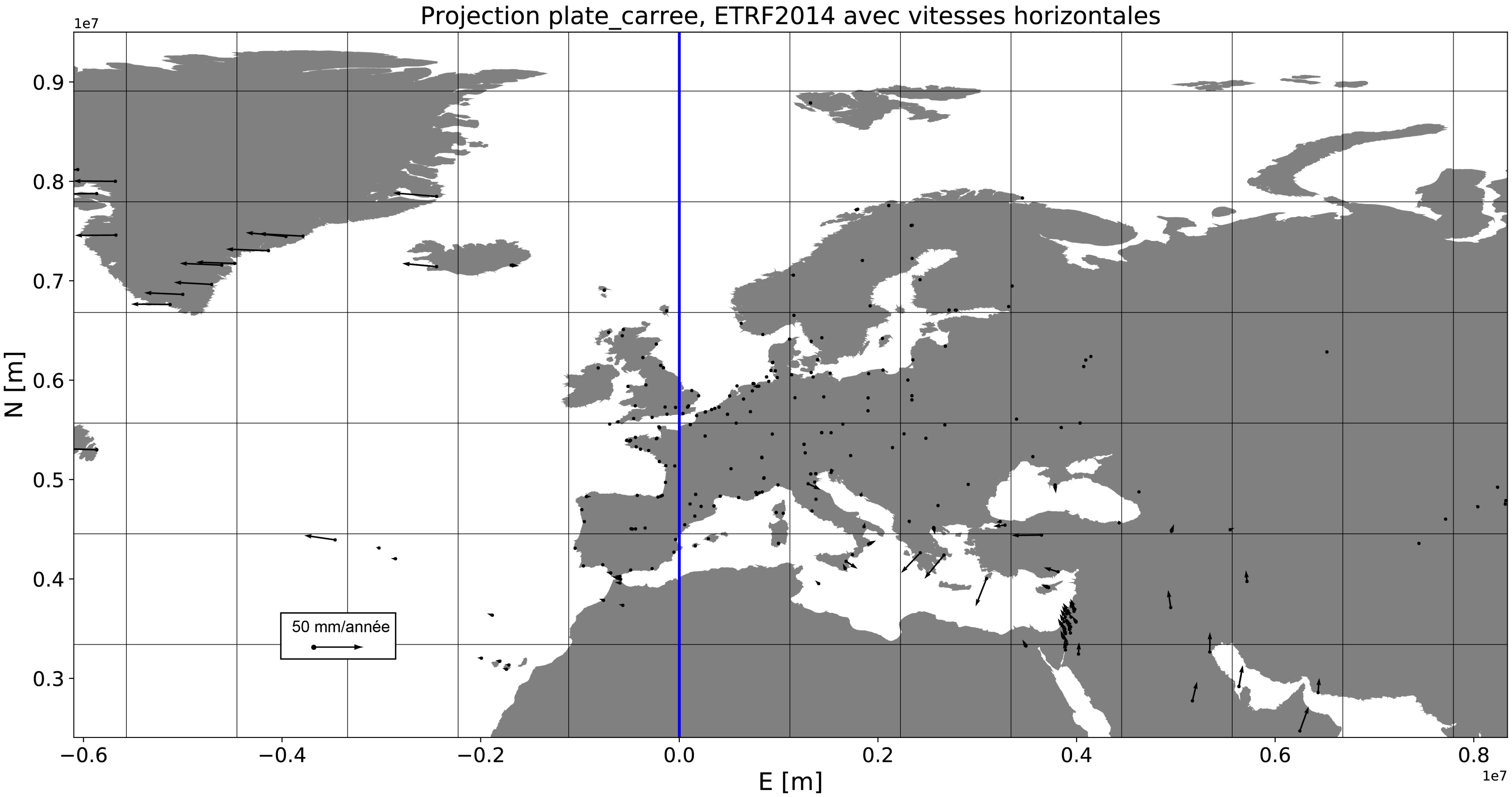
ETRF2014 STATION POSITIONS AT EPOCH 2010.0 AND VELOCITIES

| DOMES NB. | SITE NAME | TECH. | ID. | X/Vx | Y/Vy | Z/Vz | Sigmas | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | m/m/y | | | | | |
| 14001M004 | Zimmerwald | GNSS | ZIMM | 4331297.2877 | 567555.5866 | 4633133.7614 | 0.0006 | 0.0006 | 0.0005 |
| 14001M004 | | | | 0.00012 | 0.00001 | 0.00077 | .00003 | .00004 | .00003 |

# TRS / TRF – Terrestrial Reference System/Frame



# TRS / TRF – Terrestrial Reference System/Frame



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# TRS / TRF – Terrestrial Reference System/Frame

Transformation between 2 **T**errestrial **R**eference **S**ystems

=> Time dependent similarity transformation :

$$\mathbf{x}^{\mathsf{TRS2}}(t) = \mathbf{t}_{\mathsf{TRS1}}^{\mathsf{TRS2}}(t) + m_{\mathsf{TRS1}}^{\mathsf{TRS2}}(t) \cdot \mathbf{R}_{\mathsf{TRS1}}^{\mathsf{TRS2}}(t) \cdot \mathbf{x}^{\mathsf{TRS1}}(t)$$

Small scale factor & small rotation angles :

$$m_{\mathsf{TRS1}}^{\mathsf{TRS2}}(t) \approx 1 + \delta m_{\mathsf{TRS1}}^{\mathsf{TRS2}}(t)$$
$$\mathbf{R}_{\mathsf{TRS1}}^{\mathsf{TRS2}}(t) \approx \mathbf{I} + \delta \mathbf{R}_{\mathsf{TRS1}}^{\mathsf{TRS2}}(t)$$

Position and velocity vectors from **TRS1 to TRS2**

$$\mathbf{x}^{\mathsf{TRS2}}(t) = \mathbf{x}^{\mathsf{TRS1}}(t) + \mathbf{t}_{\mathsf{TRS1}}^{\mathsf{TRS2}}(t) + \delta m_{\mathsf{TRS1}}^{\mathsf{TRS2}}(t) \cdot \mathbf{x}^{\mathsf{TRS1}}(t) + \delta \mathbf{R}_{\mathsf{TRS1}}^{\mathsf{TRS2}}(t) \cdot \mathbf{x}^{\mathsf{TRS1}}(t)$$

$$\dot{\mathbf{x}}^{\mathsf{TRS2}}(t) = \dot{\mathbf{x}}^{\mathsf{TRS1}}(t) + \dot{\mathbf{t}}_{\mathsf{TRS1}}^{\mathsf{TRS2}}(t) + \delta \dot{m}_{\mathsf{TRS1}}^{\mathsf{TRS2}}(t) \cdot \mathbf{x}^{\mathsf{TRS1}}(t) + \delta \dot{\mathbf{R}}_{\mathsf{TRS1}}^{\mathsf{TRS2}}(t) \cdot \mathbf{x}^{\mathsf{TRS1}}(t)$$

# TRS / TRF – Terrestrial Reference System/Frame

Position and velocity vectors from **TRS1 to TRS2**

$$\mathbf{x}^{\mathsf{TRS2}}(t) = \mathbf{x}^{\mathsf{TRS1}}(t) + \mathbf{t}^{\mathsf{TRS2}}_{\mathsf{TRS1}}(t) + \delta m^{\mathsf{TRS2}}_{\mathsf{TRS1}}(t) \cdot \mathbf{x}^{\mathsf{TRS1}}(t) + \delta \mathbf{R}^{\mathsf{TRS2}}_{\mathsf{TRS1}}(t) \cdot \mathbf{x}^{\mathsf{TRS1}}(t)$$

$$\dot{\mathbf{x}}^{\mathsf{TRS2}}(t) = \dot{\mathbf{x}}^{\mathsf{TRS1}}(t) + \dot{\mathbf{t}}^{\mathsf{TRS2}}_{\mathsf{TRS1}}(t) + \delta \dot{m}^{\mathsf{TRS2}}_{\mathsf{TRS1}}(t) \cdot \mathbf{x}^{\mathsf{TRS1}}(t) + \delta \dot{\mathbf{R}}^{\mathsf{TRS2}}_{\mathsf{TRS1}}(t) \cdot \mathbf{x}^{\mathsf{TRS1}}(t)$$

Time dependent translation vector :

$$\mathbf{t}^{\mathsf{TRS2}}_{\mathsf{TRS1}}(t) = \begin{pmatrix} t_x(t) \\ t_y(t) \\ t_z(t) \end{pmatrix}^{\mathsf{TRS2}}_{\mathsf{TRS1}} = \begin{pmatrix} t_x(t_0) \\ t_y(t_0) \\ t_z(t_0) \end{pmatrix}^{\mathsf{TRS2}}_{\mathsf{TRS1}} + \begin{pmatrix} \dot{t}_x(t_0) \\ \dot{t}_y(t_0) \\ \dot{t}_z(t_0) \end{pmatrix}^{\mathsf{TRS2}}_{\mathsf{TRS1}} \cdot (t - t_0)$$

Time dependent scale factor :

$$\delta m^{\mathsf{TRS2}}_{\mathsf{TRS1}}(t) = \delta m^{\mathsf{TRS2}}_{\mathsf{TRS1}}(t_0) + \delta \dot{m}^{\mathsf{TRS2}}_{\mathsf{TRS1}}(t_0) \cdot (t - t_0)$$

# TRS / TRF – Terrestrial Reference System/Frame

Position and velocity vectors from **TRS1 to TRS2**

$$\mathbf{x}^{\mathsf{TRS2}}(t) = \mathbf{x}^{\mathsf{TRS1}}(t) + \mathbf{t}_{\mathsf{TRS1}}^{\mathsf{TRS2}}(t) + \delta m_{\mathsf{TRS1}}^{\mathsf{TRS2}}(t) \cdot \mathbf{x}^{\mathsf{TRS1}}(t) + \delta \mathbf{R}_{\mathsf{TRS1}}^{\mathsf{TRS2}}(t) \cdot \mathbf{x}^{\mathsf{TRS1}}(t)$$

$$\dot{\mathbf{x}}^{\mathsf{TRS2}}(t) = \dot{\mathbf{x}}^{\mathsf{TRS1}}(t) + \dot{\mathbf{t}}_{\mathsf{TRS1}}^{\mathsf{TRS2}}(t) + \delta \dot{m}_{\mathsf{TRS1}}^{\mathsf{TRS2}}(t) \cdot \mathbf{x}^{\mathsf{TRS1}}(t) + \delta \dot{\mathbf{R}}_{\mathsf{TRS1}}^{\mathsf{TRS2}}(t) \cdot \mathbf{x}^{\mathsf{TRS1}}(t)$$

Time dependent rotation matrix :

$$\delta \mathbf{R}_{\mathsf{TRS1}}^{\mathsf{TRS2}}(t) = \begin{pmatrix} 0 & -\delta\gamma(t) & +\delta\beta(t) \\ +\delta\gamma(t) & 0 & -\delta\alpha(t) \\ -\delta\beta(t) & +\delta\alpha(t) & 0 \end{pmatrix}_{\substack{\mathsf{TRS2} \\ \mathsf{TRS1}}}$$

$$= \begin{pmatrix} 0 & -\delta\gamma(t_0) & +\delta\beta(t_0) \\ +\delta\gamma(t_0) & 0 & -\delta\alpha(t_0) \\ -\delta\beta(t_0) & +\delta\alpha(t_0) & 0 \end{pmatrix}_{\substack{\mathsf{TRS2} \\ \mathsf{TRS1}}} + \begin{pmatrix} 0 & -\delta\dot{\gamma}(t_0) & +\delta\dot{\beta}(t_0) \\ +\delta\dot{\gamma}(t_0) & 0 & -\delta\dot{\alpha}(t_0) \\ -\delta\dot{\beta}(t_0) & +\delta\dot{\alpha}(t_0) & 0 \end{pmatrix}_{\substack{\mathsf{TRS2} \\ \mathsf{TRS1}}} \cdot (t - t_0)$$

# TRS / TRF – Terrestrial Reference System/Frame

| TRS1/TRF1 | TRS2/TRF2 | $t_0$ [année] | $t_x$ [mm] / $\dot{t}_x$ $\left[\frac{mm}{année}\right]$ | $t_y$ [mm] / $\dot{t}_y$ $\left[\frac{mm}{année}\right]$ | $t_z$ [mm] / $\dot{t}_z$ $\left[\frac{mm}{année}\right]$ | $\delta\alpha$ [mas] / $\delta\dot{\alpha}$ $\left[\frac{mas}{année}\right]$ | $\delta\beta$ [mas] / $\delta\dot{\beta}$ $\left[\frac{mas}{année}\right]$ | $\delta\gamma$ [mas] / $\delta\dot{\gamma}$ $\left[\frac{mas}{année}\right]$ | $\delta m$ [ppb] / $\delta\dot{m}$ $\left[\frac{ppb}{année}\right]$ |
|---|---|---|---|---|---|---|---|---|---|
| **ITRF2014** | **ITRF2008** | 2010.0 | 1.6 | 1.9 | 2.4 | 0.000 | 0.000 | 0.000 | -0.02 |
| | | | 0.0 | 0.0 | -0.1 | 0.000 | 0.000 | 0.000 | 0.030 |
| **ITRF2014** | **ITRF1993** | 2010.0 | -50.4 | 3.3 | -60.2 | -2.810 | -3.380 | 0.400 | 4.290 |
| | | | -2.8 | -0.1 | -2.5 | -0.110 | -0.190 | 0.070 | 0.12 |
| **ITRF2014** | **ETRF2000** | 1989.0 | 54.0 | 51.0 | -48.0 | 0.000 | 0.000 | 0.000 | 0.000 |
| | | | 0.0 | 0.0 | 0.0 | 0.081 | 0.490 | -0.792 | 0.000 |
| **ITRF1993** | **ETRF1993** | 1989.0 | 19.0 | 53.0 | -21.0 | 0.000 | 0.000 | 0.000 | 0.000 |
| | | | 0.0 | 0.0 | 0.0 | 0.320 | 0.780 | -0.670 | 0.000 |
| **ETRF2000** | **CHTRF2016** | 1993.0 | 0.0 | 0.0 | 0.0 | 0.000 | 0.000 | 0.000 | 0.000 |
| | | | 0.0 | 0.0 | 0.0 | 0.000 | 0.000 | 0.000 | 0.000 |
| **ETRF1993** | **CHTRS95** | 1993.0 | 0.0 | 0.0 | 0.0 | 0.000 | 0.000 | 0.000 | 0.000 |
| | | | 0.0 | 0.0 | 0.0 | 0.000 | 0.000 | 0.000 | 0.000 |
| **CHTRF2016** | **CHTRS95** | 1993.0 | 0.0 | 0.0 | 0.0 | 0.000 | 0.000 | 0.000 | 0.000 |
| | | | 0.0 | 0.0 | 0.0 | 0.000 | 0.000 | 0.000 | 0.000 |
| **CHTRS95** | **CH1903+** | 1993.0 | - 674’374.0 | - 15’056.0 | - 405’346.0 | 0.000 | 0.000 | 0.000 | 0.000 |
| | | | 0.0 | 0.0 | 0.0 | 0.000 | 0.000 | 0.000 | 0.000 |

# TRS / TRF – Terrestrial Reference System/Frame

Home / Products & Services / Services / ETRF/ITRF Transformation

## ETRF/ITRF Transformation

Google Translate

17

The form below allows to transform on-line coordinates (position and velocity) between realisations (ETRFxx) of the European Terrestrial Reference System (ETRS89) and realizations (ITRFyy) of the International Terrestrial Reference System (ITRS). In case input and output coordinates are requested at different epochs, then station velocities are mandatory.
For transformations to and from the Galileo Terrestrial Reference Frame (GTRF), use ITRF. GTRF is aligned to current versions of the ITRF.

Explanation and examples are available from the following tutorial.

### Input

Frame : ETRF89

Epoch : 2000 . 00

```
# Lines starting by # are treated as comments
# Fields (in decimal format) should be separated by at least one space
#
# --> Example without velocity - StationName(no space character) X[m] Y[m] Z[m] :
StationName 4027894.006 307045.600 4919474.910
#
# --> Example with velocity - StationName(no space character) X[m] Y[m] Z[m] VX[m/yr] VY[m/yr] VZ[m/yr] :
StationName 4027894.006 307045.600 4919474.910 0.01 0.2 0.03
```

### Output

Frame : ETRF89

Epoch : 2000 . 00

# TRS / TRF – Terrestrial Reference System/Frame

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## ETRF/ITRF Transformation

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The form below allows to transform on-line coordinates (position and velocity) between realisations (ETRFxx) of the European Terrestrial Reference System (ETRS89) and realizations (ITRFyy) of the International Terrestrial Reference System (ITRS). In case input and output coordinates are requested at different epochs, then station velocities are mandatory.
For transformations to and from the Galileo Terrestrial Reference Frame (GTRF), use ITRF. GTRF is aligned to current versions of the ITRF.

Explanation and examples are available from the following tutorial.

### Input

Frame : ETRF89

Epoch : 2000 . 00

```
# Lines starting by # are treated as comments
# Fields (in decimal format) should be separated by at least one space
#
# --> Example without velocity - StationName(no space character) X[m] Y[m] Z[m] :
StationName 4027894.006 307045.600 4919474.910
#
# --> Example with velocity - StationName(no space character) X[m] Y[m] Z[m] VX[m/yr] VY[m/
StationName 4027894.006 307045.600 4919474.910 0.01 0.2 0.03
```

### Output

Frame : ETRF89

Epoch : 2000 . 00

### Intermediate steps

| MarkerName | Frame | Epoch | X | Y | Z | $V_X$ | $V_Y$ | $V_Z$ |
|---|---|---|---|---|---|---|---|---|
| StationName | ETRF89 | 2000.0 | 4027894.0060 | 307045.6000 | 4919474.9100 | 0.010000 | 0.200000 | 0.030000 |
| StationName | ITRF89 | 2000.0 | 4027893.8448 | 307045.7814 | 4919475.0306 | -0.004652 | 0.216488 | 0.040967 |
| StationName | ITRF2000 | 2000.0 | 4027893.7912 | 307045.7381 | 4919475.0920 | -0.004662 | 0.216695 | 0.042318 |
| StationName | ITRF94 | 2000.0 | 4027893.8042 | 307045.7440 | 4919475.0770 | -0.004652 | 0.216488 | 0.040967 |
| StationName | ITRF94 | 2023.0 | 4027893.6972 | 307050.7233 | 4919476.0193 | -0.004652 | 0.216488 | 0.040967 |

# Reference Systems and Frames

## Topocentric Reference Systems/Frames

# Topocentric Reference Systems/Frames

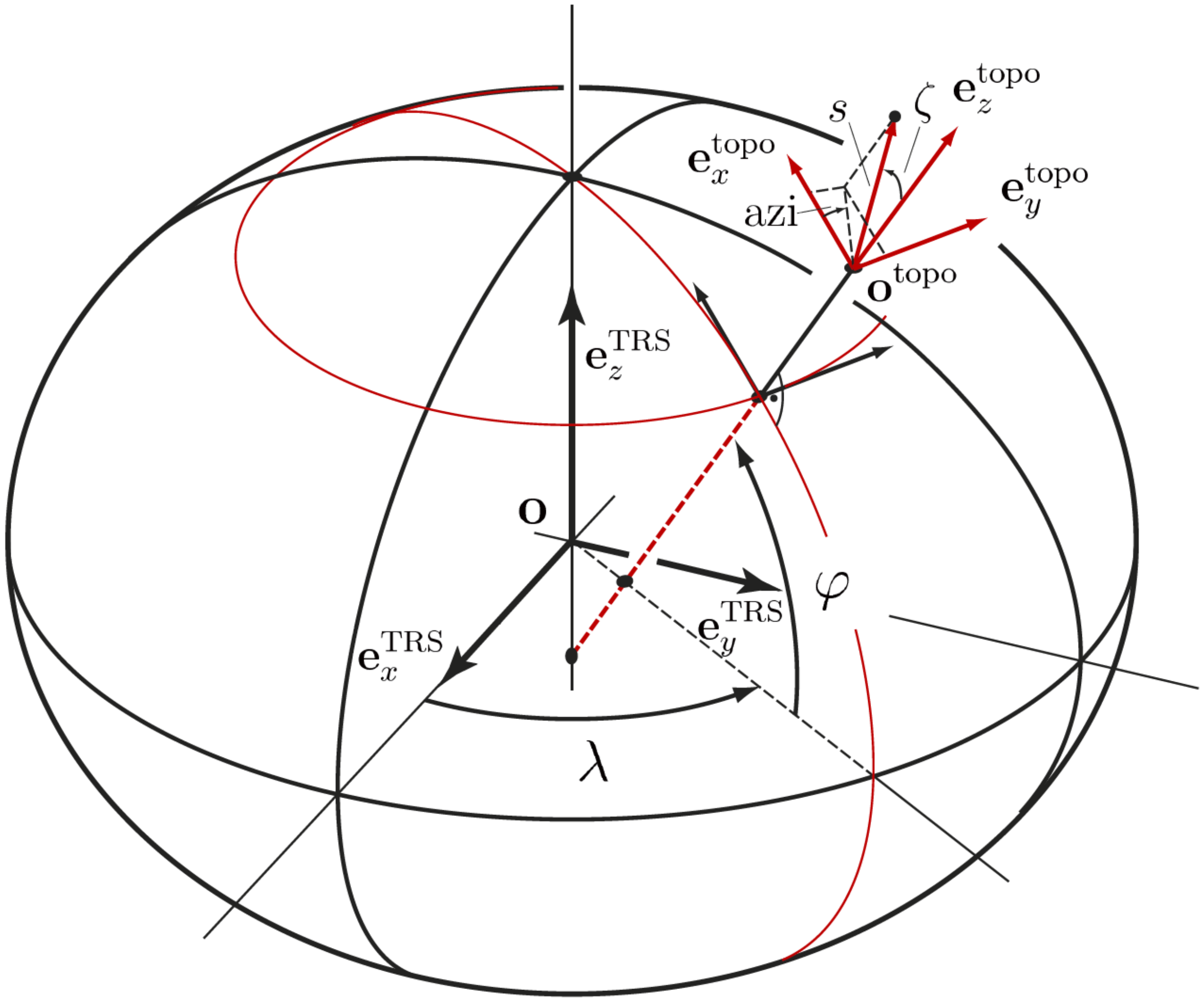
## Topocentric Ellipsoidal System

$$\mathbf{o}^{\text{topo}} = \begin{pmatrix} x_{\mathbf{o}}^{\text{TRS}} \\ y_{\mathbf{o}}^{\text{TRS}} \\ z_{\mathbf{o}}^{\text{TRS}} \end{pmatrix}$$

$$\mathbf{e}_z^{\text{topo}} = \mathbf{n} = \begin{pmatrix} \cos\varphi \cdot \cos\lambda \\ \cos\varphi \cdot \sin\lambda \\ \sin\varphi \end{pmatrix}$$

$$\mathbf{e}_x^{\text{topo}} = \frac{\mathbf{t}_\varphi}{|\mathbf{t}_\varphi|} = \begin{pmatrix} \frac{\partial x(\lambda,\varphi)}{\partial\varphi} \\ \frac{\partial y(\lambda,\varphi)}{\partial\varphi} \\ \frac{\partial z(\lambda,\varphi)}{\partial\varphi} \end{pmatrix} = \begin{pmatrix} -\sin\varphi \cdot \cos\lambda \\ -\sin\varphi \cdot \sin\lambda \\ \cos\varphi \end{pmatrix}$$

$$\mathbf{e}_y^{\text{topo}} = \frac{\mathbf{t}_\lambda}{|\mathbf{t}_\lambda|} = \begin{pmatrix} \frac{\partial x(\lambda,\varphi)}{\partial\lambda} \\ \frac{\partial y(\lambda,\varphi)}{\partial\lambda} \\ \frac{\partial z(\lambda,\varphi)}{\partial\lambda} \end{pmatrix} = \begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix}$$



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# Topocentric Reference Systems/Frames

## Topocentric Ellipsoidal System

Ellipsoidal Topocentric Observables :

$$
\begin{aligned}
s^{\mathrm{ell}} &= \sqrt{(x^{\mathsf{topo}})^2 + (y^{\mathsf{topo}})^2 + (z^{\mathsf{topo}})^2} \\
\mathrm{azi}^{\mathrm{ell}} &= \mathtt{atan2}(y^{\mathsf{topo}}, x^{\mathsf{topo}}) \\
\zeta^{\mathrm{ell}} &= \mathtt{atan2}\left(\sqrt{(x^{\mathsf{topo}})^2 + (y^{\mathsf{topo}})^2}, z^{\mathsf{topo}}\right)
\end{aligned}
$$

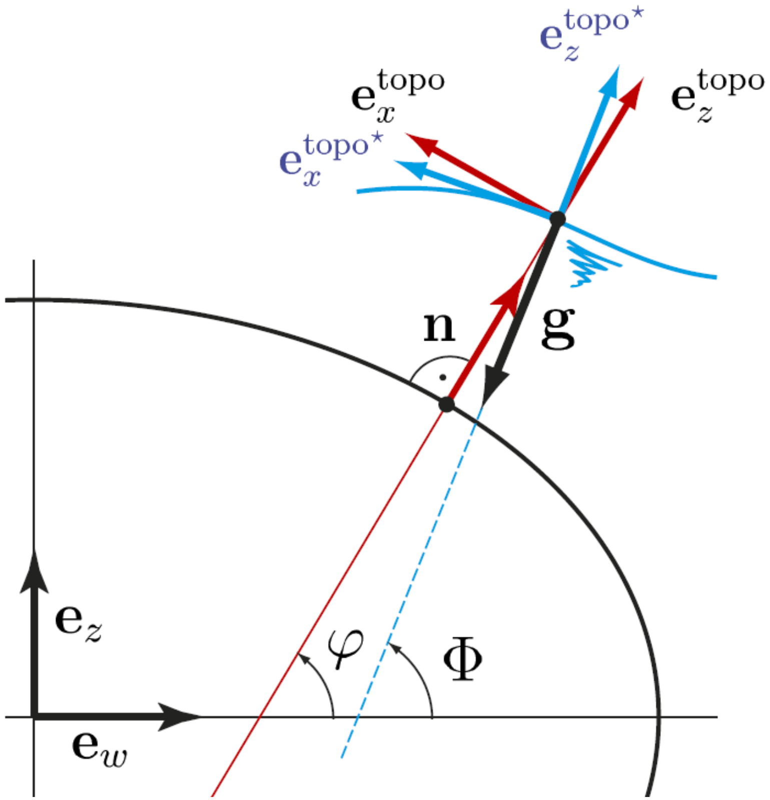
Link to **TRS** :

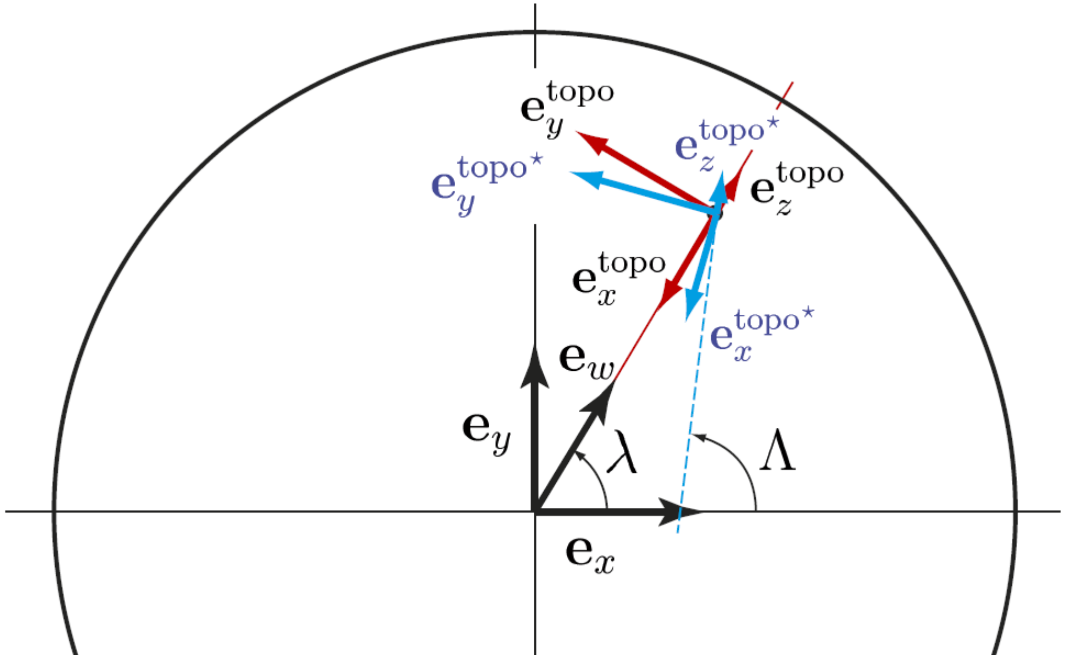
$$
\begin{pmatrix} x^{\mathsf{TRS}} \\ y^{\mathsf{TRS}} \\ z^{\mathsf{TRS}} \end{pmatrix} = \begin{pmatrix} x_{\mathsf{o}}^{\mathsf{TRS}} \\ y_{\mathsf{o}}^{\mathsf{TRS}} \\ z_{\mathsf{o}}^{\mathsf{TRS}} \end{pmatrix} + \begin{pmatrix} -\sin\varphi \cdot \cos\lambda & -\sin\lambda & \cos\varphi \cdot \cos\lambda \\ -\sin\varphi \cdot \sin\lambda & \cos\lambda & \cos\varphi \cdot \sin\lambda \\ \cos\varphi & 0 & \sin\varphi \end{pmatrix} \cdot \begin{pmatrix} x^{\mathsf{topo}} \\ y^{\mathsf{topo}} \\ z^{\mathsf{topo}} \end{pmatrix}
$$

# Topocentric Reference Systems/Frames

## Topocentric Astronomical System

Astronomical Topocentric Observables :



# Topocentric Reference Systems/Frames

## Topocentric Astronomical System

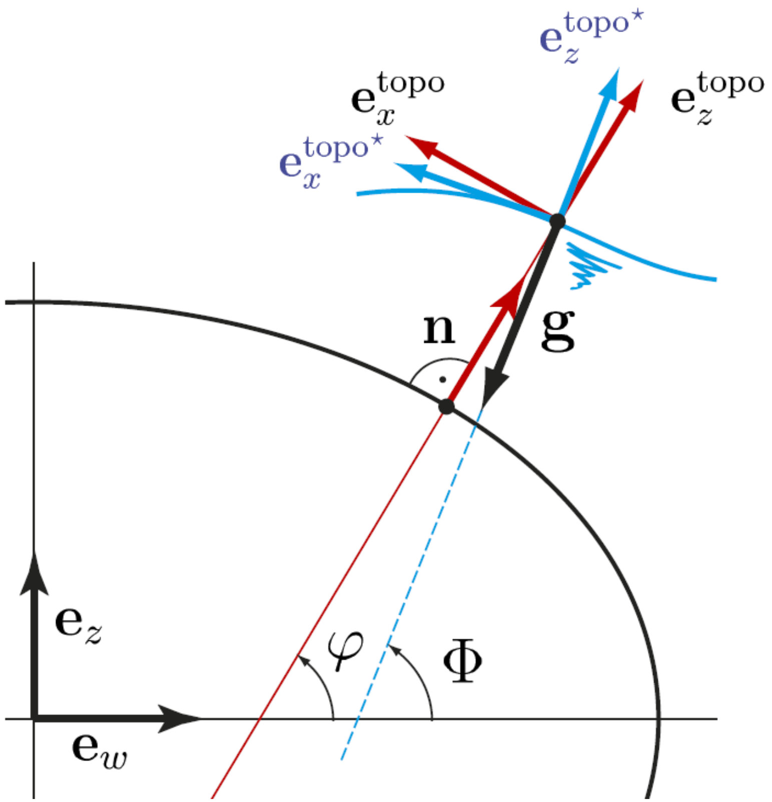
Astronomical Topocentric Observables :

$$\mathbf{o}^{\mathsf{topo}\star} = \begin{pmatrix} x_{\mathbf{o}}^{\mathsf{TRS}} \\ y_{\mathbf{o}}^{\mathsf{TRS}} \\ z_{\mathbf{o}}^{\mathsf{TRS}} \end{pmatrix}$$

$$\mathbf{e}_z^{\mathsf{topo}\star} = -\frac{\mathbf{g}}{|\mathbf{g}|} = \begin{pmatrix} \cos\Phi \cdot \cos\Lambda \\ \cos\Phi \cdot \sin\Lambda \\ \sin\Phi \end{pmatrix}$$

$$\mathbf{e}_x^{\mathsf{topo}\star} = \frac{\mathbf{t}_\Phi}{|\mathbf{t}_\Phi|} = \begin{pmatrix} \frac{\partial x(\Lambda,\Phi)}{\partial\Phi} \\ \frac{\partial y(\lambda,\Phi)}{\partial\Phi} \\ \frac{\partial z(\Lambda,\Phi)}{\partial\Phi} \end{pmatrix} = \begin{pmatrix} -\sin\Phi \cdot \cos\Lambda \\ -\sin\Phi \cdot \sin\Lambda \\ \cos\Phi \end{pmatrix}$$

$$\mathbf{e}_y^{\mathsf{topo}\star} = \mathbf{e}_x^{\mathsf{topo}\star} \times \mathbf{e}_z^{\mathsf{topo}\star} = \begin{pmatrix} -\sin\Lambda \\ \cos\Lambda \\ 0 \end{pmatrix}$$



# Topocentric Reference Systems/Frames

## Topocentric Astronomical System

Astronomical Topocentric Observables :

$$
\begin{aligned}
s^{\text{astro}} &= \sqrt{(x^{\textbf{topo}^\star})^2 + (y^{\textbf{topo}^\star})^2 + (z^{\textbf{topo}^\star})^2} \\
\text{azi}^{\text{astro}} &= \mathtt{atan2}(y^{\textbf{topo}^\star}, x^{\textbf{topo}^\star}) \\
\zeta^{\text{astro}} &= \mathtt{atan2}\left(\sqrt{(x^{\textbf{topo}^\star})^2 + (y^{\textbf{topo}^\star})^2}, z^{\textbf{topo}^\star}\right)
\end{aligned}
$$

Link to **TRS** :

$$
\begin{pmatrix} x^{\textbf{TRS}} \\ y^{\textbf{TRS}} \\ z^{\textbf{TRS}} \end{pmatrix} = \begin{pmatrix} x_{\textbf{o}}^{\textbf{TRS}} \\ y_{\textbf{o}}^{\textbf{TRS}} \\ z_{\textbf{o}}^{\textbf{TRS}} \end{pmatrix} + \begin{pmatrix} -\sin\Phi\cdot\cos\Lambda & -\sin\Lambda & \cos\Phi\cdot\cos\Lambda \\ -\sin\Phi\cdot\sin\Lambda & \cos\Lambda & \cos\Phi\cdot\sin\Lambda \\ \cos\Phi & 0 & \sin\Phi \end{pmatrix} \cdot \begin{pmatrix} x^{\textbf{topo}^\star} \\ y^{\textbf{topo}^\star} \\ z^{\textbf{topo}^\star} \end{pmatrix}
$$

# Topocentric Reference Systems/Frames

## Transformation between Astronomical and Ellipsoidal Topocentric Systems

$$\begin{pmatrix} x^{\mathrm{TRS}} \\ y^{\mathrm{TRS}} \\ z^{\mathrm{TRS}} \end{pmatrix} = \begin{pmatrix} x_{\mathrm{o}}^{\mathrm{TRS}} \\ y_{\mathrm{o}}^{\mathrm{TRS}} \\ z_{\mathrm{o}}^{\mathrm{TRS}} \end{pmatrix} + \begin{pmatrix} -\sin\Phi\cdot\cos\Lambda & -\sin\Lambda & \cos\Phi\cdot\cos\Lambda \\ -\sin\Phi\cdot\sin\Lambda & \cos\Lambda & \cos\Phi\cdot\sin\Lambda \\ \cos\Phi & 0 & \sin\Phi \end{pmatrix} \cdot \begin{pmatrix} x^{\mathrm{topo}^\star} \\ y^{\mathrm{topo}^\star} \\ z^{\mathrm{topo}^\star} \end{pmatrix}$$

Astronomical Topocentric System

$$\begin{pmatrix} x^{\mathrm{TRS}} \\ y^{\mathrm{TRS}} \\ z^{\mathrm{TRS}} \end{pmatrix} = \begin{pmatrix} x_{\mathrm{o}}^{\mathrm{TRS}} \\ y_{\mathrm{o}}^{\mathrm{TRS}} \\ z_{\mathrm{o}}^{\mathrm{TRS}} \end{pmatrix} + \begin{pmatrix} -\sin\varphi\cdot\cos\lambda & -\sin\lambda & \cos\varphi\cdot\cos\lambda \\ -\sin\varphi\cdot\sin\lambda & \cos\lambda & \cos\varphi\cdot\sin\lambda \\ \cos\varphi & 0 & \sin\varphi \end{pmatrix} \cdot \begin{pmatrix} x^{\mathrm{topo}} \\ y^{\mathrm{topo}} \\ z^{\mathrm{topo}} \end{pmatrix}$$

Ellipsoidal Topocentric System

Connection :

$$\begin{pmatrix} -\sin\Phi\cdot\cos\Lambda & -\sin\Lambda & \cos\Phi\cdot\cos\Lambda \\ -\sin\Phi\cdot\sin\Lambda & \cos\Lambda & \cos\Phi\cdot\sin\Lambda \\ \cos\Phi & 0 & \sin\Phi \end{pmatrix} \cdot \begin{pmatrix} x^{\mathrm{topo}^\star} \\ y^{\mathrm{topo}^\star} \\ z^{\mathrm{topo}^\star} \end{pmatrix} = \begin{pmatrix} -\sin\varphi\cdot\cos\lambda & -\sin\lambda & \cos\varphi\cdot\cos\lambda \\ -\sin\varphi\cdot\sin\lambda & \cos\lambda & \cos\varphi\cdot\sin\lambda \\ \cos\varphi & 0 & \sin\varphi \end{pmatrix} \cdot \begin{pmatrix} x^{\mathrm{topo}} \\ y^{\mathrm{topo}} \\ z^{\mathrm{topo}} \end{pmatrix}$$

# Topocentric Reference Systems/Frames

## Transformation between Astronomical and Ellipsoidal Topocentric Systems

Connection :

$$\begin{pmatrix} -\sin\Phi\cdot\cos\Lambda & -\sin\Lambda & \cos\Phi\cdot\cos\Lambda \\ -\sin\Phi\cdot\sin\Lambda & \cos\Lambda & \cos\Phi\cdot\sin\Lambda \\ \cos\Phi & 0 & \sin\Phi \end{pmatrix}\cdot\begin{pmatrix} x^{\textbf{topo}^\star} \\ y^{\textbf{topo}^\star} \\ z^{\textbf{topo}^\star} \end{pmatrix} = \begin{pmatrix} -\sin\varphi\cdot\cos\lambda & -\sin\lambda & \cos\varphi\cdot\cos\lambda \\ -\sin\varphi\cdot\sin\lambda & \cos\lambda & \cos\varphi\cdot\sin\lambda \\ \cos\varphi & 0 & \sin\varphi \end{pmatrix}\cdot\begin{pmatrix} x^{\textbf{topo}} \\ y^{\textbf{topo}} \\ z^{\textbf{topo}} \end{pmatrix}$$

Astronomical Topocentric Coordinates in function of Ellipsoidal Topocentric Coordinates :

$$\begin{pmatrix} x^{\textbf{topo}^\star} \\ y^{\textbf{topo}^\star} \\ z^{\textbf{topo}^\star} \end{pmatrix} = \begin{pmatrix} -\sin\Phi\cdot\cos\Lambda & -\sin\Phi\cdot\sin\Lambda & \cos\Phi \\ -\sin\Lambda & \cos\Lambda & 0 \\ \cos\Phi\cdot\cos\Lambda & \cos\Phi\cdot\sin\Lambda & \sin\Phi \end{pmatrix}\cdot\begin{pmatrix} -\sin\varphi\cdot\cos\lambda & -\sin\lambda & \cos\varphi\cdot\cos\lambda \\ -\sin\varphi\cdot\sin\lambda & \cos\lambda & \cos\varphi\cdot\sin\lambda \\ \cos\varphi & 0 & \sin\varphi \end{pmatrix}\cdot\begin{pmatrix} x^{\textbf{topo}} \\ y^{\textbf{topo}} \\ z^{\textbf{topo}} \end{pmatrix}$$

$$= \mathbf{T}_{\textbf{topo}}^{\textbf{topo}^\star}(\lambda,\varphi,\Lambda,\Phi)\cdot\begin{pmatrix} x^{\textbf{topo}} \\ y^{\textbf{topo}} \\ z^{\textbf{topo}} \end{pmatrix}$$

# Topocentric Reference Systems/Frames

## Transformation between Astronomical and Ellipsoidal Topocentric Systems

Astronomical Topocentric Coordinates in function of Ellipsoidal Topocentric Coordinates :

$$\begin{pmatrix} x^{\mathbf{topo}^\star} \\ y^{\mathbf{topo}^\star} \\ z^{\mathbf{topo}^\star} \end{pmatrix} = \begin{pmatrix} -\sin\Phi\cdot\cos\Lambda & -\sin\Phi\cdot\sin\Lambda & \cos\Phi \\ -\sin\Lambda & \cos\Lambda & 0 \\ \cos\Phi\cdot\cos\Lambda & \cos\Phi\cdot\sin\Lambda & \sin\Phi \end{pmatrix} \cdot \begin{pmatrix} -\sin\varphi\cdot\cos\lambda & -\sin\lambda & \cos\varphi\cdot\cos\lambda \\ -\sin\varphi\cdot\sin\lambda & \cos\lambda & \cos\varphi\cdot\sin\lambda \\ \cos\varphi & 0 & \sin\varphi \end{pmatrix} \cdot \begin{pmatrix} x^{\mathbf{topo}} \\ y^{\mathbf{topo}} \\ z^{\mathbf{topo}} \end{pmatrix}$$

$$= \mathbf{T}_{\mathbf{topo}}^{\mathbf{topo}^\star}(\lambda,\varphi,\Lambda,\Phi)\cdot\begin{pmatrix} x^{\mathbf{topo}} \\ y^{\mathbf{topo}} \\ z^{\mathbf{topo}} \end{pmatrix}$$

After 1st order Taylor series around ( $\lambda, \varphi$ )

$$\begin{pmatrix} x^{\mathbf{topo}^\star} \\ y^{\mathbf{topo}^\star} \\ z^{\mathbf{topo}^\star} \end{pmatrix} \approx \begin{pmatrix} 1 & -(\Lambda-\lambda)\cdot\sin\varphi & -(\Phi-\varphi) \\ (\Lambda-\lambda)\cdot\sin\varphi & 1 & -(\Lambda-\lambda)\cdot\cos\varphi \\ (\Phi-\varphi) & (\Lambda-\lambda)\cdot\cos\varphi & 1 \end{pmatrix} \cdot \begin{pmatrix} x^{\mathbf{topo}} \\ y^{\mathbf{topo}} \\ z^{\mathbf{topo}} \end{pmatrix}$$

# Topocentric Reference Systems/Frames

## Transformation between Astronomical and Ellipsoidal Topocentric Systems

$$\begin{pmatrix} x^{\text{topo}\star} \\ y^{\text{topo}\star} \\ z^{\text{topo}\star} \end{pmatrix} \approx \begin{pmatrix} 1 & -(\Lambda - \lambda) \cdot \sin\varphi & -(\Phi - \varphi) \\ (\Lambda - \lambda) \cdot \sin\varphi & 1 & -(\Lambda - \lambda) \cdot \cos\varphi \\ (\Phi - \varphi) & (\Lambda - \lambda) \cdot \cos\varphi & 1 \end{pmatrix} \cdot \begin{pmatrix} x^{\text{topo}} \\ y^{\text{topo}} \\ z^{\text{topo}} \end{pmatrix}$$

$$\begin{pmatrix} x^{\text{topo}\star} \\ y^{\text{topo}\star} \\ z^{\text{topo}\star} \end{pmatrix} \approx \begin{pmatrix} 1 & -\psi & -\xi \\ \psi & 1 & -\eta \\ \xi & \eta & 1 \end{pmatrix} \cdot \begin{pmatrix} x^{\text{topo}} \\ y^{\text{topo}} \\ z^{\text{topo}} \end{pmatrix}$$

with :

$$\eta = (\Lambda - \lambda) \cdot \cos\varphi$$

$$\xi = \Phi - \varphi$$

$$\psi = (\Lambda - \lambda) \cdot \sin\varphi$$

# Topocentric Reference Systems/Frames

## Transformation between Astronomical and Ellipsoidal Topocentric Systems

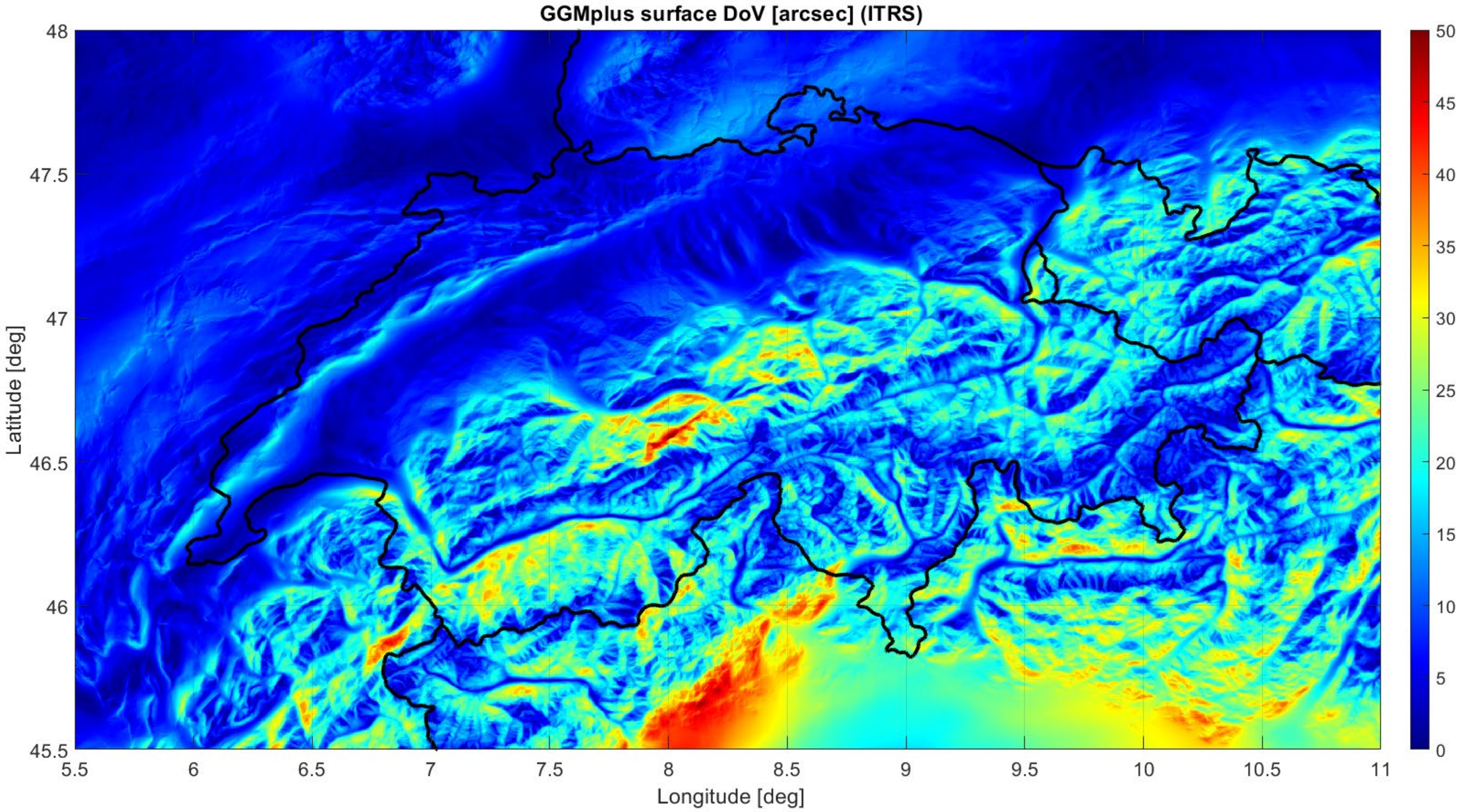
$$\begin{pmatrix} x^{\text{topo}^\star} \\ y^{\text{topo}^\star} \\ z^{\text{topo}^\star} \end{pmatrix} \approx \begin{pmatrix} 1 & -\psi & -\xi \\ \psi & 1 & -\eta \\ \xi & \eta & 1 \end{pmatrix} \cdot \begin{pmatrix} x^{\text{topo}} \\ y^{\text{topo}} \\ z^{\text{topo}} \end{pmatrix}$$

Example :

$$\begin{pmatrix} -\xi \\ -\eta \\ 1 \end{pmatrix}_{\text{topo}^\star} \approx \begin{pmatrix} 1 & -\psi & -\xi \\ \psi & 1 & -\eta \\ \xi & \eta & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{\text{topo}}$$

$$\begin{pmatrix} \xi \\ \eta \\ 1 \end{pmatrix}_{\text{topo}} \approx \begin{pmatrix} 1 & \psi & \xi \\ -\psi & 1 & \eta \\ -\xi & -\eta & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{\text{topo}^\star}$$

# Topocentric Reference Systems/Frames




$$\sqrt{\eta^2 + \xi^2}$$

Computed with GGMPlus (Ch. Hirt)

# Gravity Field

very short introduction

# Gravity Field

**Physical Fundaments of Gravity (classical mechanics)**

1. Newton's 2nd Law in non-inertial reference systems :

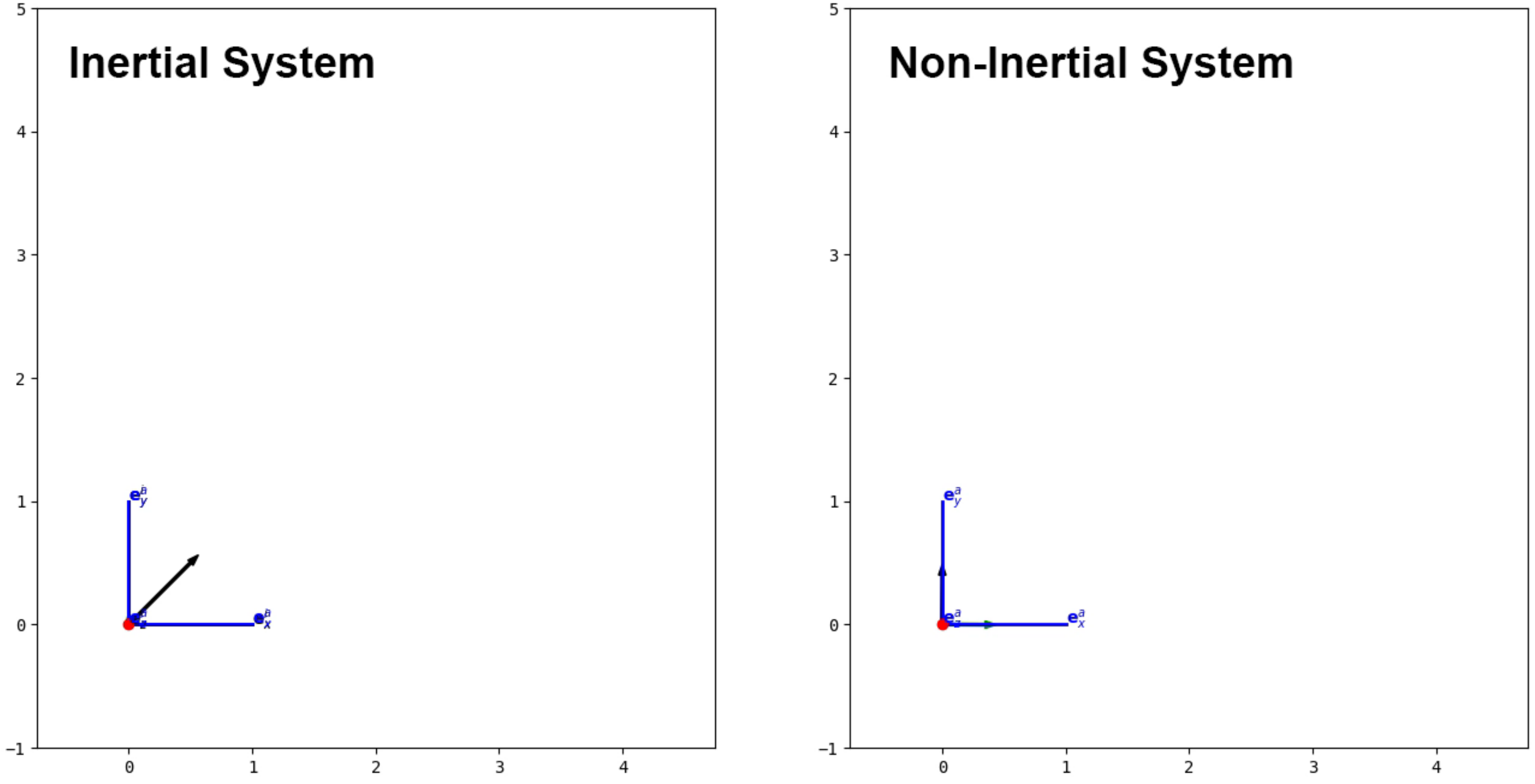
$$\ddot{\mathbf{x}}^a = \frac{1}{m}\sum \mathbf{F}^a - \mathbf{R}_i^a \cdot \ddot{\mathbf{o}}^a - \mathbf{\Omega}_{ia}^a \cdot \mathbf{\Omega}_{ia}^a \cdot \mathbf{x}^a - \dot{\mathbf{\Omega}}_{ia}^a \cdot \mathbf{x}^a - 2 \cdot \mathbf{\Omega}_{ia}^a \cdot \dot{\mathbf{x}}^a$$

2. Newton's Law of Gravitation

$$\mathbf{F}_{12} = G \cdot \frac{m_1 \cdot m_2}{|\mathbf{x}_2 - \mathbf{x}_1|^2} \cdot \frac{\mathbf{x}_2 - \mathbf{x}_1}{|\mathbf{x}_2 - \mathbf{x}_1|}$$

# Gravity Field

## Newton's 2nd Law in non-inertial reference systems :



# Gravity Field

**Newton's 2nd Law in non-inertial reference systems :**

Newton's 2nd Law in inertial reference systems

$$m \cdot \ddot{\mathbf{x}}^i = \sum \mathbf{F}^i$$

Link between inertial and non-inertial systems

$$\mathbf{x}^i(t) = \mathbf{o}^a(t) + \mathbf{R}_a^i(t) \cdot \mathbf{x}^a(t)$$

Newton's 2nd Law in inertial reference systems

$$\ddot{\mathbf{x}}^a = \frac{1}{m} \sum \mathbf{F}^a - \mathbf{R}_i^a \cdot \ddot{\mathbf{o}}^a - \mathbf{\Omega}_{ia}^a \cdot \mathbf{\Omega}_{ia}^a \cdot \mathbf{x}^a - \dot{\mathbf{\Omega}}_{ia}^a \cdot \mathbf{x}^a - 2 \cdot \mathbf{\Omega}_{ia}^a \cdot \dot{\mathbf{x}}^a$$

acceleration | external forces | acceleration of origin | centrifugal acceleration | Euler acceleration | Coriolis acceleration

# Gravity Field

**Observed acceleration = specific "force" = proper acceleration :**

Observed acceleration in Inertial System

$$\mathbf{a}^i_{\text{obs}} = \frac{1}{m}\sum \mathbf{F}^i - \frac{1}{m}\sum \mathbf{F}^i_{\text{grav}}$$
$$= \ddot{\mathbf{x}}^i - \frac{1}{m}\sum \mathbf{F}^i_{\text{grav}}$$

Observed acceleration in Non-Inertial System

$$\mathbf{a}^a_{\text{obs}} = \frac{1}{m}\sum \mathbf{F}^a - \frac{1}{m}\sum \mathbf{F}^a_{\text{grav}}$$
$$= \mathbf{R}^a_i \cdot \mathbf{a}^i_{\text{obs}}$$

$$\mathbf{a}^a_{\text{obs}} = \ddot{\mathbf{x}}^a - \frac{1}{m}\sum \mathbf{F}^a_{\text{grav}} + \mathbf{R}^a_i \cdot \ddot{\mathbf{o}}^a + \mathbf{\Omega}^a_{ia} \cdot \mathbf{\Omega}^a_{ia} \cdot \mathbf{x}^a + \dot{\mathbf{\Omega}}^a_{ia} \cdot \mathbf{x}^a + 2 \cdot \mathbf{\Omega}^a_{ia} \cdot \dot{\mathbf{x}}^a$$

# Gravity Field

**Observed acceleration = specific "force" = proper acceleration :**

Observed acceleration in Inertial System

$$\mathbf{a}^i_{\text{obs}} = \frac{1}{m}\sum \mathbf{F}^i - \frac{1}{m}\sum \mathbf{F}^i_{\text{grav}}$$
$$= \ddot{\mathbf{x}}^i - \frac{1}{m}\sum \mathbf{F}^i_{\text{grav}}$$

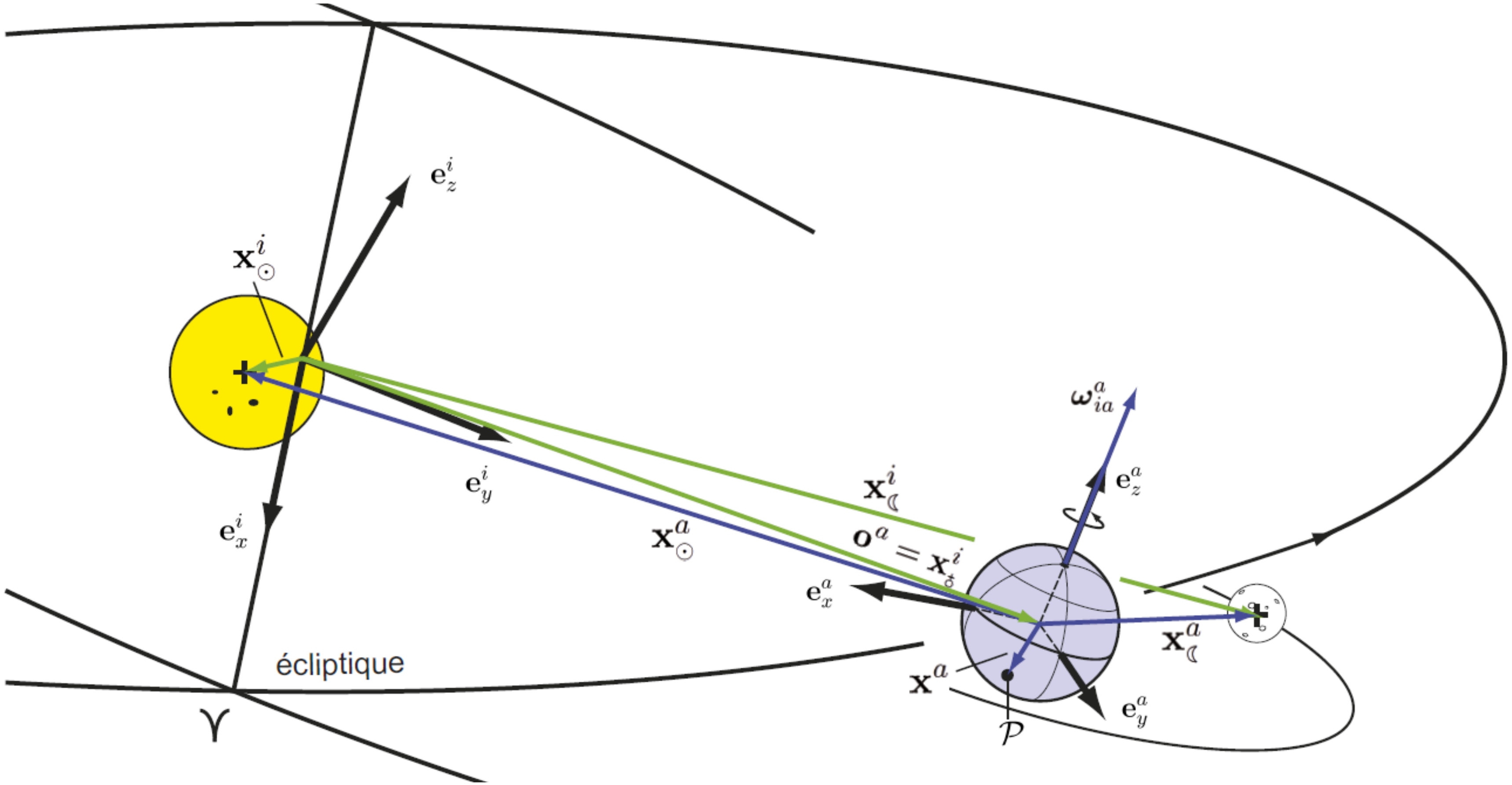
Observed acceleration in Non-Inertial System

$$\mathbf{a}^a_{\text{obs}} = \frac{1}{m}\sum \mathbf{F}^a - \frac{1}{m}\sum \mathbf{F}^a_{\text{grav}}$$
$$= \mathbf{R}^a_i \cdot \mathbf{a}^i_{\text{obs}}$$

$$\mathbf{a}^a_{\text{obs}} = \ddot{\mathbf{x}}^a - \frac{1}{m}\sum \mathbf{F}^a_{\text{grav}} + \mathbf{R}^a_i \cdot \ddot{\mathbf{o}}^a + \boldsymbol{\Omega}^a_{ia} \cdot \boldsymbol{\Omega}^a_{ia} \cdot \mathbf{x}^a + \dot{\boldsymbol{\Omega}}^a_{ia} \cdot \mathbf{x}^a + 2 \cdot \boldsymbol{\Omega}^a_{ia} \cdot \dot{\mathbf{x}}^a$$

# Gravity Field

## Observed acceleration : fix Point on the Earth, in a Sun – Moon – Earth Universe



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# Gravity Field

## Observed acceleration : fix Point on the Earth, in a Sun – Moon – Earth Universe

$$\mathbf{a}^a_{\text{obs}} = \underbrace{\ddot{\mathbf{x}}^a}_{=\mathbf{0}} - \boxed{\frac{1}{m}\sum \mathbf{F}^a_{\text{grav}}} + \boxed{\mathbf{R}^a_i \cdot \ddot{\mathbf{o}}^a} + \boxed{\mathbf{\Omega}^a_{ia} \cdot \mathbf{\Omega}^a_{ia} \cdot \mathbf{x}^a} + \underbrace{\dot{\mathbf{\Omega}}^a_{ia}}_{=\mathbf{0}} \cdot \mathbf{x}^a + 2 \cdot \mathbf{\Omega}^a_{ia} \cdot \underbrace{\dot{\mathbf{x}}^a}_{=\mathbf{0}}$$

point is fix on Earth

Earth constant angular velocity

point is fix on Earth

$$\frac{1}{m}\sum \mathbf{F}^a_{\text{grav}} = G \cdot \iiint_{♁} \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \cdot \rho(\mathbf{x}') \cdot \mathrm{d}V + GM_{\odot}\frac{\mathbf{x}_{\odot} - \mathbf{x}}{|\mathbf{x}_{\odot} - \mathbf{x}|^3} + GM_{☾}\frac{\mathbf{x}_{☾} - \mathbf{x}}{|\mathbf{x}_{☾} - \mathbf{x}|^3}$$

$$GM_{\odot}\frac{\mathbf{x}^a_{\odot}}{|\mathbf{x}^a_{\odot}|^3} + GM_{☾}\frac{\mathbf{x}^a_{☾}}{|\mathbf{x}^a_{☾}|^3}$$

$$-\omega^2_{♁} \cdot \begin{pmatrix} x^a \\ y^a \\ 0 \end{pmatrix}$$

# Gravity Field

## Observed acceleration : fix Point on the Earth, in a Sun – Moon – Earth Universe

$$\mathbf{a}^a_{\text{obs}} = \underbrace{-G \cdot \iiint_{\oplus} \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \cdot \rho(\mathbf{x}') \cdot \mathrm{d}V - GM_\odot \frac{\mathbf{x}_\odot - \mathbf{x}}{|\mathbf{x}_\odot - \mathbf{x}|^3} - GM_☾ \frac{\mathbf{x}_☾ - \mathbf{x}}{|\mathbf{x}_☾ - \mathbf{x}|^3}}_{\text{forces gravitationnelles}}$$

$$\underbrace{+GM_\odot \frac{\mathbf{x}_\odot}{|\mathbf{x}_\odot|^3} + GM_☾ \frac{\mathbf{x}_☾}{|\mathbf{x}_☾|^3}}_{\text{origine du système accéléré } a}$$

$$\underbrace{-\omega_{\oplus}^2 \cdot \begin{pmatrix} x^a \\ y^a \\ 0 \end{pmatrix}}_{\text{force centrifuge}}$$

# Gravity Field

## Observed acceleration : fix Point on the Earth, in a Sun – Moon – Earth Universe

$$\mathbf{a}^a_{\text{obs}} = \underbrace{-G \cdot \iiint_{\oplus} \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \cdot \rho(\mathbf{x}') \cdot \mathrm{d}V}_{\text{forces gravitationnelles de la Terre}}$$

$$\underbrace{-\omega^2_{\oplus} \cdot \begin{pmatrix} x^a \\ y^a \\ 0 \end{pmatrix}}_{\text{force centrifuge}}$$

$$\underbrace{-GM_{\odot} \left\{ \frac{\mathbf{x}_{\odot} - \mathbf{x}}{|\mathbf{x}_{\odot} - \mathbf{x}|^3} - \frac{\mathbf{x}_{\odot}}{|\mathbf{x}_{\odot}|^3} \right\} - GM_{☾} \left\{ \frac{\mathbf{x}_{☾} - \mathbf{x}}{|\mathbf{x}_{☾} - \mathbf{x}|^3} - \frac{\mathbf{x}_{☾}}{|\mathbf{x}_{☾}|^3} \right\}}_{\text{forces de marée}}$$

# Gravity Field

## Observed Free-Falling : on the Earth, in a Sun – Moon – Earth Universe

$$\underbrace{\mathbf{a}^a_{\text{obs}}}_{=\mathbf{0}} = \ddot{\mathbf{x}}^a - \frac{1}{m}\sum \mathbf{F}^a_{\text{grav}} + \mathbf{R}^a_i \cdot \ddot{\mathbf{o}}^a + \boldsymbol{\Omega}^a_{ia} \cdot \boldsymbol{\Omega}^a_{ia} \cdot \mathbf{x}^a + \underbrace{\dot{\boldsymbol{\Omega}}^a_{ia}}_{=\mathbf{0}} \cdot \mathbf{x}^a + 2 \cdot \boldsymbol{\Omega}^a_{ia} \cdot \underbrace{\dot{\mathbf{x}}^a}_{=\mathbf{0}}$$

$$\ddot{\mathbf{x}}^a = \underbrace{+G \cdot \iiint_{♁} \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \cdot \rho(\mathbf{x}') \cdot \mathrm{d}V}_{\text{forces gravitationnelles de la Terre}}$$

$$\underbrace{+\omega^2_{♁} \cdot \begin{pmatrix} x^a \\ y^a \\ 0 \end{pmatrix}}_{\text{force centrifuge}}$$

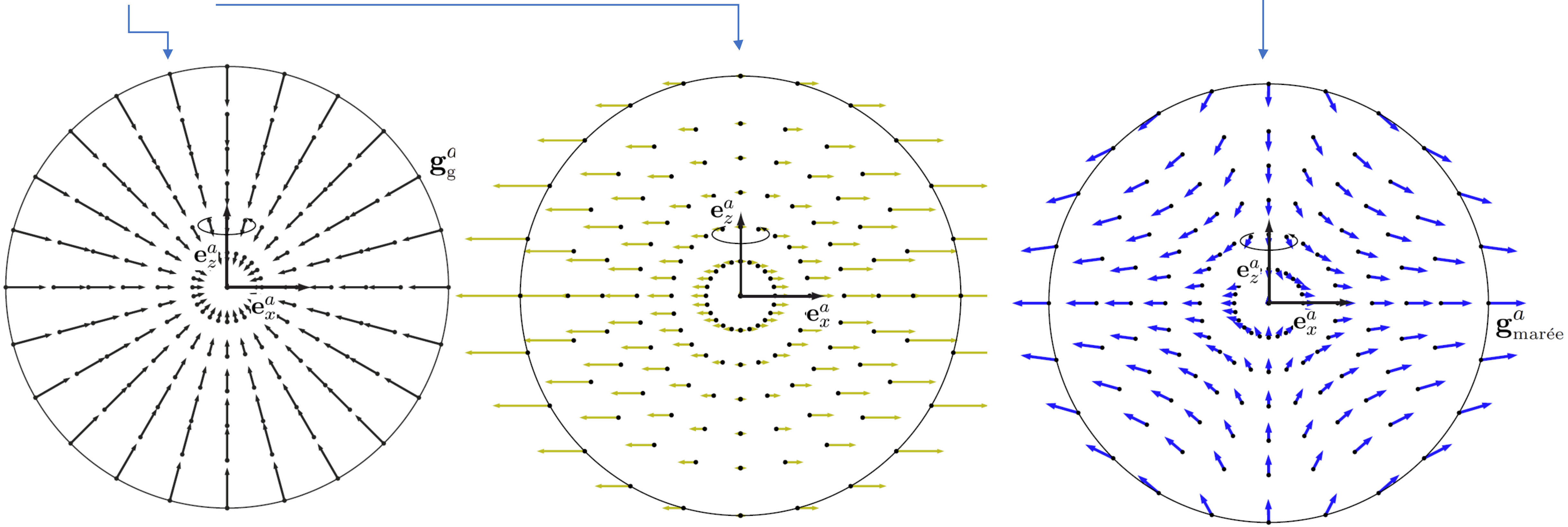
$$\underbrace{+GM_{\odot}\left\{\frac{\mathbf{x}_{\odot} - \mathbf{x}}{|\mathbf{x}_{\odot} - \mathbf{x}|^3} - \frac{\mathbf{x}_{\odot}}{|\mathbf{x}_{\odot}|^3}\right\} + GM_{☾}\left\{\frac{\mathbf{x}_{☾} - \mathbf{x}}{|\mathbf{x}_{☾} - \mathbf{x}|^3} - \frac{\mathbf{x}_{☾}}{|\mathbf{x}_{☾}|^3}\right\}}_{\text{forces de marée}}$$

$$\ddot{\mathbf{x}}^a = \mathbf{g}^a_{\text{grav}} + \mathbf{g}^a_{\text{cent}} + \mathbf{g}^a_{\text{marée}}$$

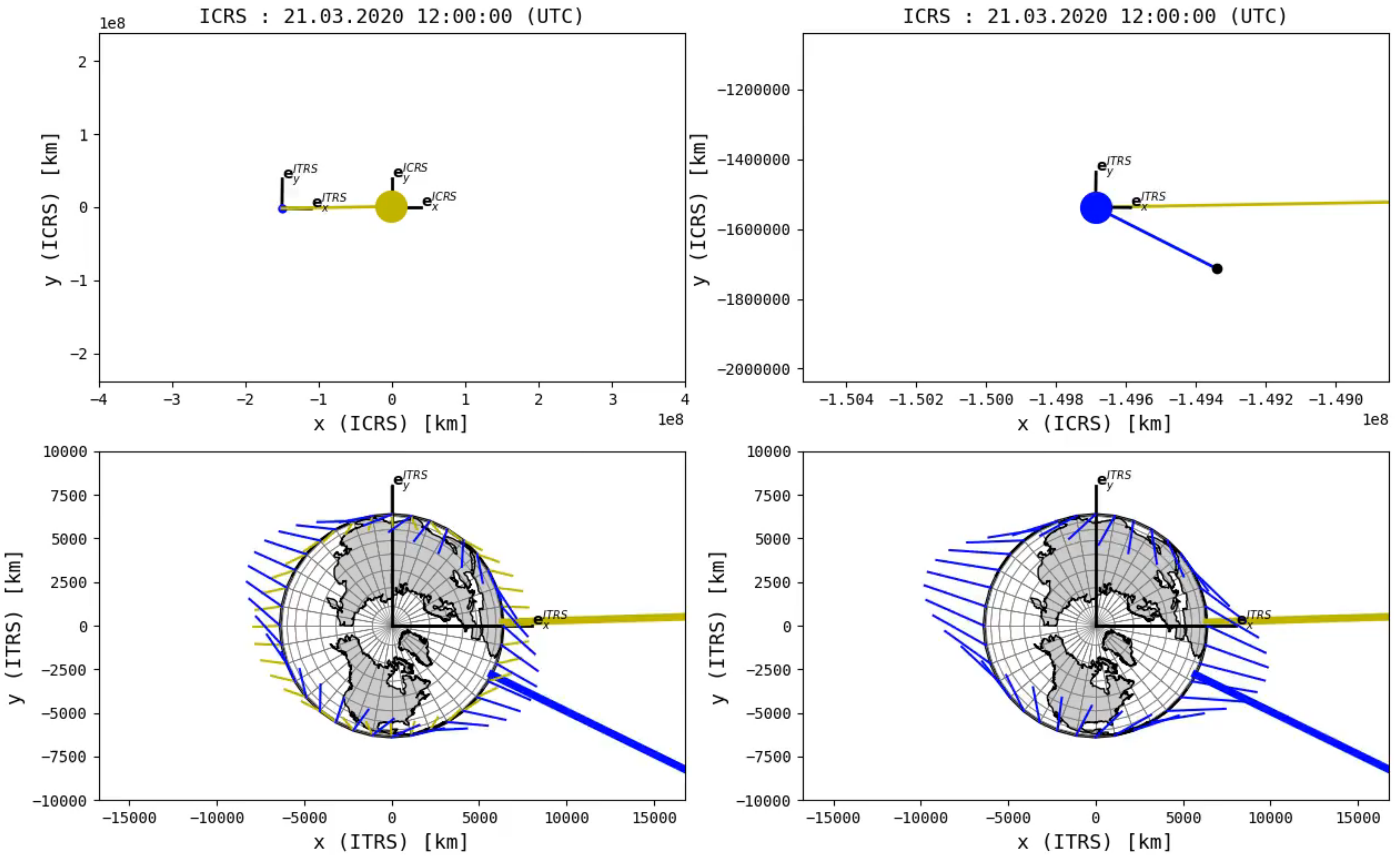
# Gravity Field

## Observed Free-Falling : on the Earth, in a Sun – Moon – Earth Universe

$$\ddot{\mathbf{x}}^a = \mathbf{g}^a_{\mathrm{grav}} + \mathbf{g}^a_{\mathrm{cent}} + \mathbf{g}^a_{\mathrm{marée}}$$



# Gravity Field



# Gravity Field

## Definition of Gravity in Physical Geodesy

$$\mathbf{g}(\mathbf{x}) = \mathbf{g}_{\text{grav}}(\mathbf{x}) + \mathbf{g}_{\text{cent}}(\mathbf{x})$$

$$= \underbrace{G \cdot \iiint_{♁} \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \cdot \rho(\mathbf{x}') \cdot \mathrm{d}V}_{\text{l'accélération gravitationnelle}} \quad \underbrace{+\omega_{♁}^2 \cdot \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}}_{\text{l'accélération centrifuge}}$$

# Gravity Field

## Definition of Gravity Potential in Physical Geodesy

$$\nabla W(\mathbf{x}) = \mathbf{g}(\mathbf{x})$$
$$= \mathbf{g}_{\text{grav}}(\mathbf{x}) + \mathbf{g}_{\text{cent}}(\mathbf{x}) \quad \left[\frac{\text{m}}{\text{s}^2}\right]$$

$$W(\mathbf{x}) = G \iiint_{♁} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \cdot \rho(\mathbf{x}') \cdot \mathrm{d}V + \frac{1}{2} \cdot \omega_{♁}^2 \cdot (x^2 + y^2) \quad \left[\frac{\text{m}^2}{\text{s}^2}\right]$$

# Gravity Field

## Definition of Gravity Potential in Physical Geodesy

Example : Homogeneous Spherical Earth

$$W(\mathbf{x}) = V(\mathbf{x}) + Z(\mathbf{x})$$

$$= 62'702'851.186 + 108'222.703 = 62'811'073.889 \quad \left[\frac{\mathrm{m}^2}{\mathrm{s}^2}\right]$$

with :

$$R_{\mathcal{B}} = 6'380'000 \ [\mathrm{m}]$$

$$\rho_{\mathcal{B}} = 5.51 \cdot 10^3 \left[\frac{\mathrm{kg}}{\mathrm{m}^3}\right]$$

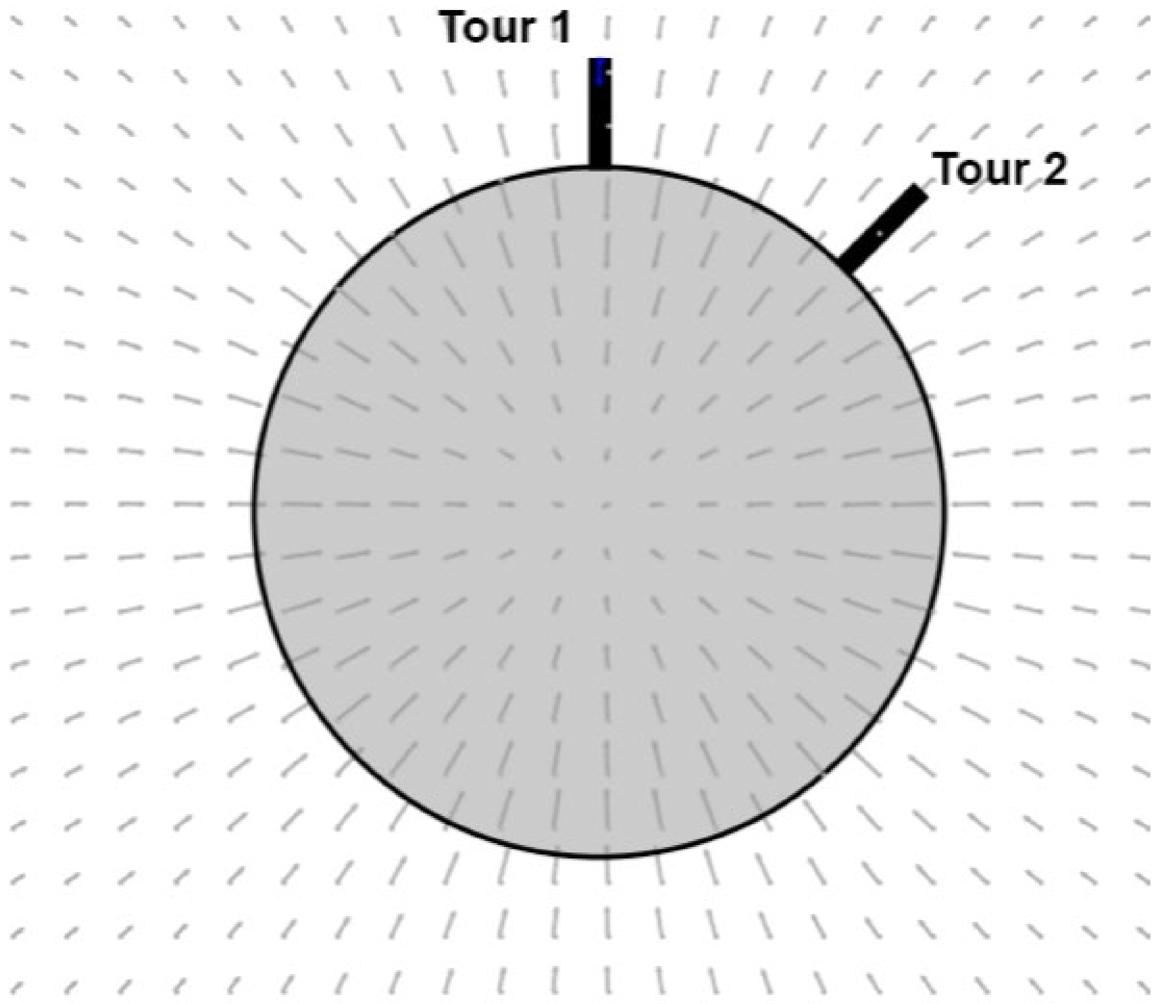
$$\omega_{♁} = 7.292115 \cdot 10^{-5} \left[\frac{\mathrm{rad}}{\mathrm{s}}\right]$$

# Height Systems

very short introduction

# Height Systems

**Homogeneous Spherical Earth**



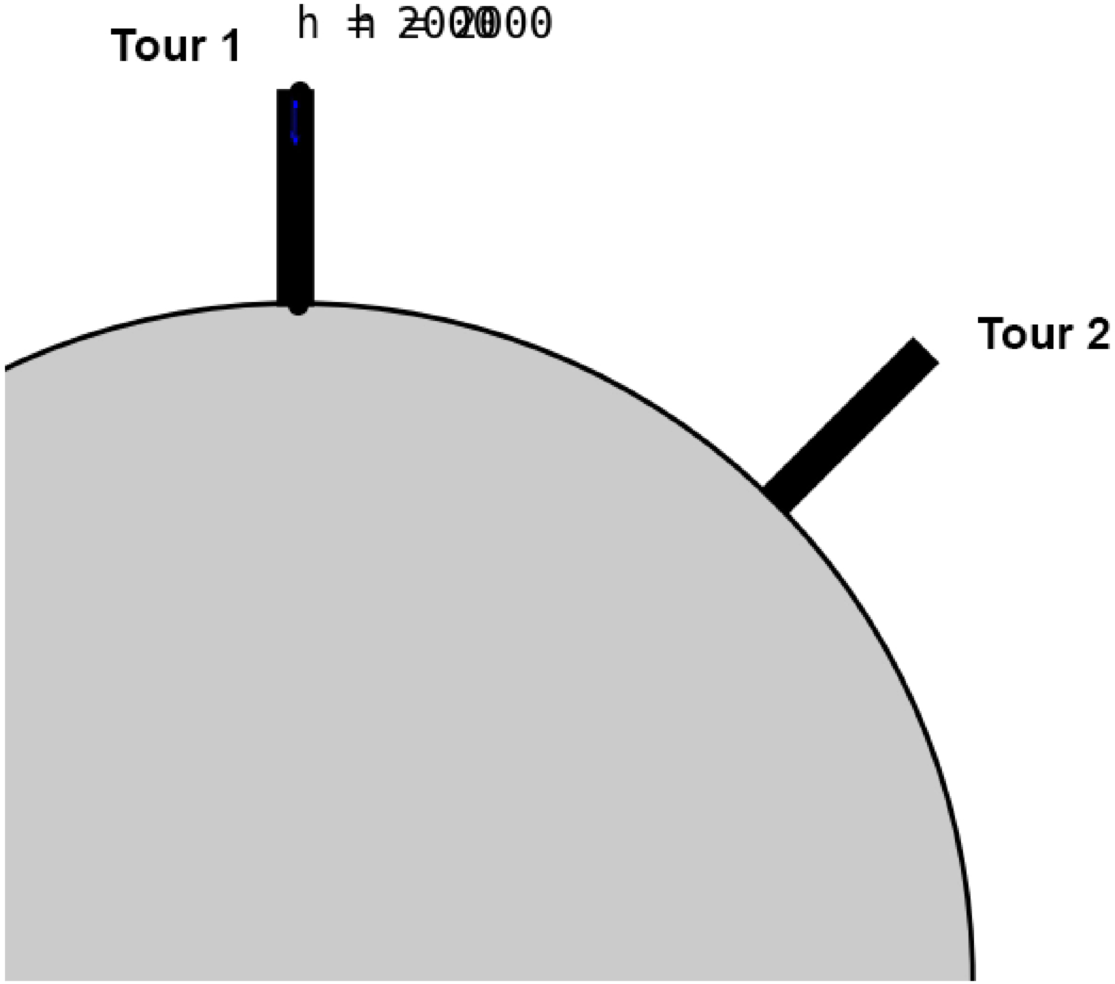
# Height Systems

**Homogeneous Spherical Earth**

**Same Geometrical Height**

**=**

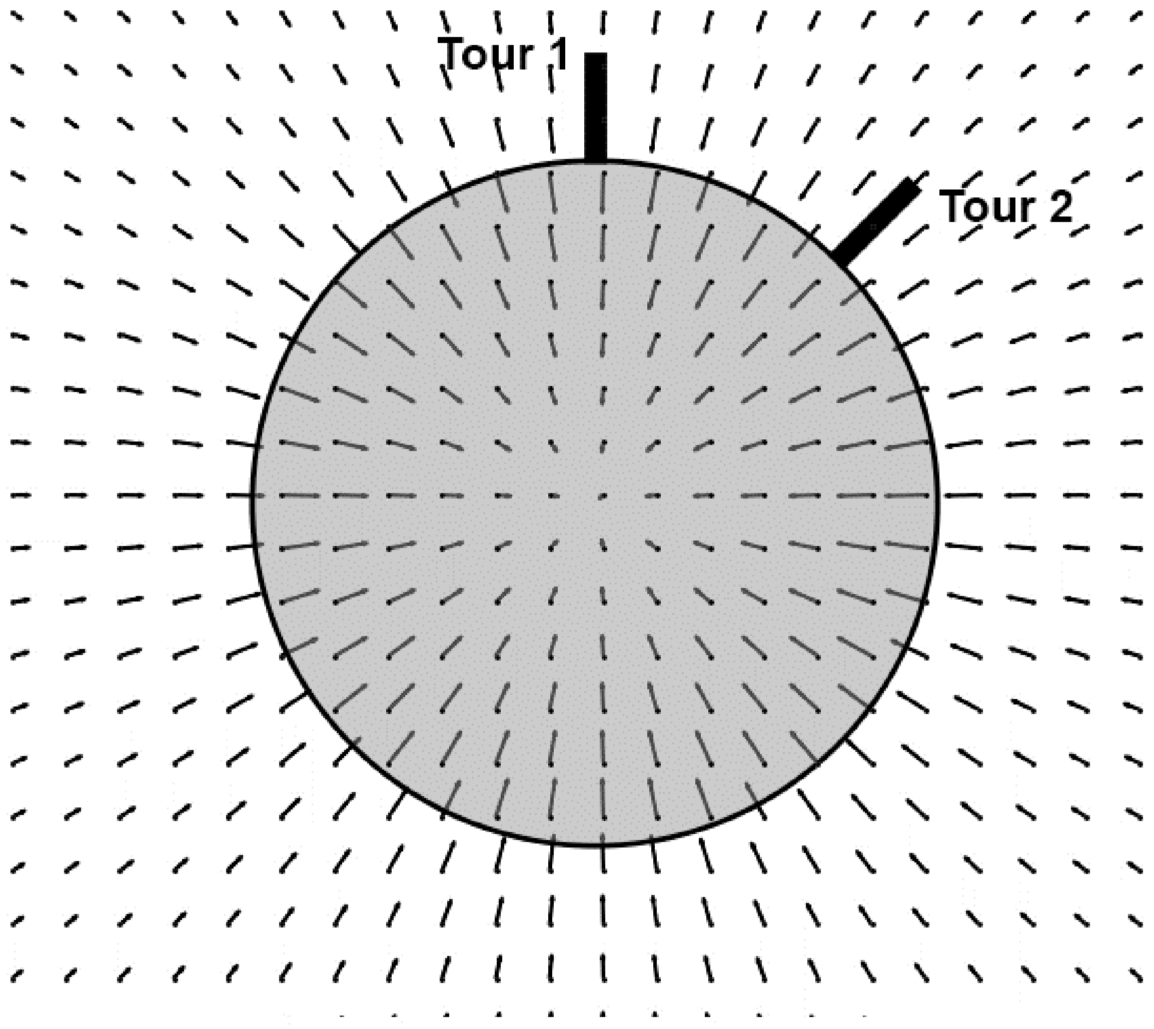
**Same equipotential**



# Height Systems

**Homogeneous Spherical Earth**

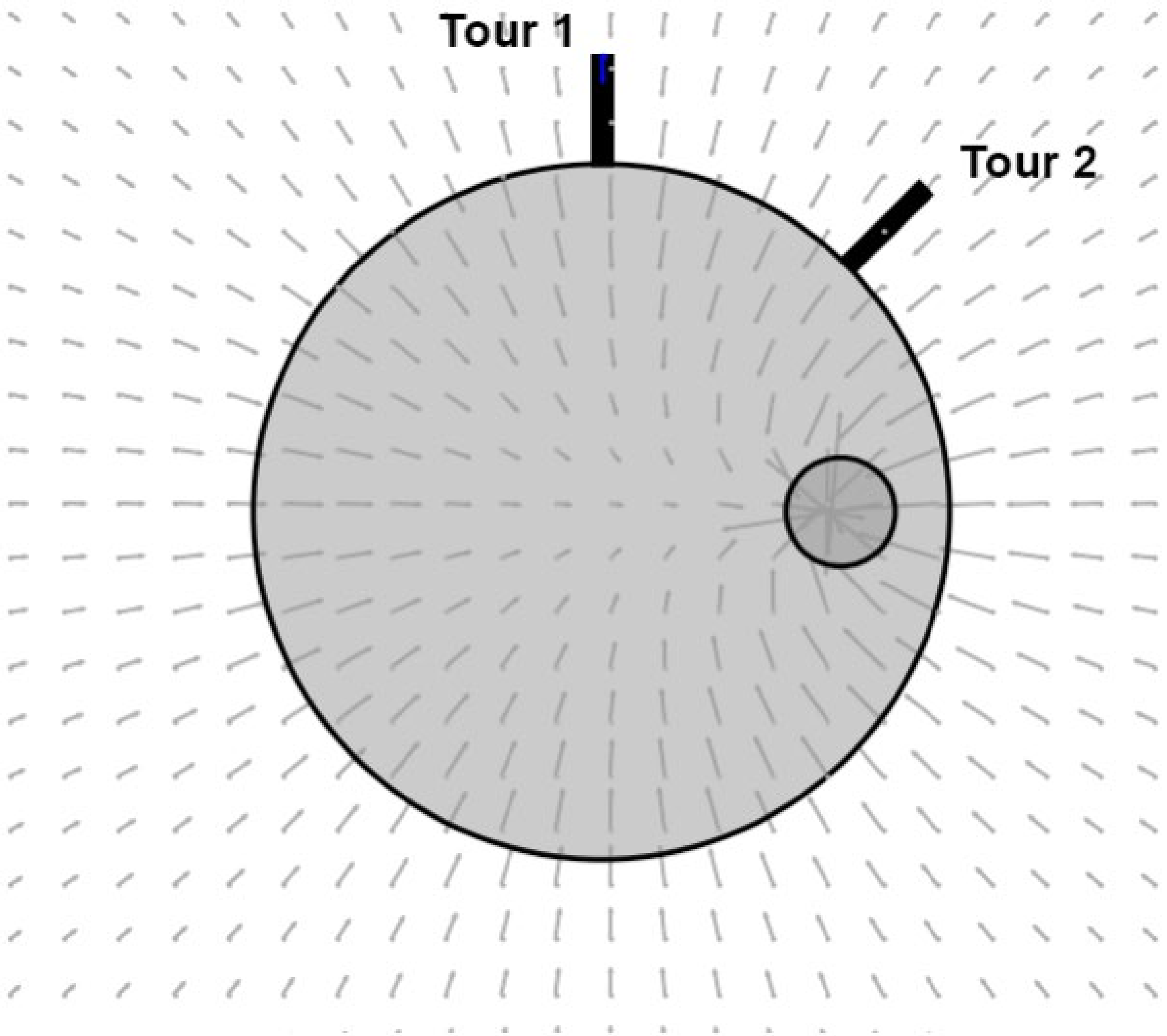
**+ Density Anomaly**



# Height Systems

**Homogeneous Spherical Earth**

**+ Density Anomaly**



# Height Systems

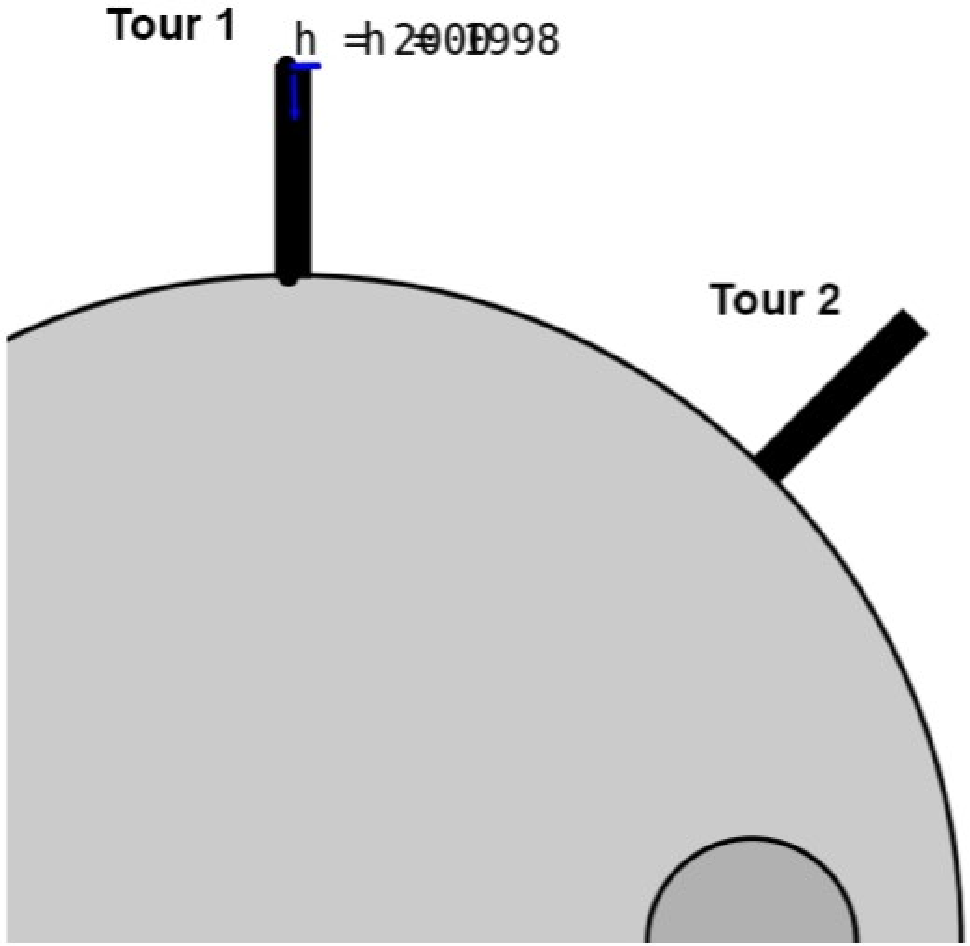
**Homogeneous Spherical Earth**

**+ Density Anomaly**

**Same Geometrical Height**

$\neq$

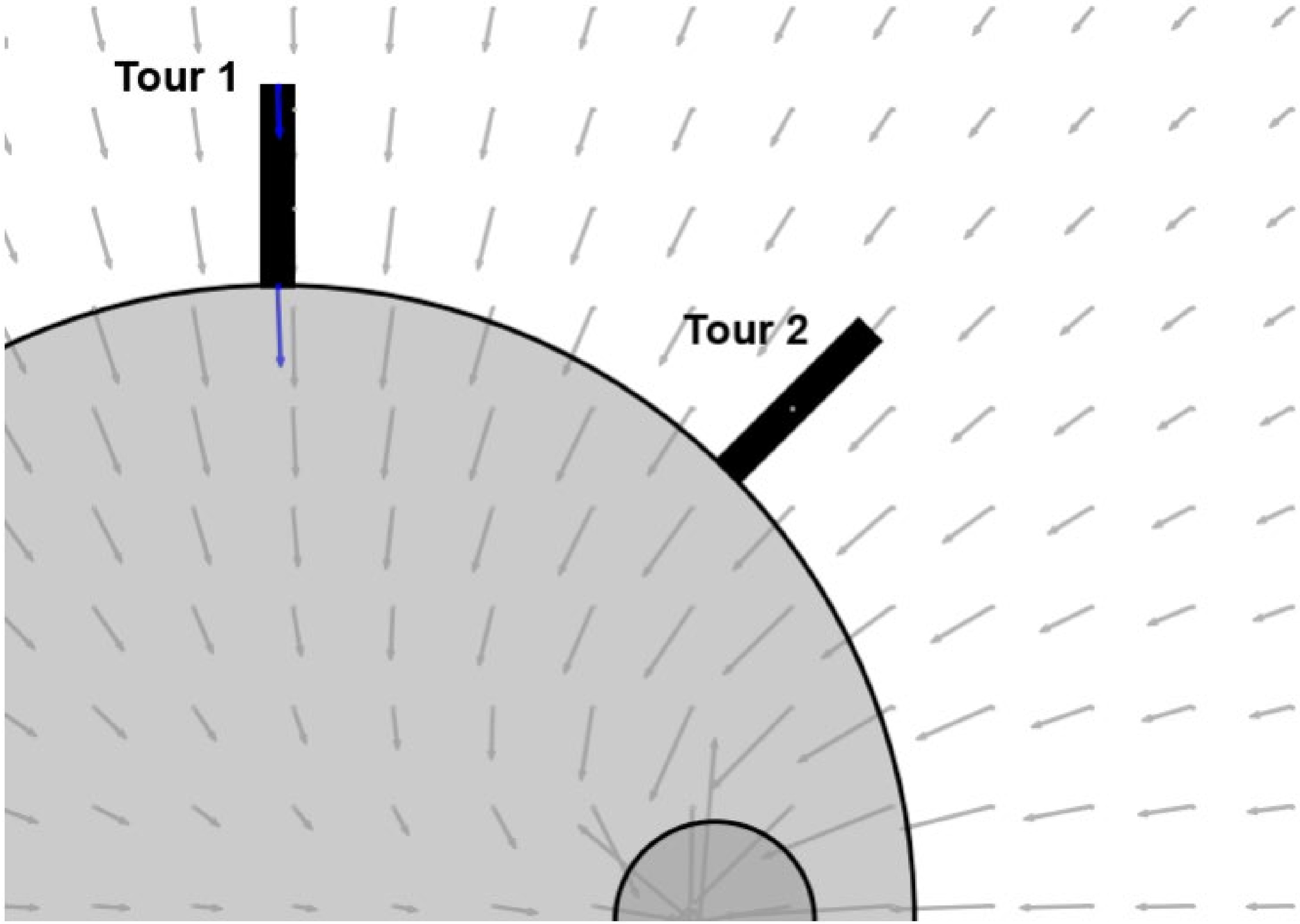
**Same equipotential**



# Height Systems

**Homogeneous Spherical Earth**

**+ Density Anomaly**




**Introducing a new reference surface for Heights**

⬇

**Geoid**

# Height Systems

**Homogeneous Spherical Earth**

**+ Density Anomaly**

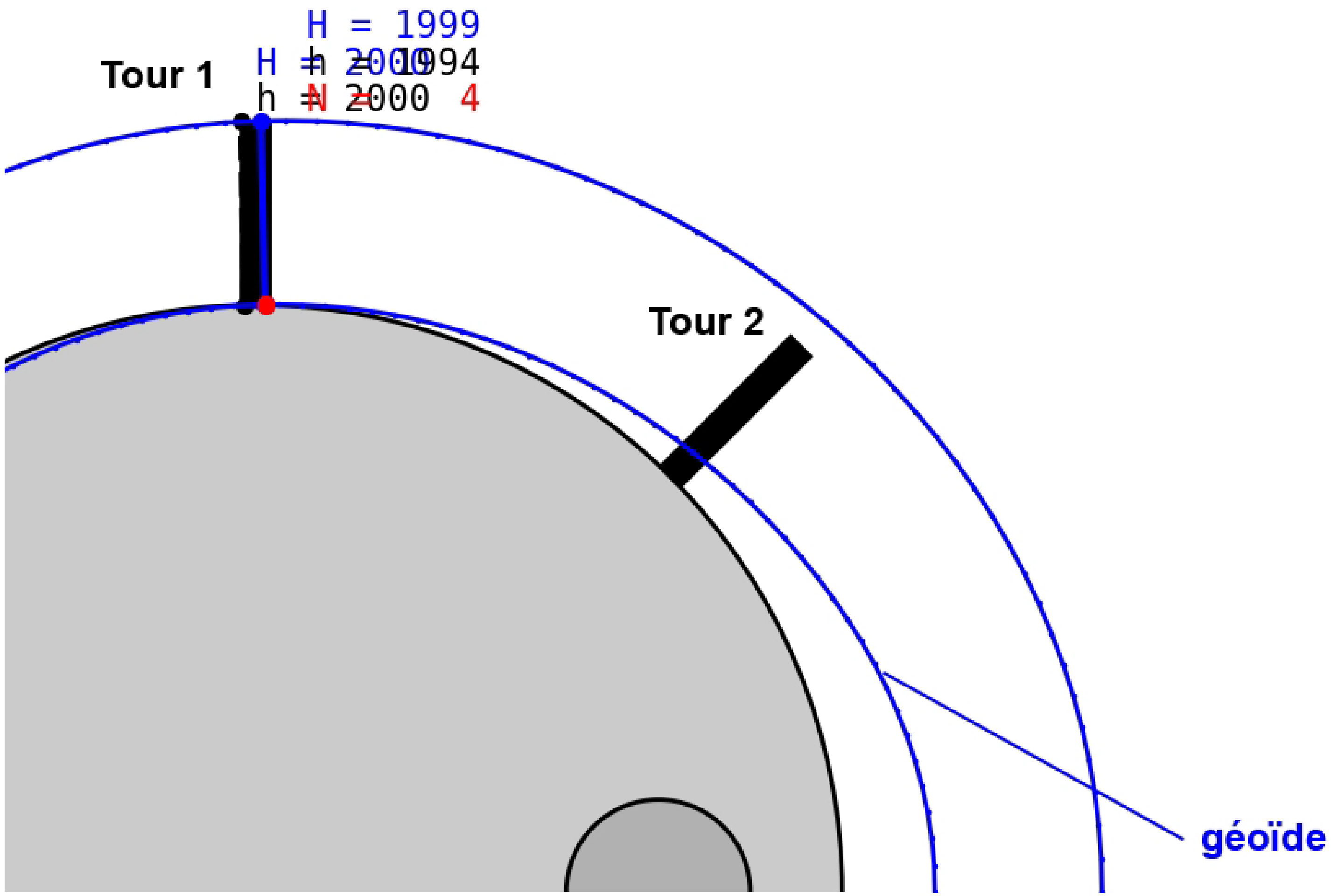
**Introducing a new reference surface for Heights**

⬇

**Geoid**

⬇

**Orthometric Height**



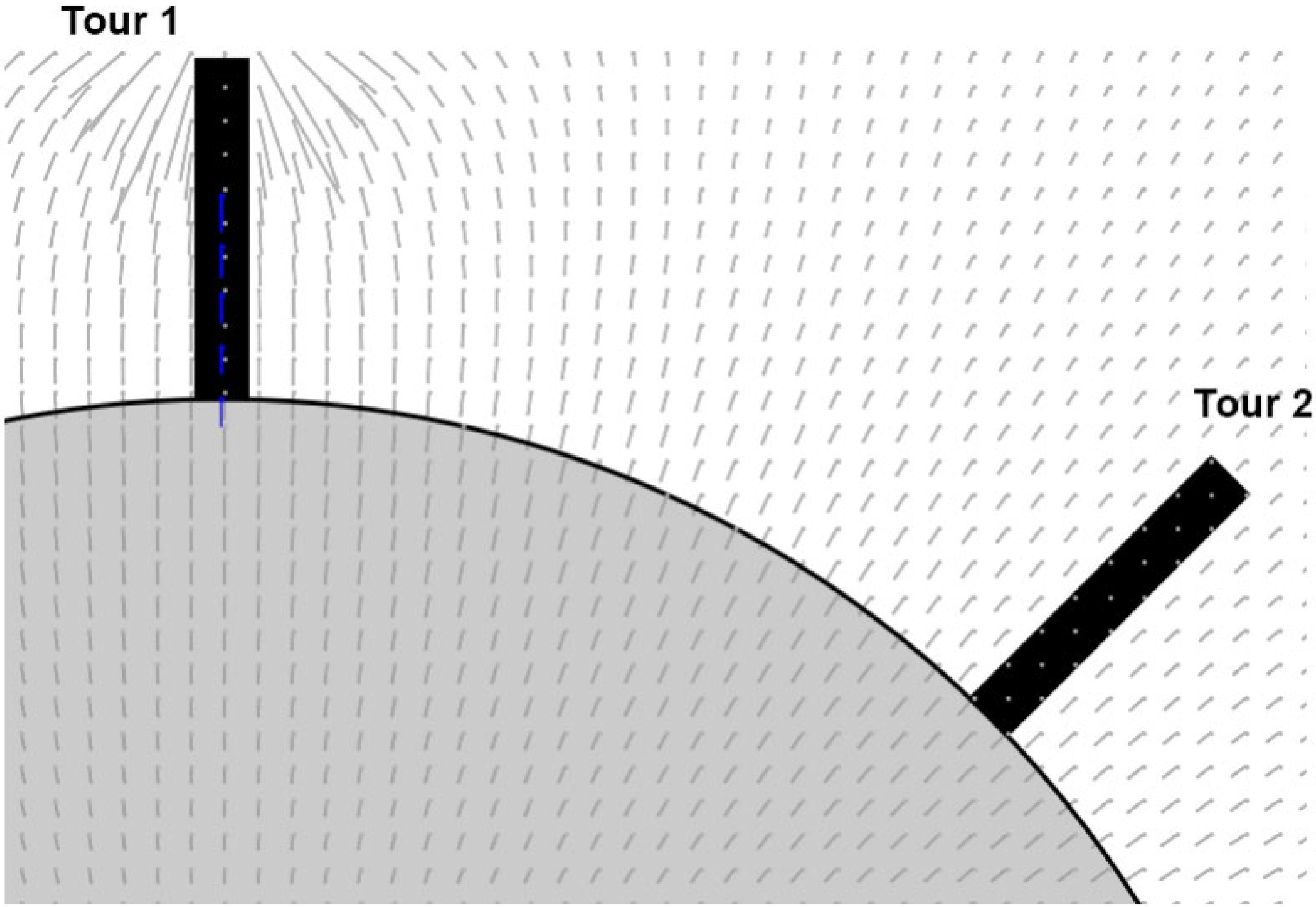
# Height Systems

**Homogeneous Spherical Earth**

**+ Density Anomaly**

**How to measure**

**Orthometric Heights?**



# Height Systems

## Modern Height Systems

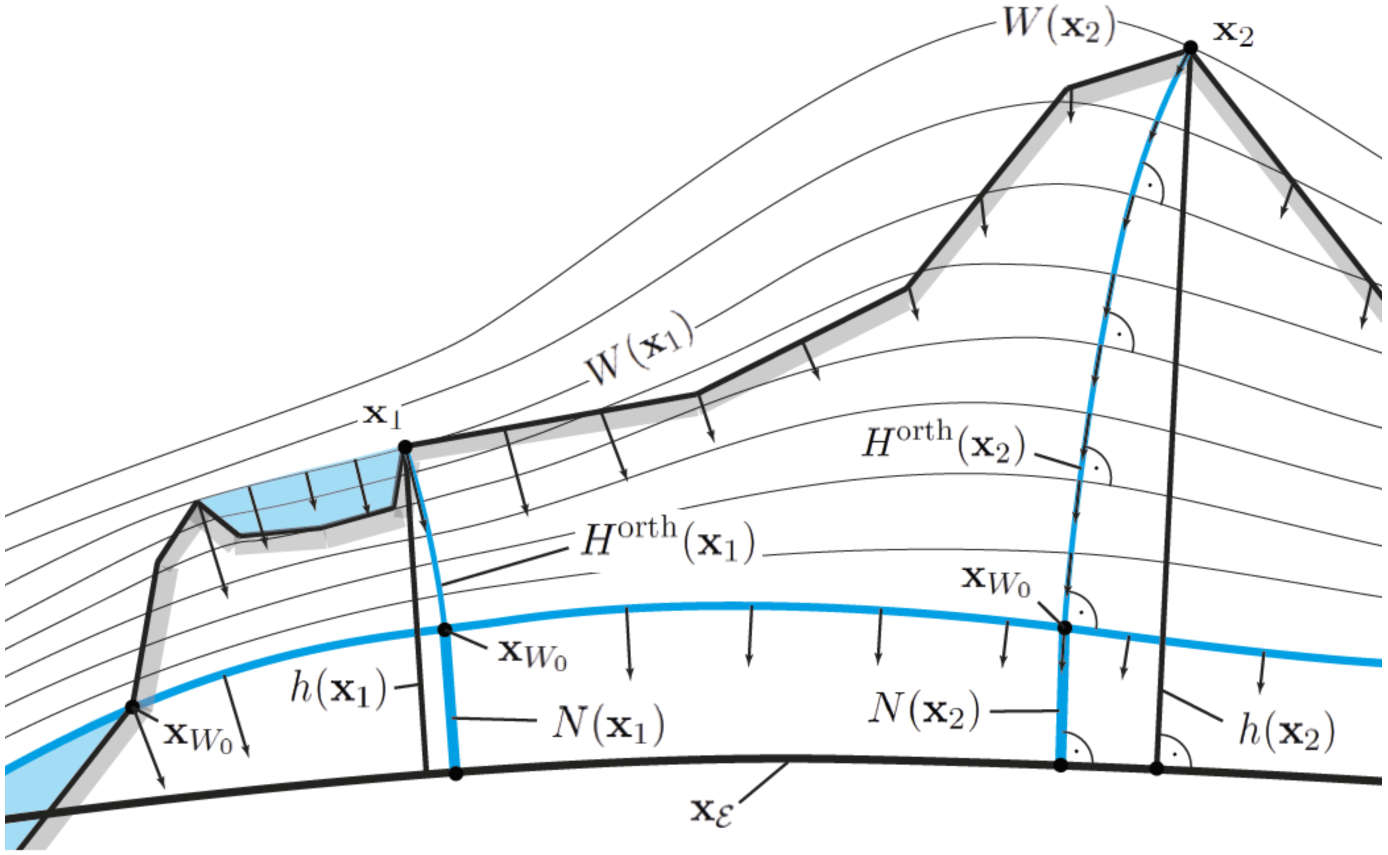
all based on the geopotential number

$$C(\mathbf{x}) = W(\mathbf{x}_{W_0}) - W(\mathbf{x})$$

Dynamic Height : $$H^{\mathrm{dyn}}(\mathbf{x}) = \frac{C(\mathbf{x})}{\gamma_0}$$

Orthometric Height : $$H^{\mathrm{orth}}(\mathbf{x}) = \frac{C(\mathbf{x})}{\overline{g}(\mathbf{x})}$$

Normal Height : $$H^{\mathrm{norm}}(\mathbf{x}) = \frac{C(\mathbf{x})}{\overline{\gamma}(\mathbf{x})}$$



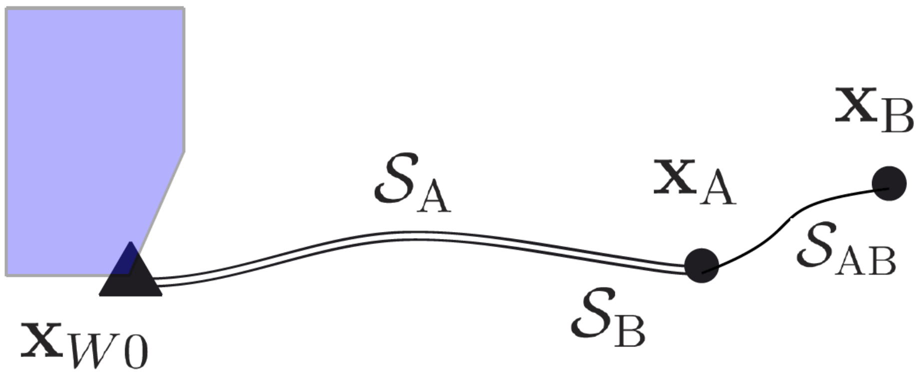
## Orthometric Height Difference from Levelling

$$H^{\text{orth}}(\mathbf{x}_\text{B}) - H^{\text{orth}}(\mathbf{x}_\text{A}) = \boxed{H^{\text{u}}(\mathbf{x}_\text{B}, \mathcal{S}_\text{B}) - H^{\text{u}}(\mathbf{x}_\text{A}, \mathcal{S}_\text{A})}$$

pure levelling

$$+ \int_{\mathbf{x}_A, \mathcal{S}_\text{AB}}^{\mathbf{x}_\text{B}} \frac{g(\mathbf{x}) - \gamma_0}{\gamma_0} \cdot \text{d}n + \frac{\overline{g}(\mathbf{x}_\text{A}) - \gamma_0}{\gamma_0} \cdot H^{\text{orth}}(\mathbf{x}_\text{A}) - \frac{\overline{g}(\mathbf{x}_\text{B}) - \gamma_0}{\gamma_0} \cdot H^{\text{orth}}(\mathbf{x}_\text{B})$$

orthometric correction




$$\mathcal{S}_\text{B} = \mathcal{S}_\text{A} + \mathcal{S}_\text{AB}$$