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# Elementary construction of the Nagel point

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## Abstract

In this short note, we give an easy construction of the Nagel point of a triangle.

## 1 Introduction

If you continue the sides of a triangle beyond every vertex at the distances equaling to the length of the opposite side, the resulting six points lie on a circle, which is called Conway's circle (figure 1).

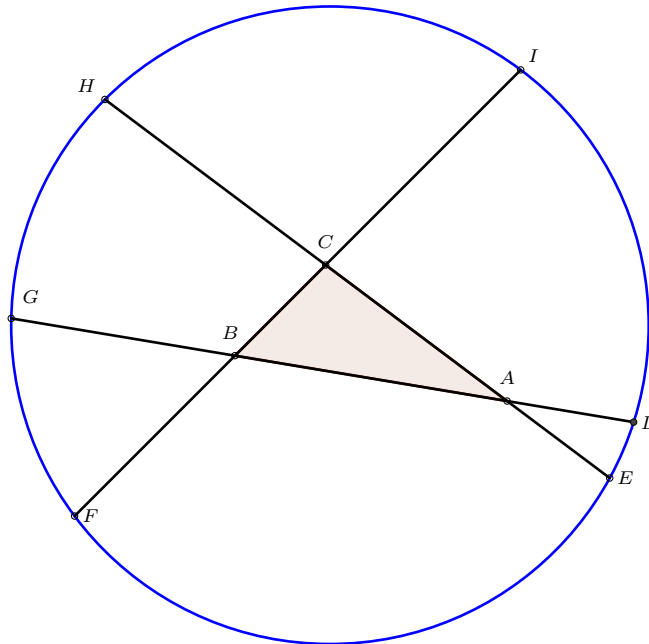


Figure 1. The Conway circle

In this note, we investigate a similar construction which characterizes the Nagel point of a triangle  $ABC$ . Starting from point  $A$ , we transport the length of the opposite edge on each half-line starting from  $A$ . We obtain two points  $A'$  and  $A''$  which satisfy  $\overrightarrow{AA'} = \frac{a}{c}\overrightarrow{AB}$  and  $\overrightarrow{AA''} = \frac{a}{b}\overrightarrow{AC}$  (figure 2). Repeating this construction with vertices  $B$  and  $C$ , we have four more points  $B'$ ,  $B''$ ,  $C'$  and  $C''$  which satisfy  $\overrightarrow{BB'} = \frac{b}{a}\overrightarrow{BC}$ ,  $\overrightarrow{BB''} = \frac{b}{c}\overrightarrow{BA}$ ,  $\overrightarrow{CC'} = \frac{c}{b}\overrightarrow{CA}$  and  $\overrightarrow{CC''} = \frac{c}{a}\overrightarrow{CB}$ .

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By construction,

$$AA' = a, \quad BB' = b, \quad CC' = c,$$

$$AA'' = a, \quad BB'' = b, \quad CC'' = c.$$

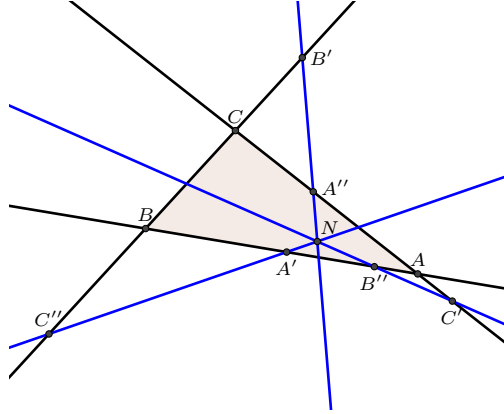


Figure 2. Easy construction of the Nagel point

Using barycentric coordinates, we prove that the three lines  $A''B'$ ,  $A'C''$  and  $C'B''$  are concurrent (figure 2) when  $ABC$  is a scalene triangle (a triangle with edges of different lengths).

Moreover, it turns out that the intersection point is the Nagel point of the triangle  $ABC$ .

Figure 3 gives the original construction of the Nagel point which is defined by the following result "the lines joining the points of contact of an excircle with the sides of a triangle to the vertices opposite the respective sides, are concurrent".

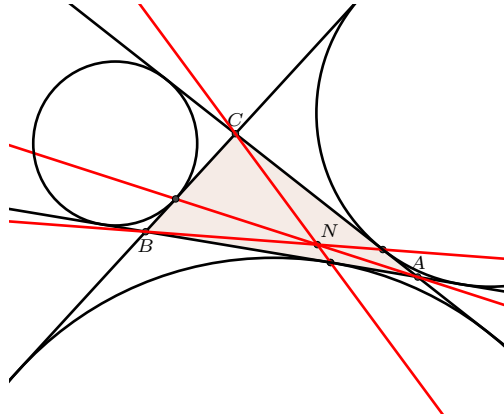


Figure 3. Original construction of the Nagel Point

A construction of the Nagel point requiring only the incircle of the triangle can be found in Hoehn [Hoe07].

## 2 Main result

Our theorem will use basic facts of barycentric coordinates in the plane. In particular, a line, in barycentric coordinates, is given by a linear homogeneous equation  $\alpha X + \beta Y + \gamma W = 0$  for some coefficients  $\alpha$ ,  $\beta$ , and  $\gamma$  not all of which are 0.

**Theorem 1.** *Let  $ABC$  be a scalene triangle and the six points  $A'$ ,  $A''$ ,  $B'$ ,  $B''$ ,  $C'$  and  $C''$  given by the relations:*

$$\overrightarrow{AA'} = \frac{a}{c}\overrightarrow{AB}, \quad \overrightarrow{AA''} = \frac{a}{b}\overrightarrow{AC}, \quad \overrightarrow{BB'} = \frac{b}{a}\overrightarrow{BC},$$

$$\overrightarrow{BB''} = \frac{b}{c}\overrightarrow{BA}, \quad \overrightarrow{CC'} = \frac{c}{b}\overrightarrow{CA}, \quad \overrightarrow{CC''} = \frac{c}{a}\overrightarrow{CB}.$$

Then, the three lines  $A'B'$ ,  $A'C''$  and  $C'B''$  are concurrent and the intersection point is the Nagel point.

*Proof.* The condition  $\overrightarrow{AA'} = \frac{a}{c}\overrightarrow{AB}$  implies that  $A'$  has barycentric coordinates  $(a-c, -a, 0)$ . We also deduce that  $A''(a-b, 0, -a)$ ,  $B'(0, b-a, b)$ ,  $B''(-b, b-c, 0)$ ,  $C'(-c, 0, c-b)$  and  $C''(0, c, c-a)$ .

Let  $\alpha X + \beta Y + \gamma Z = 0$  be an equation of the line  $A'B'$ . Fix  $\gamma = 1$ . The barycentric coordinates of  $A''$  and  $B'$  satisfy this equation and we obtain  $\beta = \frac{b}{b-a}$  and  $\alpha = \frac{a}{a-b}$ . Multiplying by  $a-b$ , we deduce that  $aX - bY + (a-b)Z = 0$  is an equation of the line  $A''B'$ .

We can now proceed analogously for lines  $A'C''$  and  $C'B''$ . It follows immediately that  $aX + (a-c)Y - cZ = 0$  is an equation of  $A'C''$  and  $(b-c)X + bY - cZ = 0$  is an equation of  $C'B''$ .

Easy computations show that

$$\begin{vmatrix} a & -b & a-b \\ a & a-c & -c \\ b-c & b & -c \end{vmatrix} = 0.$$

This determinant is 0 hence the three lines  $A'B'$ ,  $A'C''$  and  $C'B''$  are concurrent or parallel. We leave it to the reader to verify that the coordinates  $(-a+b+c, a-b+c, a+b-c)$  satisfy the three equations. Hence, the three lines are concurrent. The barycentric coordinates of the Nagel point are  $(-a+b+c, a-b+c, a+b-c)$  (see [Kim]) so we can conclude that the intersection point of these three lines is the Nagel point.  $\square$

## References

[Hoe07] Larry Hoehn. A new characterization of the nagel point. *Missouri J. Math. Sci.*, 19:45–48, 2007.

[Kim] C. Kimberling. Encyclopaedia of triangle centers, <https://faculty.evansville.edu/ck6/encyclopedia/etc.html>.