Elementary construction of the Nagel point

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Abstract

In this short note, we give an easy contruction of the Nagel point of a triangle.

1 Introduction

If you continue the sides of a triangle beyond every vertex at the distances equaling to the length of the opposite side, the resulting six points lie on a circle, which is called Conway's circle (figure 1).

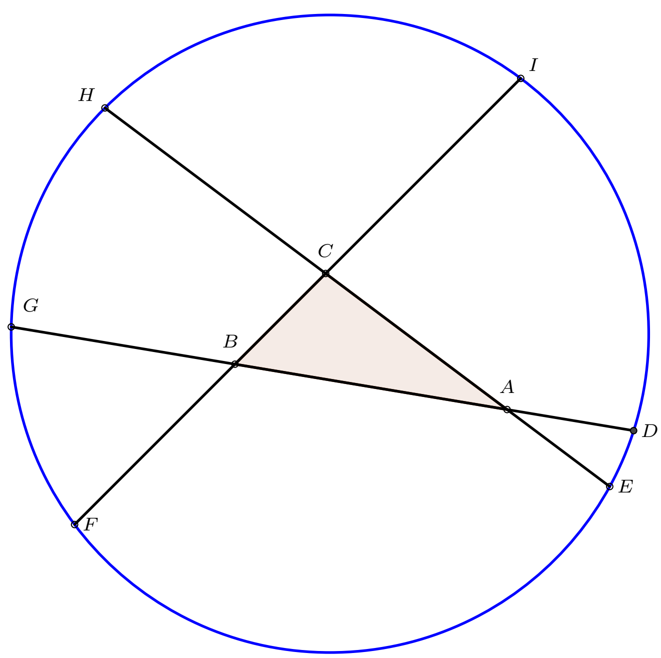

Figure 1. The Conway circle

In this note, we investigate a similar construction which charaterizes the Nagel point of a triangle ABC. Starting from point A, we transport the length of the opposite edge on each half-line starting from A. We obtain two points A' and A'' which satisfy $\overrightarrow{AA'} = \frac{a}{c}\overrightarrow{AB}$ and $\overrightarrow{AA''} = \frac{a}{b}\overrightarrow{AC}$ (figure 2). Repeating this construction with vertices B and C, we have four more points B', B'', C' and C'' which satisfy $\overrightarrow{BB'} = \frac{b}{a}\overrightarrow{BC}$, $\overrightarrow{BB''} = \frac{b}{c}\overrightarrow{BA}$, $\overrightarrow{CC'} = \frac{c}{b}\overrightarrow{CA}$ and $\overrightarrow{CC''} = \frac{c}{a}\overrightarrow{CB}$.

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By construction,

$$AA' = a, \quad BB' = b, \quad CC' = c,$$
$$AA'' = a, \quad BB'' = b, \quad CC'' = c.$$

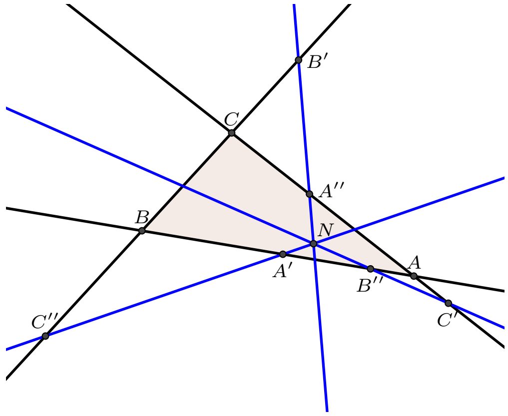

Figure 2. Easy construction of the Nagel point

Using barycentric coordinates, we prove that the three lines $A''B'$, $A'C''$ and $C'B''$ are concurrent (figure 2) when ABC is a scalene triangle (a triangle with edges of different lengths).

Moreover, it turns out that the intersection point is the Nagel point of the triangle ABC.

Figure 3 gives the original construction of the Nagel point which is defined by the following result "the lines joining the points of contact of an excircle with the sides of a triangle to the vertices opposite the respective sides, are concurrent".

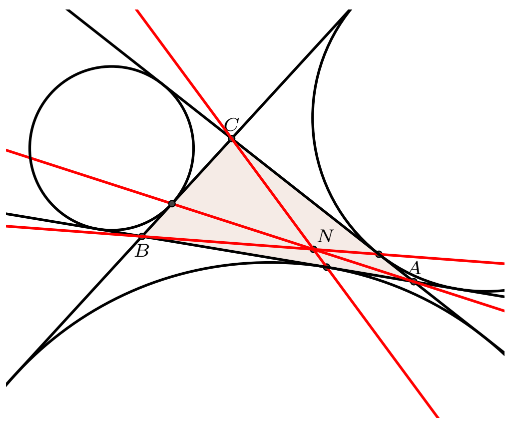

Figure 3. Original construction of the Nagel Point

A construction of the Nagel point requiring only the incircle of the triangle can be found in Hoehn [Hoe07].

2 Main result

Our theorem will use basic facts of barycentric coordinates in the plane. In particular, a line, in barycentric coordinates, is given by a linear homogeneous equation $\alpha X + \beta Y + \gamma W = 0$ for some coefficients α, β, and γ not all of which are 0.

Theorem 1. *Let ABC be a scalene triangle and the six points A', A'', B', B'', C' and C'' given by the relations:*

$$\overrightarrow{AA'} = \frac{a}{c}\overrightarrow{AB}, \quad \overrightarrow{AA''} = \frac{a}{b}\overrightarrow{AC}, \quad \overrightarrow{BB'} = \frac{b}{a}\overrightarrow{BC},$$

$$\overrightarrow{BB''} = \frac{b}{c}\overrightarrow{BA}, \quad \overrightarrow{CC'} = \frac{c}{b}\overrightarrow{CA}, \quad \overrightarrow{CC''} = \frac{c}{a}\overrightarrow{CB}.$$

Then, the three lines $A''B'$, $A'C''$ and $C'B''$ are concurrent and the intersection point is the Nagel point.

Proof. The condition $\overrightarrow{AA'} = \frac{a}{c}\overrightarrow{AB}$ implies that A' has barycentric coordinates $(a-c, -a, 0)$. We also deduce that $A''(a-b, 0, -a)$, $B'(0, b-a, b)$, $B''(-b, b-c, 0)$, $C'(-c, 0, c-b)$ and $C''(0, c, c-a)$.

Let $\alpha X + \beta Y + \gamma Z = 0$ be an equation of the line $A''B'$. Fix $\gamma = 1$. The barycentric coordinates of A'' and B' satisfy this equation and we obtain $\beta = \frac{b}{b-a}$ and $\alpha = \frac{a}{a-b}$. Multiplying by $a-b$, we deduce that $aX - bY + (a-b)Z = 0$ is an equation of the line $A''B'$.

We can now proceed analogously for lines $A'C''$ and $C'B''$. It follows immediately that $aX + (a-c)Y - cZ = 0$ is an equation of $A'C''$ and $(b-c)X + bY - cZ = 0$ is an equation of $C'B'$.

Easy computations show that

$$\begin{vmatrix} a & -b & a-b \\ a & a-c & -c \\ b-c & b & -c \end{vmatrix} = 0.$$

This determinant is 0 hence the three lines $A''B'$, $A'C''$ and $C'B''$ are concurrent or parallel. We leave it to the reader to verify that the coordinates $(-a+b+c, a-b+c, a+b-c)$ satisfy the three equations. Hence, the three lines are concurrent. The barycentric coordinates of the Nagel point are $(-a+b+c, a-b+c, a+b-c)$ (see [Kim]) so we can conclude that the intersection point of these three lines is the Nagel point. □

References

[Hoe07] Larry Hoehn. A new characterization of the nagel point. *Missouri J. Math. Sci.*, 19:45–48, 2007.

[Kim] C. Kimberling. Encyclopaedia of triangle centers, https://faculty.evansville.edu/ck6/encyclopedia/etc.html.