

The Nonlinear Filter Model of Stream Cipher Redivivus

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Abstract

The nonlinear filter model is an old and well understood approach to the design of secure stream ciphers. Extensive research over several decades has shown how to attack stream ciphers based on this model and has identified the security properties required of the Boolean function used as the filtering function to resist such attacks. This led to the problem of constructing Boolean functions which provide adequate security *and* at the same time are efficient to implement. Unfortunately, over the last two decades no good solutions to this problem appeared in the literature. The lack of good solutions has effectively led to nonlinear filter model becoming more or less obsolete. This is a big loss to the cryptographic design toolkit, since the great advantages of the nonlinear filter model are its simplicity, well understood security and the potential to provide low cost solutions for hardware oriented stream ciphers. In this paper we construct balanced functions on an odd number $n \geq 5$ of variables with the following provable properties: linear bias equal to $2^{-\lfloor n/2 \rfloor - 1}$, algebraic degree equal to $2^{\lfloor \log_2 \lfloor n/2 \rfloor \rfloor}$, algebraic immunity at least $\lceil (n-1)/4 \rceil$, fast algebraic immunity at least $1 + \lceil (n-1)/4 \rceil$, *and* can be implemented using $O(n)$ NAND gates. The functions are obtained from a simple modification of the well known class of Maiorana-McFarland bent functions. By appropriately choosing n and the length L of the linear feedback shift register, we show that it is possible to obtain examples of stream ciphers which are *provably* κ -bit secure against well known classes of attacks for various values of κ . We provide concrete proposals for $\kappa = 80, 128, 160, 192, 224$ and 256 . For the 80-bit, 128-bit, and the 256-bit security levels, the circuits for the corresponding stream ciphers require about 1743.5, 2771.5, and 5607.5 NAND gates respectively. For the 80-bit and the 128-bit security levels, the gate count estimates compare quite well to the famous ciphers Trivium and Grain-128a respectively, while for the 256-bit security level, we do not know of any other stream cipher design which has such a low gate count.

Keywords: Boolean function, stream cipher, nonlinearity, algebraic immunity, efficient implementation.

1 Introduction

The nonlinear filter model of stream ciphers is several decades old; one may note that the model was extensively discussed in the book by Rueppel [37] which was published in the mid-1980s. The nonlinear filter model consists of two components, namely a linear feedback shift register (LFSR) and a Boolean

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function f which is applied to a subset of the bits of the LFSR. At each clock, f is applied to the present state of the LFSR to produce a single keystream bit and simultaneously the LFSR also moves to the next state. LFSRs are very efficient to implement, particularly in hardware. So the implementation efficiency of the nonlinear filter model is essentially determined by the efficiency of implementation of f .

Extensive research carried out over the last few decades has shown several approaches to crypt-analysing the nonlinear filter model of stream ciphers. The initial line of attack was based upon determining the linear complexity of the produced keystream. For an LFSR of length L and a Boolean function of algebraic degree d , under certain reasonable and easy-to-ensure conditions, the linear complexity of the keystream is known to be at least $\binom{L}{d}$ (see [37]). Using large enough values of L and d , linear complexity based attacks can be made infeasible. The second phase of attacks consisted of various kinds of (fast) correlation attacks. Starting with the first such attack in 1985 [41], a long line of papers [35, 23, 17, 28, 29, 15, 12, 30, 11, 16, 42, 45, 44, 31, 32] explored various avenues for mounting correlation attacks. Surveys of some of the older attacks appear in [9, 10, 33, 2]. The resistance to correlation attacks is mainly determined by the linear bias ε of the Boolean function f . The linear bias is determined by the nonlinearity of the function f ; the higher the nonlinearity, the lower the linear bias. The third phase of attacks started in 2003 with the publication of the algebraic attack [19] and was soon followed by the publication of the fast algebraic attack [18]. Resistance to these attacks requires the function f to possess high (fast) algebraic immunity.

The various attacks mentioned above have posed the following design challenge for a Boolean function to be used in the nonlinear filter model of stream ciphers. Construct balanced Boolean functions which achieve a good combination of nonlinearity and algebraic resistance *and* are also very efficient to implement. Unfortunately, since the time algebraic attacks were proposed in the early-2000s, no good solutions to the design challenge for Boolean functions have appeared in the literature. A consequence of not being able to find good solutions to the design challenge is that the nonlinear filter model of stream ciphers became obsolete. This is somewhat unfortunate since the model is very old, well studied with well understood security, and the potential to provide low gate count solutions in hardware.

We note that even though the nonlinear filter model became obsolete, use of LFSRs in the design of stream cipher has continued for both hardware and software oriented proposals (see for example [26, 4, 22]). Instead of using a Boolean function, such designs typically use a nonlinear finite state machine to filter the output of the LFSR. Some ciphers such as [8, 3] have gone further and replaced the LFSR with one or more nonlinear feedback shift registers (NFSRs).

In this paper, we revisit the above mentioned design problem for Boolean functions towards the goal of reviving the nonlinear filter model. Bent functions [36] are a very well studied class of Boolean functions. They exist for even number of variables and provide the highest possible nonlinearity. However, their use in cryptography is not clear. The introduction of Chapter 6 of [13] summarises the state-of-the-art as follows: “we do not know an efficient construction using bent functions which would provide Boolean functions having all the necessary features for being used in stream ciphers.” Even though this is discouraging, nonetheless, we start our construction from bent functions.

The well known Maiorana-McFarland class of bent functions is defined as follows. For $m \geq 1$, let \mathbf{X} and \mathbf{Y} be two vectors of m variables. Then a $2m$ -variable Maiorana-McFarland bent function is defined to be $\langle \pi(\mathbf{X}), \mathbf{Y} \rangle \oplus h(\mathbf{X})$, where π is a bijection from m -bit strings to m -bit strings and h is any m -variable Boolean function. It is well-known that the nonlinearity of Maiorana-McFarland functions does not depend on the choice of h nor on that of the permutation π . Bad choices of π and h (for example, π to be the identity permutation and h to be a constant function) provide functions whose algebraic immunity is low. On the bright side, we observe that it may be possible to improve the algebraic

resistance by properly choosing π and h . Assume that π is chosen to be the identity permutation. It is known [20] that the majority function possesses the best possible algebraic immunity. Motivated by this fact, we choose h to be the majority function on m variables. We *prove* that the resulting bent function on $2m$ variables has algebraic immunity at least $\lceil m/2 \rceil$, and hence fast algebraic immunity at least $1 + \lceil m/2 \rceil$. On the implementation aspect, we show that the majority function can be computed using $O(m)$ NAND gates. So the obtained bent function has maximum nonlinearity, good algebraic immunity and at the same time is quite efficient to implement. The algebraic immunity is not the maximum possible, but due to the implementation efficiency, it is possible to increase the number of variables to achieve the desired level of algebraic resistance. One small problem is that a bent function is not balanced. This problem is easily rectified by XORing a new variable to the bent function, obtaining a function on an odd number of variables. This modification does not affect the linear bias or the algebraic resistance and requires only one extra XOR gate for implementation. A novelty of our work is *the observation and the proof* that choosing h to be the majority function improves the algebraic immunity. While Maiorana-McFarland bent functions have been extensively studied in the literature, this simple observation has escaped the notice of earlier researchers.

We perform a detailed concrete security analysis of some of the well known attacks on the nonlinear filter model. As the outcome of this analysis, for various security levels, we provide concrete proposals for stream ciphers based on the nonlinear filter model using the Boolean functions described above as the filtering functions. A strong point in favour of these proposals is that at the appropriate security levels they provide *provable* assurance against well known classes of attacks. Further, we provide concrete gate count estimates for the entire circuit to implement the stream ciphers. For the 80-bit, 128-bit, 160-bit, 192-bit, 224-bit and the 256-bit security levels, we obtain gate count estimates of 1743.5, 2771.5, 3520.5, 4188.5, 4854.5, and 5607.5 NAND gates respectively. The gate count estimates for the 80-bit and the 128-bit security levels compare quite well¹ with famous ciphers such as Trivium [8] and Grain-128a [1] which offer 80-bit and 128-bit security respectively. For the other security levels, we are not aware of other stream ciphers which have such low gate counts. So our revival of the nonlinear filter model of stream ciphers leads to concrete proposals which offer a combination of both provable security against well known classes of attacks at a desired level of security and also low gate count.

We note that there are some old (prior to the advent of algebraic attacks) works [39, 40, 24] on efficient implementation of Boolean functions on a large number of variables targeted towards the nonlinear combiner model of stream ciphers. In the context of the nonlinear filter model, however, we are not aware of any previous work on the design and implementation of Boolean functions on a large number of variables. So this aspect is also new to the present work.

Comparison between the nonlinear filter model and some modern stream ciphers. Well known stream ciphers such as Trivium [8], Mickey [3], Grain-128a [1], SNOW [22] and Sosemanuk [4], use novel and ingenious ideas. Nonetheless, these are standalone designs. The nonlinear filter model, on the other hand, is a *model* for stream cipher design. Due to the simplicity of the nonlinear filter model, provable properties of the filtering function can be translated into provable resistance of the stream cipher against well known classes of attacks. In particular, for the proposals that we put forward, the provable linear bias of the filtering function translates to provable protection against a large class of fast correlation attacks, and the provable lower bound on the (fast) algebraic immunity translates to provable resistance against algebraic attacks. Stream ciphers based on either nonlinear finite state machines, or

¹Following [1] we estimated 8 NAND gates for a flip-flop, whereas Trivium estimated 12 NAND gates for a flip-flop. Using 12 NAND gates for a flip-flop, our proposal at the 80-bit security level requires about 2395.5 NAND gates, whereas Trivium requires about 3488 NAND gates.

nonlinear feedback shift registers do not enjoy this advantage, i.e. for such stream ciphers it is very hard to obtain provable guarantees against various well known types of attacks. As an example, our proposal at the 128-bit security level generates keystream for which the best linear approximation has *provable* correlation of 2^{-60} , while at the same security level, for Sosemanuk [4] the best known [31] linear approximation has correlation $2^{-20.84}$, and it is not known whether there are approximations with higher correlations. Similarly, the time complexities of fast algebraic attacks against Trivium, Mickey, Grain-128a, SNOW and Sosemanuk are not known. It is believed that these stream ciphers can withstand fast algebraic attacks. In contrast, we are able to *prove* that at the appropriate security levels, the fast algebraic attack is ineffective against the new nonlinear filter model based stream cipher proposals that we put forward.

One advantage of using LFSRs is that it is possible to provably ensure that the LFSR has a maximum period. For stream ciphers based on NFSRs, such as Trivium and Mickey, such provable assurance is not available. For stream ciphers which use a combination of LFSR and NFSR such as Grain-128a, it is possible to mount a divide-and-conquer attack. For example the attack in [42] on Grain-128a finds the state of the LFSR independently of the state of the NFSR. (In some ways this is reminiscent of the attack by Siegenthaler [41] on the nonlinear combiner model.) So even though Grain-128a uses a 256-bit state, due to the divide-and-conquer strategy the full protection of the large state is not achieved. On the other hand, for the nonlinear filter model, there is no known way to mount a divide-and-conquer attack. Our proposal at the 128-bit security level uses a 257-bit LFSR, and there is no way to estimate half the state of the LFSR without involving the other half.

Lastly, we note that the nonlinear filter model provides a scalable design, while it is not clear how to scale the ideas behind standalone designs such as those in [8, 3, 1, 22, 4]. By properly choosing the LFSR and the filtering function, the nonlinear filter model can be instantiated to various security levels. This provides a *family* of stream ciphers rather than a single stream cipher. The scalability of the design makes it easier to target different security levels and also to ramp up parameters in response to improvements of known attacks. For example, at the 128-bit security level, the complexity of the fast algebraic attack on the proposal that we put forward is more than $2^{130.12}$, and the correlation of the best linear approximation of the keystream is 2^{-60} . By increasing the gate count by about 47 gates, it is possible to ensure that the complexity of the fast algebraic attack is more than $2^{135.83}$ and the best linear approximation of the keystream is 2^{-64} .

The paper is organised as follows. In Section 2 we describe the preliminaries. The Boolean function construction is described in Section 3. Section 4 performs the concrete security analysis and Section 5 provides the gate count estimates. Finally, Section 6 concludes the paper.

2 Preliminaries

This section provides the notation and the basic definitions. For details on Boolean functions we refer to [13].

By $\#S$ we will denote the cardinality of a finite set S . The finite field of two elements will be denoted by \mathbb{F}_2 , and for a positive integer n , \mathbb{F}_2^n will denote the vector space of dimension n over \mathbb{F}_2 . By \oplus , we will denote the addition operator over both \mathbb{F}_2 and \mathbb{F}_2^n . An element of \mathbb{F}_2^n will be considered to be an n -bit binary string.

For $n \geq 0$, let $\mathbf{x} = (x_1, \dots, x_n)$ be an n -bit binary string. The weight of \mathbf{x} is $\text{wt}(\mathbf{x}) = \#\{i : x_i = 1\}$. By $\mathbf{0}_n$ and $\mathbf{1}_n$ we will denote the all-zero and all-one strings of length n respectively. Let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ be two n -bit strings. The distance between \mathbf{x} and \mathbf{y} is $d(\mathbf{x}, \mathbf{y}) = \#\{i : x_i \neq y_i\}$; the inner product of \mathbf{x} and \mathbf{y} is $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 \oplus \dots \oplus x_ny_n$; and we define $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i$ for $i = 1, \dots, n$.

An n -variable Boolean function f is a map $f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$. The weight of f is $\text{wt}(f) = \#\{\mathbf{x} \in \mathbb{F}_2^n : f(\mathbf{x}) = 1\}$; f is said to be *balanced* if $\text{wt}(f) = 2^{n-1}$.

Algebraic normal form. Let f be an n -variable function. The *algebraic normal form (ANF) representation* of f is the following: $f(X_1, \dots, X_n) = \bigoplus_{\alpha \in \mathbb{F}_2^n} a_\alpha \mathbf{X}^\alpha$, where $\mathbf{X} = (X_1, \dots, X_n)$; for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{F}_2^n$, \mathbf{X}^α denotes $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$; and $a_\alpha \in \mathbb{F}_2$. The (algebraic) degree of f is $\text{deg}(f) = \max\{\text{wt}(\alpha) : a_\alpha = 1\}$. Functions of degree at most 1 are said to be affine functions. Affine functions with $a_{\mathbf{0}_n} = 0$ are said to be linear functions. It is known that if f is balanced, then $\text{deg}(f) \leq n - 1$.

We have the following relations between the coefficients a_α in the ANF of f and the values of f (see for example Section 2.2 of [13]). For $\mathbf{x}, \alpha \in \mathbb{F}_2^n$,

$$f(\mathbf{x}) = \bigoplus_{\beta \leq \mathbf{x}} a_\beta \quad \text{and} \quad a_\alpha = \bigoplus_{\mathbf{z} \leq \alpha} f(\mathbf{z}). \quad (1)$$

Nonlinearity and Walsh transform. For two n -variable functions f and g , the distance between them is $d(f, g) = \#\{\mathbf{x} \in \mathbb{F}_2^n : f(\mathbf{x}) \neq g(\mathbf{x})\}$. The *nonlinearity* of an n -variable function f is $\text{nl}(f) = \min d(f, g)$, where the minimum is over all n -variable affine functions g .

The Walsh transform of an n -variable function f is the map $W_f : \mathbb{F}_2^n \rightarrow \mathbb{Z}$, where for $\alpha \in \mathbb{F}_2^n$, $W_f(\alpha) = \sum_{\mathbf{x} \in \mathbb{F}_2^n} (-1)^{f(\mathbf{x}) \oplus (\alpha, \mathbf{x})}$. The nonlinearity of a function f is given by its Walsh transform as follows: $\text{nl}(f) = 2^{n-1} - \frac{1}{2} \max_{\alpha \in \mathbb{F}_2^n} |W_f(\alpha)|$.

A function f such that $W_f(\alpha) = \pm 2^{n/2}$ for all $\alpha \in \mathbb{F}_2^n$ is said to be a bent function [36]. Clearly such functions can exist only if n is even. The nonlinearity of an n -variable bent function is $2^{n-1} - 2^{n/2-1}$ and this is the maximum nonlinearity that can be attained by n -variable functions.

A function is said to be plateaued if its Walsh transform takes only the values $0, \pm v$, for non-zero v . We define the linear bias of an n -variable Boolean function f to be $\text{LB}(f) = 1/2 - \text{nl}(f)/2^n$.

Algebraic resistance. Let f be an n -variable function. The *algebraic immunity* of f is defined [19, 34] as follows: $\text{AI}(f) = \min_{g \neq 0} \{\text{deg}(g) : \text{either } gf = 0, \text{ or } g(f \oplus 1) = 0\}$. It is known [19] that $\text{AI}(f) \leq \lceil n/2 \rceil$.

The fast algebraic attack (FAA) was introduced in [18]. The idea of the attack is based on the following observation. Let f be an n -variable function and suppose g is another n -variable function of degree e such that gf has degree d . If both e and d are small, then f is susceptible to an FAA. Given f , and for e and d satisfying $e + d \geq n$, it is known [18] that there exists functions g and h with $\text{deg}(g) = e$ and $\text{deg}(h) \leq d$ such that $gf = h$. Based on this result, we provide the following definition. For each $e \in \{1, \dots, \text{AI}(f) - 1\}$, let $d \leq n - 1 - e$ be the maximum integer such that there do not exist n -variable functions g and h with $\text{deg}(g) = e$, $\text{deg}(h) = d$ and $gf = h$. We call the list of all such pairs (e, d) to be the *FAA-profile* of f .

The *fast algebraic immunity (FAI)* of f is a combined measure of resistance to both algebraic and fast algebraic attacks: $\text{FAI}(f) = \min(2\text{AI}(f), \min_{g \neq 0} \{\text{deg}(g) + \text{deg}(fg) : 1 \leq \text{deg}(g) < \text{AI}(f)\})$. So $\text{FAI}(f) = \min(2\text{AI}(f), \min\{e + d + 1\})$, where the second minimum is taken over all pairs (e, d) in the FAA-profile of f . Further, for any function f , $1 + \text{AI}(f) \leq \text{FAI}(f) \leq 2\text{AI}(f)$.

Majority function. For $n \geq 1$, let $\text{Maj} : \{0, 1\}^n \rightarrow \{0, 1\}$ be the majority function defined in the following manner. For $\mathbf{x} \in \{0, 1\}^n$, $\text{Maj}(\mathbf{x}) = 1$ if and only if $\text{wt}(\mathbf{x}) \geq \lceil n/2 \rceil$. It is known [20] that Maj_n has the maximum possible AI of $\lceil n/2 \rceil$.

3 Construction from Maiorana-McFarland Bent Functions

The Maiorana-McFarland class of bent functions is defined as follows. For $m \geq 1$, let $\pi : \{0, 1\}^m \rightarrow \{0, 1\}^m$ be a bijection and $h : \{0, 1\}^m \rightarrow \{0, 1\}$ be a Boolean function. Let π_1, \dots, π_m be the coordinate functions of π . Let $\mathbf{X} = (X_1, \dots, X_m)$ and $\mathbf{Y} = (Y_1, \dots, Y_m)$. For $m \geq 1$, MM_{2m} is defined to be the following.

$$\begin{aligned} \text{MM}_{2m}(\mathbf{X}, \mathbf{Y}) &= \langle \pi(\mathbf{X}), \mathbf{Y} \rangle \oplus h(\mathbf{X}) \\ &= \pi_1(\mathbf{X})Y_1 \oplus \dots \oplus \pi_m(\mathbf{X})Y_m \oplus h(\mathbf{X}). \end{aligned} \quad (2)$$

Since MM_{2m} is bent, $\text{nl}(\text{MM}_{2m}) = 2^{2m-1} - 2^{m-1}$, and $\text{LB}(\text{MM}_{2m}) = 2^{-m-1}$. Note that the nonlinearity of MM_n does not depend on the choices of the bijection π and the function h . The degree of MM_{2m} is given by the following result.

Proposition 1 For $m \geq 1$, $\deg(\text{MM}_{2m}) = \max(\deg(\pi_1) + 1, \dots, \deg(\pi_m) + 1, \deg(h))$.

To the best of our knowledge the following result on the algebraic immunity of MM_{2m} is new.

Theorem 1 Suppose $m \geq 1$. There is an $\omega^* \in \mathbb{F}_2^m$ such that

$$\text{Al}(\text{MM}_{2m}) \geq \text{wt}(\omega^*) + \text{Al}(\langle \omega^*, \pi(\mathbf{X}) \rangle \oplus h(\mathbf{X})).$$

Suppose that π is a bit permutation, i.e. $\pi(X_1, \dots, X_m) = (X_{\rho(1)}, \dots, X_{\rho(m)})$, for some permutation ρ of $\{1, \dots, m\}$. Then

$$\text{Al}(\text{MM}_{2m}) \geq \text{Al}(h).$$

Proof: Suppose $g(\mathbf{X}, \mathbf{Y})$ is an annihilator for $\text{MM}_{2m}(\mathbf{X}, \mathbf{Y})$. Recall that for $\omega = (\omega_1, \dots, \omega_m) \in \mathbb{F}_2^m$, by \mathbf{Y}^ω we denote the monomial $Y_1^{\omega_1} \dots Y_m^{\omega_m}$. Using this notation, we write $g(\mathbf{X}, \mathbf{Y}) = \bigoplus_{\omega \in \mathbb{F}_2^m} \mathbf{Y}^\omega g_\omega(\mathbf{X})$, for some functions $g_\omega(\mathbf{X})$'s. We have

$$\begin{aligned} 0 &= g(\mathbf{X}, \mathbf{Y})\text{MM}_{2m}(\mathbf{X}, \mathbf{Y}) \\ &= \left(\bigoplus_{\omega \in \mathbb{F}_2^m} \mathbf{Y}^\omega g_\omega(\mathbf{X}) \right) (\pi_1(\mathbf{X})Y_1 \oplus \dots \oplus \pi_m(\mathbf{X})Y_m \oplus h(\mathbf{X})). \end{aligned} \quad (3)$$

Since the right hand side of (3) is equal to 0, for $\omega \in \mathbb{F}_2^m$, the coefficient of \mathbf{Y}^ω in the expansion on the right hand side of (3) must be equal to 0. Since $g(\mathbf{X}, \mathbf{Y}) \neq 0$, let $w \geq 0$ be the minimum integer such that there is an ω^* with $\text{wt}(\omega^*) = w$ and $g_{\omega^*}(\mathbf{X}) \neq 0$. In (3), equating the coefficient of \mathbf{Y}^{ω^*} to 0, we have

$$\begin{aligned} 0 &= g_{\omega^*}(\mathbf{X}) \left(h(\mathbf{X}) \oplus \left(\bigoplus_{i \in \text{supp}(\omega^*)} \pi_i(\mathbf{X}) \right) \right) \\ &= g_{\omega^*}(\mathbf{X}) (\langle \omega^*, \pi(\mathbf{X}) \rangle \oplus h(\mathbf{X})). \end{aligned}$$

So $g_{\omega^*}(\mathbf{X})$ is an annihilator for $\langle \omega^*, \pi(\mathbf{X}) \rangle \oplus h(\mathbf{X})$. Consequently, $\deg(g) \geq \text{wt}(\omega^*) + \deg(g_{\omega^*}) \geq \text{wt}(\omega^*) + \text{Al}(\langle \omega^*, \pi(\mathbf{X}) \rangle \oplus h(\mathbf{X}))$.

If, on the other hand, $g(\mathbf{X}, \mathbf{Y})$ is an annihilator for $1 \oplus \text{MM}_{2m}(\mathbf{X}, \mathbf{Y})$, then a similar argument shows that $g_{\omega^*}(\mathbf{X})$ is an annihilator for $\langle \omega^*, \pi(\mathbf{X}) \rangle \oplus 1 \oplus h(\mathbf{X})$, and again we have $\deg(g) \geq \text{wt}(\omega^*) + \text{Al}(\langle \omega^*, \pi(\mathbf{X}) \rangle \oplus h(\mathbf{X}))$.

Suppose now that π is a bit permutation. Then $\ell(\mathbf{X}) = \langle \omega^*, \pi(\mathbf{X}) \rangle$ is a linear function. If $w = 0$, then $\langle \omega^*, \pi(\mathbf{X}) \rangle \oplus h(\mathbf{X}) = h(\mathbf{X})$, and so we have the result. So suppose $w > 0$. From Lemma 1 of [14], we have $\text{Al}(\ell(\mathbf{X}) \oplus h(\mathbf{X})) \geq \text{Al}(h(\mathbf{X})) - 1$. So

$$\begin{aligned} \text{Al}(\text{MM}_{2m}) &\geq \text{wt}(\omega^*) + \text{Al}(\ell(\mathbf{X}) \oplus h(\mathbf{X})) \\ &\geq w + \text{Al}(h(\mathbf{X})) - 1 \geq \text{Al}(h(\mathbf{X})). \end{aligned}$$

□

Extension to MM_n , for odd n . We consider a very well known extension of MM_{2m} to odd number of variables.

$$\begin{aligned} \text{MM}_1(W) &= W, \\ \text{MM}_{2m+1}(W, \mathbf{X}, \mathbf{Y}) &= W \oplus \text{MM}_{2m}(\mathbf{X}, \mathbf{Y}), \quad \text{for } m \geq 1. \end{aligned} \tag{4}$$

The following result states the properties of MM_{2m+1} .

Proposition 2 For $m \geq 1$, MM_{2m+1} is balanced, and

1. $\text{nl}(\text{MM}_{2m+1}) = 2^{2m} - 2^m$. In particular, $\text{LB}(\text{MM}_{2m+1}) = \text{LB}(\text{MM}_{2m}) = 2^{-(m+1)}$.
2. $\text{deg}(\text{MM}_{2m+1}) = \text{deg}(\text{MM}_{2m})$.
3. $\text{Al}(\text{MM}_{2m}) \leq \text{Al}(\text{MM}_{2m+1}) \leq 1 + \text{Al}(\text{MM}_{2m})$.

Proof: The first point is well known. The second point is immediate from the definition of MM_{2m+1} . We provide the proof of the third point. For brevity, let us write $\mathbf{Z} = (\mathbf{X}, \mathbf{Y})$. Clearly if $g(\mathbf{Z})$ is an annihilator for MM_{2m} (resp. $1 \oplus \text{MM}_{2m}$), then $(1 \oplus W)g(\mathbf{Z})$ is an annihilator for MM_{2m+1} (resp. $1 \oplus \text{MM}_{2m+1}$). This shows the upper bound. Next we consider the lower bound. Suppose $g(W, \mathbf{Z}) \neq 0$ is an annihilator for $\text{MM}_{2m+1}(W, \mathbf{Z})$. We write $g(W, \mathbf{Z})$ as $g(W, \mathbf{Z}) = Wg_1(\mathbf{Z}) + g_0(\mathbf{Z})$. Noting that $\text{MM}_{2m+1}(W, \mathbf{Z}) = W \oplus \text{MM}_{2m}(\mathbf{X}, \mathbf{Y})$, we obtain

$$\begin{aligned} 0 &= g(W, \mathbf{Z})\text{MM}_{2m+1}(W, \mathbf{Z}) \\ &= g_0(\mathbf{Z})\text{MM}_{2m}(\mathbf{Z}) \oplus W(g_0(\mathbf{Z}) \oplus g_1(\mathbf{Z})(1 \oplus \text{MM}_{2m}(\mathbf{Z}))). \end{aligned}$$

So $g_0(\mathbf{Z})\text{MM}_{2m}(\mathbf{Z}) = 0$ and $g_0(\mathbf{Z}) \oplus g_1(\mathbf{Z})(1 \oplus \text{MM}_{2m}(\mathbf{Z})) = 0$. If g_0 is non-zero, then g_0 is an annihilator for MM_{2m} and so $\text{deg}(g) \geq \text{deg}(g_0) \geq \text{Al}(\text{MM}_{2m})$. If $g_0 = 0$, then since $g \neq 0$, it follows that $g_1 \neq 0$. In this case, g_1 is an annihilator for $1 \oplus \text{MM}_{2m}$, and so $\text{deg}(g) \geq 1 + \text{deg}(g_1) \geq 1 + \text{Al}(\text{MM}_{2m})$. Consequently, in both cases $\text{deg}(g) \geq \text{Al}(\text{MM}_{2m})$.

On the other hand, if $g(W, \mathbf{Z}) \neq 0$ is an annihilator for $1 \oplus \text{MM}_{2m+1}(W, \mathbf{Z})$, then noting that $W(1 \oplus W) = 0$, we obtain $g_0(\mathbf{Z})(1 \oplus \text{MM}_{2m}(\mathbf{Z})) = 0$ and $g_0(\mathbf{Z}) \oplus g_1(\mathbf{Z})\text{MM}_{2m}(\mathbf{Z}) = 0$. If $g_0 \neq 0$, then g_0 is an annihilator for $1 \oplus \text{MM}_{2m}$, and if $g_0 = 0$, then g_1 is an annihilator for MM_{2m} . So again we have $\text{deg}(g) \geq \text{Al}(\text{MM}_{2m})$. □

From Theorem 1, choosing π to be the identity function results in the AI of MM_{2m} to be lower bounded by the AI of h . Motivated by this result, we make the following concrete choices.

Concrete choice of π in MM_{2m} : Choose the m -bit to m -bit permutation π in the construction of MM_{2m} given by (2) to be the identity permutation.

Concrete choice of h in MM_{2m} : Choose the m -variable function h in the construction of MM_{2m} given by (2) to be Maj_m , which is the m -variable majority function.

Our choice of Maj_m for h is motivated by the fact that the majority function has the maximum possible algebraic immunity. There are other functions which achieve maximum algebraic immunity [14] and these could also be chosen to instantiate h . Our choice of Maj_m is the simplest choice of a function achieving maximum algebraic immunity.

Remark 1 *Theorem 1 assures us of the lower bound on MM_{2m} when π is any bit permutation. Our choice of π as the identity permutation is the simplest choice. In fact, all the $m!$ bit permutations π result in functions with the same level of security and similar implementation efficiency.*

By $(\text{Maj}, \text{id})\text{-MM}_{2m}$ we denote the function obtained by instantiating the definition of MM_{2m} given by (2) with $h = \text{Maj}_m$ and $\pi = \text{id}$. Similarly, by $(\text{Maj}, \text{id})\text{-MM}_{2m+1}$, we denote the function obtained from $(\text{Maj}, \text{id})\text{-MM}_{2m}$ using (4).

Proposition 3 *For $n \geq 4$, the degree of $(\text{Maj}, \text{id})\text{-MM}_n$ is $2^{\lceil \log_2 \lfloor n/2 \rfloor \rceil}$.*

Proof: The degree of Maj_m is $2^{\lceil \log_2 m \rceil}$ (see [20]). Applying Proposition 1, we obtain the required result for even n . For odd n , the result follows from the second point of Proposition 2. \square

Proposition 4 *For $n \geq 4$, if n is even, then $\text{AI}((\text{Maj}, \text{id})\text{-MM}_n) \geq \lceil n/4 \rceil$, and if n is odd then $\text{AI}((\text{Maj}, \text{id})\text{-MM}_n) \geq \lceil (n-1)/4 \rceil$.*

Proof: Suppose $n = 2m$ is even. From Theorem 1 we have $\text{AI}((\text{Maj}, \text{id})\text{-MM}_{2m}) \geq \text{AI}(\text{Maj}_m)$. Since $\text{AI}(\text{Maj}_m) = \lceil m/2 \rceil$, we have the result. For odd $n = 2m + 1$, from Proposition 2, we have $\text{AI}((\text{Maj}, \text{id})\text{-MM}_{2m+1}) \geq \text{AI}((\text{Maj}, \text{id})\text{-MM}_{2m})$, which shows the result. \square

We computed the algebraic immunity of $(\text{Maj}, \text{id})\text{-MM}_n$ for small values of n and observed that for $n \geq 4$, $\text{AI}((\text{Maj}, \text{id})\text{-MM}_n) = 1 + \lfloor n/4 \rfloor$. This suggests that the lower bound in Proposition 4 is exact if $n \not\equiv 0 \pmod{4}$ and if $n \equiv 0 \pmod{4}$, the actual algebraic immunity is one more than the lower bound. As mentioned in Section 2, for any Boolean function f , $\text{FAI}(f) \geq 1 + \text{AI}(f)$. Our experiments suggest that for $(\text{Maj}, \text{id})\text{-MM}_n$, the fast algebraic immunity is actually one more than the algebraic immunity.

So $(\text{Maj}, \text{id})\text{-MM}_n$ does not have the best possible algebraic immunity. On the other hand, it can be implemented using $O(m)$ gates (as we show in Section 5). So it is possible to increase the number of variables to achieve the desired level of algebraic resistance without increasing the circuit size too much.

4 Concrete Proposals

From Proposition 2 we have MM_{2m+1} is balanced, and the cryptographic resistance provided by MM_{2m+1} is very similar to that provided by MM_{2m} . In terms of implementation efficiency, MM_{2m+1} requires one extra XOR gate in addition to the circuit to implement MM_{2m} . An LFSR is a linear map, so the all zero-state will lead to all-zero output of the LFSR. If MM_{2m+1} maps the all-zero string to 0, then the keystream generated from the all zero-state of the LFSR will be the all-zero keystream. To avoid this scenario, we define the function $f_{2m+1} : \{0, 1\}^{2m+1} \rightarrow \{0, 1\}$ as follows.

$$f_{2m+1}(W, \mathbf{X}, \mathbf{Y}) = \begin{cases} \text{MM}_{2m+1}(W, \mathbf{X}, \mathbf{Y}) & \text{if } \text{MM}_{2m+1}(\mathbf{0}_{2m+1}) = 1, \\ 1 \oplus \text{MM}_{2m+1}(W, \mathbf{X}, \mathbf{Y}) & \text{if } \text{MM}_{2m+1}(\mathbf{0}_{2m+1}) = 0. \end{cases} \quad (5)$$

The cryptographic properties of f_{2m+1} are exactly the same as those of MM_{2m+1} , with the only exception being that f_{2m+1} maps $\mathbf{0}_{2m+1}$ to 1, while MM_{2m+1} may or may not do so. We construct f_{2m+1} from $(\text{Maj}, \text{id})\text{-MM}_{2m+1}$, i.e. where h is chosen to be Maj_m and π is chosen to be id .

We revisit the filter generator model where the state of an LFSR of length L is filtered using a Boolean function. By $\mathcal{S}(L, m)$, we denote the class of stream ciphers obtained from an LFSR of length L with a primitive connection polynomial, and with f_{2m+1} as the filtering function. For the κ -bit security level, we assume that the stream cipher supports a κ -bit secret key and a κ -bit initialisation vector (IV), and so $L \geq 2\kappa$. Suppose that the state (s_{L-1}, \dots, s_0) of the LFSR leads to the next state (s_L, \dots, s_1) , i.e. the state is right shifted once and the next bit s_L becomes the leftmost bit. The initial state of the LFSR is constructed as $(k_{\kappa-1}, \dots, k_0, v_{\kappa-1}, \dots, v_0, b_1, \dots, b_{L-2\kappa})$, where $(k_{\kappa-1}, \dots, k_0)$ is the key, $(v_{\kappa-1}, \dots, v_0)$ is the IV, and $(b_1, \dots, b_{L-2\kappa}) \neq \mathbf{0}_{L-2\kappa}$ is an arbitrary constant bit string. We next specify which state bits of the LFSR are fed into the filtering function. From the state (s_{L-1}, \dots, s_0) of the LFSR, the value of W in $f_{2m+1}(W, \mathbf{X}, \mathbf{Y})$ is the state bit $s_{L-\kappa+m}$, the values of the variables in \mathbf{X} are the state bits $s_{L-\kappa+m-1}, \dots, s_{L-\kappa}$, and the values of the variables in \mathbf{Y} are the state bits $s_{L-2\kappa+m-1}, \dots, s_{L-2\kappa}$. We assume that the stream cipher goes through an initialisation phase of 2κ steps, where no keystream bit is produced, and instead the keystream bit is fed back by XORing it with the next bit of the LFSR. So after the initialisation phase the leftmost 2κ bits of the LFSR are complex nonlinear functions of the key and the IV. Keystream generation starts after the initialisation phase.

The parameters L and m are the *design parameters* of the stream cipher. We consider two basic *attack parameters*, namely the number N of keystream bits that is required for a successful attack, and the time complexity T of the attack. To ensure security at level κ any attack should require $T > 2^\kappa$. The condition $L \geq 2\kappa$ ensures that the size of the state is at least twice the size of the key. This prevents certain (theoretical) time/memory trade-off attacks [27]. We assume that at most 2^B keystream bits are generated from a single key and IV pair. So N is at most 2^B , as otherwise the attack cannot be mounted. Further, a basic condition is that $N \leq T$, i.e. some operation is performed on each keystream bit that is required. So for κ -bit security using 2^B keystream bits, we have $B \leq \kappa$. In the analysis below, our goal is to determine values of L and m such that the stream cipher is not vulnerable to the known types of attacks. For this concrete analysis, we ignore small constants appearing in the expressions for N and T (i.e. we consider these constants to be 1) and focus only on the exponential components of these quantities.

Linear complexity attack. The degree of (Maj, id)- MM_{2m+1} is $d = 2^{\lceil \log_2 m \rceil}$. If L is a prime, then the linear complexity of the generated keystream sequence is $\binom{L}{d}$ (see [37]) and so at least these many keystream bits are required to determine the linear complexity. So if

$$\alpha = \binom{L}{2^{\lceil \log_2 m \rceil}} > 2^B, \quad (6)$$

then the linear complexity attack is not applicable.

Fast correlation attacks. The basic correlation attack [41] is applicable to the combiner generator model. Applying this attack to the filter generator model results in going through all the possible 2^L states of the LFSR. Since by our choice $L \geq 2\kappa$, the basic correlation attack does not defeat the κ -bit security level.

Fast correlation attacks do not require exhaustive search on the states of the LFSR. There is a large literature on fast correlation attacks including older papers such as [35, 28, 29, 15, 12, 30, 11, 16] as well as more recent papers such as [42, 45, 44, 31, 32]. See [9, 10, 33, 2] for surveys of the area. In the following, we evaluate security against some representative fast correlation attacks and show how to choose the values of the design parameters so as to resist these attack. Recall from Proposition 2 that for MM_{2m+1} , and hence for f_{2m+1} , the linear bias $\varepsilon = 2^{-m-1}$.

Type of attack. For attacks based on low-weight parity-check equations [35, 12], the number of keystream bits required is about $N = (2\varepsilon)^{-2(d-2)/(d-1)} \cdot 2^{L/(d-1)}$, the pre-computation step requires about $N^{d-2}/(d-2)!$ operations, and the (online) time for decoding is about $(2\varepsilon)^{-2d(d-2)/(d-1)} \cdot 2^{L/(d-1)}$, where $d \geq 3$ is the number of non-zero terms in (some multiple of the) LFSR connection polynomial, and ε is the linear bias.

Evaluation against $\mathcal{S}(L, m)$. We have $\varepsilon = 2^{-m-1}$ and so $N = 2^{(2(m+2)(d-2)+L)/(d-1)}$. We set $N = 2^B$ and so $(d-1)(B-2(m+2)) = L-2(m+2)$. If $B = 2(m+2)$, then $L = 2(m+2)$. We choose L and m to ensure that $L > 2(m+2)$, and so $B \neq 2(m+2)$. In this case, we solve for d to obtain $d = 1 + (L-2(m+2))/(B-2(m+2))$. If $B < 2(m+2)$, then $d < 1$ which violates the condition $d \geq 3$ and the attack does not work. So let us consider $B > 2(m+2)$. In this case, substituting the obtained expression for d in the expression for the decoding time, we find the decoding time to be $2^{(2(m+2)L+B^2)/(B-2(m+2))}$. So the following condition ensures κ -bit security.

$$L > (2m+2) \text{ and either } B < 2(m+2) \text{ or } \kappa < \frac{2(m+2)L+B^2}{B-2(m+2)}. \quad (7)$$

Note that (7) ensures κ -bit security for all values of $d \geq 3$. In particular, it does not matter whether the feedback polynomial of the LFSR is sparse, or whether it has a sparse multiple.

Type of attack. For attacks based on general decoding, [9] identifies the key idea to be from [15]. The attack in [15] requires N to be about $\varepsilon^{-2} \cdot 2^{(L-k)/w}$ under the condition $N \gg (2\varepsilon)^{-2w}$, and the time complexity of the decoding step is about $2^k \cdot (2\varepsilon)^{-2w}$, where ε is the linear bias of the filtering function, and $k \in \{1, \dots, L\}$ and $w \geq 2$ are algorithm parameters.

Evaluation against $\mathcal{S}(L, m)$. Again we have $\varepsilon = 2^{-m-1}$. Setting $N = 2^B$, we obtain $k = L + 2w(m+1) - wB$. Setting T to be the time complexity of the decoding step, we obtain $\log_2 T = L + 2w(2m+3) - wB$. Since $L \geq 2\kappa$, ensuring $2(2m+3) \geq B$ is (more than) sufficient to ensure κ -bit security, irrespective of the values of w and k . We record this condition as follows.

$$B \leq 2(2m+3). \quad (8)$$

Type of attack. The attacks in [30, 11] apply specifically to the filter generator model. These two attacks are essentially the same when the filtering function is plateaued (which is the case for f_{2m+1}). For the attack, N is about $2^{(L-k)/w}$, and T is about $2^k \cdot F^w$, where w and k are as in the attack in [15] (see above) and F is the size of the support of the Walsh transform of the filtering function.

Evaluation against $\mathcal{S}(L, m)$. The size of the support of the Walsh transform of MM_{2m+1} is equal to the size of the support of the Walsh transform of MM_{2m} . Since MM_{2m} is bent, it follows that the size of the support of MM_{2m} is 2^{2m} . So $F = 2^{2m}$. Setting $N = 2^B$, we obtain $k = L - wB$. Substituting k in the expression for T , we obtain $\log_2 T = L + w(2m - B)$. Since $L \geq 2\kappa$ and $w \geq 2$, we have $\log_2 T = L + w(2m - B) \geq 2\kappa + 2(2m - B)$. So if $2\kappa + 2(2m - B) > \kappa$, or equivalently, $\kappa + 4m > 2B$, then κ -bit security is achieved against the attack. We record this condition as follows.

$$B < (\kappa + 4m)/2. \quad (9)$$

Other fast correlation attacks. We next consider some of the more recent attacks. The attack in [42] is based on using $M > 1$ linear approximations. For this attack, both N and T are about $2^{L-\beta}$, where β is an algorithm parameter. A necessary condition for the attack to succeed is that $M > 2^\beta$.

For the attack to succeed at the κ -bit security level, i.e. $T \leq 2^\kappa$, it is required to have $\beta \geq L - \kappa$. From the bound on M , it follows that more than $2^{L-\kappa}$ linear approximations with sufficiently high correlations are required. For Grain-128a [1], L equals 128 and about $2^{26.58}$ (i.e. M is about $2^{26.58}$) linear approximations with absolute correlations at least $2^{-54.2381}$ were identified in [42]. For $\kappa = 128$, Table 1 recommends $L = 257$ and $m = 58$. To apply the attack in [42] to $\mathcal{S}(257, 59)$, more than 2^{129} linear approximations with sufficiently high correlations are required. The linear bias of the filtering function itself is 2^{-60} . Finding multiple linear approximations requires combining keystream bits which further lowers the linear bias. So there is no approximation with linear bias greater than 2^{-60} . Further, going through the details of the attack in [1], we could not identify any method to obtain more than 2^{129} linear approximations. So there does not seem to be any way to apply the attack in [42] to $\mathcal{S}(257, 59)$.

Subsequent works (such as [45, 44, 31, 32]) on fast correlation attacks use vectorial decoding technique along with multiple linear approximations, use of the BKW algorithm [5], and multivariate correlation attack. The attacks are quite complex and there are no simple closed form expressions for the values of N and T . The stream ciphers to which these attacks are applied are Grain-128a and Sosemanuk [4], which is another cipher providing 128-bit security. From the above discussion, we already know that the linear bias of $\mathcal{S}(257, 59)$ is substantially lower than that of Grain-128a. For Sosemanuk, the best known [31] linear approximation has correlation $2^{-20.84}$, which is far greater than the linear bias 2^{-60} for $\mathcal{S}(257, 59)$. The very low linear bias of $\mathcal{S}(257, 59)$ and in general of the other stream cipher proposals in Table 1 make the attacks in [45, 44, 31, 32] inapplicable at the stated security levels.

(Fast) Algebraic attacks. For a filtering function f having $\text{Al}(f) = a$, an algebraic attack requires about $N = \sum_{i=0}^a \binom{L}{i}$ keystream bits and has time complexity T to be about N^ω , where ω is the exponent of matrix multiplication (see Section 3.1.5 of [13]). We assume $\omega = 2.8$. From Proposition 4, we conservatively assume that $\text{Al}((\text{Maj}, \text{id})\text{-MM}_{2m+1}) = \lceil m/2 \rceil$, i.e. the algebraic immunity is equal to its lower bound. Let

$$\beta = \left(\sum_{i=0}^{\lceil m/2 \rceil} \binom{L}{i} \right)^{2.8}. \quad (10)$$

So choosing L and m such that $T = \beta > 2^\kappa$ prevents algebraic attacks at the κ -bit security level.

For any (e, d) in the FAA-profile of $(\text{Maj}, \text{id})\text{-MM}_{2m}$, the Berlekamp-Massey step in a fast algebraic attack on the filter generator model has time complexity $\mathcal{O}(ED \log D)$, where $E = \sum_{i=0}^e \binom{L}{i}$ and $D = \sum_{i=0}^d \binom{L}{i}$ (see [25] and Section 3.1.5 of [13]). This complexity dominates the overall time complexity of a fast algebraic attack and ignoring the logarithm term, and we take T to be equal to ED . The maximum value of e is one less than the algebraic immunity. The number N of keystream bits required is about $2E$. From Proposition 4, we again assume that $\text{Al}((\text{Maj}, \text{id})\text{-MM}_{2m+1}) = \lceil (2m+1)/4 \rceil$. Recall that for any Boolean function, its fast algebraic immunity is at least one more than its algebraic immunity, and so $\text{FAI}((\text{Maj}, \text{id})\text{-MM}_{2m+1}) \geq 1 + \text{Al}((\text{Maj}, \text{id})\text{-MM}_{2m+1}) = 1 + \lceil (2m+1)/4 \rceil$. For the concrete analysis, we use the lower bound $1 + \lceil m/2 \rceil$ as the actual value of $\text{FAI}((\text{Maj}, \text{id})\text{-MM}_{2m+1})$. Define

$$\gamma = \min_{\substack{1 \leq e \leq \lceil m/2 \rceil, \\ d = a - e}} \left(\sum_{i=0}^e \binom{L}{i} \right) \left(\sum_{i=0}^d \binom{L}{i} \right). \quad (11)$$

So choosing L and m such that $T = \gamma > 2^\kappa$ prevents fast algebraic attacks at the κ -bit security level.

Given κ , we obtained representative values of L and m . The procedure we followed is to choose the value of L to be the first prime number greater than 2κ , and then for the chosen value of L , choose m

κ	L	m	$2m + 1$	deg	LB	AI	FAI
80	163	37	75	32	2^{-38}	19	20
128	257	59	119	32	2^{-60}	30	31
160	331	71	143	64	2^{-72}	36	37
192	389	87	175	64	2^{-88}	44	45
224	449	101	203	64	2^{-102}	51	52
256	521	115	231	64	2^{-116}	58	59

Table 1: Values of L and m which provide κ -bit security against the attacks analysed in this section when the attacker has access to at most 2^B keystream bits.

to be the least integer such that $\beta, \gamma > 2^\kappa$. This ensures security against algebraic attacks considered above. Next, using κ and the corresponding values of L and m , we computed the maximum value of B such that $B \leq \kappa$, $\alpha > 2^B$, and (7), (8) and (9) hold. In each of the cases that we considered, it turns out that this maximum value of B is in fact κ . So for the chosen values of L and m , the corresponding $\mathcal{S}(L, m)$ ensures κ -bit security against the above analysed correlation attacks even when the adversary has access to 2^κ keystream bits. We note, however, that generating 2^κ keystream bits from a single key and IV pair is meaningless from a practical point of view. Instead, we set $B = 64$, i.e. from a single key and IV pair at most 2^{64} keystream bits are to be generated. Table 1 shows the values of κ , L , m and other parameters of the filtering function. We note the following points regarding the entries in Table 1.

1. The number of variables of the filtering function f_{2m+1} is $2m + 1$ (and not m).
2. For each κ , the value of L is the first prime number greater than 2κ . Our choice of a prime number for the value of L is to ensure that the linear complexity of the generated keystream is indeed equal to α .
3. The table provides representative values of L and m . It is possible to have other pairs of values for (L, m) which provide κ -bit security. Different values of L and m lead to different sizes of the circuits implementing the corresponding stream ciphers. For a fixed value of κ , a practical designer may need to consider various values of (L, m) to determine which pair provides the smallest circuit. In Section 5, we provide gate count estimates for the stream ciphers corresponding to the values of L and m in the table.
4. For each entry in the table, our calculation shows that $\beta \gg \gamma$. For example, for $\mathcal{S}(257, 59)$, we obtained $\beta = 2^{364.38}$ while $\gamma = 2^{130.12}$. This is not surprising, since the fast algebraic attack is expected to be much more faster than the basic algebraic attack.
5. The feedback polynomial for the LFSR has to be a primitive polynomial of degree L . The above analysis of correlation attacks shows that for the obtained values of L and m , it does not matter whether the polynomial is sparse, or whether it has a sparse multiple. So from an efficiency point of view, one may choose a low weight primitive polynomial. Examples of primitive polynomials for the values of L in Table 1 are shown in Table 2.

5 Efficiency of Computing MM_n

In Section 4, we proposed using f_{2m+1} (which is either MM_{2m+1} or $1 \oplus \text{MM}_{2m+1}$) as the filtering function in the nonlinear filter model. Further, Table 1 provides specific suggestions of values of m for achieving different security levels. In this section, we consider the complexity of implementing MM_{2m+1} . Since MM_{2m+1} is constructed from MM_{2m} using one XOR gate, we need to consider the complexity of implementing MM_{2m} .

L	prim poly
163	$x^{163} \oplus x^7 \oplus x^6 \oplus x^3 \oplus 1$
257	$x^{257} \oplus x^7 \oplus x^5 \oplus x^4 \oplus x^3 \oplus x^2 \oplus 1$
331	$x^{331} \oplus x^7 \oplus x^6 \oplus x^5 \oplus x^4 \oplus x^2 \oplus 1$
389	$x^{389} \oplus x^7 \oplus x^6 \oplus x^3 \oplus x^2 \oplus x \oplus 1$
449	$x^{449} \oplus x^9 \oplus x^6 \oplus x^5 \oplus x^4 \oplus x^3 \oplus x^2 \oplus x \oplus 1$
521	$x^{521} \oplus x^9 \oplus x^6 \oplus x^5 \oplus x^3 \oplus x \oplus 1$

Table 2: Examples of primitive polynomials for the values of L shown in Table 1.

The computation of MM_{2m} requires computing h and an inner product of two m -bit strings. The computation of h requires the computation of the weight of an m -bit string and the computation of a threshold function. We discuss basic strategies for implementing these operations which provide estimates of the number of gates required.

Inner product of two m -bit strings. This operation requires m AND gates and $m - 1$ XOR gates.

Weight of an m -bit string. A half-adder takes two input bits and outputs two bits which represent sum of the two input bits. A full adder takes three input bits and outputs two bits which represent the sum of the three input bits. We estimate the numbers of half and full adders that are required for computing the weight of an m -bit string \mathbf{x} . The algorithm for computing weight that we use is from [6]. If $m = 1$, then no adders are required, if $m = 2$, a half adder computes the weight, and if $m = 3$, a full adder computes the weight. For $m > 3$, write $m = m_1 + m_2 + 1$, where $m_1 + 1$ is the highest power of two that is at most m . The algorithm computes the weight of the first m_1 bits of \mathbf{x} , the weight of the next m_2 bits of \mathbf{x} , and then adds these two weights together together with the last bit of \mathbf{x} .

Using the above algorithm, it is easy to show that the computation of the weight of an m -bit string, when $m = 2^r - 1$ with $r \geq 1$, requires $2^r - r - 1$ full adders. From this it easily follows that the number of full adders required to compute the weight of an m -bit string for arbitrary m is $O(m)$. Since a full adder can be implemented using a constant number of NAND gates, the number of NAND gates required to compute the weight of an m -bit string is also $O(m)$.

We are interested in the exact counts of full and half adders required for the values of m in Table 1. Let us denote a full adder by [F] and a half adder by [H]. In Table 1, one of the choices is $m = 37$. Writing $37 = 31 + 5 + 1$, it is required to find the weight of one 31-bit string, one 5-bit string and then perform the final addition. Computation of the weight of the 31-bit string requires 26[F]. Writing $5 = 3 + 1 + 1$, the computation of the weight of a 5-bit string requires 3[F] (a full adder to compute the weight of a 3-bit string, and 1[F]+1[H] to add the remaining two bits to this weight). The weight of a 5-bit string is a 3-bit quantity, while the weight of a 31-bit string is a 5-bit quantity. So the final addition of the weights along with the remaining bit requires 3[F]+2[H]. So a total of 31[F]+3[H] is required to compute the weight of a 37-bit string. In a similar manner, it is possible to obtain the numbers of full and half adders required to compute the weights of m -bit strings for the values of m given in Table 1 and these counts are given below.

$$\begin{array}{lll}
37: 31[\text{F}]+3[\text{H}]; & 59: 53[\text{F}]+1[\text{H}]; & 71: 64[\text{F}]+3[\text{H}]; \\
87: 80[\text{F}]+2[\text{H}]; & 101: 94[\text{F}]+3[\text{H}]; & 115: 108[\text{F}]+2[\text{H}].
\end{array}$$

Threshold function on the weight of an m -bit string. The weight of a 37-bit string is a 6-bit quantity, say $w_5w_4w_3w_2w_1w_0$. It is required to determine whether the value represented by this string

	$\mathcal{S}(163, 36)$	$\mathcal{S}(257, 59)$	$\mathcal{S}(331, 70)$	$\mathcal{S}(389, 87)$	$\mathcal{S}(449, 100)$	$\mathcal{S}(521, 114)$
LFSR	1304	2056	2648	3112	3592	4168
f_{2m+1}	439.5	715.5	872.5	1075.5	1262.5	1439.5
total	1743.5	2771.5	3520.5	4187.5	4854.5	5607.5

Table 3: Estimates of the number of NAND gates required to implement $\mathcal{S}(L, m)$ for values of L and m in Table 1.

is at least 19. This is computed by the Boolean formula $w_5 \vee (w_4 \wedge (w_3 \vee w_2 \vee (w_1 \wedge w_0)))$, requiring $3[\text{OR}] + 2[\text{AND}]$ gates. In general, the weight of an m -bit string is an ω -bit value, where $\omega = \lceil \log_2 m \rceil$. To compute the threshold function, ω_1 OR and ω_2 AND gates are required for some values of ω_1 and ω_2 satisfying $\omega_1 + \omega_2 \leq \omega$. So the number of gates for computing the threshold function on the weight of an m -bit string requires a logarithmic (in m) number of gates.

Circuit size for computing MM_n . The inner product requires $O(m)$ OR and AND gates and the weight computation requires $O(m)$ full adders. So the circuit size for computing MM_n is $O(n)$.

Next we consider concrete estimates. For such estimates, we ignore the at most $\lceil \log_2 m \rceil$ AND and OR gates required for computing the threshold function from the weight, and also the gates required to compute the next bit of the LFSR. Similar assumptions were made in [1] to obtain the gate count estimate for Grain-128a. To obtain concrete estimates, it is convenient to convert the various gate counts into a single unit. Previous works [1, 8] have taken a single NAND gate as the basic unit and translated other gates in terms of this unit. A half-adder can be implemented using 5 NAND gates, while a full adder can be implemented using 9 NAND gates. In [1, 8], a XOR gate was taken to be 2.5 units and an AND gate was taken to be 1.5 units. Between the papers [1] and [8] there is a difference in the number of units required for a flip-flop: [1] takes a flip-flop to be 8 units, while [8] takes a flip-flop to be 12 units. In Table 3, we provide estimates of the circuit sizes of $\mathcal{S}(L, m)$ for the values of (L, m) in Table 1. These estimates consider a flip-flop to be 8 units.

If we consider a flip-flop to be 12 units as done for Trivium [8], then the gate count estimate for $\mathcal{S}(163, 36)$ is 2395.5. The gate estimate for Trivium obtained in [8] is 3488. Both Trivium and $\mathcal{S}(163, 36)$ are targeted at the 80-bit security level, and $\mathcal{S}(163, 36)$ is substantially smaller than Trivium. The lower size of $\mathcal{S}(163, 36)$ is due to the smaller state size; $\mathcal{S}(163, 36)$ uses a 163-bit state, while Trivium uses a 288-bit state.

The gate count estimate for Grain-128a obtained in [1] is 2145.5. Grain-128a is targeted at the 128-bit security level. Comparing to $\mathcal{S}(257, 59)$, which is also targeted at the 128-bit security level, we find the gate count estimate of $\mathcal{S}(257, 59)$ to be 2771.5. The state sizes of both Grain-128a and $\mathcal{S}(257, 59)$ are almost the same. The greater size of $\mathcal{S}(257, 59)$ is due to the greater gate size requirement of the filtering function f_{2m+1} in comparison to the gate size requirement of the nonlinear components of Grain-128a. Even though $\mathcal{S}(257, 59)$ is larger than Grain-128a, its size of about 2771.5 gates is small enough for $\mathcal{S}(257, 59)$ to be considered as a small cipher. Finally we note that $\mathcal{S}(521, 114)$ which is targeted at the 256-bit security level requires about 5607.5 gates. We know of no other 256-bit secure stream cipher which has such a small gate count.

We note that it is possible to ramp up security at the cost of a reasonable increase in gate count. For example, from Table 1, at the 128-bit security level, our proposal has $L = 257$ and $m = 59$, resulting in linear bias equal to 2^{-60} and the fast algebraic attack requiring more than $2^{130.12}$ operations. If we increase m from 59 to 63, then the linear bias drops to 2^{-64} , and the fast algebraic attack now requires more than $2^{135.83}$ operations. The gate estimate for $\mathcal{S}(257, 63)$, i.e. the stream cipher with $L = 257$ and

$m = 63$, is 2818.5 gates (the LFSR requires 2056 gates, and f_{127} requires 762.5 gates). So the security increases by about 5 bits at the cost of an increase of only 47 gates in the circuit size.

6 Conclusion

We described a construction of balanced Boolean functions which has several provable properties, namely very high nonlinearity, acceptable algebraic resistance, *and* is efficient to implement in hardware. Using such Boolean functions, we proposed concrete constructions of the nonlinear filter model for stream ciphers targeted at different security levels. Gate count estimates for the stream cipher proposals shows that the circuit sizes compare well with well known ciphers at the 80-bit and the 128-bit security levels, while for higher security levels, we do not know of any stream cipher with lower gate count estimates.

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A Improved Degree and Algebraic Resistance

With the choice of π to be the identity permutation and h to be Maj_m in (2), the degree of MM_n is $2^{\lceil \log_2 \lfloor n/2 \rfloor \rceil}$ and the algebraic immunity is about $\lfloor n/4 \rfloor$. We discuss ways to improve these parameters.

For $n \geq 1$, let $\text{HWB}_n : \{0, 1\}^n \rightarrow \{0, 1\}$ be the hidden weight bit function [7] defined as follows. For $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{F}_2^n$, $\text{HWB}_n(\mathbf{x}) = 0$ if $\text{wt}(\mathbf{x}) = 0$, and $\text{HWB}_n(\mathbf{x}) = x_{\text{wt}(\mathbf{x})}$, if $\text{wt}(\mathbf{x}) > 0$. In [43], it was shown that the AI of HWB_n is at least $\lfloor n/3 \rfloor + 1$. We extend HWB_n to n -HWBP : $\mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$ as follows. For $(x_1, \dots, x_n) \in \mathbb{F}_2^n$, let $w = \text{wt}(x_1, \dots, x_n)$. Then n -HWBP (x_1, \dots, x_n) is defined to be the following. n -HWBP $(x_1, \dots, x_n) = (0, \dots, 0)$ if $w = 0$, and n -HWBP $(x_1, \dots, x_n) = (x_w, x_{w+1}, \dots, x_n, x_1, \dots, x_{w-1})$, if $\text{wt}(\mathbf{x}) > 0$. It is easy to verify that n -HWBP is a bijection.

Suppose we set π in the definition of MM_{2m} to be m -HWBP (along with choosing h to be Maj_m), and define MM_{2m+1} from MM_{2m} as in (4). This leads to an improvement of algebraic resistance. We conducted experiments to compute the AI and FAI of MM_n for n up to 20. Our experiments suggest that $\lfloor n/3 \rfloor \leq \text{AI}(\text{MM}_n) \leq 1 + \lfloor n/3 \rfloor$, and $\lfloor n/2 \rfloor \leq \text{FAI}(\text{MM}_n) \leq 1 + \lfloor n/2 \rfloor$. (This is only a conjecture, and we do not have a proof.)

MM_{2m+1} is a balanced function having nonlinearity $2^{2m} - 2^m$ and degree at most m . A construction of balanced functions on $2m + 1$ variables which has nonlinearity $2^{2m} - 2^m$ and the maximum possible degree $2m$ is given by Theorem 10(a) of [38]. Dobbertin [21] gave a general construction of balanced Boolean functions on an even number of variables. The proposal was to modify a normal bent function on $2m$ variables by inserting a balanced function on m variables on the flat where the bent function is constant. This is a recursive procedure, where the recursion terminates with a balanced function on an odd number of variables and possessing the maximum possible nonlinearity. A simple modification of this recursive procedure is to require the recursion to terminate with a balanced function on an odd number of variables possessing the maximum possible degree and as high nonlinearity as possible. Then the resulting function on $2m$ variables is balanced and has maximum degree. The terminating function can be chosen to be the maximum degree balanced function on odd number of variables obtained from the construction given by Theorem 10(a) of [38]. The net result is that for any n (either odd or even), it is possible to obtain balanced functions with maximum degree and very high nonlinearity. The procedure requires bent functions at several stages. One may instantiate these bent functions using MM_{2m} with π chosen to be m -HWBP and h chosen to be Maj_m . We have implemented this construction and computed the algebraic immunities of the resulting functions. From our experiments for n up to 20, we observed

that (as above) the algebraic immunity is either $\lfloor n/3 \rfloor$, or $1 + \lfloor n/3 \rfloor$, and the fast algebraic immunity is either $\lfloor n/2 \rfloor$ or $1 + \lfloor n/2 \rfloor$.

Even though the above modifications result in function with higher degrees and higher algebraic immunities, we did not opt for these functions. The reason is the implementation efficiency. The modifications make the functions more complex, which results in more gate requirements. In particular, the implementation of m -HWBP requires $O(m^2)$ gates, whereas the implementation of Maj_m requires $O(m)$ gates. Since implementation efficiency is a primary concern, we choose the simpler option and compensate by increasing the number of variables.