The Zeros of Zeta Function Revisited

Zhengjun Cao, Lihua Liu

Abstract. Let $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$, $\psi(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}, z \in \mathbb{C}$. We show that $\psi(z) \neq (1-2^{1-z})\zeta(z)$, if $0 < z < 1$. Besides, we clarify that the known zeros are not for the original series, but very probably for the alternating series.

Keywords: Zeta function, alternating series, partial sum, absolute convergence.

1 Introduction

The zeta function plays a key role in modern public key cryptography, due to its importance to primality testing and factorization. In this note, we show a new result about its zeros.

The below complex series is called zeta function

$$\begin{aligned} \zeta(z) &= \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} e^{-z\ln n} \xlongequal[a,b\in\mathbb{R}]{z=a+ib} \sum_{n=1}^{\infty} e^{-(a+ib)\ln n} \\ &= \sum_{n=1}^{\infty} e^{-a\ln n} e^{-ib\ln n} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^a \cos(b\ln n) - i\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^a \sin(b\ln n) \end{aligned} \tag{1}$$

If $a > 1$, then

$$\left|\left(\frac{1}{n}\right)^a \cos(b\ln n)\right| \le \left(\frac{1}{n}\right)^a, \quad \left|\left(\frac{1}{n}\right)^a \sin(b\ln n)\right| \le \left(\frac{1}{n}\right)^a.$$

Both $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^a \cos(b\ln n)$ and $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^a \sin(b\ln n)$ are absolutely convergent. Hence, $\zeta(z)$ has no zeros for $a > 1$. Notice that

$$\zeta(\bar{z}) = \sum_{n=1}^{\infty} \frac{1}{n^{\bar{z}}} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^a \cos(b\ln n) + i\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^a \sin(b\ln n), \quad \overline{\zeta(z)} = \zeta(\bar{z}).$$

2 Numerical calculations for zeta function

If $z = a + bi$, denote the kth partial sum as

$$\sum_{n=1}^{k} \left(\frac{1}{n}\right)^a \cos(b\ln n) - i\sum_{n=1}^{k} \left(\frac{1}{n}\right)^a \sin(b\ln n) = c + di, \quad c, d \in \mathbb{R}.$$

Z. Cao is with Department of Mathematics, Shanghai University, Shanghai, 200444, China.
L. Liu is with Department of Mathematics, Shanghai Maritime University, Shanghai, 201306, China. Email: liulh@shmtu.edu.cn

The values for $\zeta(3+14.1347251\,i), \zeta(-3+14.1347251\,i)$ are listed below (Table 1). We find both convergence and divergence can be certified with great confidences.

Table 1: The numerical calculations for two complex series

Partial-Sum	$\zeta(3 + 14.1347251\,i)$	$\lvert c\rvert + \lvert d\rvert$	$\zeta(-3 + 14.1347251\,i)$	$\lvert c\rvert + \lvert d\rvert$
1	1	1	1	1
2	0.88358 -0.0455126 I	0.929093	-6.45088-2.9128 I	9.36368
3	0.847137 -0.0389044 I	0.886042	-33.0176+1.90454 I	34.9222
4	0.85862 -0.0283073 I	0.886927	14.0135 +45.3104 I	59.3239
5	0.852809 -0.033806 I	0.886615	-76.779-40.6065 I	117.386
6	0.857352 -0.0329167 I	0.890269	135.199 +0.883902 I	136.083
7	0.855258 -0.0308883 I	0.886146	-111.181+239.518 I	350.699
8	0.854404 -0.0326446 I	0.887048	-335.171-220.887 I	556.058
9	0.855688 -0.0331263 I	0.888814	347.415 -476.849 I	824.264
10	0.856114 -0.0322216 I	0.888336	773.64 +427.768 I	1201.41
11	0.855522 -0.0317587 I	0.887281	-274.626+1247.95 I	1522.58
12	0.855034 -0.032069 I	0.887103	-1733.19+321.362 I	2054.55
13	0.855091 -0.0325205 I	0.887612	-1455.93-1858.07 I	3314.
14	0.855428 -0.0326613 I	0.888089	1074.91 -2918.45 I	3993.36
15	0.855676 -0.0324994 I	0.888175	3900.87 -1073.29 I	4974.16
16	0.855695 -0.032256 I	0.887951	4228.72 +3009.57 I	7238.29
17	0.855553 -0.0321108 I	0.887663	784.762 +6513.38 I	7298.14
18	0.855381 -0.0321132 I	0.887494	-5046.67+6432.28 I	11479.
19	0.855277 -0.0322156 I	0.887493	-9931.84+1617.61 I	11549.4
20	0.855269 -0.0323403 I	0.887609	-10472.6-6364.09 I	16836.7
21	0.855332 -0.032428 I	0.88776	-5076.68-13890.7 I	18967.4
22	0.855422 -0.032455 I	0.887877	5122.85 -16948.4 I	22071.3
23	0.855499 -0.0324278 I	0.887927	16605.3 -12924.7 I	29530.
24	0.855542 -0.0323695 I	0.887911	24773.9 -1772.28 I	26546.2
25	0.855545 -0.0323056 I	0.887851	25635.5 +13828.9 I	39464.4
26	0.855518 -0.0322556 I	0.887774	17221.3 +29260. I	46481.4
27	0.855475 -0.0322296 I	0.887704	320.275 +39348.4 I	39668.7
28	0.855429 -0.0322285 I	0.887658	-21625.4+39877.3 I	61502.6
29	0.855393 -0.0322471 I	0.88764	-43348.7+28790.3 I	72138.9
30	0.855371 -0.0322773 I	0.887648	-59029.9+6810.8 I	65840.7
31	0.855366 -0.0323104 I	0.887676	-63665.-22617.4 I	86282.4
32	0.855375 -0.0323396 I	0.887714	-54215.3-53993.3 I	108209.
33	0.855393 -0.0323604 I	0.887754	-30317.3-80832.7 I	111150.
34	0.855416 -0.0323708 I	0.887787	5549.11 -96907.6 I	102457.
35	0.85544 -0.0323711 I	0.887811	48421.2 -97405.6 I	145827.
36	0.85546 -0.032363 I	0.887823	91634.3 -79815.6 I	171450.

37	0.855474 -0.0323492 I	0.887823	127863. -44414.5 I	172277.
38	0.855481 -0.0323326 I	0.887814	150237. +5688.56 I	155926.
39	0.855482 -0.0323157 I	0.887798	153370. +64924.8 I	218295.
40	0.855477 -0.0323008 I	0.887778	134150. +125971. I	260121.
41	0.855468 -0.0322893 I	0.887758	92218.9 +180668. I	272887.
42	0.855457 -0.032282 I	0.887739	30090.9 +221031. I	251122.
43	0.855445 -0.0322789 I	0.887724	-47069.7+240204. I	287274.
44	0.855433 -0.0322799 I	0.887713	-131972.+233277. I	365249.
45	0.855423 -0.0322842 I	0.887707	-215937.+197871. I	413808.
46	0.855415 -0.0322909 I	0.887706	-289771.+134445. I	424215.
47	0.85541 -0.032299 I	0.887709	-344664.+46320.3 I	390984.
48	0.855408 -0.0323078 I	0.887716	-373043.-60568.7 I	433611.
49	0.855408 -0.0323163 I	0.887724	-369286.-178158. I	547444.
50	0.855411 -0.0323239 I	0.887735	-330262.-296910. I	627172.
100	0.855442 -0.0323112 I	0.887754	$3.54196 \times 10^6 + 5.97972 \times 10^6$I	9.52168×10^6
1000	0.855437 -0.0323152 I	0.887752	$-3.46413 \times 10^{10} + 5.87601 \times 10^{10}$I	9.34014×10^{10}
10000	0.855437 -0.0323152 I	0.887752	$-6.78363 \times 10^{14} - 5.84977 \times 10^{13}$I	7.3686×10^{14}
20000	0.855437 -0.0323152 I	0.887752	$9.76874 \times 10^{15} + 4.81966 \times 10^{15}$I	1.45884×10^{16}

3 Approximations of zeta function

The following approximation of zeta function was due to Hardy, Poisson, et al., [1] which involves an improper integral

$$\zeta(z) \sim z \int_1^{\infty} \frac{[x] - x + \frac{1}{2}}{x^{z+1}} \mathrm{d}x + \frac{1}{z-1} + \frac{1}{2}, \quad \mathrm{Re}(z) > 1, \tag{2}$$

where $[x]$ denotes the integral part of real number x. Besides,

$$\zeta(z) \sim z \int_0^{\infty} \frac{[x] - x + \frac{1}{2}}{x^{z+1}} \mathrm{d}x, \quad -1 < \mathrm{Re}(z) < 0, \tag{3}$$

$$\zeta(z) \sim z \int_0^{\infty} \frac{[x] - x}{x^{z+1}} \mathrm{d}x, \quad 0 < \mathrm{Re}(z) < 1. \tag{4}$$

The below approximation of zeta function was due to Riemann, which involves two improper integrals

$$\zeta(z) \sim \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{x^{z-1}}{e^x - 1} \mathrm{d}x, \quad \mathrm{Re}(z) > 1, \tag{5}$$

where $\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} \mathrm{d}x$. Besides,

$$\zeta(z) \sim \frac{1}{\Gamma(z)} \int_0^{\infty} \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right) x^{z-1} \mathrm{d}x, \quad -1 < \mathrm{Re}(z) < 0, \tag{6}$$

$$\zeta(z) \sim \frac{1}{\Gamma(z)} \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) x^{z-1} \mathrm{d}x, \quad 0 < \mathrm{Re}(z) < 1. \tag{7}$$

In history, the zeros of $\zeta(z)$ were very hard to calculate. The Riemann hypothesis claims that all nontrivial zeros are on the straight line $\mathrm{Re}(z) = \frac{1}{2}$. Nowadays, several million zeros have been obtained. We refer to the address https://www-users.cse.umn.edu/~odlyzko/zeta_tables/index.html

4 The method to calculate zeros

The method to calculate zeros of zeta function is based on the functional equation [3]:

$$\zeta(z) = 2^z \pi^{z-1} \sin \frac{z\pi}{2} \Gamma(1-z) \zeta(1-z). \tag{8}$$

Define the functions

$$\chi(z) = 2^{z-1} \pi^z \sec \frac{z\pi}{2} \,/\, \Gamma(z), \quad \vartheta = \vartheta(t) = -\frac{|\chi(\frac{1}{2} + it)|}{2} \arg \chi(\frac{1}{2} + it),$$

and the function $Z(t) = e^{i\vartheta(t)} \zeta(\frac{1}{2} + it)$ which is real for real values of t. Hence, if $Z(t_1)$ and $Z(t_2)$ have opposite signs, $Z(t)$ vanishes between t_1 and t_2, and so $\zeta(z)$ has a zero on the critical line between $\frac{1}{2} + it_1$ and $\frac{1}{2} + it_2$.

To calculate the first nontrivial zero, one needs to determine the sign of $Z(0) = e^{i\vartheta(\frac{1}{2})} \zeta(\frac{1}{2})$. Define the psi function as

$$\psi(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}.$$

Some arguments (Hardy, Poisson, et al.) for Eq.(8) are based on the equality

$$\psi(z) = (1 - 2^{1-z}) \zeta(z). \tag{9}$$

It claims that this series is convergent for all values of z such that $\mathrm{Re}(z) > 0$ (see pages 16-17, Ref.[1]).

If $z = 1/2$, the left side of Eq.(9) is

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots,$$

which converges to a positive number. Since the factor $(1 - \sqrt{2})$ is a negative number, it claims that $\zeta(1/2) < 0$ (see page 388, Ref.[1]).

Define

$$\xi(z) = \frac{1}{2} z(z-1) \pi^{-\frac{z}{2}} \Gamma(\frac{z}{2}) \zeta(z). \tag{10}$$

Hence, $\xi(1/2) = -\frac{1}{8} \pi^{-\frac{1}{4}} \Gamma(\frac{1}{4}) \zeta(1/2)$. Since $\zeta(\frac{1}{2}) < 0$ and $\Gamma(\frac{1}{4}) > 0$, then $\xi(\frac{1}{2}) > 0$, which implies $Z(0) < 0$.

By numerical analysis, it shows that $Z(6\pi) > 0$. Therefore, there is one zero at least on the critical line between $t = 0$ and $t = 6\pi$. We currently know that the first zero approximates to $1/2 + 14.1347251\, i$.

5 The failure to certify zeros

We know, it is hard to prove the convergence or divergence of

$$\zeta(a+bi)=\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^{a}\cos(b\ln n)-i\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^{a}\sin(b\ln n),\quad 0<a<1.$$

Taking the first two Riemann zeros $1/2+14.1347251\,i$ and $1/2+21.02203963877\,i$, for instance, we have the following numerical calculations (see Fig.2).

Table 2: Numerical calculations for two zeros of zeta function

Partial-sum	$\zeta(1/2+14.1347251\,i)$	$\lvert c\rvert+\lvert d\rvert$	$\psi(1/2+14.1347251\,i)$	$\lvert c\rvert+\lvert d\rvert$
1	1	1	1	1
2	0.341429 -0.257458 I	0.598887	1.65857 +0.257458 I	1.91603
3	-0.226657-0.154447 I	0.381104	1.09048 +0.360469 I	1.45095
4	0.140774 +0.184661 I	0.325435	0.723054 +0.0213603 I	0.744414
5	-0.184056-0.122724 I	0.30678	0.398224 -0.286025 I	0.68425
6	0.21659 -0.0443058 I	0.260896	-0.00242168-0.364444 I	0.366865
7	-0.0549045+0.218654 I	0.273558	-0.273917-0.101484 I	0.375401
8	-0.209577-0.0992714 I	0.308849	-0.119244+0.216441 I	0.335684
9	0.102533 -0.21631 I	0.318843	0.192867 +0.0994024 I	0.29227
10	0.237318 +0.0697555 I	0.307073	0.058083 -0.186663 I	0.244746
11	-0.000145971+0.255552 I	0.255698	-0.179381-0.000866626 I	0.180247
12	-0.24381+0.100758 I	0.344568	0.0642838 +0.153927 I	0.218211
13	-0.208808-0.174375 I	0.383183	0.0992857 -0.121206 I	0.220491
14	0.0376913 -0.277653 I	0.315345	-0.147214-0.0179269 I	0.165141
15	0.253886 -0.136493 I	0.390379	0.0689812 +0.123234 I	0.192215
16	0.273897 +0.112705 I	0.386602	0.0489704 -0.125964 I	0.174934
17	0.103882 +0.285675 I	0.389557	-0.121044+0.0470055 I	0.16805
18	-0.131797+0.282397 I	0.414194	0.114635 +0.050283 I	0.164918
19	-0.295193+0.121359 I	0.416552	-0.048761-0.110755 I	0.159516
20	-0.310308-0.101736 I	0.412045	-0.0336458+0.11234 I	0.145986
21	-0.183163-0.279087 I	0.46225	0.0934991 -0.0650105 I	0.15851
22	0.0210577 -0.34031 I	0.361368	-0.110722-0.00378751 I	0.11451
23	0.21784 -0.271353 I	0.489193	0.08606 +0.0651695 I	0.151229
24	0.338457 -0.106677 I	0.445134	-0.0345573-0.0995064 I	0.134064
25	0.349485 +0.0930186 I	0.442504	-0.0235292+0.100189 I	0.123719
26	0.255599 +0.265201 I	0.5208	0.0703571 -0.0719934 I	0.14235
27	0.0903493 +0.36384 I	0.454189	-0.0948926+0.0266452 I	0.121538
28	-0.0985781+0.368393 I	0.466971	0.0940348 +0.0220921 I	0.116127
29	-0.263977+0.283978 I	0.547955	-0.0713643-0.062323 I	0.133687

30	-0.370014+0.135353 I	0.505367	0.0346726 +0.0863024 I	0.120975
31	-0.397958-0.0420656 I	0.440024	0.00672845 -0.0911157 I	0.0978441
32	-0.346979-0.211332 I	0.558311	-0.044251+0.0781506 I	0.122402
33	-0.231218-0.341341 I	0.572559	0.0715097 -0.0518589 I	0.123369
34	-0.0747187-0.411482 I	0.486201	-0.0849895+0.0182821 I	0.103272
35	0.0943008 -0.413446 I	0.507746	0.08403 +0.0163189 I	0.100349
36	0.248669 -0.35061 I	0.599278	-0.0703378-0.0465171 I	0.116855
37	0.366251 -0.235712 I	0.601963	0.0472446 +0.0683803 I	0.115625
38	0.432398 -0.0875896 I	0.519988	-0.0189027-0.0797424 I	0.0986451
39	0.440856 +0.072315 I	0.513171	-0.0104451+0.0801623 I	0.0906074
40	0.393372 +0.223131 I	0.616503	0.0370381 -0.0706533 I	0.107691
41	0.298356 +0.347075 I	0.645432	-0.0579778+0.0532911 I	0.111269
42	0.168962 +0.431138 I	0.600101	0.0714164 -0.0307723 I	0.102189
43	0.0209641 +0.467913 I	0.488877	-0.0765818+0.00600191 I	0.0825837
44	-0.129292+0.455654 I	0.584946	0.0736747 +0.0182606 I	0.0919353
45	-0.266651+0.397733 I	0.664384	-0.0636839-0.0396603 I	0.103344
46	-0.378492+0.301657 I	0.680149	0.0481575 +0.0564158 I	0.104573
47	-0.455614+0.177847 I	0.633462	-0.0289645-0.0673937 I	0.0963582
48	-0.492652+0.0383427 I	0.530995	0.00807347 +0.0721109 I	0.0801843
49	-0.48809-0.104442 I	0.592532	0.0126354 -0.0706734 I	0.0833088
50	-0.44394-0.238795 I	0.682734	-0.0315151+0.0636796 I	0.0951947
100	0.496254 +0.506912 I	1.00317	0.0344818 -0.0362057 I	0.0706875
1000	-0.644382+2.14158 I	2.78596	0.015289 +0.00401687 I	0.0193058
10000	-6.98642+1.08736 I	8.07378	0.000943084 +0.00491021 I	0.00585329
100000	-12.5386-18.5117 I	31.0504	-0.00127694+0.000932504 I	0.00220945
200000	6.98425 +30.8386 I	37.8229	0.001081 -0.00028527 I	0.00136627
300000	27.0911 +27.6726 I	54.7637	0.000629326 -0.000661189 I	0.00129052
400000	6.67927 -44.2152 I	50.8945	-0.000785391-0.000090299 I	0.00087569
500000	-8.08382+49.3371 I	57.4209	0.000701403 +0.0000896724 I	0.000791075
600000	-21.4343-50.3981 I	71.8324	-0.000584709+0.00027355 I	0.00085826
700000	57.9047 +12.0973 I	70.002	0.000101452 -0.000588861 I	0.000690313
800000	-31.5642+54.7987 I	86.3629	0.000493962 +0.000261798 I	0.000755759
900000	-54.7225-38.7884 I	93.5109	-0.000289395+0.000440568 I	0.000729963
Partial-sum	$\zeta(1/2 + 21.02203963877\, i)$	$\lvert c\rvert + \lvert d\rvert$	$\psi(1/2 + 21.02203963877\, i)$	$\lvert c\rvert + \lvert d\rvert$
1	1	1	1	1
2	0.702531 +0.641492 I	1.34402	1.29747 -0.641492 I	1.93896
3	0.442667 +0.12593 I	0.568597	1.03761 -1.15705 I	2.19466
4	0.119643 -0.255719 I	0.375362	1.36063 -0.775406 I	2.13604
5	-0.215436+0.04046 I	0.255896	1.02555 -0.479227 I	1.50478
6	0.192594 +0.0271235 I	0.219718	0.61752 -0.465891 I	1.08341
7	-0.18454+0.00208444 I	0.186625	0.240386 -0.49093 I	0.731315
8	0.156374 -0.091604 I	0.247978	-0.100528-0.397241 I	0.49777

9	-0.0419011+0.176348 I	0.218249	-0.298804-0.12929 I	0.428093
10	-0.132221-0.126707 I	0.258929	-0.208483+0.173765 I	0.382249
11	0.166205 -0.0836881 I	0.249893	0.0899432 +0.216784 I	0.306728
12	0.0533838 +0.182027 I	0.235411	0.202765 -0.048931 I	0.251696
13	-0.188211+0.045811 I	0.234022	-0.0388305-0.185147 I	0.223978
14	-0.059963-0.188669 I	0.248632	-0.167079+0.0493329 I	0.216412
15	0.17981 -0.0928809 I	0.272691	0.0726946 +0.145121 I	0.217816
16	0.138499 +0.153682 I	0.292181	0.114006 -0.101442 I	0.215448
17	-0.101979+0.185204 I	0.287183	-0.126473-0.06992 I	0.196393
18	-0.214887-0.0216951 I	0.236582	-0.0135648+0.136979 I	0.150544
19	-0.0784235-0.206111 I	0.284535	0.122899 -0.0474365 I	0.170336
20	0.142851 -0.173901 I	0.316753	-0.0983758-0.0796464 I	0.178022
21	0.227946 +0.0270416 I	0.254987	-0.0132816+0.121296 I	0.134578
22	0.111576 +0.205683 I	0.317259	0.103088 -0.0573449 I	0.160433
23	-0.0965758+0.21797 I	0.314546	-0.105064-0.0450576 I	0.150122
24	-0.233469+0.0665541 I	0.300023	0.031829 +0.106359 I	0.138188
25	-0.208913-0.131933 I	0.340845	0.0563855 -0.0921281 I	0.148514
26	-0.0496639-0.246394 I	0.296058	-0.102863+0.022333 I	0.125196
27	0.140006 -0.213801 I	0.353808	0.0868069 +0.0549254 I	0.141732
28	0.252274 -0.0617805 I	0.314054	-0.0254604-0.0970956 I	0.122556
29	0.233441 +0.122957 I	0.356398	-0.044293+0.0876423 I	0.131935
30	0.100668 +0.248276 I	0.348944	0.0884796 -0.0376763 I	0.126156
31	-0.0785315+0.260339 I	0.338871	-0.0907201-0.025613 I	0.116333
32	-0.224411+0.160493 I	0.384905	0.0551594 +0.0742327 I	0.129392
33	-0.279782-0.00454309 I	0.284325	-0.000211774-0.0908038 I	0.0910156
34	-0.228468-0.168185 I	0.396653	-0.0515257+0.072838 I	0.124364
35	-0.0946823-0.271494 I	0.366176	0.0822603 -0.030471 I	0.112731
36	0.0716286 -0.282377 I	0.354006	-0.0840506-0.0195876 I	0.103638
37	0.215056 -0.20203 I	0.417086	0.0593766 +0.0607594 I	0.120136
38	0.292763 -0.0596317 I	0.352395	-0.0183309-0.0816391 I	0.09997
39	0.285317 +0.100323 I	0.38564	-0.0257771+0.0783159 I	0.104093
40	0.198832 +0.232688 I	0.43152	0.0607077 -0.0540487 I	0.114756
41	0.0598026 +0.303828 I	0.363631	-0.0783221+0.0170917 I	0.0954138
42	-0.0944135+0.298641 I	0.393055	0.0758941 +0.0222788 I	0.0981728
43	-0.226123+0.221774 I	0.447897	-0.0558151-0.0545881 I	0.110403
44	-0.306103+0.0939839 I	0.400087	0.0241656 +0.0732022 I	0.0973678
45	-0.319027-0.054526 I	0.373553	0.0112421 -0.0753078 I	0.0865498
46	-0.26499-0.191709 I	0.456699	-0.0427946+0.0618752 I	0.10467
47	-0.157614-0.290435 I	0.448049	0.0645818 -0.0368512 I	0.101433
48	-0.01976-0.33321 I	0.35297	-0.0732721+0.00592312 I	0.0791952
49	0.121843 -0.314324 I	0.436167	0.0683312 +0.0248093 I	0.0931405

50	0.241866 -0.239527 I	0.481393	-0.0516917-0.0499872 I	0.101679
100	0.208036 +0.429927 I	0.637962	0.0446217 -0.0228872 I	0.0675089
1000	1.00999 -1.11475 I	2.12474	-0.0119641-0.0103326 I	0.0222967
10000	-4.30873-2.01282 I	6.32154	-0.00200786+0.004579 I	0.00658686
100000	-2.19417+14.8775 I	17.0717	0.00156926 +0.000193434 I	0.00176269
200000	-17.7828-11.6654 I	29.4482	-0.000590844+0.000949158 I	0.00154
300000	24.7208 -8.20642 I	32.9272	-0.000308126-0.000859296 I	0.00116742
400000	25.5457 -15.8756 I	41.4213	-0.000433137-0.000661356 I	0.00109449
500000	-18.3583-28.1735 I	46.5318	-0.000583085+0.000400014 I	0.000983099
600000	-4.17803+36.5988 I	40.7768	0.000642891 +0.0000579427 I	0.000700834
700000	8.39659 -38.8919 I	47.2885	-0.000586989-0.000112191 I	0.00069918
800000	5.17052 +42.2197 I	47.3902	0.000553099 -0.0000811279 I	0.000634226
900000	-31.9656-31.8369 I	63.8025	-0.000362941+0.000382167 I	0.000745108

Frankly, we have no faith in that $1/2 + 14.1347251\,i$, $1/2 + 21.02203963877\,i$ are two zeros of $\zeta(z)$, but very possibly zeros of $\psi(z)$. What are the causes of this inconsistency?

6 A false series equality

For the sum, difference, and product of two series, we have the following theorem.

Theorem 1 (Rearrangement of Terms [2]). *The terms of an absolutely convergent series can be rearranged in any order, and all such rearranged series will converge to the same sum. The sum, difference, and product of two absolutely convergent series is absolutely convergent. However, if the terms of a conditionally convergent series are suitably rearranged, the resulting series may diverge or converge to any desired sum.*

It is well known that the series $\psi(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}$ is only **conditionally convergent** for $0 < z < 1$. By the definition of zeta function, we have

$$\frac{2}{2^z}\zeta(z) = \frac{2}{2^z}\left(1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots\right) = \frac{2}{2^z} + \frac{2}{4^z} + \frac{2}{6^z} + \frac{2}{8^z} + \cdots,$$

$$\left(1 - \frac{2}{2^z}\right)\zeta(z) = \left(1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots\right) - \left(\frac{2}{2^z} + \frac{2}{4^z} + \frac{2}{6^z} + \frac{2}{8^z} + \cdots\right)$$

$$\overset{\text{rearranged}}{=\!=\!=} 1 - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z} = \psi(z).$$

But $\zeta(z)$ is divergent if $0 < z < 1$. In this case, the above process tries to rearrange the difference of two divergent series. It is irrational. So, we have

$$\psi(z) \neq (1 - 2^{1-z})\zeta(z), \quad 0 < z < 1. \tag{9'}$$

Therefore, the sign of $\psi(1/2)$ cannot be used to determine the sign of $\zeta(1/2)$. Finally, we want to stress that

$$\zeta(1/2) = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

which is not convergent. Its sign, needless to say, is meaningless. The below equality makes no sense

$$\xi(1/2) = -\frac{1}{8}\pi^{-\frac{1}{4}}\Gamma(\frac{1}{4})\zeta(1/2), \tag{10'}$$

due to the divergence of $\zeta(1/2)$.

7 Conclusion

The arguments for functional equation involve rearrangement of terms, without checking the absolute convergence of respective series. We want to stress that the equation cannot be logically extended from the domain $\mathrm{Re}(z) > 1$ to domain $0 < \mathrm{Re}(z) < 1$. The numerical calculations for some values imply that the zeros could be for another alternating series, not for the usual zeta function. It's worth noting that the arithmetic of divergent series can throw us into total confusion.

References

[1] E. Titchmarsh, The Theory of the Riemann Zeta-Function, Oxford University Press, 1986.

[2] R. WredeE, Theory and Problems of Advanced Calculus, McGraw-Hill, 2002.

[3] R. Pérez-Marco, Notes on the Riemann Hypothesis, arXiv:1707.01770v2, 2018.