### CRYPTOGRAPHIC GROUP AND SEMIGROUP ACTIONS

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Dedicated to Joachim Rosenthal on the occasion of his 60th birthday

ABSTRACT. We consider actions of a group or a semigroup on a set, which generalize the setup of discrete logarithm based cryptosystems. Such cryptographic group actions have gained increasing attention recently in the context of isogeny-based cryptography. We introduce generic algorithms for the semigroup action problem and discuss lower and upper bounds. Also, we investigate Pohlig-Hellman type attacks in a general sense. In particular, we consider reductions provided by non-invertible elements in a semigroup, and we deal with subgroups in the case of group actions.

**Keywords**: Discrete logarithm problem; cryptographic group action; semigroup action problem.

# 1. Introduction

The discrete logarithm problem has a long and profound history (see [8] for a recent survey). In cryptography it has been playing a key role ever since Diffie and Hellman have based the security of their famous protocol [6] on the hardness of computing discrete logarithms modulo a large prime p. The underlying group  $\mathbb{F}_p^*$  has then been generalized, most notably to the  $\mathbb{F}_q$ -rational points on an elliptic curve, due to Miller [18] and Koblitz [12]. In fact, while the discrete logarithm problem in the unit group  $\mathbb{F}_q^*$  of a finite field can be solved in subexponential time by index calculus algorithms (for an overview, see [9]), the fastest known algorithm in a general elliptic curve is basically a generic one that requires exponential time.

However, Shor's quantum algorithm [24] constitutes a polynomial time attack on the discrete logarithm problem in any group (as well as on the integer factorization problem). These observations, and reports on the progress in building quantum computers now achieving a "quantum supremacy" [2], underline the need for new concepts to build cryptosystems resistant to quantum attacks. An interesting approach is based on isogenies of supersingular elliptic curves (SIDH, [10]), which however is broken due to the Castryck-Decru attack [4].

More recently, a commutative supersingular isogeny-based Diffie-Hellman scheme (CSIDH, [3]) has been proposed as a more efficient variant, which is based on the action of the class group of an endomorphism ring on isomorphism classes of elliptic curves. This is an example for an action of

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an abelian group on a set, which as a framework suffices to build a Diffie-Hellman protocol, as has been observed by Couveignes [5] and independently by Rostovtsev and Stolbunov [22, 26].

With a somewhat different background, Maze, Monico and Rosenthal have introduced actions of commutative semigroups on sets [19, 17] in order to generalize the discrete logarithm problem, one motivation being to find examples that do not allow even an (exponential) square-root attack. The setup was further investigated by the theses of the present authors, in which Zumbrägel [27] considered a generalization to non-commutative semigroups and Gnilke [7] showed how a Pohlig-Hellman like reduction applies to semigroups with non-invertible elements.

In this work, we revisit the concept of semigroup actions for discrete logarithm based cryptosystems and connect it to recent proposals of isogeny-based cryptography. In the case of abelian group actions, with a view towards isogenies, this has been considered by Couveignes [5] and, more recently, by Smith [25] and Alamati et al. [1]. Here we aim to take a slightly more abstract viewpoint and introduce generic algorithms for the semigroup action problem. We also investigate Pohlig-Hellman type reductions and consider those provided by non-units in a semigroup and by subgroups in the context of a group action.

### 2. Cryptographic semigroup actions

In this section we briefly recall the notion of a semigroup action and its application to cryptography [19, 17].

By a *semigroup* we mean a set S with an associative binary operation (written multiplicatively). It is called a *monoid* if a neutral element exists. A *semigroup action* with respect to a semigroup S and a set X is a map

$$S \times X \to X$$
,  $(s, x) \mapsto s.x$ 

such that st.x = s.(t.x) for all  $s,t \in S$  and  $x \in X$ . Considering for  $s \in S$  the transformation  $\Phi_s: X \to X$ ,  $x \mapsto s.x$ , this means that  $\Phi_{st} = \Phi_s \circ \Phi_t$  for  $s,t \in S$ . When there is a semigroup action, then X is also called S-set.

**Definition 2.1.** Consider a semigroup S acting on a set X. The *semigroup action problem* is the problem, for given  $x, y \in X$  to find some  $s \in S$  such that y = s.x.

So the above problem asks to find preimages of the "orbit map"

$$\Psi_r \colon S \to X \,, \quad s \mapsto s.x \,.$$

Suppose that S is a group and let  $S_x = \{s \in S \mid s.x = x\}$  be the *stabilizer* subgroup of  $x \in X$ . Then the orbit map induces a bijection  $S/_{\ell} S_x \xrightarrow{\sim} S.x$  of the left cosets and the orbit. Thus solutions to the semigroup action problem (which we also call *group action problem*) are unique up to left congruence modulo the stabilizer.

Since we deal with cryptographic applications, we assume all structures to be finite and that the semigroup action is efficiently computable. This means that elements of the semigroup S and the set X are encoded by bit strings, and both the semigroup operation and the action are computable in polynomial time.

**Example 2.2.** Consider a finite cyclic group  $(G, \cdot)$  of order n, which may be seen as a  $\mathbb{Z}_n$ -module. If we "forget" its additive structure, we have

$$(\mathbb{Z}_n,\cdot)\times G\to G,\quad (s,g)\mapsto g^s,$$

and the semigroup action problem is just the discrete logarithm problem in the group G. Note that the action is efficiently computable by a square-and-multiply method, provided the group operation is efficient.

In order to set up a Diffie-Hellman like key agreement, we need some way to generate commuting elements of the semigroup S. For simplicity we assume here the semigroup to be commutative and have the following key agreement scheme:

Alice 
$$public$$
 Bob
$$x \in X$$

$$a \in S \rightarrow a.x \in X$$

$$b.x \in X \leftarrow b \in S$$

$$k_A = a.(b.x) \qquad k_B = b.(a.x)$$

Observe that both parties compute the same key ab.x = ba.x. Also notice that in the case of Example 2.2 the scheme amounts to classical Diffie-Hellman.

**Definition 2.3.** Consider a commutative semigroup S acting on a set X. The *semigroup Diffie-Hellman problem* is the problem, for given  $x, y, z \in X$  to find some  $k \in X$  such that y = a.x, z = b.x and k = ab.x = ba.x for some  $a, b \in S$ .

It is clear that if one can solve the semigroup action problem, then one can break the Diffie-Hellman protocol, while the converse direction is not obvious. There have been results on subexponential reductions in the classical case of group exponentiation [13, 15], and recently on polynomial quantum reduction for abelian group actions [11, 20].

**Example 2.4.** In isogeny-based cryptography and CSIDH, a particular case of interest is the action of an abelian group (the class group in an endomorphism ring) on a set X (of isomorphism classes of elliptic curves), cf. [5, 3, 25]. In Couveignes' work [5] the group action is also assumed to be simply transitive, and the semigroup action problem and the semigroup Diffie-Hellman problem are called *vectorization problem* and *parallelization problem*, respectively; if those are intractable, the set X is referred to as a hard homogeneous space.

It would be interesting to view SIDH also in the framework of a group action, but this seems not to be obvious, cf. [25, Sec. 15].

### 3. Generic algorithms

We use Maurer's abstract model of computation [14] to describe generic algorithms for a semigroup action. Recall that in this model one specifies

- $\bullet$  a ground set M,
- a set  $\Pi$  of certain operations  $f: M^t \to M$  of arity  $t \in \{0, 1, 2, \dots\}$ ,
- a set  $\Sigma$  of certain relations  $\rho \subseteq M^t$  of arity  $t \in \{1, 2, \dots\}$ .

There are internal state variables  $V_1, V_2, \ldots$  storing elements in M, which cannot be read directly. A generic algorithm  $A(M, \Pi, \Sigma)$  is then allowed to perform computation operations and relation queries for  $f \in \Pi$  and  $\rho \in \Sigma$ , using the internal state variables as input (and operation output). In the analysis we usually discard the relation queries and only count the number of operations performed.

**Example 3.1.** Generic algorithms in a cyclic group of order n can be modeled using  $M = \mathbb{Z}_n$ ,  $\Pi = \{+\}$  and  $\Sigma = \{=\}$ . For the discrete logarithm problem the state variables are initialized with  $V_1 = 1$  and  $V_2 = s \in \mathbb{Z}_n$  random, and the goal is to solve the *extraction problem* for the value  $s \in V_2$ .

Now let  $S \times X \to X$  be a semigroup action and fix  $x \in X$ . In order to address the semigroup action problem, we define the ground set as

$$M := S/\sim_x$$
, where  $s \sim_x t :\Leftrightarrow s.x = t.x$ ,

and denote the class of  $s \in S$  by [s]. We allow for any  $a \in S$  to perform the unary operation  $\Phi_a \colon M \to M$ ,  $[s] \mapsto a.[s] := [as]$ . The state variables are initialized with  $V_1 = [1]$  (assuming that  $1 \in S$ , otherwise we adjoin it) and  $V_2 = [s] \in M$ , which describes the solutions s to a semigroup action problem y = s.x. The goal is to solve the extraction problem for  $V_2$ , i.e., to find some  $s' \in S$  such that [s] = [s'].

If we are dealing with a group action, the set  $S/\sim_x = S/\ell S_x$  consists of the left cosets  $[s] = sS_x$  of the stabilizer subgroup of x. In this case, we can show the following result, which provides a generic lower bound of  $\Omega(\sqrt{n})$  for the group action problem in a set X of size n.

**Theorem 3.2.** Let  $S \times X \to X$  be a transitive group action, which is abelian or free. Fix  $x \in X$  and let  $M := S/_{\ell} S_x$ ,  $\Pi = \{\Phi_a \mid a \in S\}$ ,  $\Sigma = \{=\}$  as above. If  $s \in S$  is uniformly random, the success probability of a generic algorithm  $A(M, \Pi, \Sigma)$  for the group action problem y = s.x using m operation queries is at most  $\frac{1}{4}m^2/|X|$ .

*Proof.* Following the proof of [14, Thm. 1] it suffices to upper bound the probability that a collision in the state variables occurs. These entries are either of the form a.[1] = [a] or of the form b.[s], for known  $a, b \in S$ . A collision of type [a] = [a'] is independent of s and can be discarded. Moreover, a collision of type b.[s] = b'.[s] means that  $b^{-1}b' \in S_{s.x} = sS_x s^{-1}$ , but since the action is abelian or free we have  $sS_x s^{-1} = S_x$ , again independent of s.

This means that the only collisions related to s are of the form

$$a.[1] = b.[s]$$

for some  $a,b \in S$ . Such an event is equivalent to  $[s] = [b^{-1}a]$ , and since  $[s] \in M$  is uniformly random it occurs with probability 1/n, where n := |M| = |X|. So if the algorithm computes u and v values of the form a.[1] and b.[s] respectively, the probability for a collision is at most uv/n. But since  $u+v \le m$  we have  $uv \le \frac{1}{4}m^2$ , from which the result follows.

The following example shows that there may be much faster generic algorithms in case the group action is neither abelian nor free.

**Example 3.3.** Let X be a set of size n and let the symmetric group  $S := \operatorname{Sym} X$  act on X. Fix  $x \in X$  and let  $s \in S$  be random. A generic algorithm to find the coset  $[s] \in S/_{\ell} S_x$  can employ either "usual" collisions [a] = b.[s], from which a solution  $s' := b^{-1}a$  can be obtained, or collisions of the form

$$b.[s] = b'.[s],$$

i.e., c.[s] = [s] for  $c := b^{-1}b'$ . If we choose  $c \in S$  having k fixed points, then the event probability is k/n, and thus we may apply a divide-and-conquer strategy to obtain [s] in  $O(\log n)$  steps.

Furthermore, for proper semigroup actions the difficulty of the generic semigroup action problem very much depends on the structure of the semigroup, and ranges from efficient algorithms in  $O(\log n)$  to lower bounds of  $\Omega(n)$ , see the examples below. From a cryptography perspective there are however issues with applying those actions, as discussed in Section 5.

**Example 3.4.** Let S be a semigroup, X a set and  $\varphi \colon S \to X$  a bijection. Then we can make X an S-set by letting

$$s.\varphi(t) = \varphi(st)$$

for  $s,t\in S$ . We assume this action to be efficiently computable and think of the inverse map  $\varphi^{-1}$  as "hidden". For example, if G is a cyclic group with generator g, we use the bijection  $\varphi\colon (\mathbb{Z}_n,\cdot)\to G,\, s\mapsto g^s$ . Let us look at two further cases.

(1) Suppose that  $(S, \cdot) = (\{1, \dots, n\}, \min)$  and we are given a semigroup action problem instance  $x, y \in X$  where y = s.x. Let  $a = \varphi^{-1}(x)$  which we suppose is known, e.g., a = n if  $x = \varphi(n)$ . Since  $y = s.x = s.\varphi(a) = \varphi(sa) = \varphi(saa) = sa.\varphi(a) = sa.x$ , we may assume that s = sa, i.e.,  $s \leq a$ . Then for any  $t \in S$ ,  $t \leq a$ , there holds

$$t \le s \Leftrightarrow t = ts \Leftrightarrow ta = tsa \Leftrightarrow \varphi(ta) = \varphi(tsa)$$
  
  $\Leftrightarrow t.\varphi(a) = t.(s.\varphi(a)) \Leftrightarrow t.x = t.y.$ 

Hence, we can find s using binary search in  $O(\log n)$  steps.

(2) On the other hand, define  $(S, \cdot) = (\{0, s_1, \dots, s_m, 1\}, \wedge)$ , where  $\wedge$  is a semilattice operation such that

$$0 \wedge s_i = 0$$
,  $1 \wedge s_i = s_i$ , and  $s_i \wedge s_j = 0$  whenever  $i \neq j$ ,

and let  $o = \varphi(0)$  and  $e = \varphi(1)$  in X. Consider a semigroup action problem instance  $e, y \in X$  where y = s.e. As  $s.e = s.\varphi(1) = \varphi(s1) = \varphi(s)$  and  $\varphi$  is bijective, there is a unique solution s. When using a generic algorithm we may for  $t \in S$  compute  $t.e = \varphi(t)$  and  $t.y = t.(s.e) = ts.e = \varphi(ts)$ , where a collision occurs only if t = ts, which if  $s = s_i$  means that  $t \in \{0, s_i\}$ . So it requires  $\Omega(n)$  steps to find any collision and thus information about s.

Note that this example is not interesting for a Diffie-Hellman type key agreement, because the key k = a.(b.e) = b.(a.e) will usually be o.

We should mention here that Shoup's model for generic algorithms [23] is also widely used, which is based on representations of algebraic objects as random bitstrings. In the case of a cyclic group of order n one has an injective "encoding function"  $\sigma \colon \mathbb{Z}_n \to \{0,1\}^*$ , and the algorithm maintains a list of encodings  $(\sigma(x_1), \ldots, \sigma(x_k))$  to which entries  $\sigma(x_i + x_j)$  may be appended by performing oracle queries. The bitstring representation allows a generic algorithm to perform additional operations, like sorting or hashing of algebraic objects, and thus enables algorithms based on pseudorandom functions such as Pollard's rho method. It is possible to show a square-root lower bound for the group action problem in Shoup's model by adapting the proof of Theorem 3.2.

Regarding upper bounds for the complexity of the semigroup action problem, generic algorithms often aim at finding a collision in square-root time. However, in order to deduce a solution from it one needs the ability to invert some elements in the semigroup. Below we present a version of Shanks' baby-step-giant-step method as well as a Pollard-rho type attack for the case of group actions.

Note that in the quantum world generic attacks on group actions may be faster than square-root time. Indeed, there are subexponential algorithms solving the group action problem for (free) abelian group actions based on Kuperberg's quantum algorithm, cf. [20, Sec. 7].

# 4. Collision attacks

Let X be an S-set of size n and let  $x \in X$ ,  $y \in S.x$ . We now discuss generic upper bounds for the corresponding semigroup action problem, i.e., to find  $s \in S$  such that y = s.x. An important class of generic algorithms relies on finding a collision a.x = b.y for some  $a, b \in S$ , from which we can deduce a solution  $s := b^{-1}a$ , provided that b is invertible.

For analyzing such algorithms, the following combinatorial result on the probability that two random subsets are disjoint is useful (cf. [7, Lem. 68]).

**Lemma 4.1.** Let X be a set of size n and let  $A, B \subseteq X$  be uniformly chosen random subsets of size  $k, \ell$  respectively. Then

$$1 - \frac{k\ell}{n} \le \Pr(A \cap B = \varnothing) \le \exp(-\frac{k\ell}{n}).$$

*Proof.* Let us assume w.l.o.g. that B is fixed. For the lower bound we have

$$\Pr(A \cap B \neq \varnothing) = \Pr(\bigcup_{x \in B} \{x \in A\}) \le \sum_{x \in B} \Pr(x \in A) = \ell \cdot \frac{k}{n}$$

by the union bound. For the upper bound we may assume that A consists of k randomly chosen elements of X, possibly with repetition, by which the probability of being disjoint to B only can increase. Let  $T_i$  be the event that the i-th element of A does not lie in B, for  $1 \le i \le k$ . Then these events are independent of probability  $1 - \frac{\ell}{n}$ , whence

$$\Pr(A \cap B = \varnothing) \le \Pr(\bigcap_i T_i) = (1 - \frac{\ell}{n})^k \le \exp(-\frac{\ell k}{n}),$$

since  $1+x \leq \exp x$  holds for all  $x \in \mathbb{R}$ .

A time-memory trade-off. We first present a simple time-memory trade-off attack adopting Shanks' baby-step-giant-step method (cf. [7, Sec. 4.2]). Let the semigroup S be a monoid with group of invertible elements  $S^*$ . The algorithm is described as follows.

- (1) precompute a table of entries  $(b_j, y, b_j)$  for  $b_j \in S^*$  random
- (2) for  $a \in S$  random, check if  $a.x = b_j.y$  for some j, if yes output  $s := b_j^{-1}a$

For the analysis, according to the proof of Lemma 4.1 we may bound the collision probability even if the set  $B := \{b_1.y, b_2.y, ...\}$  is of a special nature (as the  $b_j$  have to be invertible). It is enough to require that the set  $A := \{a_1.x, a_2.x, ...\}$  generated by the algorithm behaves as random, which holds if the orbit map  $\Psi_x \colon S \to X$ ,  $s \mapsto s.x$  has (nearly) constant-sized preimages.

Suppose the algorithm generates k := |A| elements  $a_i.x$  with  $a_i \in S$  and  $\ell := |B|$  elements  $b_j.y$  with  $b_j \in S^*$ , which is upper bounded by  $|S^*|$ . Then we infer from Lemma 4.1 that the success probability of the algorithm is at least  $\frac{1}{2}$  provided that  $\exp(-\frac{k\ell}{n}) \leq \frac{1}{2}$ , or equivalently,

$$k\ell > n \ln 2$$
.

Therefore, an optimal choice of parameters is  $k, \ell \in \Theta(\sqrt{n})$ , which is possible if a sufficient amount of invertible elements is available and there is enough memory. In such a case the complexity of this algorithm is seen to be  $O(\sqrt{n})$ . Next we show an approach how to drastically reduce the memory requirement.

A Pollard-rho attack for group actions. Suppose now that a group G acts transitively on a set X of size n. Given  $x, y \in X$  the group action problem asks to find  $g \in G$  such that y = g.x. We describe a Pollard-rho type birthday attack for solving this problem, which is a slight adaption and simplification of an algorithm given by Monico [19, Sec. 4.2].

As before, the idea is to generate elements  $a_i.x$ ,  $b_j.y \in X$  in a pseudorandom way and to provoke a collision  $a_i.x = b_j.y$  from which a solution  $g := b_j^{-1}a_i$  to the group action problem is deduced. But here, this is to be done using very little memory while still maintaining a heuristic square-root complexity  $O(\sqrt{n})$ .

The algorithm depends on a pseudorandom function  $f: X \to G$ . Define a recursive sequence  $(a_1, a_2, a_3, \dots)$  in G by

$$a_1 := a \in G \text{ random}, \qquad a_{i+1} := f(a_i.x) a_i \text{ for } i \ge 1,$$

hence  $a_2 = f(a.x) a$ ,  $a_3 = f(f(a.x) a.x) f(a.x) a$ , etc. Observe that if  $a_i.x = a_j.x$  for some i, j then  $a_{i+r}.x = a_{j+r}.x$  for all r. Similarly, define a recursive sequence  $(b_1, b_2, b_3, ...)$  in G by

$$b_1 := b \in G \text{ random}, \quad b_{j+1} := f(b_j, y) b_j \text{ for } j \ge 1,$$

and note that if  $a_i.x = b_j.y$  then  $a_{i+r}.x = b_{j+r}.y$  for all r. The algorithm now consists of two steps:

(1) construct a-loop, i.e., find smallest k such that  $a_k.x = a_{2k}.x$  construct b-loop, i.e., find smallest  $\ell$  such that  $b_{\ell}.y = b_{2\ell}.y$ 

(2) find a collision in the sets

$$A := \{a_{k+i}.x \mid 0 \leq i < k\} \quad \text{and} \quad B := \{b_{\ell+j}.y \mid 0 \leq j < \ell\}$$

by checking  $a_k.x = b_{\ell+s}.y$  for s = 0, 1, 2, ...; if successful output  $g := b_{\ell+s}^{-1} a_k$ 

Notice here that if  $a_{k+i}.x = b_{\ell+j}.y$  for some i, j, then  $a_k.x = a_{2k}.x = b_{\ell+s}.y$  for s := j+k-i.

We sketch a heuristic analysis of this algorithm. With good probability we have |A| = k and  $|B| = \ell$  (if the preperiod does not exceed the period) and we expect  $k, \ell \in O(\sqrt{n})$  by the birthday paradox. Assuming that A, B behave as random subsets of X we can estimate the probability of a collision using Lemma 4.1 by

$$\Pr(A\cap B\neq\varnothing)\geq 1\!-\!\exp\left(-\tfrac{k\ell}{n}\right)\in\Omega(1)\,,$$

provided that  $k, \ell \in \Omega(\sqrt{n})$ . These arguments show that the algorithm has an expected running time of  $O(\sqrt{n})$  and succeeds with non-negligible probability.

This algorithm may be adapted to work also for proper semigroup actions, in case the semigroup S has sufficiently many invertible elements  $S^*$  and we are able to define a sequence  $(b_1, b_2, b_3, \dots)$  in  $S^*$ . Alternatively, as pointed out by Maze [16], semigroups with a large subgroup can be attacked by excluding all non-units first and then employing the above algorithm on the unit subgroup.

# 5. Pohlig-Hellman type reductions

In this section we examine how the hardness of the semigroup action problem is affected by exploiting certain substructures. We recollect the framework of Pohlig-Hellman type reductions from [7, Sec. 4.3] and discuss a few special cases for cryptographic group actions.

Recall that the classical Pohlig-Hellman algorithm [21] essentially reduces the difficulty of the discrete logarithm problem in a cyclic group G of order n to that in a group of order the largest prime factor of n. The algorithm can be viewed as applying multiplication-by-m maps

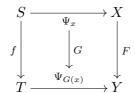
$$\lambda_m \colon \mathbb{Z}_n \to \mathbb{Z}_n \,, \quad s \mapsto ms$$

in order to reduce the problem to the action of the (smaller) ideal  $m\mathbb{Z}_n$ . The following general concept captures this scenario and many others.

**Definition 5.1.** Let S and T be semigroups, let X be an S-set and Y be a T-set. A reduction (f, F, G) consists of maps  $f: S \to T$  and  $F, G: X \to Y$  such that

$$f(s).G(x) = F(s.x)$$

for all  $s \in S$  and  $x \in X$ , see the diagram below.



For a general reduction (f, F, G) the map  $f: S \to T$  is not required to be a semigroup homomorphism. However, if F = G it is reasonable to assume this, since f(st).F(x) = F(st.x) = F(s.t.x) = f(s).F(t.x) = f(s).f(t).F(x) = f(s)f(t).F(x) for all  $s, t \in S$  and  $x \in X$ .

**Example 5.2.** Let G be a cyclic group with generator  $\alpha$  of composite order n = km. In the Pohlig-Hellman setup above we may apply the isomorphism  $m\mathbb{Z}_n \cong \mathbb{Z}_k$  after the multiplication-by-m map  $\lambda_m$ , which results in the natural map  $\pi \colon \mathbb{Z}_n \to \mathbb{Z}_k$ . Then one has a reduction  $(\pi, \Phi_m, \Phi_m)$ , where  $\Phi_m \colon \langle \alpha \rangle \to \langle \alpha^m \rangle$ ,  $g \mapsto g^m$ . Indeed, there holds  $(g^m)^{\pi(s)} = (g^s)^m$  for any  $s \in \mathbb{Z}_n$  and  $g \in G$ , see below.

$$\mathbb{Z}_{n} \xrightarrow{\Psi_{g}} \langle \alpha \rangle$$

$$\downarrow^{\Phi_{m}} \qquad \downarrow^{\Phi_{m}}$$

$$\mathbb{Z}_{k} \xrightarrow{\Psi_{g^{m}}} \langle \alpha^{m} \rangle$$

Given any reduction (f, F, G), an adversary who can solve the semigroup action problem in T can restrict the search in S to preimages of the solutions in T under the map f. Indeed, given a semigroup action problem instance  $x, y \in X$  where y = s.x, one reduces it to the instance  $G(x), F(y) \in Y$  where F(y) = f(s).G(x). Nevertheless one should note the following caveats:

- (1) In general, for a single solution  $t \in \text{im } f \subseteq T$  of F(y) = t.G(x) it is not clear whether there always exists  $s \in f^{-1}(t)$  such that y = s.x.
- (2) If the above solution  $t \in \text{im } f$  is unique, then t = f(s) and we can deduce that  $s \in f^{-1}(t)$ , which may be a much smaller set than S. However, the preimage under f could be hard to obtain, and often it does not admit a useful semigroup action structure itself.

Similarly to the Pohlig-Hellman algorithm we may apply several reductions  $(f_1, F_1, G_1), \ldots, (f_r, F_r, G_r)$  in parallel to further narrow down the search space to  $s \in f_1^{-1}(t_1) \cap \cdots \cap f_r^{-1}(t_r)$  if suitable solutions  $t_1, \ldots, t_r$  of the reduced semigroup action problems are found.

**Examples based on non-units.** We call a reduction *effective* if the maps f, F, G are efficiently computable and there holds 1 < |T| < |S|. The next result describes a very general class of reductions.

**Proposition 5.3.** Let S be a monoid, X an S-set and  $m \in S$ . Then the triple  $(\lambda_m, \Phi_m, \mathrm{id})$  forms a reduction, which is effective iff m is not left-absorbing and not invertible, provided the semigroup operation and action are efficient.

*Proof.* For all  $x \in X$  and  $s \in S$  there holds that

$$\lambda_m(s).x = ms.x = m.(s.x) = \Phi_m(s.x).$$

The reduction maps the semigroup S onto its right ideal mS, which is a non-trivial proper subsemigroup of S iff  $\lambda_m$  is not constant or surjective.  $\square$ 

Akin to the Pohlig-Hellman approach for groups of prime power order, we may apply this reduction recursively. The idea for solving the semigroup action problem y = s.x is to find  $t_1 := m_1 s$  from  $m_1.y = t_1.x$  (bearing in mind the caveats above), and for obtaining  $t_1$  to find  $t_2 := m_2 m_1 s$  from  $m_2 m_1.y = t_2.x$  etc., for suitable non-units  $m_1, m_2, ...$  of S.

Hence, the result constitutes a considerable threat to the security of a cryptosystem based on proper semigroup actions. On the other hand, the rather degenerate semigroup S of Example 3.4 (2) contains many non-absorbing non-units and yet has a generic complexity of  $\Omega(n)$  for the semigroup action problem. In this case, one has |mS|=2 for all non-absorbing non-units m and the corresponding preimage sets are not useful (see also [7, Ex. 79]).

In conclusion, while it is conceivable that certain semigroup actions avoid the attacks outlined in this section and are in fact interesting for cryptography, possible candidates have to be chosen very carefully.

**Examples based on automorphisms.** Now we consider a second family of reductions, which also applies to group actions. Let us start with a general semigroup S and an S-set X. Its automorphisms are the bijective maps  $\varphi \colon X \to X$  such that  $\varphi(s.x) = s.\varphi(x)$  for all  $s \in S, x \in X$ . We use automorphism groups to construct equivalence relations on X compatible with the action as follows. Suppose that a group H acts on X by automorphisms, i.e., we have  $H \times X \to X$ ,  $(h,x) \mapsto h.x$ , such that h.s.x = s.h.x for  $h \in H$ ,  $s \in S$ ,  $x \in X$ . Let  $X/\sim$  be the set of its orbits  $[x] = \{h.x \mid h \in H\}$ . This induces an action

$$S \times X/\sim \to X/\sim$$
,  $(s,[x]) \mapsto [s.x]$ .

Suppose next that the semigroup S is a commutative monoid with unit group  $S^*$ . Then each  $h \in S^*$  defines an S-automorphism  $\Psi_h \colon X \to X$  by  $x \mapsto h.x$ , so we may consider any subgroup H of  $S^*$  as a group of automorphisms of X. The relation  $s \approx t$  iff sH = tH then provides a semigroup congruence on S, and we denote by  $S/H := \{sH \mid s \in S\}$  its classes. Moreover,  $s \approx t$  implies  $s.x \sim t.x$ , for any  $s,t \in S$  and  $x \in X$ , so we can define an action

$$S/H \times X/\sim \to X/\sim$$
,  $([s]_{\approx}, [x]_{\sim}) \mapsto [s.x]_{\sim}$ .

We hence obtain a reduction (f, F, F) with  $f: S \to S/\approx$  and  $F: X \to X/\sim$  being the natural maps, see below.

$$S \xrightarrow{\Psi_x} X$$

$$\downarrow f \qquad \qquad \downarrow F \qquad \qquad \downarrow F$$

$$S/H \xrightarrow{\Psi_{[x]}} X/\sim$$

In particular, if S = G is an abelian group, so that  $S^* = G$ , we can employ any subgroup H of G and thus  $X/\sim$  becomes a G/H-set. Regarding the practical implications however, the reduced action may not be efficiently computable, because the equality of orbits could be difficult to check. In any case, an effective reduction attack cannot be of generic type, since we are guaranteed a lower square-root complexity for such algorithms due to Theorem 3.2.

This approach was also described in the context of CSIDH, cf. [25, Sec. 12].

Remark 5.4. As in Example 2.2 consider the discrete logarithm setup of an abelian group X of order n, i.e., we have the action of  $(\mathbb{Z}_n, \cdot)$  on X by exponentiation. Every group automorphism is also an automorphism of X as an  $\mathbb{Z}_n$ -set, thus any subgroup H of the automorphism group of X induces an action

$$(\mathbb{Z}_n,\cdot)/H \times X/\sim \to X/\sim, \quad ([s],[x]) \mapsto [x^s].$$

In the special case of  $H = \{\pm 1\}$  we have the orbits  $[x] = \{x, x^{-1}\}$ , reflecting the practice in elliptic curve cryptography to identify the points  $\pm P$  and thus use only the x-coordinate [18].

Such reductions could potentially weaken the security of the discrete logarithm problem in groups for which the automorphism group has several subgroups, e.g., in cyclic groups of order n where  $\varphi(n) = |\mathbb{Z}_n^*|$  is smooth. However, as mentioned above these reductions appear to be not effective in general.

Let us state a concrete example illustrating this phenomenon.

**Example 5.5.** Consider the discrete logarithm problem in a group X of prime order n=29 (e.g., a subgroup of  $\mathbb{Z}_{59}^*$ ). Then we have  $S=(\mathbb{Z}_{29},\cdot)$  and hence

$$S^* = \mathbb{Z}_{29}^* \cong \mathbb{Z}_{28} \cong \mathbb{Z}_4 \times \mathbb{Z}_7$$
.

Therefore, one could try to exploit the subgroups  $H_1 = \langle \alpha_1 \rangle$ ,  $H_2 = \langle \alpha_2 \rangle$  of order 4 and 7 respectively, say  $\alpha_1 = 12 \in \mathbb{Z}_{29}^*$  and  $\alpha_2 = 7 \in \mathbb{Z}_{29}^*$ , to attack to problem. But the reduced group actions seem to be more difficult to compute, e.g., we have  $\mathbb{Z}_{29}^*/\langle 7 \rangle \times X/\sim \to X/\sim$  where  $[x] = \{x, x^7, x^{20}, x^{24}, x^{23}, x^{16}, x^{25}\} \in X/\sim$ .

# 6. Conclusion

Cryptographic group actions or semigroup actions provide a framework that encompasses both the classical discrete logarithm problem as well as interesting proposals for post-quantum cryptography. In this article we have examined this framework from a theoretical viewpoint and studied the semigroup action problem as an analog of the discrete logarithm problem.

In the case of group actions, the generic complexity can be considered well understood, as there is both a square-root lower bound and a square-root upper bound for the group action problem. On the other hand, for proper semigroup actions the situation appears to be less clear. The generic lower bound may exceed the square-root barrier, however such instances tend to be degenerate and not interesting for cryptography applications.

We also have discussed the potential of certain substructures to weaken the hardness of the semigroup action problem, in particular in the presence of non-units. While such substructures do not guarantee to practically break the semigroup action problem, they should be taken into account when designing cryptosystems based on semigroup or group actions.

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