Luby-Rackoff Ciphers from Weak Round Functions?

(full version)

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Abstract. The Feistel-network is a popular structure underlying many block-ciphers where the cipher is constructed from many simpler rounds, each defined by some function which is derived from the secret key.

Luby and Rackoff showed that the three-round Feistel-network – each round instantiated with a pseudorandom function secure against adaptive chosen plaintext attacks (CPA) – is a CPA secure pseudorandom permutation, thus giving some confidence in the soundness of using a Feistel-network to design block-ciphers.

But the round functions used in actual block-ciphers are – for efficiency reasons – far from being pseudorandom. We investigate the security of the Feistel-network against CPA distinguishers when the only security guarantee we have for the round functions is that they are secure against non-adaptive chosen plaintext attacks (nCPA). We show that in the information-theoretic setting, four rounds with nCPA secure round functions are sufficient (and necessary) to get a CPA secure permutation. Unfortunately, this result does not translate into the more interesting pseudorandom setting. In fact, under the so-called Inverse Decisional Diffie-Hellman assumption the Feistel-network with four rounds, each instantiated with a nCPA secure pseudorandom function, is in general not a CPA secure pseudorandom permutation.

We also consider other relaxations of the Luby-Rackoff construction and prove their (in)security against different classes of attacks.

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1 Introduction

Feistel-network. The Feistel-network is a popular design approach for block-ciphers where the cipher over $\{0,1\}^{2n}$ is constructed by cascading simpler permutations, each constructed from a round function $f : \{0,1\}^n \to \{0,1\}^n$. The secret key of the cipher is only used to choose the particular round functions.

Luby-Rackoff Ciphers. In their celebrated paper [LR86] Luby and Rackoff prove that the three-round Feistel-network is an adaptive chosen plaintext (CPA) secure block-cipher – i.e. a pseudorandom permutation (PRP) – if each round is instantiated with an independent CPA secure pseudorandom function (PRF), and with one extra round even adaptive chosen ciphertext (CCA) security is achieved. Besides reducing PRPs to PRFs, this result also gives some confidence in the soundness of using a Feistel-network to design block-ciphers. But unlike in the Luby-Rackoff ciphers, in most block-ciphers based on Feistel-networks the round functions are not independent (in order to keep the secret key short) and also far from being pseudorandom (for efficiency reasons). Instead, the number of rounds is much larger than four (which was sufficient for the Luby-Rackoff constructions).

In order to achieve more efficient constructions of PRPs from PRFs, many researchers have investigated the security of weakened versions of the Luby-Rackoff ciphers. Several variations of the ciphers were proven to be pseudorandom where for example the round functions were not required to be independent [Pie90], some round functions were replaced by weaker primitives than PRFs [Luc96,NR99] or the distinguisher was given direct oracle access to some of the round functions [RR00]. These results further fortify the confidence in using Feistel-networks to design block ciphers.

All these relaxed constructions need at least some of the round functions to be CPA secure PRFs in order to get a CPA secure PRP. In this paper, we investigate for the first time – to the best of our knowledge – the CPA security of the permutation one gets by a Feistel-network where none of the round functions is guaranteed to be CPA secure. In particular, we investigate the security of the Feistel-network where each round is instantiated with a *non-adaptive* chosen plaintext (nCPA) secure round function. Although nCPA security is still a strong requirement, this was the weakest natural class of attacks we could imagine which does not make the Feistel-network trivially insecure against CPA attackers. For example round functions which are only secure against known-plaintext attacks (KPA), i.e. look random on random inputs, are too weak.[1]

Pseudo- and Quasirandomness. Informally, a pseudorandom function PRF is a family of functions which can be efficiently computed, and where a random member from the family cannot be distinguished from a uniform random func-

[1] Just consider a function f which satisfies $f(0\dots0) = 0\dots0$ but otherwise looks random. This f is KPA secure as a random query is unlikely to be the all zero string. But a Feistel-network build from such functions will output $0\dots0$ on input $0\dots0$ and thus is easily seen not to be CPA (or even nCPA) secure.

tion (URF) by any efficient adversary. Pseudorandom permutations (PRP) are defined analogously. As usual in cryptography, an adversary is efficient if he is in P/poly, i.e. in non-uniform polynomial time (but almost all our results also hold when considering uniform adversaries; the only exception is addressed in Footnote 13). A *quasirandom* function (QRF) (similarly for a quasirandom permutation (QRP)) is defined similar to a *pseudorandom* one but where one does not require the distinguisher or the function to be efficient, only the number of queries the distinguisher is allowed to make is bounded. Quasirandomness can be seen as an extension of the concept of statistically close distributions to systems which can be queried interactively.

In order to prove that some system – which is built from *pseudorandom* components – is pseudorandom itself, it is often enough to prove it to be *quasirandom* when the components are replaced by the corresponding ideal systems. In particular, to prove the security of the original three-round Luby-Rackoff cipher it is enough to prove – the purely information-theoretic result – that the network instantiated with URFs is a CPA secure QRP. It then immediately follows that the construction is a CPA secure PRP when the URFs are replaced by CPA secure PRFs, since if it was not a CPA secure PRP, we could use the distinguisher for it to build a distinguisher for the CPA secure PRF (via a standard hybrid argument). Similarly one can easily show that if the round functions are only nCPA or only KPA secure PRFs, the construction is a secure PRF, but only against the class of attacks nCCA (hence also nCPA) and KPA, respectively.

2 Contributions

Our results and related work are summarized in Fig. 2 on page 5.

(In)secure Relaxations of the Three-Round Luby-Rackoff Cipher. In the pseudo- and quasirandom setting, the three-round Feistel-network is – as mentioned above – $\mathsf{ATK} \in \{\mathsf{CPA}, \mathsf{nCPA}, \mathsf{KPA}\}$ secure, when the round functions are ATK-secure. Moreover it is known that one can replace the first round with a pairwise independent permutation [Luc96,NR99].[2] We further relax this by showing that the function in the last round only needs to be secure against known plaintext attacks (KPA). This resolves an open question posed by Minematsu and Tsunoo in [MT05]. Furthermore, for $\mathsf{ATK} = \mathsf{KPA}$ we show that the first round is not necessary – as opposed to when $\mathsf{ATK} \in \{\mathsf{CPA}, \mathsf{nCPA}\}$ – and that it is sufficient to instantiate the (two) round functions with a single instantiation of a KPA secure function.

But the second round seems to be the crucial one for $\mathsf{ATK} \in \{\mathsf{CPA}, \mathsf{nCPA}\}$. We show that for constructing a CPA secure permutation – i.e. PRP or QRP depending on the setting – one cannot in general instantiate the second round with a function which is only nCPA secure by constructing a counter-example,

[2] In fact, the permutation must only be such that on any two values, the collision probability on one half of the domain is small. For example one can use one normal Feistel round instantiated with an almost XOR-universal function.

i.e. a nCPA secure function such that the three-round Feistel-network with this function in the second, and any random functions in the first and third round can easily be distinguished from a uniformly random permutation (URP) with only three adaptively chosen queries. Similarly, if one instantiates the second round with a KPA secure function, then the construction will in general not even be nCPA secure.

Four Rounds with non-adaptive Round Functions. As a consequence, three rounds with nCPA secure round functions are not enough to get CPA security. On the positive side, we show that one extra nCPA secure round is sufficient (and necessary) in the quasirandom setting. Note that for the translation of a security proof from quasi- to pseudorandom systems – as described at the end of the previous section – it is crucial that we can construct a distinguisher for the components from a distinguisher for the whole system. But here the components have a weaker security guarantee (i.e. nCPA) than what we prove for the whole system (i.e. CPA). So even when we have a CPA distinguisher for the four-round Feistel-network, we cannot construct a nCPA distinguisher for any round function. This is not just a shortcoming of the used approach, but indeed, in the pseudorandom setting the situation is different: we show that here four rounds are not enough to get CPA security. To show this we construct a nCPA secure PRF, such that the four-round Feistel-network with such round functions can easily be distinguished from URP with only three adaptive queries.

This phenomenon – i.e. that some construction implies adaptive security for quasirandom but not for pseudorandom systems – has already been proven [MP04,MPR06,Pie05] for two simple constructions: the sequential composition $f \triangleright g(.) \overset{\text{def}}{=} g(f(.))$ and the parallel composition $f \star g(.) \overset{\text{def}}{=} f(.) \star g(.)$ (where $\star$ stands for any group operation). The security proofs from [MP04] in the quasirandom setting crucially use the fact that the sequential composition of two permutations is a URPs whenever at least one of the permutations is a URP, similarly the parallel composition of two functions is a URF whenever one of the components is a URF. The Feistel-network does not have this nice property of being ideal whenever one of the components is ideal, and we have to work harder here (using a more general approach from [MPR06]). Our counter-example for the pseudorandom setting – i.e. a four-round Feistel-network with nCPA secure PRFs as round functions that is not a CPA secure PRP – is similar to the counter-examples for sequential and parallel composition shown in [Pie05,Ple05]. In [Ple05], it is shown that the sequential composition of arbitrarily many nCPA secure PRFs will not be a CPA secure PRF in general, whereas for the parallel composition only a counter-example with two components is known [Pie05]. For the Feistel-network we also could only find a counter-example for four rounds. So we cannot rule out the possibility that five or more rounds imply adaptive security. However, if this was the case, then it seems likely that – like for sequential composition [Mye04] – there is no black-box proof for this fact.[3]

[3] Myers [Mye04] constructs an oracle relative to which there exist PRPs that are nCPA secure, but for which their sequential composition is not a CPA secure PRP.

<table>
<tr><th>Construction</th><th>Quasirandom</th><th>Pseudorandom</th><th>Reference</th></tr>
<tr><td>$\psi[RRR]$</td><td colspan="2">CPA, nCCA, ¬ CCA</td><td>[LR86,Mau02]</td></tr>
<tr><td>$H \triangleright \psi[RR]$</td><td colspan="2">CPA, ¬ nCCA</td><td>[Luc96,NR99]</td></tr>
<tr><td>$H \triangleright \psi[RK]$</td><td colspan="2">CPA , ¬ nCCA</td><td>§4</td></tr>
<tr><td>$\psi[RNR]$</td><td colspan="2">¬ CPA</td><td>§5</td></tr>
<tr><td>$\psi[NNNN]$</td><td>CPA</td><td>¬ CPA (under IDDH)</td><td>§6 and §7</td></tr>
<tr><td>$\psi[RRRR]$</td><td colspan="2">CCA</td><td>[LR86,Mau02]</td></tr>
<tr><td>$H \triangleright \psi[RR] \triangleright H^{-1}$</td><td colspan="2">CCA</td><td>[Luc96,NR99]</td></tr>
<tr><td>$\psi[NNNNN]$</td><td>CCA</td><td>?</td><td>§8</td></tr>
<tr><td>$\psi[RR]$</td><td colspan="2">KPA, ¬ nCPA</td><td>[MT05]</td></tr>
<tr><td>$\psi[K^2]$</td><td colspan="2">KPA, ¬ nCPA</td><td>§4</td></tr>
<tr><td>$\psi[NNN]$</td><td colspan="2">nCCA, ¬ CPA</td><td>§4</td></tr>
<tr><td>$H \triangleright \psi[NK]$</td><td colspan="2">nCPA, ¬ nCCA</td><td>§4</td></tr>
<tr><td>$\psi[RKR]$</td><td colspan="2">¬ nCPA</td><td>§5</td></tr>
</table>

Fig. 1. Security of the Feistel-network ψ with various security guarantees on the round functions. Here $\psi[f_1 \cdots f_k](\cdot)$ denotes the k-round Feistel-network with f_i in the i'th round, and $\psi[f^2] \stackrel{\text{def}}{=} \psi[ff]$ – i.e. the same function f in both rounds. Each occurrence of R, N, and K stands for an independent CPA, nCPA, and KPA secure function (i.e. a PRF or a QRF depending on the setting) respectively. The same holds for H which is any "lightweight" permutation from which we only require that the collision probability be small on the left half of the output, an almost pairwise independent permutation or a Feistel round instantiated with an almost XOR-universal function is thus sufficient. The results in gray are implied by other results in the table.

What about CCA Security? While it seems unlikely in the pseudorandom setting to achieve CPA security (and hence also CCA security) of the Feistel-network with nCPA secure round functions, we show that (even) CCA security can be achieved in the quasirandom setting. In particular, we show that the five-round Feistel-network with nCPA secure QRFs is a CCA secure QRP.

Unconditional vs. Conditional Counter-examples. The counter-example showing that the three-round Feistel-network with a nCPA secure PRF F in the second round is not adaptively secure is unconditional[4] and black-box; with this we mean that we can construct F starting from any (nCPA secure) PRF via a

The idea behind this oracle is quite general, and we see no reason (besides being technically challenging) why one should not be able to construct a similar oracle for the Feistel-network, and thus also rule out a black-box proof for showing that the Feistel-network with nCPA secure PRFs as round functions is a CPA secure PRP.

[4] I.e. we make no other assumption besides the trivially necessary one that pseudorandom functions – which are equivalent to one-way functions [HILL99,GGM86] – exist at all.

reduction which uses this PRF only as a black-box.[5] As four rounds are enough to get adaptive security for quasirandom systems, there cannot be a black-box counter-example (like for three rounds) for the four (or more) round case. Thus it is not surprising that our counter-example for four rounds is not unconditional. It relies on the so-called Inverse Decisional Diffie-Hellman assumption. The fact that there is no black-box counter-example can be used to show that there is in some sense no "generic" adversary which breaks the adaptive security of the four-round Feistel-network with non-adaptive round functions. What "generic" actually means will not be the topic of this paper, but see Sect. 4 from [Pie06] for the corresponding statement for sequential composition.

3 Basic Definitions and Random Systems

We use capital calligraphic letters like $\mathcal{X}$ to denote sets, capital letters like X to denote random variables and small letters like x denote concrete values. To save on notation we write X^i for $(X_1, X_2, \ldots, X_i)$.

For $x \in \{0,1\}^{2n}$ we denote with ${}_Lx$ and ${}_Rx$ the left and right half of x respectively, so $x = {}_Lx \| {}_Rx$. Similarly for any function f with range $\{0,1\}^{2n}$, we denote with ${}_Lf$ (${}_Rf$) the function one gets by ignoring the right (left) half of the output of f. For two functions $f(.)$ and $g(.)$ we denote with $f \triangleright g(.) \overset{\text{def}}{=} g(f(.))$ the sequential composition of f and g.[6] For a (randomized) function f we denote with $\mathsf{coll}_k(f)$ the collision probability of any fixed k-tuple of distinct inputs, i.e.

$$\mathsf{coll}_k(f) = \max_{x_1,\ldots,x_k} \mathsf{P}(\exists i,j; 1 \le i < j \le k : f(x_i) = f(x_j)).$$

If f denotes a uniform random function with range $\{0,1\}^n$, then $\mathsf{coll}_k(f) \le k^2/2^{n+1}$, this is called the *birthday bound* which we will use quite often.

Definition 1 (Feistel-network) *The (one-round) Feistel-network $\psi[f] : \{0,1\}^{2n} \to \{0,1\}^{2n}$ is a permutation based on a function $f : \{0,1\}^n \to \{0,1\}^n$, and is defined as follows*

$$\psi[f](x) \overset{def}{=} (f({}_Lx) \oplus {}_Rx) \| {}_Lx.$$

With $\psi[f_1 \cdots f_k] \overset{def}{=} \psi[f_1] \triangleright \psi[f_2] \triangleright \cdots \triangleright \psi[f_k]$ we denote the k-round Feistel-network based on (randomized) round functions $f_1, \ldots, f_k$, here the randomness used by any function is always assumed to be independent of the randomness of the other round functions. The k round Feistel-network where the same instantiation of a function f is used for all rounds is denoted by $\psi[f^k] \overset{def}{=} \psi[\underbrace{f \cdots f}_{k \text{ times}}]$.

[5] We build F from a pseudorandom involution (PRI), how to construct a PRI from a PRP (via a black-box reduction) has been shown in [NR02].

[6] Note that $f \triangleright g$ is usually denoted with $g \circ f$.

Random Systems: Many results from this paper are stated and proven in the random systems framework of [Mau02]. A *random system* is a system which takes inputs $X_1, X_2, \ldots$ and generates, for each new input X_i, an output Y_i which depends probabilistically on the inputs and outputs seen so far. We define random systems in terms of the distribution of the outputs Y_i conditioned on $X^i Y^{i-1}$ (i.e. the actual query X_i and all previous input/output pairs $X_1 Y_1, \ldots, X_{i-1} Y_{i-1}$).

Definition 2 (Random systems) *An* $(\mathcal{X}, \mathcal{Y})$-random system $\mathbf{F}$ *is a sequence of conditional probability distributions* $\mathsf{P}^{\mathbf{F}}_{Y_i|X^iY^{i-1}}$ *for* $i \geq 1$. *Here we denote by* $\mathsf{P}^{\mathbf{F}}_{Y_i|X^iY^{i-1}}(y_i, x^i, y^{i-1})$ *the probability that* $\mathbf{F}$ *will output* y_i *on input* x_i *conditioned on the fact that* $\mathbf{F}$ *did output* y_j *on input* x_j *for* $j = 1, \ldots, i-1$.

As special classes of random systems we will consider *random functions* (which are exactly the stateless random systems) and *random permutations*.

Definition 3 (Random functions and permutations) *A* random function $\mathcal{X} \to \mathcal{Y}$ *(*random permutation *on* $\mathcal{X}$*) is a random variable which takes as values functions* $\mathcal{X} \to \mathcal{Y}$ *(permutations on* $\mathcal{X}$*).*

A uniform random function (URF) $\mathbf{R} : \mathcal{X} \to \mathcal{Y}$ *(A* uniform random permutation (URP) $\mathbf{P}$ *on* $\mathcal{X}$*) is a random function with uniform distribution over all functions from* $\mathcal{X}$ *to* $\mathcal{Y}$ *(permutations on* $\mathcal{X}$*). Throughout, the symbols* $\mathbf{R}$ *and* $\mathbf{P}$ *are used for the systems defined above (*$\mathcal{X}, \mathcal{Y}$ *to be understood).*

Indistinguishability of random systems. The distinguishing advantage of a computationally unbounded distinguisher for two random variables A and B is simply the statistical distance of A and B. It is more intricate to define what we mean by the indistinguishability of random systems as here one must specify how the systems can be accessed. For this we define the concept of a distinguisher.

Definition 4 *A* $(\mathcal{Y}, \mathcal{X})$-distinguisher *is a* $(\mathcal{Y}, \mathcal{X})$*-random system which is one query ahead; i.e. it is defined by* $\mathsf{P}^{\mathbf{D}}_{X_i|Y^{i-1}X^{i-1}}$ *instead of* $\mathsf{P}^{\mathbf{D}}_{X_i|Y^iX^{i-1}}$ *for all* i. *In particular the first output* $\mathsf{P}^{\mathbf{D}}_{X_1}$ *is defined before* $\mathbf{D}$ *is fed with any input.*

We can now consider the random experiment where a $(\mathcal{Y}, \mathcal{X})$-distinguisher queries a $(\mathcal{X}, \mathcal{Y})$-random system

Definition 5 *With* $\mathbf{D}\Diamond\mathbf{F}$ *we denote the random experiment where a distinguisher* $\mathbf{D}$ *interactively queries a compatible random system* $\mathbf{F}$.

We divide distinguishers into classes by posing restrictions on how the distinguisher can access its inputs and produce its queries. In particular the following attacks will be of interest to us:

- CPA: Adaptively Chosen Plaintext Attack; here the adversary can choose the i'th query after receiving the $(i-1)$'th output.
- nCPA: Non-Adaptively Chosen Plaintext Attack; here the distinguisher must choose all queries in advance.
- KPA: Known Plaintext Attack; the queries are chosen uniformly at random.

If $\mathbf{F}$ is a permutation, its inverse $\mathbf{F}^{-1}$ is well-defined and we can consider the attacks

- – CCA: Adaptively Chosen Ciphertext Attack
- – nCCA: Non-Adaptively Chosen Ciphertext Attack

which are defined like a CPA and nCPA, respectively, but where the attacker can additionally make queries from the inverse direction.

Definition 6 *For $k \geq 1$, the two random experiments $\mathbf{D}\Diamond\mathbf{F}$ and $\mathbf{D}\Diamond\mathbf{G}$ define a distribution over $\mathcal{X}^k \times \mathcal{Y}^k$. The* advantage *of $\mathbf{D}$ after k queries in distinguishing $\mathbf{F}$ from $\mathbf{G}$, denoted $\Delta_k^{\mathsf{D}}(\mathbf{F},\mathbf{G})$, is the statistical difference between those distributions*[7]

$$\Delta_k^{\mathsf{D}}(\mathbf{F},\mathbf{G}) \overset{def}{=} \frac{1}{2} \sum_{\mathcal{X}^k \times \mathcal{Y}^k} \left| \mathsf{P}_{X^kY^k}^{\mathbf{D}\Diamond\mathbf{F}} - \mathsf{P}_{X^kY^k}^{\mathbf{D}\Diamond\mathbf{G}} \right| . \qquad (1)$$

The advantage of the best ATK*-distinguisher making k queries for $\mathbf{F}$ and $\mathbf{G}$ is*

$$\Delta_k^{\mathsf{ATK}}(\mathbf{F},\mathbf{G}) \overset{def}{=} \max_{\mathsf{ATK}-distinguisher\ \mathbf{D}} \Delta_k^{\mathsf{D}}(\mathbf{F},\mathbf{G}).$$

Informally, a family of random functions indexed by a security parameter ($\gamma \in \mathbb{N}$) is ATK-secure QRF, if for any polynomial $p(.)$ the distinguishing advantage $\Delta_{p(\gamma)}^{\mathsf{ATK}}(\mathbf{F},\mathbf{R})$ is negligible (in γ). QRPs are defined similarly but using $\mathbf{P}$ instead of $\mathbf{R}$, and where we additionally require that $\mathbf{F}$ (for any value of the security parameter) is a permutation.

Pseudorandomness. We denote with $\mathbf{Adv}_{t,k}^{\mathsf{ATK}}(\mathbf{F},\mathbf{G})$ the distinguishing advantage of the best oracle circuit for random systems $\mathbf{F}$ and $\mathbf{G}$ where the circuit must be of size at most t and make at most k ATK-queries to its oracle. So $\mathbf{Adv}$ is defined similarly to Δ but with an additional restriction on the size of the distinguisher. In particular $\mathbf{Adv}_{\infty,k}^{\mathsf{ATK}}(\mathbf{F},\mathbf{G}) = \Delta_k^{\mathsf{ATK}}(\mathbf{F},\mathbf{G})$.

Informally, a family of keyed functions F indexed by a security parameter $\gamma \in \mathbb{N}$ is an ATK-secure pseudorandom function (PRF) if F (with security parameter γ) can be computed in uniform polynomial (in γ) time, and for any polynomial $p(.)$ the distinguishing advantage $\mathbf{Adv}_{p(\gamma),p(\gamma)}^{\mathsf{ATK}}(\mathsf{F},\mathbf{R})$ is negligible in γ (for a key chosen uniformly at random). Pseudorandom permutations (PRP) are defined similarly but using $\mathbf{P}$ instead of $\mathbf{R}$, and where we additionally require that F (for any value of the security parameter and key) is a permutation.

We usually use sans-serif fonts like F to denote systems which can be efficiently computed (in particular pseudorandom systems), and bold fonts like $\mathbf{F}$ to denote quasirandom and ideal systems.

[7] This definition has a natural interpretation in the random experiment where we first toss a uniform random coin $C \in \{0,1\}$. Then we let $\mathbf{D}$ (which has no a priori information on C) make k queries to a system $\mathbf{H}$ where $\mathbf{H} \equiv \mathbf{F}$ if $C = 0$ and $\mathbf{H} \equiv \mathbf{G}$ if $C = 1$. Here the expected probability that an optimal guess on C based on the k inputs and outputs of $\mathbf{H}$ will be correct is $1/2 + \Delta_k^{\mathsf{D}}(\mathbf{F},\mathbf{G})/2$.

4 Relaxations of the Three-Round Luby-Rackoff Cipher

Let us first state some results for the three-round Feistel-network.

Proposition 1 *For any* $(\mathsf{ATK},\mathsf{ATK}')\in\{(\mathsf{CPA},\mathsf{CPA}),(\mathsf{nCCA},\mathsf{nCPA}),(\mathsf{KPA},\mathsf{KPA})\}$ *and random function* $\mathbf{F}$

$$\varDelta_k^{\mathsf{ATK}}(\psi_{2n}[\mathbf{FFF}],\mathbf{P}) \leq 3\cdot\varDelta_k^{\mathsf{ATK}'}(\mathbf{F},\mathbf{R})+2\cdot\frac{k^2}{2^{n+1}}. \tag{2}$$

The analogous statement also holds in the computational case – i.e. for any efficient random function F

$$\mathbf{Adv}_{t,k}^{\mathsf{ATK}}(\psi_{2n}[\mathsf{FFF}],\mathbf{P}) \leq 3\cdot\mathbf{Adv}_{t',k}^{\mathsf{ATK}'}(\mathsf{F},\mathbf{R})+2\cdot\frac{k^2}{2^{n+1}}, \tag{3}$$

where $t'=\mathsf{poly}(t,k)$ *for some polynomial* poly *which accounts for the overhead implied by the reduction we make.*

The classical result of Luby and Rackoff [LR86], states that the Feistel-network with three independent PRF rounds is a CPA secure PRP – i.e (3) for $(\mathsf{ATK},\mathsf{ATK}')=(\mathsf{CPA},\mathsf{CPA})$.

Luby and Rackoff proved this result directly. One gets a simpler proof by first showing that the three-round Feistel-network with URFs $\mathbf{R}$ is a CPA secure QRP as this is a purely information-theoretic statement. In particular it was shown in [Mau02] that[8]

$$\varDelta_k^{\mathsf{CPA}}(\psi_{2n}[\mathbf{RRR}],\mathbf{P}) \leq 2\cdot\frac{k^2}{2^{n+1}}. \tag{4}$$

This bound also holds for nCCA distinguishers (as we show in Appendix B). These results directly imply Proposition 1 by a standard hybrid argument.[9]

Lucks showed [Luc96] (see also [NR99]) that the first round in the three-round Luby-Rackoff cipher can be replaced with a much weaker primitive which only must provide some guarantee on the collision probability on the left half of the output (for any two fixed inputs). In particular, an almost pairwise independent permutation or a Feistel-round with an almost XOR-universal function will do.

[8] This bound has been improved – using larger number of rounds – in a series of papers. The latest [Pat04] by Patarin claims (optimal) security up to $k \ll 2^n$ (and not just $k \ll 2^{n/2}$) queries, using five rounds (five rounds are also necessary to get such optimal security).

[9] The argument goes as follows for pseudorandom systems: let $(\mathsf{ATK},\mathsf{ATK}')\in\{(\mathsf{CPA},\mathsf{CPA}),(\mathsf{nCCA},\mathsf{nCPA}),(\mathsf{KPA},\mathsf{KPA})\}$ and suppose there is an efficient ATK-distinguisher A for $\psi_{2n}[\mathsf{FFF}]$ and $\mathbf{P}$. Then by (4) this A will also distinguish $\psi_{2n}[\mathsf{FFF}]$ from $\psi_{2n}[\mathbf{RRR}]$. Consider the hybrids $H_0=\psi_{2n}[\mathsf{FFF}], H_1=\psi_{2n}[\mathbf{R}\mathsf{FF}],\ldots,H_3=\psi_{2n}[\mathbf{RRR}]$. By the triangle inequality there is an $0\leq i\leq 2$ (say $i=1$) such that A can distinguish H_i from H_{i+1}. Now, the distinguisher which – with access to an oracle G (implementing either F or $\mathbf{R}$) – simulates $A\lozenge\psi_{2n}[\mathbf{R}\mathsf{GF}]$ and outputs the output of A is an efficient ATK'-distinguisher for F with the same advantage as A's advantage for H_1 and H_2. The corresponding argument also holds in the quasirandom setting.

Proposition 2 *For any* $\mathsf{ATK} \in \{\mathsf{CPA}, \mathsf{nCPA}, \mathsf{KPA}\}$*, any random functions* $\mathbf{F}$*,* $\mathbf{G}$*, and any permutation* $\mathbf{H}$

$$\Delta_k^{\mathsf{ATK}}(\mathbf{H} \triangleright \psi_{2n}[\mathbf{FG}], \mathbf{P}) \leq \Delta_k^{\mathsf{ATK}}(\mathbf{F}, \mathbf{R}) + 2 \cdot \Delta_k^{\mathsf{KPA}}(\mathbf{G}, \mathbf{R}) + \mathsf{coll}_k({}_L\mathbf{H}) + \frac{k^2}{2^n}. \quad (5)$$

The analogous statement also holds in the computational case: for any $\mathsf{ATK} \in \{\mathsf{CPA}, \mathsf{nCPA}, \mathsf{KPA}\}$*, any efficient random functions* F, G*, and any efficient permutation* H

$$\begin{aligned} &\mathbf{Adv}_{t,k}^{\mathsf{ATK}}(\mathsf{H} \triangleright \psi_{2n}[\mathsf{FG}], \mathbf{P}) & (6) \\ &\leq \mathbf{Adv}_{t',k}^{\mathsf{ATK}}(\mathsf{F}, \mathbf{R}) + \mathbf{Adv}_{t',k}^{\mathsf{KPA}}(\mathsf{G}, \mathbf{R}) + \mathsf{coll}_k({}_L\mathsf{H}) + \frac{k^2}{2^{n+1}}, \end{aligned}$$

where $t' = t + \mathsf{poly}(n, k)$ *for some polynomial* poly *which accounts for the overhead implied by the reduction we make.*

Let us stress that (6) does *not* directly follow from (5).[10] The proof of Proposition 2 is given in Appendix C.

We relax the construction further for $\mathsf{ATK} = \mathsf{KPA}$ by showing that the first round can be removed completely (as opposed to when $\mathsf{ATK} \in \{\mathsf{CPA}, \mathsf{nCPA}\}$[11]). Moreover, the round functions can be replaced by a *single* instantiation of a KPA secure function. Note that if one in addition interchange the left and the right part of the output, the resulting construction is an involution, i.e. has the structural property of being self inverse. This result also generalizes Lemma 2.2 of [MT05] which states that the two round Feistel-network with CPA secure PRFs is a KPA secure PRP.

Proposition 3 *For any random function* $\mathbf{F}$

$$\Delta_k^{\mathsf{KPA}}(\psi_{2n}[\mathbf{F}^2], \mathbf{P}) \leq \Delta_{2k}^{\mathsf{KPA}}(\mathbf{F}, \mathbf{R}) + 4 \cdot \frac{k^2}{2^{n+1}}. \quad (7)$$

The analogous statement also holds in the computational case: for any (in particular efficient) random function $\mathbf{F}$

$$\mathbf{Adv}_{t,k}^{\mathsf{KPA}}(\psi_{2n}[\mathbf{F}^2], \mathbf{P}) \leq \mathbf{Adv}_{t',2k}^{\mathsf{KPA}}(\mathbf{F}, \mathbf{R}) + 4 \cdot \frac{k^2}{2^{n+1}}, \quad (8)$$

where $t' = t + \mathsf{poly}(n, k)$ *for some polynomial* poly *which accounts for the overhead implied by the reduction we make.*

The proof is given in Appendix C. Note that unlike in the previous propositions, here we do not require the round function $\mathbf{F}$ to be efficient in the computational case (the reason is that in the proof we do not need the distinguisher to simulate any round function).

[10] The reason why a reduction – like the simple argument to show that Proposition 1 follows from (4) – fails here, is that the KPA security guarantee for one of the components is weaker than the CPA security for the whole construction. But fortunately the *proof* of (5) is such that it easily translates to the pseudorandom setting.

[11] $\psi_{2n}[\mathbf{RR}]$ can be distinguish from $\mathbf{P}$ with two non-adaptively chosen queries: query $0^n \| 0^n \mapsto {}_Ly \| {}_Ry$ and $0^n \| 1^n \mapsto {}_Ly' \| {}_Ry'$, and output 1 if ${}_Ry \oplus {}_Ry' = 1^n$ and 0 otherwise.

5 The Second Round is Crucial

In the previous section we have seen that in the classical three-round Luby-Rackoff cipher the first and third round function need not be CPA secure. In this section we will see that the security requirements for the second round cannot be relaxed. We only give proof sketches for the propositions of this section.

The following proposition states that to achieve CPA security in general it is not sufficient that the second round function is nCPA secure. There exists a nCPA secure function, such that the three-round Feistel-network with this function in the second, and any random functions in the first and third round, is not CPA secure.

Proposition 4 *There exists a random function* $\mathbf{F}$ *such that for any random functions* $\mathbf{G}$ *and* $\mathbf{G}'$ *(in particular for* $\mathbf{G} = \mathbf{R}$ *and* $\mathbf{G}' = \mathbf{R}$*)*

$$\Delta_k^{\mathsf{nCPA}}(\mathbf{F}, \mathbf{R}) \leq \frac{k^2}{2^{n-1}} \quad \textit{and} \quad \Delta_2^{\mathsf{CPA}}(\psi_{2n}[\mathbf{GFG}'], \mathbf{P}) \geq 1 - 2^{-n+1}.$$

The analogous statement also holds in the computational case: (informal) there is a nCPA *secure PRF* F *such that* $\psi_{2n}[\mathbf{G}\mathsf{F}\mathbf{G}']$ *is not a* CPA *secure PRP for any (not necessarily efficient) functions* $\mathbf{G}$ *and* $\mathbf{G}'$*.*

Proof. Let us first consider the quasirandom statement. Let $\mathbf{I}$ be a uniform random involution, i.e. $\mathbf{I}(\mathbf{I}(x)) = x$ for all x. Now, $\mathbf{F}$ is simply defined as $\mathbf{F}(x) = x \oplus \mathbf{I}(x)$, note that this $\mathbf{F}$ satisfies $\mathbf{F}(x) = \mathbf{F}(x \oplus \mathbf{F}(x))$ for all x.

The nCPA security of $\mathbf{F}$ (which is simply the nCPA security of $\mathbf{I}$) can be bounded as stated in the proposition by standard techniques (see Appendix C). Furthermore, $\psi_{2n}[\mathbf{GFG}']$ can easily be distinguished from $\mathbf{P}$ with two adaptively chosen queries as follows. After a first query $0^n \| 0^n$, the output ${}_LY \| Z$ contains the output Z of the internal function $\mathbf{F}$. Now make a second query $0^n \| Z$. If the (unknown) input to $\mathbf{F}$ in the first query was some value V, then in this query it will be $V \oplus Z$, and as $\mathbf{F}$ satisfies $\mathbf{F}(V) = \mathbf{F}(V \oplus \mathbf{F}(V)) = \mathbf{F}(V \oplus Z)$, the output of $\mathbf{F}$ will again be Z, and the overall output will be $({}_LY \oplus Z) \| Z$. The proposition follows as the output of $\mathbf{P}$ will satisfy such a relation with probability at most $\frac{1}{2^n} + \frac{2^n - 1}{2^{2n} - 1} \leq 2^{-n+1}$.

The corresponding statement for the pseudorandom setting is proven almost identically. The only difference is that we need to use a CPA secure pseudorandom involution I instead of the uniform random involution $\mathbf{I}$. It is shown in [NR02] how to construct a pseudorandom involution from any CPA secure PRF. □

The next proposition states that the network will in general not (even) be nCPA secure when the second round function is only secure against KPAs.

Proposition 5 *There exists a random function* $\mathbf{F}$ *such that for any random functions* $\mathbf{G}$ *and* $\mathbf{G}'$

$$\Delta_k^{\mathsf{KPA}}(\mathbf{F}, \mathbf{R}) \leq \frac{k^2}{2^{n+1}}, \quad \textit{and} \quad \Delta_2^{\mathsf{nCPA}}(\psi_{2n}[\mathbf{GFG}'], \mathbf{P}) \geq 1 - 2^{-n}.$$

The analogous statement also holds in the computational case: (informal) there is a KPA *secure PRF* F *such that* $\psi_{2n}[\mathbf{G}\mathsf{F}\mathbf{G}']$ *is not a* nCPA *secure PRP for any (not necessarily efficient) functions* $\mathbf{G}$ *and* $\mathbf{G}'$*.*

Proof. Let us first consider the statement in the quasirandom setting. Let $\mathbf{F}$ be a URF which ignores the first input bit, i.e. for all $x \in \{0,1\}^{n-1}$ we have $\mathbf{F}(0\|x) = \mathbf{F}(1\|x)$. The KPA security of $\mathbf{F}$ follows from the fact that $\mathbf{F}$ looks completely random unless we happen to query two queries of the form $0\|x$ and $1\|x$. By the birthday bound the probability that this happens after k queries is at most $\frac{k^2}{2^{n+1}}$ (see Appendix C). Furthermore, $\psi_{2n}[\mathbf{GFG'}]$ can be distinguished from $\mathbf{P}$ with two non-adaptively chosen queries. For instance on input $0^n\|0^n$ and $0^n\|(1\|0^{n-1})$, the right half of the output will be identical. However, for $\mathbf{P}$ this only happens with probability at most $\frac{2^n-1}{2^{2n}-1} \leq 2^{-n}$.

The corresponding statement in the pseudorandom setting is proven exactly as above, except that we have to use a PRF F instead of $\mathbf{F}$. □

6 Four nCPA Secure Rounds, the Quasirandom Case

In this section we will show that the four-round Feistel-network with nCPA secure QRFs is a CPA secure QRP. This is also the best possible as in Sect. 5 we showed that four rounds are also necessary. The theorem is even stronger as the third and fourth round function must only be KPA secure QRFs.

Theorem 1 *For any random functions* $\mathbf{F}$ *and* $\mathbf{G}$

$$\Delta_k^{\mathsf{CPA}}(\psi_{2n}[\mathbf{FFGG}], \mathbf{P}) \leq 3 \cdot \left(\Delta_k^{\mathsf{nCPA}}(\mathbf{F}, \mathbf{R}) + \Delta_k^{\mathsf{KPA}}(\mathbf{G}, \mathbf{R})\right) + \frac{k^2}{2^{n-2}}.$$

To prove this theorem we use Theorem 2 from [MPR06] which, for the special case of the four-round Feistel-network, is given as Proposition 6 below. The proposition bounds the security of a composition against a "strong" attacker sATK (in particular CPA) in terms of the security of the components against "weak" attackers wATK_i (in particular nCPA or KPA).

The proposition uses the concept of conditions defined for random systems as defined in Appendix A, for now we only give an informal definition: With $\mathbf{F}^{\mathcal{A}}$ we denote the random system $\mathbf{F}$, but which additionally defines an internal binary random variable after each query (called a condition). Let $A_i \in \{0,1\}$ denote the condition after the i'th query. We set $A_0 = 0$ and require the condition to be monotone which means that $A_i = 1 \Rightarrow A_{i+1} = 1$ (i.e. when the condition failed, it will never hold again). Let $\overline{a}_i$ denote the event $A_i = 1$, then with

$$\nu^{\mathsf{ATK}}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) \stackrel{\text{def}}{=} \max_{\mathsf{ATK}-\mathit{distinguisher}\ \mathbf{D}} \mathsf{P}_{\overline{a}_k}^{\mathbf{D}\diamond\mathbf{F}^{\mathcal{A}}}, \tag{9}$$

we denote the advantage of the best ATK distinguisher to make the condition fail after at most k queries to $\mathbf{F}^{\mathcal{A}}$.

Proposition 6 *If for any* $(\{0,1\}^n, \{0,1\}^n)$*-random system with a condition* $\mathbf{F}^{\mathcal{A}}$

$$\nu^{\mathsf{sATK}}(\psi_{2n}[\mathbf{F}^{\mathcal{A}}\mathbf{RRR}], \overline{a}_k) \leq \nu^{\mathsf{wATK}_1}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) + \alpha_1 \tag{10}$$

$$\nu^{\mathsf{sATK}}(\psi_{2n}[\mathbf{RF}^{\mathcal{A}}\mathbf{RR}], \overline{a}_k) \leq \nu^{\mathsf{wATK}_2}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) + \alpha_2 \tag{11}$$

$$\nu^{\mathsf{sATK}}(\psi_{2n}[\mathbf{RRF}^{\mathcal{A}}\mathbf{R}], \overline{a}_k) \leq \nu^{\mathsf{wATK}_3}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) + \alpha_3 \tag{12}$$

$$\nu^{\mathsf{sATK}}(\psi_{2n}[\mathbf{RRRF}^{\mathcal{A}}], \overline{a}_k) \leq \nu^{\mathsf{wATK}_4}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) + \alpha_4 \tag{13}$$

for some attacks wATK_1*,* wATK_2*,* wATK_3*,* wATK_4*,* sATK *and some* $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \geq 0$*, then for any* $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4$

$$\Delta_k^{\mathsf{sATK}}(\psi_{2n}[\mathbf{F}_1\mathbf{F}_2\mathbf{F}_3\mathbf{F}_4], \psi_{2n}[\mathbf{RRRR}]) \leq \sum_{i=1}^{4}(\Delta_k^{\mathsf{wATK}_i}(\mathbf{F}_i, \mathbf{R}) + \alpha_i).$$

To apply this proposition we must show that equations (10), (11), (12) and (13) hold for some attack wATK_i and α_i for $i = 1, 2, 3, 4$.

In Appendix D we prove the following claim, from which Theorem 1 now follows.

Claim 1 *Equation (10) - (13) are satisfied for any random function with a condition* $\mathbf{F}^{\mathcal{A}}$*,* $\mathsf{sATK} = \mathsf{CPA}$*, and*

$$\Big(\mathsf{wATK}_i, \alpha_i\Big) = \begin{cases} \Big(\mathsf{nCPA},\ 2 \cdot \frac{k^2}{2^{n+1}}\Big) & \textit{if} \quad i = 1 \\ \Big(\mathsf{nCPA},\ 2 \cdot \frac{k^2}{2^{n+1}} + \Delta_k^{\mathsf{nCPA}}(\mathbf{F}, \mathbf{R})\Big) & \textit{if} \quad i = 2 \\ \Big(\mathsf{KPA},\ 2 \cdot \frac{k^2}{2^{n+1}} + \Delta_k^{\mathsf{KPA}}(\mathbf{F}, \mathbf{R})\Big) & \textit{if} \quad i = 3 \\ \Big(\mathsf{KPA},\ 2 \cdot \frac{k^2}{2^{n+1}}\Big) & \textit{if} \quad i = 4\ . \end{cases}$$

7 Four nCPA Secure Rounds, the Pseudorandom Case

In this section we again investigate the CPA security of the four-round Feistel-network with nCPA secure round functions, but this time for *pseudorandom* systems. We show that here the situation is dramatically different from the quasirandom setting by constructing a nCPA secure PRF where the four-round Feistel-network with this PRF as round function is not CPA secure.

This PRF is defined over some group (of prime order), and to prove the nCPA security we assume that the so-called *inverse decisional Diffie-Hellman* (IDDH) is hard in this group. Informally, the IDDH assumption requires that for a generator g and random x, y it is hard do distinguish the triple (g, g^x, g^y) from $(g, g^x, g^{x^{-1}})$.

Theorem 2 *(Informal) Under the IDDH assumption there exists a* nCPA *secure PRF* F *such that the four-round Feistel-network where each round is instantiated with* F *(with independent keys) is* not *a* CPA *secure pseudorandom permutation.*

This theorem follows from Lemma 1 below which states that there exist nCPA secure PRFs $\mathsf{F}_1, \mathsf{F}_2, \mathsf{F}_3$ such that the left half of the *three* round Feistel-network ${}_L\psi_{2n}[\mathsf{F}_1\mathsf{F}_2\mathsf{F}_3]$ is not a CPA secure PRF. This implies that also $\psi_{2n}[\mathsf{F}_1\mathsf{F}_2\mathsf{F}_3\mathsf{G}]$ is not a CPA secure PRP for any G (and thus proves Theorem 2) as follows. By the so-called PRF/PRP Switching Lemma any CPA secure PRP P is also a CPA secure PRF. Clearly, then also ${}_L\mathsf{P}$ must be a CPA secure PRF. Now, by Lemma 1 ${}_L\psi_{2n}[\mathsf{F}_1\mathsf{F}_2\mathsf{F}_3] = {}_L\psi_{2n}[\mathsf{F}_1\mathsf{F}_2\mathsf{F}_3\mathsf{G}]$ is not a CPA secure PRF, so $\psi_{2n}[\mathsf{F}_1\mathsf{F}_2\mathsf{F}_3\mathsf{G}]$ cannot be a CPA secure PRP.[12]

Lemma 1 *Under the IDDH-assumption there exist* nCPA *secure PRFs* $\mathsf{F}_1, \mathsf{F}_2, \mathsf{F}_3$ *such that* ${}_L\psi_{2n}[\mathsf{F}_1\mathsf{F}_2\mathsf{F}_3]$ *is not a* CPA *secure PRF: it can be distinguished efficiently from a URF with only three (adaptive) queries with high probability.*

Outline For this Section. In §7.1 we give a more formal definition of the IDDH assumption. Then, in §7.2 we first show the construction from [Ple05] of a nCPA secure PRF whose sequential composition will not be CPA secure. This extremely simple and intuitive construction is the basis for the (more involved) counter-example for the Feistel-network (i.e. Lemma 1) given in §7.3.

7.1 The Non-uniform IDDH Assumption

Below we define the IDDH assumption which is similar (and easily seen to imply) the well known decisional Diffie-Hellman assumption. Throughout, we will work with hardness assumptions in a non-uniform model of computation (i.e. we require hardness against polynomial size circuit families and not just any fixed Turing machine).[13]

Let $\mathcal{G}$ denote an efficiently computable family of groups indexed by a security parameter $n \in \mathbb{N}$. By efficiently computable we mean that one can efficiently (i.e. in time polynomial in n) sample a group (together with a generator) from the family, and efficiently compute the group operations therein. Abusing notation we denote with $(G, g) = \mathcal{G}(n)$ any group/generator pair for security parameter n.

The IDDH assumption is hard in $\mathcal{G}$ if for $(G, g) = \mathcal{G}(n)$ polynomial size circuits have negligible advantage guessing whether for a given triple (g, g^x, g^y) the y is random or computed as $y = x^{-1}$, more formally

[12] The lemma talks about three different F_i's (and in the proof we really construct a different F_i for every round), but the theorem is stated for a single F. This does not really make a difference. For example this single F can be defined as behaving like F_i with probability 1/3 for $i \in \{1, 2, 3\}$. Then with constant probability 3^{-3} the $\psi_{2n}[\mathsf{FFF}]$ behaves like $\psi_{2n}[\mathsf{F}_1\mathsf{F}_2\mathsf{F}_3]$.

[13] In cryptography security usually means security against non-uniform (and not just uniform) adversaries, and thus also the hardness assumptions used are usually non-uniform, though this is sometimes not explicitly stated as the security proofs work in both settings – i.e. a uniform (non-uniform) assumption implies hardness against uniform (non-uniform) adversaries. But here this is not quite the case, we do not know how to prove a uniform version of Lemma 1. (But one can do so under a somewhat stronger assumption than IDDH. Informally, this assumption is IDDH but where the attacker can also choose the generators to be used in the challenge.)

Definition 7 (non-uniform IDDH) *For a group G and a generator g of G*

$$\mathbf{Adv}_s^{IDDH}(G,g) \overset{def}{=} \max_{C,|C|\leq s} \left| \Pr_x \left[C(g,g^x,g^{x^{-1}}) = true \right] - \Pr_{x,y} \left[C(g,g^x,g^y) = true \right] \right|,$$

where the probability is over the random choice of $x, y \in [1, \ldots, |G|]$. We say that IDDH is hard in $\mathcal{G}$ if for any polynomial $p(.)$

$$\mathbf{Adv}_{p(n)}^{IDDH}(\mathcal{G}(n)) = \mathsf{negl}(n).$$

7.2 Counter-example for Sequential Composition from [Ple05]

In this section we construct a simple PRF F, but where the sequential composition of (arbitrary many) such F (with independent keys) is not CPA secure.

F is based on some prime order cyclic group $(G, g) = \mathcal{G}(n)$ where the IDDH problem is hard and where the elements of the group can be efficiently and densely encoded into $\{0,1\}^n$ (with dense we mean that all but a negligible fraction of the strings should correspond to an element of the group).[14] For example we can let G be a subgroup of prime order q of $\mathbb{Z}_p^*$, where p is a safe prime (i.e. $2q+1$) and q is close to 2^n ([Dam04] describes how to embed such a G into $\{0,1\}^n$).

Let $[.] : \mathcal{G}(n) \rightarrow \{0,1\}^n$ denote an (efficient) embedding of $\mathcal{G}$ into bitstrings (to save on notation we let $[a,b]$ denote the concatenation of $[a]$ and $[b]$). Let $\mathsf{R} : \mathcal{K} \times \{0,1\}^{4n} \rightarrow \mathbb{Z}_q^4$ be any nCPA secure PRF. Now consider the following definition of a nCPA secure PRF $\mathsf{F} : \{0,1\}^{4n} \rightarrow \{0,1\}^{4n}$ with secret key $(\kappa \in \mathcal{K}, x \in \mathbb{Z}_q^*)$.

The first thing F does on input $(\alpha, \beta, \gamma, \delta) \in \{0,1\}^{4n}$ is to generate some pseudorandom values using R, i.e.

$$(r_1, r_2, r_3, r_4) \leftarrow \mathsf{R}(\kappa, \alpha, \beta, \gamma, \delta). \tag{14}$$

Further, if there exists $(a,b,c,d) \in G^4$ s.t. $\alpha = [a], \beta = [b], \gamma = [c], \delta = [d]$ then F outputs (here x^{-1} is the inverse of x in $\mathbb{Z}_q^*$)

$$\mathsf{F}([a,b,c,d]) \rightarrow ([a^{xr_1}, b^{r_1}, c^{x^{-1}r_2}, d^{r_2}]), \tag{15}$$

with r_1, r_2 generated as in (14). On the remaining inputs (which are a negligible fraction of $\{0,1\}^{4n}$) F outputs just the pseudorandom values $[g^{r_1}, g^{r_2}, g^{r_3}, g^{r_4}]$.

Now consider the cascade $\mathsf{F}' \triangleright \mathsf{F}'' \triangleright \mathsf{F}'''$ of three independent F's (with corresponding keys (x_1, κ_1), (x_2, κ_2), and (x_3, κ_3)). Make a first query $[g,g,g,g]$

$$\mathsf{F}' \triangleright \mathsf{F}'' \triangleright \mathsf{F}'''([g,g,g,g]) \rightarrow [g^{x_1x_2x_3r}, g^r, g^{x_1^{-1}x_2^{-1}x_3^{-1}r'}, g^{r'}].$$

[14] For this construction we actually do not need this embedding, we could define F directly over the group. But we will need it (or more precisely, the fact that if X is in the range of F, also $X \oplus R$ for a random bitstring R is in the range with overwhelming probability) when we extend this construction to get the counter-example for the Feistel-network in the next section.

Then the output will have the form $g^{x_1 x_2 x_3 r}, g^r, g^{x_1^{-1} x_2^{-1} x_3^{-1} r'}, g^{r'}$ for some r, r'. Now exchange the right and the left half of this output and use it as the second query

$$\mathsf{F}' \triangleright \mathsf{F}'' \triangleright \mathsf{F}'''([g^{x_1^{-1} x_2^{-1} x_3^{-1} r'}, g^{r'}, g^{x_1 x_2 x_3 r}, g^r]) \to [g^{r''}, g^{r''}, g^{r'''}, g^{r'''}]$$

so the output is of the form $[u, u, v, v]$ for some u, v and thus can be distinguished from random. Therefore $\mathsf{F}' \triangleright \mathsf{F}'' \triangleright \mathsf{F}'''$ is not a CPA secure PRF. This proves that the sequential composition of nCPA secure PRFs does not yield a CPA secure function in general. Note that this distinguishing attack works for any number of rounds, not just three. The following lemma states that F is an nCPA secure PRF if IDDH is hard in $\mathcal{G}$, R is a nCPA secure PRF and the encoding $[.]$ is dense (as then $(2^n - |G|)/2^n$ is negligible).

Lemma 2 *For* F *over* $(G, g) = \mathcal{G}(n)$ *we have*

$$\mathbf{Adv}_{k,s}^{\mathsf{nCPA}}(\mathsf{F}, \mathbf{R}) \leq 6 \cdot k \cdot \mathbf{Adv}_{s'}^{IDDH}(G, g) + \mathbf{Adv}_{k,s'}^{\mathsf{nCPA}}(\mathsf{R}, \mathbf{R}) + 4 \cdot k \cdot \frac{2^n - |G|}{2^n}, \quad (16)$$

where $s' = s + \mathsf{poly}(k, n)$ *for some polynomial* poly *which accounts for the overhead implied by the reduction we make.*

Proof. The Lemma follows from the Lemmata 3 and 4 below. □

Instead of proving this lemma directly, we consider a function $\widetilde{\mathsf{F}}^{\mathbf{R}'} : \mathbb{Z}_{|G|} \times G^4 \to G^4$ (defined below) which will be easier to analyze. $\widetilde{\mathsf{F}}^{\mathbf{R}'}$ is defined almost like F but with two differences. First, the PRF R used by F is replaced by a uniformly random function $\mathbf{R}'$, and second we do not embed the output of $\widetilde{\mathsf{F}}^{\mathbf{R}'}$ into $\{0,1\}^n$ as in F (using the embedding $[.]$).

We define $\widetilde{\mathsf{F}}^{\mathbf{R}'}$, with key $x \in Z_{|G|}$ and oracle access to $\mathbf{R}' : G^4 \to Z_{|G|}^2$ as

$$\widetilde{\mathsf{F}}^{\mathbf{R}'}(x, a, b, c, d) \to (a^{xr}, b^r, c^{x^{-1}r'}, d^{r'}) \quad \text{where} \quad \mathbf{R}'(a, b, c, d) \to (r, r').$$

By the following lemma, distinguishing $\widetilde{\mathsf{F}}^{\mathbf{R}'}$ from a URF is basically as hard as distinguishing F.

Lemma 3 *For URFs* $\mathbf{R} : G^4 \to G^4$, $\mathbf{R}' : G^4 \to Z_{|G|}^2$, $\mathbf{R}'' : \{0,1\}^{4n} \to \{0,1\}^{4n}$ *and* R *from the definition of* F*,*

$$\mathbf{Adv}_{k,s}^{\mathsf{nCPA}}(\mathsf{F}, \mathbf{R}'') \leq \mathbf{Adv}_{k,s'}^{\mathsf{nCPA}}(\widetilde{\mathsf{F}}^{\mathbf{R}'}, \mathbf{R}) + \mathbf{Adv}_{k,s'}^{\mathsf{nCPA}}(\mathsf{R}, \mathbf{R}') + 4 \cdot k \cdot \frac{2^n - |G|}{2^n},$$

where $s' = s + \mathsf{poly}(k, n)$ *for some polynomial* poly *which accounts for the overhead implied by the reduction we make.*

Proof. Let $\mathsf{F}^{\mathbf{R}'}$ be F, but where one uses the URF $\mathbf{R}'$ instead of R. Then

$$\mathbf{Adv}_{k,s}^{\mathsf{nCPA}}(\mathsf{F}, \mathbf{R}'') \leq \mathbf{Adv}_{k,s}^{\mathsf{nCPA}}(\mathsf{F}^{\mathbf{R}'}, \mathbf{R}'') + \mathbf{Adv}_{k,s'}^{\mathsf{nCPA}}(\mathsf{R}, \mathbf{R}').$$

$\mathsf{F}^{\mathbf{R}'}$ only differs from $\widetilde{\mathsf{F}}^{\mathbf{R}'}$ by the use of the embedding $[.]$, as for a random $x \in G$, $[x]$ is $|G|/2^n$ close to uniform we further get

$$\mathbf{Adv}^{\mathsf{nCPA}}_{k,s}(\mathsf{F}^{\mathbf{R}'}, \mathbf{R}'') \leq \mathbf{Adv}^{\mathsf{nCPA}}_{k,s'}(\widetilde{\mathsf{F}}^{\mathbf{R}'}, \mathbf{R}) + 4 \cdot k \cdot \frac{2^n - |G|}{2^n}.$$

□

We will now bound the indistinguishability of $\widetilde{\mathsf{F}}^{\mathbf{R}'}$ from random in terms of the hardness of the IDDH problem.

Lemma 4

$$\mathbf{Adv}^{\mathsf{nCPA}}_{k,s}(\widetilde{\mathsf{F}}^{\mathbf{R}'}, \mathbf{R}) \leq 6 \cdot k \cdot \mathbf{Adv}^{IDDH}_{s+\mathsf{poly}(k,n)}(G, g).$$

Proof. First we observe that for any nCPA and non-uniform distinguishers C, there is a distinguisher C' with[15] $|C'| \leq |C| + O(k \cdot \log(|G|)) = |C| + \mathsf{poly}(k,n)$ which behaves exactly as C, but which additionally “knows” all the discrete logarithms to basis g of its inputs, i.e. when C' makes a query (a_1, a_2, a_3, a_4) where $a_i = g^{z_i}$, then the $z_1, \ldots, z_4$ are somehow hardwired into C'.[16]

The task of our distinguisher C' is to distinguish k quadruples with uniform distribution over G^4 from k quadruples of the form

$$(a_1^{xr}, a_2^r, a_3^{x^{-1}r'}, a_4^{r'}) = (g^{z_1 xr}, g^{z_2 r}, g^{z_3 x^{-1} r'}, g^{z_4 r'}). \qquad (17)$$

where $(a_1, \ldots, a_4)$ is a query chosen (non-adaptively) by C' and x, r, r' are uniformly random (note that x, which is part of the key of $\widetilde{\mathsf{F}}^{\mathbf{R}'}$, is the same for all k quadruples, but the r, r' are independently generated by $\mathbf{R}'$ for each of the k quadruples). As we do assume that C' knows the $z_1, \ldots, z_4$, this is equivalent[17] to distinguish

$$(g^{xr}, g^r, g^{x^{-1}r'}, g^{r'}) \text{ from } (g^r, g^{r'}, g^{r''}, g^{r'''}), \qquad (18)$$

where x and r, r', r'', r''' are uniformly random.

We make the task for C' even simpler and additionally provide g^x and $g^{x^{-1}}$, i.e. C' must distinguish

$$(g^x, g^{x^{-1}}, g^{xr}, g^r, g^{x^{-1}r'}, g^{r'}) \text{ from } (g^x, g^{x^{-1}}, g^r, g^{r'}, g^{r''}, g^{r'''}). \qquad (19)$$

[15] As we require that group operations can be done in time polynomial in n, the representation of elements of $|G|$ — which is at least $\log(|G|)$ bits long — must also be polynomial (as otherwise one could not even read an element in polynomial time).

[16] This observation may seem silly, but this “knowledge” seems necessary in the following reduction. This is also the reason why we can only prove this lemma in the non-uniform setting.

[17] Here and below with problem A being “equivalent” or “easier” than problem B, we mean that if there is a distinguisher C with advantage ϵ for B, then there's a distinguisher $\tilde{C}$ with the same advantage ϵ for A, where $|\tilde{C}| \leq |C| + \mathsf{poly}(k,n)$.

Clearly the task given by (19) is at most as difficult as (18) as one can always ignore the first two elements. We call the corresponding problem EDDH, i.e.

$$\mathbf{Adv}_s^{EDDH}(G,g) = \max_{C,|C|\leq s} \left| \Pr_{x,r,r'}\left[C(g^x, g^{x^{-1}}, g^{xr}, g^r, g^{x^{-1}r'}, g^{r'}) = true\right] - \Pr_{x,r,r',r'',r'''}\left[C(g^x, g^{x^{-1}}, g^r, g^{r'}, g^{r''}, g^{r'''}) = true\right] \right|.$$

Thus distinguishing $\widetilde{\mathsf{F}}^{\mathbf{R}'}$ from $\mathbf{R}$ is at most as hard as distinguishing

$$(g^x, g^{x^{-1}}, g^{xr_1}, g^{r_1}, g^{x^{-1}r_1'}, g^{r_1'}), \ldots, (g^x, g^{x^{-1}}, g^{xr_k}, g^{r_k}, g^{x^{-1}r_k'}, g^{r_k'}) \tag{20}$$

from

$$(g^x, g^{x^{-1}}, g^{r_1}, g^{r_1'}, g^{r_1''}, g^{r_1'''}), \ldots, (g^x, g^{x^{-1}}, g^{r_k}, g^{r_k'}, g^{r_k''}, g^{r_k'''}), \tag{21}$$

where x and all the $r_i, \ldots, r_i'''$ are uniformly random. We can use a hybrid argument to bound this distinguishing advantage in terms of the hardness of the EDDH problem. Let H_i denote the i'th hybrid given by

$$\begin{gathered}(g^x, g^{x^{-1}}, g^{xr_1}, g^{r_1}, g^{x^{-1}r_1'}, g^{r_1'})\\ \vdots \\ (g^x, g^{x^{-1}}, g^{xr_i}, g^{r_i}, g^{x^{-1}r_i'}, g^{r_i'})\\ (g^x, g^{x^{-1}}, g^{r_{i+1}}, g^{r_{i+1}'}, g^{r_{i+1}''}, g^{r_{i+1}'''})\\ \vdots \\ (g^x, g^{x^{-1}}, g^{r_k}, g^{r_k'}, g^{r_k''}, g^{r_k'''}).\end{gathered}$$

Note that the distribution (21) is just H_0 and the distribution (20) is H_k. Thus there is a j such that C' can distinguish H_{j-1} from H_j with advantage at least ϵ/k. Now consider the following distinguisher C'' for EDDH: on input $(a_1, \ldots, a_6)$ (which always satisfies $a_1 = g^x$ and $a_2 = g^{x^{-1}}$ for a random x) C'' generates the distribution

$$\begin{gathered}(g^x, g^{x^{-1}}, g^{xr_1}, g^{r_1}, g^{x^{-1}r_1'}, g^{r_1'})\\ \vdots \\ (g^x, g^{x^{-1}}, g^{xr_{j-1}}, g^{r_{j-1}}, g^{x^{-1}r_{j-1}'}, g^{r_{j-1}'})\\ (g^x, g^{x^{-1}}, a_3, a_4, a_5, a_6)\\ (g^x, g^{x^{-1}}, g^{r_{j+1}}, g^{r_{j+1}'}, g^{r_{j+1}''}, g^{r_{j+1}'''})\\ \vdots \\ (g^x, g^{x^{-1}}, g^{r_k}, g^{r_k'}, g^{r_k''}, g^{r_k'''})\end{gathered}$$

and runs C' on this input.[18] As the above distribution is equivalent to H_j if $(a_1, \ldots, a_6)$ is of the form as shown by the left side of (19), and H_{j-1} if it's of

[18] Note that C'' really can efficiently sample this distribution as it knows g^x and $g^{x^{-1}}$ (which are given by a_1 and a_2 respectively).

the form on the right side of (19), we conclude that C'' has the same advantage ϵ/k for EDDH as C' had in distinguishing H_{j-1} from H_j, so

$$\mathbf{Adv}^{\mathsf{nCPA}}_{k,s}(\widetilde{\mathsf{F}}^{\mathbf{R}'}, \mathbf{R}) \leq k \cdot \mathbf{Adv}^{EDDH}_{s+\mathsf{poly}(k,n)}(G, g).$$

To conclude the proof of the lemma, we must now reduce IDDH to EDDH

Claim 2

$$\mathbf{Adv}^{EDDH}_{s}(G, g) \leq 6 \cdot \mathbf{Adv}^{IDDH}_{s'}(G, g),$$

where $s' = s + \mathsf{poly}(k, n)$ *for some polynomial* poly *which accounts for the overhead implied by the reduction we make.*

Proof. First we show that EDDH is equivalent to deciding whether $z = xy$ or $z = r$ in the tuple $(g, g^{x^{-1}}, g^x, g^y, g^z)$, referred to as DDH- (up to a factor of 2). Reducing EDDH to DDH- is trivial, as we can ignore the unnecessary components from an EDDH tuple. For the reverse direction, we examine the following distributions:

$$\begin{aligned} H_0 &= (g, g^{x^{-1}}, g^x, g^y, g^{xy}, g^{y'}, g^{x^{-1}y'}) \\ H_1 &= (g, g^{x^{-1}}, g^x, g^y, g^c, g^{y'}, g^{c'}) \\ H_2 &= (g, g^{x^{-1}}, g^x, g^y, g^r, g^{y'}, g^{r'}), \end{aligned}$$

where x, y, y', r, r' are chosen uniformly at random and with probability 1/2 $(c = xy \wedge c' = r')$. In the other half of the cases $(c = r \wedge c' = x^{-1}y')$. Let $\mathbf{Adv}^{A,B}_{s}$ denote the maximum advantage – over any circuit of size s – for distinguishing the distributions A and B. We can write

$$\begin{aligned} \mathbf{Adv}^{EDDH}_{s}(G, g) &\leq \mathbf{Adv}^{H_0,H_2}_{s} && (22) \\ &\leq \mathbf{Adv}^{H_0,H_1}_{s} + \mathbf{Adv}^{H_1,H_2}_{s} && (23) \\ &\leq 2 \cdot \mathbf{Adv}^{DDH-}_{s'}(G, g). && (24) \end{aligned}$$

Step (23) follows by applying the triangle inequality. Given a distinguisher $D_{0,1}$, that is able to distinguish between H_0 and H_1, we can build a distinguisher for DDH. To decide for a tuple $(g, g^{a^{-1}}, g^a, g^b, g^c)$ if $c = ab$ or $c = r$, it first generates $g^r, g^{a^{-1}r}$. Then with probability 1/2 it returns the answer $D_{0,1}$ gives to the input $(g, g^{a^{-1}}, g^a, g^b, g^c, g^r, g^{a^{-1}r})$ and otherwise $D_{0,1}$'s response to $(g, g^a, g^{a^{-1}}, g^r, g^{a^{-1}r}, g^b, g^c,)$. Hence $\mathbf{Adv}^{H_0,H_1}_{s} \leq \mathbf{Adv}^{DDH-}_{s'}(G, g)$. An analogous argument can be used to tell D_1 and D_2 apart and therefore (24) follows.

In our next step we bound the distinguishing advantage of DDH- by demonstrating that

$$\mathbf{Adv}^{DDH-}_{s}(G, g) \leq \mathbf{Adv}^{DDH}_{s'}(G, g) + 2 \cdot \mathbf{Adv}^{IDDH}_{s'}(G, g). \qquad (25)$$

Consider the following distributions

$$\begin{aligned} D_0 &= (g, g^{a^{-1}}, g^a, g^b, g^{ab}) \\ H_0 &= (g, g^r, g^a, g^b, g^{ab}) \\ H_1 &= (g, g^r, g^a, g^b, g^c) \\ D_1 &= (g, g^{a^{-1}}, g^a, g^b, g^c). \end{aligned}$$

We want to bound the distinguishing advantage of D_0 and D_1. To this purpose we use the triangle inequality

$$\begin{aligned}\mathbf{Adv}_s^{DDH-}(G,g) &= \mathbf{Adv}_s^{D_0,D_1} \\ &\leq \mathbf{Adv}_s^{D_0,H_0} + \mathbf{Adv}_s^{H_0,H_1} + \mathbf{Adv}_s^{H_1,D_1}. \end{aligned} \tag{26}$$

When distinguishing H_0 from H_1, we have to solve a plain DDH problem, as g^r carries no information on a and b. Hence

$$\mathbf{Adv}_s^{H_0,H_1} \leq \mathbf{Adv}_{s'}^{DDH}(G,g). \tag{27}$$

Moreover g^b, g^c do not help distinguishing H_1 from D_1, and thus

$$\mathbf{Adv}_s^{H_1,D_1} \leq \mathbf{Adv}_{s'}^{IDDH}(G,g). \tag{28}$$

We encounter a similar situation comparing the first two distributions. Since g^b, g^{ab} can be generated easily when knowing g^a, it follows that

$$\mathbf{Adv}_s^{D_0,H_0} \leq \mathbf{Adv}_{s'}^{IDDH}(G,g). \tag{29}$$

Combining equations (26) – (29) proves (25). Equations (22) – (25) conclude the proof of the claim. △

□

7.3 Proof of Lemma 1

The Feistel-network can be seen as a sequential composition of the round functions, but where one additionally XORs the input to the i'th round function to the output of the $(i+1)$'th round function. So it is not surprising that we can use F_i's similar to the F from the previous section to prove Lemma 1. But the $\mathsf{F}_1, \mathsf{F}_2$, and F_3 (from the statement of the lemma) are a bit more complicated as we have to "work around" this additional XORs. Like F, each F_i has a $\kappa_i \in \mathcal{K}$ as part of its secret key. Moreover F_1 has a $x \in \mathbb{Z}_q^*$ and $s,t \in \{0,1\}^n$, F_2 has a $y \in \mathbb{Z}_q^*$, and F_3 a $z \in \mathbb{Z}_q^*$ as keys. On input $(\alpha,\beta,\gamma,\delta) = [a,b,c,d]$ the F_i's are defined as (with the r_i's generated as in (14))

$$\mathsf{F}_1([a,b,c,d]) \rightarrow \begin{cases} [g^{xr_1}, g^{r_1}], s, t & \text{if } [a,b,c,d] = [0,0,0,0]; \\ [0,0,0,0] & \text{elseif } c = d^x; \\ [g^{xr_1}, g^{r_1}, ([\gamma \oplus s]^{-1})^{x^{-1}r_2}, ([\delta \oplus t]^{-1})^{r_2}] & \text{elseif } [a,b] = [0,0]; \\ [g^{r_1}, g^{r_2}, g^{r_3}, g^{r_4}] & \text{otherwise.} \end{cases}$$

$$\mathsf{F}_2([a,b,c,d]) \rightarrow \quad [c^{y^{-1}r_1}, d^{r_1}, a^{yr_2}, b^{r_2}]$$

$$\mathsf{F}_3([a,b,c,d]) \rightarrow \begin{cases} [0,0,0,0] & \text{if } b^z = a; \\ [a^{z^{-1}r_1}, b^{r_1}, c^{zr_2}, d^{r_2}] & \text{otherwise.} \end{cases}$$

Proof (of Lemma 1). The lemma follows from Claim 3 and 4 below. □

Claim 3 *One can distinguish* ${}_L\psi_{2n}[\mathsf{F}_1\mathsf{F}_2\mathsf{F}_3]$ *from a URF with three adaptively chosen queries with advantage almost* 1.

$$\begin{array}{rll}
{}_LQ_1 : & [0,0,0,0] & {}_RQ_1 : [0,0,0,0] \\
R_1^2 : & [g^{xr'_1}, g^{r'_1}], s, t & \\
R_1^3 : & *, *, [g^{xyr'_2}, g^{r'_2}] & \\
O_1 : & *, *, [g^{xyzr'_3}] \oplus s, [g^{r'_3}] \oplus t & \\
 & & \\
{}_LQ_2 : & [0,0], [g^{xyzr'_3}] \oplus s, [g^{r'_3}] \oplus t & {}_RQ_2 : [0,0,0,0] \\
R_2^2 : & [g^{xr'_4}, g^{r'_4}, g^{yzr'_5}, g^{r'_5}] & \\
R_2^3 : & [g^{zr'_6}, g^{r'_6}], *, * & \\
O_2 : & [g^{xr'_4}, g^{r'_4}, g^{yzr'_5}, g^{r'_5}] & \\
 & & \\
{}_LQ_3 : & [0,0,g^{xr'_4}, g^{r'_4}] & {}_RQ_3 : [0,0,g^{yzr'_5}, g^{r'_5}] \\
R_3^2 : & [0,0,g^{yzr'_5}, g^{r'_5}] & \\
R_3^3 : & [g^{zr'_7}, g^{r'_7}], *, * & \\
O_3 : & [0,0,g^{yzr'_5}, g^{r'_5}] &
\end{array}$$

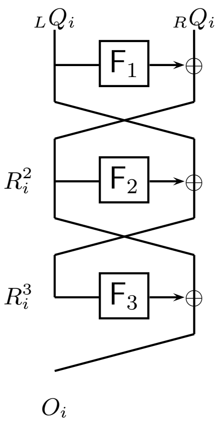

Fig. 2. An adaptive three query distinguishing attack for ${}_L\psi_{2n}[\mathsf{F}_1\mathsf{F}_2\mathsf{F}_3]$.

Proof (sketch). In Fig. 2 we demonstrate an adaptive three query distinguishing attack on ${}_L\psi_{2n}[\mathsf{F}_1\mathsf{F}_2\mathsf{F}_3]$. In the figure, values which are not relevant for the attack are denoted by $*$. All r'_i values are random, but not necessarily equal to a random value generated by a round function (i.e. as in (14)).[19] To see that this is a legal attack note that every query Q_i can be computed from the previous output O_{i-1}. That the values will really have the form as described in the attack can be verified from the definition of the F_i's.[20] Since the third output starts with $[0,0]$ it can be distinguished from a random output with high probability. □

Claim 4 $\mathsf{F}_1, \mathsf{F}_2$, *and* F_3 *are* nCPA *secure PRFs if IDDH is hard in* $\mathcal{G}$.

Proof (sketch). The nCPA security of the F_i's follows from the nCPA security of F from the previous section as stated in Lemma 2: F_2 is exactly F, so there is nothing else to prove here. The function F_3 behaves exactly as F unless it is queried on an input $[a,b,c,d]$ which satisfies $b^z = a$ for a random z. The probability that this happens on any (non-adaptive) query is just $|G|^{-1}$ (and thus exponentially small even after taking the union bound over all polynomially many queries).

To prove that F_1 is non-adaptively secure, we show how to turn any distinguisher D for F_1 into one for F_3 whose distinguishing advantage differs only by a negligible amount. First, below we completely ignore the cases where $c = d^x$ for

[19] For instance, r'_1 is the first random value generated by F_1 and r'_2 is the product of r'_1 and the second random value generated by F_2.

[20] Actually, there is an exponentially small probability that the values will not have that form, namely when the input to some round function "by chance" satisfies a condition that is checked. E.g. when R_1^3 is of the form $[b^z, b, c, d]$, then the "$b^z = a$" case of F_3 applies, which is only supposed to happen in the second and third query.

F_1 and $a = b^z$ for F_3 as they only happen with exponentially small probability. Further, as whenever $[a, b] \neq [0, 0]$ the output of F_1 is pseudorandom, we can assume that the non-adaptive distinguisher D for F_1 only makes queries where $[a, b] = [0, 0]$.

Now consider the following distinguisher D' for F_3. First D' picks some uniformly random $s, t \in \{0, 1\}^n$. D' basically simulates D, but when the query was $[0, 0, 0, 0]$ then the right half of the output is replaced with s, t. On all other queries (chosen by D) of the form $[0, 0], \gamma, \delta$, D' invokes the system at hand by $[0, 0], \gamma \oplus s, \delta \oplus t$. Finally D' outputs the decision bit of the simulated D.

If the system queried by D' is F_3 (with secret key x) then the output distribution that the simulated D gets to see is exactly as if it was generated by F_1 (with secret key x, s, t). Also note that when the system queried by D' is a URF, then also the output that D sees is uniformly random. Thus the distinguishing advantage of D' for F_1 (from a URF) is the same as the advantage of D for F_3. □

8 Some Remarks on CCA Security

We have shown that the four-round Feistel-network with nCPA secure round functions is CPA secure in the information-theoretic, but in general not in the computational setting. A natural question to ask is how many rounds are necessary/not sufficient to achieve CCA security.

In order to get a CCA secure QRP, it is enough – by the following statement (taken from [MPR06]) – to cascade two nCPA secure QRPs (the second in inverse direction)

$$\Delta_k^{\mathsf{CCA}}(\mathbf{F} \triangleright \mathbf{G}^{-1}, \mathbf{P}) \leq \Delta_k^{\mathsf{nCPA}}(\mathbf{F}, \mathbf{P}) + \Delta_k^{\mathsf{nCPA}}(\mathbf{G}, \mathbf{P}).$$

With this and Proposition 1 we directly get that six rounds with nCPA secure QRFs give a CCA secure QRP, i.e.

$$\Delta_k^{\mathsf{CCA}}(\psi_{2n}[\mathbf{FFFFFF}], \mathbf{P}) \leq 6 \cdot \Delta_k^{\mathsf{nCPA}}(\mathbf{F}, \mathbf{R}) + \frac{k^2}{2^{n-1}}.$$

So six nCPA secure round functions are sufficient to get CCA security, and by Proposition 4 we know that at least four rounds are necessary. Next we show that the five-round Feistel-network with nCPA secure QRFs is a CCA secure QRP. The theorem, which is proved in Appendix E, is stated even more generally.

Theorem 3 *For any random functions* $\mathbf{F}$ *and* $\mathbf{G}$

$$\Delta_k^{\mathsf{CCA}}(\psi_{2n}[\mathbf{FFGFF}], \mathbf{P}) \leq 8 \cdot \Delta_k^{\mathsf{nCPA}}(\mathbf{F}, \mathbf{R}) + 2 \cdot \Delta_k^{\mathsf{KPA}}(\mathbf{G}, \mathbf{R}) + 12 \cdot \frac{k^2}{2^{n+1}}.$$

It remains an open question whether four rounds are sufficient.

As to the (in)security of the Feistel-network with nCPA secure round-functions in the computational setting, we do not know anything beyond what is already implied by CPA security alone, i.e. four rounds are not enough to get CCA security (as it is not enough to get CPA security by Theorem 2).

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A Monotone Conditions for Random Systems

Monotone Conditions for Random Systems. We now define the concept of *monotone conditions* for random systems and show how they can be used to prove bounds on the indistinguishability of random systems.

A monotone condition $\mathcal{A}$ for a $(\mathcal{X},\mathcal{Y})$-random system $\mathbf{F}$ is an event sequence $A_1, A_2, \ldots$, where $A_i \in \{a_i, \overline{a}_i\}$. Here a_i ($\overline{a}_i$) denotes the event that the condition is satisfied (failed) after the i-th query to $\mathbf{F}$ has been processed. Monotone means that if the condition failed, it will never hold again (i.e. $\overline{a}_i \Rightarrow \overline{a}_{i+1}$). So the event $\overline{a}_i$ immediately implies $\overline{a}_j$ for all $j > i$.

We denote a $(\mathcal{X},\mathcal{Y})$-*random system* $\mathbf{F}$ with a monotone condition $\mathcal{A}$ as $\mathbf{F}^{\mathcal{A}}$, and model the monotone condition by an extra binary output of the system A_i (where $A_i = 0$ indicates the event a_i and $A_i = 1$ the event $\overline{a}_i$). As this output A_i is never used as an input to a distinguisher or another system, it is convenient to think of $\mathbf{F}^{\mathcal{A}}$ as of $\mathbf{F}$ with a lamp. This lamp is initially off ($A_0 = 0$), but may turn on at some point to indicate that the condition failed, i.e. the lamp is on after the i'th query iff $A_i = 1$.

Definition 8 *A* $(\mathcal{X},\mathcal{Y})$-random system $\mathbf{F}$ *with a monotone condition* $\mathcal{A}$*, denoted* $\mathbf{F}^{\mathcal{A}}$*, is the same random system but with an additional monotone binary variable sequence* $A_1, A_2, \ldots$ *defined on it. The value of* $A_i \in \{0,1\}$ *is determined after the i'th query. Monotone means* $\mathbf{F}^{\mathcal{A}}$ *is given by the infinite sequence of conditional probability distributions* $\mathsf{P}^{\mathbf{F}}_{A_iY_i|X^iY^{i-1}A_{i-1}}$ *for* $i \geq 1$ *(or equivalently by* $\mathsf{P}^{\mathbf{F}}_{A_iY^i|X^i}$ *for* $i \geq 1$*).* $A_i = 0$ *means that the condition holds after the i'th query, this event is denoted by* a_i*, the event* $A_i = 1$ *is denoted with* $\overline{a}_i$*.*

Let $\mathbf{F}$ and $\mathbf{G}$ be random systems and $\mathcal{A}$ be a condition defined for $\mathbf{F}$. We define three relations for random systems with conditions

$$\mathbf{F}^{\mathcal{A}} \equiv \mathbf{G}^{\mathcal{B}} \iff \forall i \geq 1 : \mathsf{P}^{\mathbf{F}}_{a_iY^i|X^i} = \mathsf{P}^{\mathbf{G}}_{b_iY^i|X^i}$$

$$\mathbf{F}|\mathcal{A} \equiv \mathbf{G} \iff \forall i \geq 1 : \mathsf{P}^{\mathbf{F}}_{Y^i|X^ia_i} = \mathsf{P}^{\mathbf{G}}_{Y^i|X^i}$$

$$\mathbf{F}^{\mathcal{A}} \preceq \mathbf{G} \iff \forall i \geq 1 : \mathsf{P}^{\mathbf{F}}_{a_iY^i|X^i} \leq \mathsf{P}^{\mathbf{G}}_{Y^i|X^i}.$$

It is not hard to see that $\mathbf{F}|\mathcal{A} \equiv \mathbf{G}$ implies $\mathbf{F}^{\mathcal{A}} \preceq \mathbf{G}$ but not vice versa. Proposition 7 below states that if $\mathbf{F}^{\mathcal{A}} \preceq \mathbf{G}$, then distinguishing $\mathbf{F}$ from $\mathbf{G}$ is at least as hard as making the condition fail.

Definition 9 *For a random system* $\mathbf{F}$ *with a condition* $\mathcal{A}$ *we denote with*

$$\nu^{\mathbf{D}}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) \overset{def}{=} \mathsf{P}^{\mathbf{D}\diamond\mathbf{F}^{\mathcal{A}}}_{\overline{a}_k} \tag{30}$$

the probability that the distinguisher $\mathbf{D}$ *can make* $\mathcal{A}$ *fail with* k *queries. The probability of the best* ATK*-distinguisher is denoted by*

$$\nu^{\mathsf{ATK}}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) \overset{def}{=} \max_{\mathsf{ATK}-distinguisher\ \mathbf{D}} \mathsf{P}^{\mathbf{D}\diamond\mathbf{F}^{\mathcal{A}}}_{\overline{a}_k}. \tag{31}$$

Proposition 7 *If* $\mathbf{F}^{\mathcal{A}} \preceq \mathbf{G}$ *(which is implied by* $\mathbf{F}|\mathcal{A} \equiv \mathbf{G}$ *or* $\mathbf{F}^{\mathcal{A}} \equiv \mathbf{G}^{\mathcal{B}}$*) then for any distinguisher* $\mathbf{D}$

$$\Delta^{\mathbf{D}}_k(\mathbf{F}, \mathbf{G}) \le \nu^{\mathbf{D}}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k),$$

and if $\mathbf{F}^{\mathcal{A}} \equiv \mathbf{G}^{\mathcal{B}}$

$$\nu^{\mathbf{D}}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) = \nu^{\mathbf{D}}(\mathbf{G}^{\mathcal{B}}, \overline{b}_k).$$

By this proposition we can bound $\Delta^{\mathsf{ATK}}_k(\mathbf{F}, \mathbf{G})$ by first finding a condition $\mathcal{A}$ for $\mathbf{F}$ which satisfies $\mathbf{F}^{\mathcal{A}} \preceq \mathbf{G}$ and then trying to prove an upper bound on $\nu^{\mathsf{ATK}}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k)$.

The next proposition states that if $\mathbf{F}|\mathcal{A}$ is itself a random system, then adaptivity is of no use when one want to make $\mathcal{A}$ fail. We will use this proposition many times as dealing with non-adaptive distinguishers is usually much easier than to handle adaptive ones.

Proposition 8 *For any* $i \in \mathbb{N}$*, if for a random system* $\mathbf{F}$ *with a condition* $\mathcal{A}$ *there exists a random system* $\mathbf{G}$ *such that* $\mathbf{F}|\mathcal{A} \equiv \mathbf{G}$*, i.e. for all* $i \ge 1$

$$\mathsf{P}^{\mathbf{F}}_{Y^i|X^i a_i} \equiv \mathsf{P}^{\mathbf{G}}_{Y^i|X^i}, \tag{32}$$

then adaptivity does not help in provoking $\overline{a}_i$*, i.e.*

$$\nu^{\mathsf{CPA}}(\mathbf{F}^{\mathcal{A}}, \overline{a}_i) = \nu^{\mathsf{nCPA}}(\mathbf{F}^{\mathcal{A}}, \overline{a}_i). \tag{33}$$

In the sequel we make use of a random system called beacon, denoted by $\mathbf{B}$.

Definition 10 (Beacon) *An* $\mathcal{X} \to \mathcal{Y}$-beacon $\mathbf{B}$ *is a random system for which* $Y_1, Y_2, \ldots$ *are independent and uniformly distributed over the range* $\mathcal{Y}$ *(and in particular independent of the inputs).*

Note that $\mathbf{R}|\mathcal{A} \equiv \mathbf{B}$, if $\mathcal{A}$ denotes the condition that the inputs to the URF $\mathbf{R}$ are distinct. Hence, by Proposition 7 it follows that

$$\Delta_k^{\mathsf{KPA}}(\mathbf{F},\mathbf{B}) - \Delta_k^{\mathsf{KPA}}(\mathbf{F},\mathbf{R}) \overset{tri.\,ineq.}{\leq} \Delta_k^{\mathsf{KPA}}(\mathbf{R},\mathbf{B}) \overset{Prop.\ 7}{\leq} \nu^{\mathsf{KPA}}(\mathbf{R},\overline{a}_k) \overset{b\text{-}bound}{\leq} \frac{k^2}{2^{n+1}}. \quad (34)$$

In the sequel we will frequently make use the following two arguments:

(i) Consider a monotone condition $\mathcal{A}$, defined for $\mathcal{E}(\cdot)$. Then it follows (which we show below) that

$$\nu^{\mathsf{ATK}}(\mathcal{E}^{\mathcal{A}}(\mathbf{F}),\overline{a}_k) - \nu^{\mathsf{ATK}}(\mathcal{E}^{\mathcal{A}}(\mathbf{G}),\overline{a}_k) \leq \Delta_k^{\mathsf{ATK}}(\mathcal{E}^{\mathcal{A}}(\mathbf{G}),\mathcal{E}^{\mathcal{A}}(\mathbf{F})). \quad (35)$$

Consider the ATK-distinguisher D for which it holds that

$$\nu^{D}(\mathcal{E}^{\mathcal{A}}(\mathbf{F}),\overline{a}_k) = \nu^{\mathsf{ATK}}(\mathcal{E}^{\mathcal{A}}(\mathbf{F}),\overline{a}_k),$$

and the distinguisher D' that simply runs D and outputs 1 if $\overline{a}_k$ is provoked and else 0. Clearly, D' distinguishes $\mathcal{E}^{\mathcal{A}}(\mathbf{G})$ from $\mathcal{E}^{\mathcal{A}}(\mathbf{F})$ with advantage $\nu^{D}(\mathcal{E}^{\mathcal{A}}(\mathbf{F}),\overline{a}_k) - \nu^{D}(\mathcal{E}^{\mathcal{A}}(\mathbf{G}),\overline{a}_k)$, from which (35) follows.

(ii) Suppose there is an ATK-distinguisher D for $\mathcal{E}(\mathbf{F})$ and $\mathcal{E}(\mathbf{G})$, from which we can construct a distinguisher $D \diamond \mathcal{E}(\cdot)$ for $\mathbf{F}$ and $\mathbf{G}$.
Let $\mathsf{ATK}' = \{D \diamond \mathcal{E}(\cdot) | D \in \mathsf{ATK}\}$, for some random system $\mathcal{E}(\cdot)$. Let $k' = c \cdot k$ where c is the number of invocations that $\mathcal{E}(\mathbf{E})$ makes to its component $\mathbf{E}$ on every invocation. Then it holds that

$$\Delta_k^{\mathsf{ATK}}(\mathcal{E}(\mathbf{F}),\mathcal{E}(\mathbf{G})) \leq \Delta_{k'}^{\mathsf{ATK}'}(\mathbf{F},\mathbf{G})$$

and

$$\nu^{\mathsf{ATK}}(\mathcal{E}(\mathbf{F}^{\mathcal{A}}),\overline{a}_k) \leq \nu^{\mathsf{ATK}'}(\mathbf{F}^{\mathcal{A}},\overline{a}_{k'}).$$

B Original Luby-Rackoff under nCCA

In this section we show that

$$\Delta_k^{\mathsf{nCCA}}(\psi_{2n}[\mathbf{RRR}],\mathbf{P}) \leq 2 \cdot \frac{k^2}{2^{n+1}}.$$

Let $Q_i^{\rightarrow}$ and $O_i^{\rightarrow}$ denote the set of queries and outputs in the forward direction (after i queries), respectively. Similarly, let $Q_i^{\leftarrow}$ and $O_i^{\leftarrow}$ denote the queries and outputs in the reverse direction (after i queries in total), respectively.

- Let $c_i^{\leftrightarrow}$ denote the event that the input to the second round function are all distinct (after i queries).
- Let $c_i^{\rightarrow}$ denote the event that there exist distinct $x, x' \in O_i^{\rightarrow}$ or $(x, x') \in O_i^{\rightarrow} \times Q_i^{\leftarrow}$ such that ${}_L x = {}_L x'$.
- Let $c_i^{\leftarrow}$ denote the event that there exist distinct $x, x' \in O_i^{\leftarrow}$ or $(x, x') \in O_i^{\leftarrow} \times Q_i^{\rightarrow}$ such that ${}_L x = {}_L x'$.

It holds that $\psi_{2n}[\mathbf{R}^{\mathcal{C}^{\leftarrow}}\mathbf{R}^{\mathcal{C}^{\leftrightarrow}}\mathbf{R}^{\mathcal{C}^{\rightarrow}}] \equiv \psi_{2n}[\mathbf{R}^{\mathcal{C}^{\leftarrow}}\mathbf{B}^{\mathcal{C}^{\leftrightarrow}}\mathbf{R}^{\mathcal{C}^{\rightarrow}}] \preceq \mathbf{P}$.

$$\begin{aligned}
& \Delta_k^{\mathsf{nCCA}}(\psi_{2n}[\mathbf{RRR}], \mathbf{P}) \\
\overset{\textit{Prop. 7}}{\leq} \quad & \nu^{\mathsf{nCCA}}(\psi_{2n}[\mathbf{R}^{\mathcal{C}^{\leftarrow}}\mathbf{B}^{\mathcal{C}^{\leftrightarrow}}\mathbf{R}^{\mathcal{C}^{\rightarrow}}], \overline{c}_k^{\leftarrow} \vee \overline{c}_k^{\leftrightarrow} \vee \overline{c}_k^{\rightarrow}) \\
\overset{\textit{union bound}}{\leq} \quad & \nu^{\mathsf{nCCA}}(\psi_{2n}[\mathbf{R}^{\mathcal{C}^{\leftarrow}}\mathbf{BR}], \overline{c}_k^{\leftarrow}) + \\
& \nu^{\mathsf{nCCA}}(\psi_{2n}[\mathbf{RBR}^{\mathcal{C}^{\rightarrow}}], \overline{c}_k^{\rightarrow}) + \\
& \nu^{\mathsf{nCCA}}(\psi_{2n}[\mathbf{RB}^{\mathcal{C}^{\leftrightarrow}}\mathbf{R}], \overline{c}_k^{\leftrightarrow}) \\
\overset{\textit{union bound}}{\leq} \quad & \frac{|Q_k^{\leftarrow}|^2}{2^{n+1}} + |Q_k^{\leftarrow}| \cdot \frac{|O_k^{\rightarrow}|}{2^{n+1}} + \\
& \frac{|Q_k^{\rightarrow}|^2}{2^{n+1}} + |Q_k^{\rightarrow}| \cdot \frac{|O_k^{\leftarrow}|}{2^{n+1}} + \\
& \frac{(|Q_k^{\rightarrow}| + |Q_k^{\leftarrow}|)^2}{2^{n+1}} \\
\leq \quad & 2 \cdot \frac{k^2}{2^{n+1}},
\end{aligned}$$

where the last inequality follows from the fact that $|Q^{\leftarrow}| + |Q^{\rightarrow}| \leq k$.

C Proof of Propositions 2 - 5

Without loss of generality (since we are dealing with stateless systems) we assume that the distinguisher only makes distinct queries. Let $\mathcal{C}$ ($\mathcal{C}'$) denote a monotone condition for any function defined by letting c_i (c_i') denote the event that the first i inputs of the function are distinct.

*Proof (**of Proposition 2**).* Let $\mathcal{A}$ denote a monotone condition for any function defined by letting a_i denote the event that the first i outputs of the function are distinct. Let $\mathsf{ATK} \in \{\mathsf{CPA}, \mathsf{nCPA}, \mathsf{KPA}\}$, from $\mathbf{H} \triangleright \psi_{2n}[\mathbf{R}^{\mathcal{C}}\mathbf{G}] \equiv \mathbf{H} \triangleright \psi_{2n}[\mathbf{B}^{\mathcal{C}}\mathbf{G}]$,

$\mathbf{H} \triangleright \psi_{2n}[\mathbf{BB}] \equiv \mathbf{B}$, $\mathbf{B}|\mathcal{A} \wedge \mathcal{C} \equiv \mathbf{P}$, and $\mathbf{B}^{\mathcal{C}} \equiv \mathbf{R}^{\mathcal{C}}$ it follows that

$$\begin{aligned}
& \Delta_k^{\mathsf{ATK}}(\mathbf{H} \triangleright \psi_{2n}[\mathbf{FG}], \mathbf{P}) \\
\overset{\textit{tri. ineq.}}{\leq}\; & \Delta_k^{\mathsf{ATK}}(\mathbf{H} \triangleright \psi_{2n}[\mathbf{FG}], \mathbf{H} \triangleright \psi_{2n}[\mathbf{RG}]) + \\
& \Delta_k^{\mathsf{ATK}}(\mathbf{H} \triangleright \psi_{2n}[\mathbf{RG}], \mathbf{H} \triangleright \psi_{2n}[\mathbf{BG}]) + \\
& \Delta_k^{\mathsf{ATK}}(\mathbf{H} \triangleright \psi_{2n}[\mathbf{BG}], \mathbf{H} \triangleright \psi_{2n}[\mathbf{BB}]) + \\
& \Delta_k^{\mathsf{ATK}}(\mathbf{H} \triangleright \psi_{2n}[\mathbf{BB}], \mathbf{P}) \\
\overset{(ii),\, \textit{Prop. 7}}{\leq}\; & \Delta_k^{\mathsf{ATK}}(\mathbf{F}, \mathbf{R}) + \nu^{\mathsf{ATK}}(\mathbf{H} \triangleright \psi_{2n}[\mathbf{B}^{\mathcal{C}}\mathbf{G}], \overline{c}_k) + \\
& \Delta_k^{\mathsf{KPA}}(\mathbf{G}, \mathbf{B}) + \Delta_k^{\mathsf{ATK}}(\mathbf{B}, \mathbf{P}) \\
\overset{(i), (ii)}{\leq}\; & \Delta_k^{\mathsf{ATK}}(\mathbf{F}, \mathbf{R}) + \nu^{\mathsf{ATK}}(\mathbf{H} \triangleright \psi_{2n}[\mathbf{B}^{\mathcal{C}}\mathbf{B}], \overline{c}_k) + \Delta_k^{\mathsf{KPA}}(\mathbf{G}, \mathbf{B}) + \\
& \Delta_k^{\mathsf{KPA}}(\mathbf{G}, \mathbf{B}) + \Delta_k^{\mathsf{ATK}}(\mathbf{B}, \mathbf{P}) \\
\overset{\textit{Prop. 8, tri. ineq.}}{\leq}\; & \Delta_k^{\mathsf{ATK}}(\mathbf{F}, \mathbf{R}) + \nu^{\mathsf{nCPA}}(\mathbf{H} \triangleright \psi_{2n}[\mathbf{B}^{\mathcal{C}}\mathbf{B}], \overline{c}_k) + \\
& 2 \cdot \big(\Delta_k^{\mathsf{KPA}}(\mathbf{G}, \mathbf{R}) + \Delta_k^{\mathsf{KPA}}(\mathbf{R}, \mathbf{B})\big) + \Delta_k^{\mathsf{ATK}}(\mathbf{B}, \mathbf{P}) \\
\overset{\textit{Prop. 7}}{\leq}\; & \Delta_k^{\mathsf{ATK}}(\mathbf{F}, \mathbf{R}) + \mathsf{coll}_k({}_L H) + \\
& 2 \cdot \Delta_k^{\mathsf{KPA}}(\mathbf{G}, \mathbf{R}) + 2 \cdot \nu^{\mathsf{KPA}}(\mathbf{R}^{\mathcal{C}}, \overline{c}_k) + \nu^{\mathsf{ATK}}(\mathbf{B}^{\mathcal{A} \wedge \mathcal{C}}, \overline{a}_k \vee \overline{c}_k) \\
\overset{\textit{union bound}}{\leq}\; & \Delta_k^{\mathsf{ATK}}(\mathbf{F}, \mathbf{R}) + \mathsf{coll}_k({}_L H) + 2 \cdot \Delta_k^{\mathsf{KPA}}(\mathbf{G}, \mathbf{R}) + \\
& 2 \cdot \nu^{\mathsf{KPA}}(\mathbf{R}^{\mathcal{C}}, \overline{c}_k) + \nu^{\mathsf{ATK}}(\mathbf{B}^{\mathcal{A}}, \overline{a}_k) + \nu^{\mathsf{ATK}}(\mathbf{B}^{\mathcal{C}}, \overline{c}_k) \\
\overset{\textit{b-bound}}{\leq}\; & \Delta_k^{\mathsf{ATK}}(\mathbf{F}, \mathbf{R}) + \mathsf{coll}_k({}_L H) + \\
& 2 \cdot \Delta_k^{\mathsf{KPA}}(\mathbf{G}, \mathbf{R}) + 2 \cdot \frac{k(k-1)}{2^{n+1}} + 2 \cdot \frac{k(k-1)}{2^{2n+1}}.
\end{aligned}$$

We omit the proof of the analogous statement in the pseudorandom setting, since the corresponding arguments (in the above proof) easily translates to the pseudo random setting. □

*Proof (**Proposition 3**).* Let $\mathcal{C}''$ denote the monotone condition defined by letting c_i'' denote the event that all values at the left half of the inputs and the right half of the outputs are all distinct (up to the i-th query). From $\psi_{2n}[\mathbf{R}^2]^{\mathcal{C}''} \equiv \mathbf{P}^{\mathcal{C}''}$ it follows that

$$\begin{aligned}
\Delta_k^{\mathsf{KPA}}(\psi_{2n}[\mathbf{F}^2], \mathbf{P}) \overset{\textit{tri. ineq.}}{\leq}\; & \Delta_k^{\mathsf{KPA}}(\psi_{2n}[\mathbf{F}^2], \psi_{2n}[\mathbf{R}^2]) + \Delta_k^{\mathsf{KPA}}(\psi_{2n}[\mathbf{R}^2], \mathbf{P}) \\
\overset{\textit{Prop. 7}}{\leq}\; & \Delta_k^{\mathsf{KPA}}(\psi_{2n}[\mathbf{F}^2], \psi_{2n}[\mathbf{R}^2]) + \nu^{\mathsf{KPA}}(\mathbf{P}^{\mathcal{C}''}, \overline{c}_k'') \\
\overset{(ii)}{\leq}\; & \Delta_{2k}^{\mathsf{KPA}}(\mathbf{F}, \mathbf{R}) + \nu^{\mathsf{KPA}}(\mathbf{P}^{\mathcal{C}''}, \overline{c}_k'') \\
\overset{\textit{b-bound}}{\leq}\; & \Delta_{2k}^{\mathsf{KPA}}(\mathbf{F}, \mathbf{R}) + \frac{(2k)^2}{2^{n+1}}.
\end{aligned}$$

In the third inequality, we used the fact that a KPA-distinguisher D for $\psi_{2n}[\mathbf{F}^2]$ and $\psi_{2n}[\mathbf{R}^2]$ implies a KPA-distinguisher D' for $\mathbf{F}$ and $\mathbf{R}$ with the same distinguishing advantage. D' simply runs D and answers its oracle queries with help of its own oracle. Note that given two random input-output pairs of any function f one can easily construct a random input output pair of $\psi_{2n}[f^2]$, and hence D' needs twice as many oracle queries than D.

We omit the proof of the analogous statement in the pseudorandom setting, since the corresponding arguments (in the above proof) also hold in the pseudo random setting. □

*Proof (**of Proposition 4, continued**).* Since $\mathbf{F}(x) := x \oplus \mathbf{I}(x)$ it follows that $\Delta_k^{\mathsf{nCPA}}(\mathbf{F},\mathbf{R}) \overset{(ii)}{=} \Delta_k^{\mathsf{nCPA}}(\mathbf{I},\mathbf{R})$, and hence it remains to show that

$$\Delta_k^{\mathsf{nCPA}}(\mathbf{I},\mathbf{R}) \leq \frac{k^2}{2^{n-1}}. \tag{36}$$

Let $\mathcal{A}$ denote the monotone condition that all outputs are distinct and no input is equal to a previous or subsequent output, i.e. formally

$$\overline{a}_k \iff \exists i,j \leq k,\, i \neq j : [x_i = y_j] \vee [y_i = y_j].$$

Clearly, $\mathbf{R}\,|\,\mathcal{A} \equiv \mathbf{I}$ and thus

$$\Delta_k^{\mathsf{nCPA}}(\mathbf{I},\mathbf{R}) \overset{Prop.\ 7}{\leq} \nu^{\mathsf{nCPA}}(\mathbf{R},\overline{a}_k).$$

Since we assume (with out loss of generality) that the distinguishers only issue distinct queries to $\mathbf{R}$, it follows that both $x_i = y_j$ and $y_i = y_j$ occurs with probability $\frac{1}{2^n}$, respectively. Hence, by the union bound we get

$$\nu^{\mathsf{nCPA}}(\mathbf{R},\overline{a}_k) \overset{union\,bound}{\leq} 2 \cdot k(k-1) \cdot \frac{1}{2^n},$$

which concludes the proof. □

*Proof (**of Proposition 5, continued**).* Recall that $\mathbf{F}$ be is a uniform random function which ignores the first bit (so the output does not change if one flips the first bit). Let $\mathcal{A}$ denote the monotone condition, where $\overline{a}_k$ is the event that there exists two inputs x_i and x_j (with $i < j \leq k$) for which the first bit differs and the latter $n-1$ bits are the same. As $\mathbf{F}\,|\,\mathcal{A} \equiv \mathbf{R}$, it follows that

$$\Delta_k^{\mathsf{KPA}}(\mathbf{F},\mathbf{R}) \overset{Prop.\ 7}{\leq} \nu^{\mathsf{KPA}}(\mathbf{R},\overline{a}_k) \overset{b\text{-}bound}{\leq} \frac{k^2}{2^{n+1}},$$

which concludes the proof. □

D Proof of Claim 1

Proof (Claim 1). Without loss of generality (since we are dealing with stateless systems) we assume that the distinguisher only makes distinct queries. Let $\mathcal{C}$ ($\mathcal{C}'$) denote a monotone condition for any function defined by letting c_i (c_i') denote the event that the first i inputs of the function are distinct.

$(\mathbf{i=1})$ Since $\psi_{2n}[\mathbf{F}^{\mathcal{A}}] \triangleright \mathbf{P} \,|\, \mathcal{A} \equiv \mathbf{P}$ it follows that

$$\begin{aligned}
& \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{F}^{\mathcal{A}}\mathbf{RRR}], \overline{a}_k) = \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{F}^{\mathcal{A}}] \triangleright \psi_{2n}[\mathbf{RRR}], \overline{a}_k) \\
\overset{(i)}{\leq}\ & \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{F}^{\mathcal{A}}] \triangleright \mathbf{P}, \overline{a}_k) + \varDelta_k^{\mathsf{CPA}}(\psi_{2n}[\mathbf{F}^{\mathcal{A}}] \triangleright \psi_{2n}[\mathbf{RRR}], \psi_{2n}[\mathbf{F}^{\mathcal{A}}] \triangleright \mathbf{P}) \\
\overset{(ii)}{\leq}\ & \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{F}^{\mathcal{A}}] \triangleright \mathbf{P}, \overline{a}_k) + \varDelta_k^{\mathsf{CPA}}(\psi_{2n}[\mathbf{RRR}], \mathbf{P}) \\
\overset{Prop.\ 8}{\leq}\ & \nu^{\mathsf{nCPA}}(\psi_{2n}[\mathbf{F}^{\mathcal{A}}] \triangleright \mathbf{P}, \overline{a}_k) + \varDelta_k^{\mathsf{CPA}}(\psi_{2n}[\mathbf{RRR}], \mathbf{P}) \\
\overset{(ii),(4)}{\leq}\ & \nu^{\mathsf{nCPA}}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) + 2 \cdot \frac{k^2}{2^{n+1}}.
\end{aligned}$$

$(\mathbf{i=2})$ From $\psi_{2n}[\mathbf{RF}^{\mathcal{A}}\mathbf{R}^{\mathcal{C}}\mathbf{R}^{\mathcal{C}'}] \equiv \psi_{2n}[\mathbf{RF}^{\mathcal{A}}\mathbf{B}^{\mathcal{C}}\mathbf{B}^{\mathcal{C}'}]$, $\psi_{2n}[\mathbf{RFB}^{\mathcal{C}}\mathbf{B}] \,|\, \mathcal{C} \equiv \mathbf{B}$, and $\psi_{2n}[\mathbf{RF}^{\mathcal{A}}\mathbf{BB}] \,|\, \mathcal{A} \equiv \mathbf{B}$ it follows that

$$\begin{aligned}
& \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{RF}^{\mathcal{A}}\mathbf{RR}], \overline{a}_k) \\
\leq\ & \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{RF}^{\mathcal{A}}\mathbf{R}^{\mathcal{C}}\mathbf{R}^{\mathcal{C}'}], \overline{a}_k \vee \overline{c}_k \vee \overline{c}'_k) \\
\overset{Prop.\ 7}{\leq}\ & \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{RF}^{\mathcal{A}}\mathbf{B}^{\mathcal{C}}\mathbf{B}^{\mathcal{C}'}], \overline{a}_k \vee \overline{c}_k \vee \overline{c}'_k) \\
\overset{union\ bound}{\leq}\ & \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{RF}^{\mathcal{A}}\mathbf{BB}], \overline{a}_k) + \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{RFB}^{\mathcal{C}}\mathbf{B}], \overline{c}_k) + \\
& \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{RFBB}^{\mathcal{C}'}], \overline{c}'_k) \\
\overset{Prop.\ 8}{\leq}\ & \nu^{\mathsf{nCPA}}(\psi_{2n}[\mathbf{RF}^{\mathcal{A}}\mathbf{BB}], \overline{a}_k) + \nu^{\mathsf{nCPA}}(\psi_{2n}[\mathbf{RFB}^{\mathcal{C}}\mathbf{B}], \overline{c}_k) + \\
& \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{RFBB}^{\mathcal{C}'}], \overline{c}'_k) \\
\overset{(i)}{\leq}\ & \nu^{\mathsf{nCPA}}(\psi_{2n}[\mathbf{RF}^{\mathcal{A}}\mathbf{BB}], \overline{a}_k) + \varDelta_k^{\mathsf{nCPA}}(\mathbf{F}, \mathbf{R}) + \\
& \nu^{\mathsf{nCPA}}(\psi_{2n}[\mathbf{RRB}^{\mathcal{C}}\mathbf{B}], \overline{c}_k) + \nu^{\mathsf{nCPA}}(\psi_{2n}[\mathbf{RFBB}^{\mathcal{C}'}], \overline{c}'_k) \\
\overset{(ii),\, b\text{-}bound}{\leq}\ & \nu^{\mathsf{nCPA}}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) + \varDelta_k^{\mathsf{nCPA}}(\mathbf{F}, \mathbf{R}) + 2 \cdot \frac{k^2}{2^{n+1}}.
\end{aligned}$$

$(\mathbf{i=3})$ Clearly $\psi_{2n}[\mathbf{RR}^{\mathcal{C}}\mathbf{F}^{\mathcal{A}}\mathbf{R}] \equiv \psi_{2n}[\mathbf{RB}^{\mathcal{C}}\mathbf{F}^{\mathcal{A}}\mathbf{R}]$ and thus

$$\begin{aligned}
& \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{RRF}^{\mathcal{A}}\mathbf{R}], \overline{a}_k) \\
\leq\ & \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{RR}^{\mathcal{C}}\mathbf{F}^{\mathcal{A}}\mathbf{R}], \overline{c}_k \vee \overline{a}_k) \\
\overset{Prop.\ 7}{\leq}\ & \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{RB}^{\mathcal{C}}\mathbf{F}^{\mathcal{A}}\mathbf{R}], \overline{c}_k \vee \overline{a}_k) \\
\overset{union\ bound}{\leq}\ & \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{RBF}^{\mathcal{A}}\mathbf{R}], \overline{a}_k) + \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{RB}^{\mathcal{C}}\mathbf{FR}], \overline{c}_k) \\
\overset{(i)}{\leq}\ & \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{RBF}^{\mathcal{A}}\mathbf{R}], \overline{a}_k) + \varDelta_k^{\mathsf{KPA}}(\mathbf{F}, \mathbf{B}) + \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{RB}^{\mathcal{C}}\mathbf{BR}], \overline{c}_k) \\
\overset{(ii),(34),\, b\text{-}bound}{\leq}\ & \nu^{\mathsf{KPA}}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) + \varDelta_k^{\mathsf{KPA}}(\mathbf{F}, \mathbf{R}) + 2 \cdot \frac{k^2}{2^{n+1}}
\end{aligned}$$

$(\mathbf{i}=\mathbf{4})$ From $\psi_{2n}[\mathbf{R}\mathbf{R}^{\mathcal{C}}\mathbf{R}^{\mathcal{C}'}\mathbf{F}^{\mathcal{A}}] \equiv \psi_{2n}[\mathbf{R}\mathbf{B}^{\mathcal{C}}\mathbf{B}^{\mathcal{C}'}\mathbf{F}^{\mathcal{A}}]$ and $\psi_{2n}[\mathbf{R}\mathbf{B}^{\mathcal{C}}\mathbf{B}\mathbf{F}] \,|\, \mathcal{C} \equiv \mathbf{B}$ it follows that

$$\begin{aligned}
& \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{RRRF}^{\mathcal{A}}], \overline{a}_k) \\
\leq\; & \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{R}\mathbf{R}^{\mathcal{C}}\mathbf{R}^{\mathcal{C}'}\mathbf{F}^{\mathcal{A}}], \overline{c}_k \vee \overline{c}'_k \vee \overline{a}_k) \\
\overset{\textit{Prop. } 7}{\leq}\; & \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{R}\mathbf{B}^{\mathcal{C}}\mathbf{B}^{\mathcal{C}'}\mathbf{F}^{\mathcal{A}}], \overline{c}_k \vee \overline{c}'_k \vee \overline{a}_k) \\
\overset{\textit{union bound}}{\leq}\; & \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{RBBF}^{\mathcal{A}}], \overline{a}_k) + \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{R}\mathbf{B}^{\mathcal{C}}\mathbf{B}\mathbf{F}], \overline{c}_k) + \\
& \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{R}\mathbf{B}\mathbf{B}^{\mathcal{C}'}\mathbf{F}], \overline{c}'_k) \\
\overset{\textit{Prop. } 8}{\leq}\; & \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{RBBF}^{\mathcal{A}}], \overline{a}_k) + \nu^{\mathsf{nCPA}}(\psi_{2n}[\mathbf{R}\mathbf{B}^{\mathcal{C}}\mathbf{B}\mathbf{F}], \overline{c}_k) + \\
& \nu^{\mathsf{CPA}}(\psi_{2n}[\mathbf{R}\mathbf{B}\mathbf{B}^{\mathcal{C}'}\mathbf{F}], \overline{c}'_k) \\
\overset{(ii),\, \textit{b-bound}}{\leq}\; & \nu^{\mathsf{KPA}}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) + 2 \cdot \frac{k^2}{2^{n+1}}.
\end{aligned}$$

□

E Proof of Theorem 3

To prove this Theorem 3 we use Theorem 2 from [MPR06] which, for the special case of the five-round Feistel-network, is given as Proposition 6 below.

Proposition 9 *If for any* $(\{0,1\}^n, \{0,1\}^n)$*-random system with a condition* $\mathbf{F}^{\mathcal{A}}$

$$\nu^{\mathsf{sATK}}(\psi_{2n}[\mathbf{F}^{\mathcal{A}}\mathbf{RRRR}], \overline{a}_k) \leq \nu^{\mathsf{wATK}_1}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) + \alpha_1 \tag{37}$$

$$\nu^{\mathsf{sATK}}(\psi_{2n}[\mathbf{R}\mathbf{F}^{\mathcal{A}}\mathbf{RRR}], \overline{a}_k) \leq \nu^{\mathsf{wATK}_2}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) + \alpha_2 \tag{38}$$

$$\nu^{\mathsf{sATK}}(\psi_{2n}[\mathbf{RR}\mathbf{F}^{\mathcal{A}}\mathbf{RR}], \overline{a}_k) \leq \nu^{\mathsf{wATK}_3}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) + \alpha_3 \tag{39}$$

$$\nu^{\mathsf{sATK}}(\psi_{2n}[\mathbf{RRR}\mathbf{F}^{\mathcal{A}}\mathbf{R}], \overline{a}_k) \leq \nu^{\mathsf{wATK}_4}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) + \alpha_4 \tag{40}$$

$$\nu^{\mathsf{sATK}}(\psi_{2n}[\mathbf{RRRR}\mathbf{F}^{\mathcal{A}}], \overline{a}_k) \leq \nu^{\mathsf{wATK}_5}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) + \alpha_5 \tag{41}$$

for some attacks wATK_1*,* wATK_2*,* wATK_3*,* wATK_4*,* wATK_5*,* sATK *and some* $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \geq 0$*, then for any* $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4, \mathbf{F}_5$

$$\Delta_k^{\mathsf{sATK}}(\psi_{2n}[\mathbf{F}_1\mathbf{F}_2\mathbf{F}_3\mathbf{F}_4\mathbf{F}_5], \psi_{2n}[\mathbf{RRRRR}]) \leq \sum_{i=1}^{5} (\Delta_k^{\mathsf{wATK}_i}(\mathbf{F}_i, \mathbf{R}) + \alpha_i).$$

To apply this proposition we must show that (37), (38), (39), (40), and (41) hold for some attack wATK_i and α_i for $i = 1,2,3,4,5$. This we do next.

Claim 5 *Equation (37) - (41) are satisfied for any function with a condition* $\mathbf{F}^{\mathcal{A}}$*,* $\mathsf{sATK} = \mathsf{CCA}$*, and*

$$\Big(\mathsf{wATK}_i, \alpha_i\Big) = \begin{cases} \Big(\mathsf{nCPA},\ 3 \cdot \frac{k^2}{2^{n+1}} + \Delta_k^{\mathsf{nCPA}}(\mathbf{F}, \mathbf{R})\Big) & \textit{if} \quad i = 1, 5 \\ \Big(\mathsf{nCPA},\ 2 \cdot \frac{k^2}{2^{n+1}} + \Delta_k^{\mathsf{nCPA}}(\mathbf{F}, \mathbf{R})\Big) & \textit{if} \quad i = 2, 4 \\ \Big(\mathsf{KPA},\ 2 \cdot \frac{k^2}{2^{n+1}} + \Delta_k^{\mathsf{KPA}}(\mathbf{F}, \mathbf{R})\Big) & \textit{if} \quad i = 3\ . \end{cases}$$

Proof (Claim 5). Without loss of generality (since we are dealing with stateless systems) we assume that the distinguisher only makes distinct queries. Let $\mathcal{C}$ (resp. $\mathcal{C}'$ and $\mathcal{C}''$) denote a monotone condition for any function defined by letting c_i (resp. c_i' and c_i'') denote the event that the first i inputs of the function are distinct.

For a random permutation $\mathbf{Q}$ over $\mathcal{X}$, let $\langle\mathbf{Q}\rangle$ denote the $(\mathcal{X}\times\{0,1\},\mathcal{X})$-random system defined as follows

$$\langle\mathbf{Q}(x_i,b_i)\rangle = \begin{cases}\mathbf{Q}(x_i) & \text{if } \; b_i = 0\\ \mathbf{Q}^{-1}(x_i) & \text{if } \; b_i = 1.\end{cases}$$

Note that a CCA (nCCA) attack on Q is now the same as a CPA (nCPA) attack on $\langle Q\rangle$.

Let us consider the following facts that we will frequently use in the sequel:

$$\nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{F}^{\mathcal{A}}\mathbf{BB}],\overline{a}_k) = \nu^{\mathsf{nCCA}}(\psi_{2n}[\mathbf{F}^{\mathcal{A}}\mathbf{BB}],\overline{a}_k) \overset{(ii)}{\leq} \nu^{\mathsf{nCPA}}(\mathbf{F}^{\mathcal{A}},\overline{a}_k), \tag{42}$$

where the equality follows from Theorem 2 in [Mau02] and the fact that the outputs of $\langle\psi_{2n}[\mathbf{FBB}]\rangle$ are completely independent of $\mathcal{A}$ (due to the $\mathbf{B}$'s), i.e.

$$\mathsf{P}^{\langle\psi_{2n}[\mathbf{FBB}]\rangle}_{a_i|X^iY^{i-1}a_{i-1}} = \mathsf{P}^{\langle\psi_{2n}[\mathbf{FBB}]\rangle}_{a_i|X^ia_{i-1}} \quad \text{for } \; i \geq 1.$$

Furthermore, there are random systems $\mathbf{G}$ and $\mathbf{G}'$ such that[21]

$$\langle\psi_{2n}[\mathbf{FB}^{\mathcal{C}}\mathbf{B}]\rangle \,|\, \mathcal{C} \equiv \mathbf{G} \tag{43}$$
$$\langle\psi_{2n}[\mathbf{RB}^{\mathcal{C}}\mathbf{B}]\rangle \,|\, \mathcal{C} \equiv \mathbf{G}' \tag{44}$$

and hence by Proposition 8 it holds that

$$\begin{aligned}\nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{FB}^{\mathcal{C}}\mathbf{B}],\overline{c}_k) &\overset{Prop.\ 8,\,(43)}{=} \nu^{\mathsf{nCCA}}(\psi_{2n}[\mathbf{FB}^{\mathcal{C}}\mathbf{B}],\overline{c}_k)\\ &\overset{(i)}{\leq} \nu^{\mathsf{nCCA}}(\psi_{2n}[\mathbf{RB}^{\mathcal{C}}\mathbf{B}],\overline{c}_k) + \Delta_k^{\mathsf{nCPA}}(\mathbf{F},\mathbf{R})\\ &\leq \frac{k^2}{2^{n+1}} + \Delta_k^{\mathsf{nCPA}}(\mathbf{F},\mathbf{R})\end{aligned} \tag{45}$$

and equivalently that

$$\nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{RB}^{\mathcal{C}}\mathbf{B}],\overline{c}_k) \overset{Prop.\ 8,\,(44)}{=} \nu^{\mathsf{nCCA}}(\psi_{2n}[\mathbf{FB}^{\mathcal{C}}\mathbf{B}],\overline{c}_k) \leq \frac{k^2}{2^{n+1}}. \tag{46}$$

[21] $\mathbf{G}({}_Lx_i\|{}_Rx_i,b_i) := ({}_Ly_i\|{}_Ry_i)$, where ${}_Ly_i\|{}_Ry_i$ is chosen uniformly at random from $\{0,1\}^{2n}$ if $b_i = 0$ (or $i > 2^n$), and otherwise ${}_Ly$ is chosen uniformly at random from $\{0,1\}^n$ and ${}_Ry_i = \mathbf{F}({}_Ly) \oplus \mathbf{P}(\langle i\rangle_n)$, where $\langle i\rangle_n$ is the n-bit standard binary encoding of the integer i. $\mathbf{G}'$ is defined similarly, but with $\mathbf{F}$ replaced by $\mathbf{R}$.

$(\mathbf{i=1,5})$ Since $\psi_{2n}[\mathbf{F}^{\mathcal{A}}\mathbf{R}^{\mathcal{C}}\mathbf{R}^{\mathcal{C}'}\mathbf{R}^{\mathcal{C}''}\mathbf{R}] \equiv \psi_{2n}[\mathbf{F}^{\mathcal{A}}\mathbf{B}^{\mathcal{C}}\mathbf{B}^{\mathcal{C}'}\mathbf{B}^{\mathcal{C}''}\mathbf{R}]$ it follows that

$$\begin{aligned}
& \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{RRRRF}^{\mathcal{A}}], \overline{a}_k) \overset{sym.}{=} \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{F}^{\mathcal{A}}\mathbf{RRRR}], \overline{a}_k) \\
\leq\quad & \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{F}^{\mathcal{A}}\mathbf{R}^{\mathcal{C}}\mathbf{R}^{\mathcal{C}'}\mathbf{R}^{\mathcal{C}''}\mathbf{R}], \overline{a}_k \vee \overline{c}_k \vee \overline{c}'_k \vee \overline{c}''_k) \\
\overset{Prop.\ 7}{=}\quad & \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{F}^{\mathcal{A}}\mathbf{B}^{\mathcal{C}}\mathbf{B}^{\mathcal{C}'}\mathbf{B}^{\mathcal{C}''}\mathbf{R}], \overline{a}_k \vee \overline{c}_k \vee \overline{c}'_k \vee \overline{c}''_k) \\
\overset{union\ bound}{\leq}\quad & \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{F}^{\mathcal{A}}\mathbf{BBBR}], \overline{a}_k) + \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{FB}^{\mathcal{C}}\mathbf{BBR}], \overline{c}_k) + \\
& \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{FBB}^{\mathcal{C}'}\mathbf{BR}], \overline{c}'_k) + \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{FBBB}^{\mathcal{C}'}\mathbf{R}], \overline{c}''_k) \\
\overset{(ii)}{\leq}\quad & \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{F}^{\mathcal{A}}\mathbf{BB}], \overline{a}_k) + \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{FB}^{\mathcal{C}}\mathbf{B}], \overline{c}_k) + \\
& \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{BB}^{\mathcal{C}'}\mathbf{R}], \overline{c}''_k) + \frac{k^2}{2^{n+1}} \\
\overset{(42),\,(45),\,(46)}{\leq}\quad & \nu^{\mathsf{nCPA}}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) + \varDelta_k^{\mathsf{nCPA}}(\mathbf{F}, \mathbf{R}) + 3 \cdot \frac{k^2}{2^{n+1}}
\end{aligned}$$

$(\mathbf{i=2,4})$ Clearly, $\psi_{2n}[\mathbf{RF}^{\mathcal{A}}\mathbf{R}^{\mathcal{C}}\mathbf{R}^{\mathcal{C}'}\mathbf{R}] \equiv \psi_{2n}[\mathbf{RF}^{\mathcal{A}}\mathbf{B}^{\mathcal{C}}\mathbf{B}^{\mathcal{C}'}\mathbf{R}]$ and thus

$$\begin{aligned}
& \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{RRRF}^{\mathcal{A}}\mathbf{R}], \overline{a}_k) \overset{sym.}{=} \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{RF}^{\mathcal{A}}\mathbf{RRR}], \overline{a}_k) \\
\leq\quad & \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{RF}^{\mathcal{A}}\mathbf{R}^{\mathcal{C}}\mathbf{R}^{\mathcal{C}'}\mathbf{R}], \overline{a}_k \vee \overline{c}_k \vee \overline{c}'_k) \\
\overset{Prop.\ 7}{=}\quad & \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{RF}^{\mathcal{A}}\mathbf{B}^{\mathcal{C}}\mathbf{B}^{\mathcal{C}'}\mathbf{R}], \overline{a}_k \vee \overline{c}_k \vee \overline{c}'_k) \\
\overset{union\ bound}{\leq}\quad & \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{RF}^{\mathcal{A}}\mathbf{BBR}], \overline{a}_k) + \\
& \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{RFB}^{\mathcal{C}}\mathbf{BR}], \overline{c}_k) + \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{RFBB}^{\mathcal{C}'}\mathbf{R}], \overline{c}'_k) \\
\overset{(ii)}{\leq}\quad & \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{F}^{\mathcal{A}}\mathbf{BB}], \overline{a}_k) + \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{FB}^{\mathcal{C}}\mathbf{B}], \overline{c}_k) + \\
& \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{BB}^{\mathcal{C}'}\mathbf{R}], \overline{c}'_k) \\
\overset{(42),\,(45),\,(46)}{\leq}\quad & \nu^{\mathsf{nCPA}}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) + \varDelta_k^{\mathsf{nCPA}}(\mathbf{F}, \mathbf{R}) + 2 \cdot \frac{k^2}{2^{n+1}}.
\end{aligned}$$

$(\mathbf{i=3})$ As $\psi_{2n}[\mathbf{RR}^{\mathcal{C}}\mathbf{F}^{\mathcal{A}}\mathbf{R}^{\mathcal{C}'}\mathbf{R}] \equiv \psi_{2n}[\mathbf{RB}^{\mathcal{C}}\mathbf{F}^{\mathcal{A}}\mathbf{B}^{\mathcal{C}'}\mathbf{R}]$ it follows that

$$\begin{aligned}
& \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{RRF}^{\mathcal{A}}\mathbf{RR}], \overline{a}_k) \\
\leq\quad & \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{RR}^{\mathcal{C}}\mathbf{F}^{\mathcal{A}}\mathbf{R}^{\mathcal{C}'}\mathbf{R}], \overline{c}_k \vee \overline{a}_k \vee \overline{c}'_k) \\
\overset{Prop.\ 7}{=}\quad & \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{RB}^{\mathcal{C}}\mathbf{F}^{\mathcal{A}}\mathbf{B}^{\mathcal{C}'}\mathbf{R}], \overline{c}_k \vee \overline{a}_k \vee \overline{c}'_k) \\
\overset{union\ bound}{\leq}\quad & \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{RBF}^{\mathcal{A}}\mathbf{BR}], \overline{a}_k) + \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{RB}^{\mathcal{C}}\mathbf{FB}^{\mathcal{C}'}\mathbf{R}], \overline{c}_k \vee \overline{c}'_k) \\
\overset{(i),\,(ii)}{\leq}\quad & \nu^{\mathsf{KPA}}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) + \varDelta_k^{\mathsf{KPA}}(\mathbf{F}, \mathbf{B}) + \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{RB}^{\mathcal{C}}\mathbf{BB}^{\mathcal{C}'}\mathbf{R}], \overline{c}_k \vee \overline{c}'_k) \\
\overset{sym.,\,union\,bound}{\leq}\quad & \nu^{\mathsf{KPA}}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) + \varDelta_k^{\mathsf{KPA}}(\mathbf{F}, \mathbf{B}) + 2 \cdot \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{RB}^{\mathcal{C}}\mathbf{BBR}], \overline{c}_k) \\
\overset{(ii)}{\leq}\quad & \nu^{\mathsf{KPA}}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) + \varDelta_k^{\mathsf{KPA}}(\mathbf{F}, \mathbf{B}) + 2 \cdot \nu^{\mathsf{CCA}}(\psi_{2n}[\mathbf{RB}^{\mathcal{C}}\mathbf{B}], \overline{c}_k) \\
\overset{(34),\,(46)}{\leq}\quad & \nu^{\mathsf{KPA}}(\mathbf{F}^{\mathcal{A}}, \overline{a}_k) + \varDelta_k^{\mathsf{KPA}}(\mathbf{F}, \mathbf{R}) + 2 \cdot \frac{k^2}{2^{n+1}}.
\end{aligned}$$

□

Proof (of Theorem 3). Simply apply the above claim and proposition. □