Eviction and Reception for Description Logic Ontologies (Preliminary Results)*

Ricardo Guimarães[1], Ana Ozaki[1,2] and Jandson S. Ribeiro[3]

[1]University of Bergen, Norway
[2]University of Oslo, Norway
[3]University of Hagen, Germany

Abstract
In this work, we consider the problem of modifying a knowledge base in the light of a (set of) models, while preserving the finite representation of the new knowledge base. We analyse the operation of removing models, called eviction, and the operation of adding models, called reception. Given that not all description logics (DLs) are eviction- and reception-compatible, we analyse natural restrictions of the general problem. In particular, we investigate eviction and reception in DL knowledge bases (ontologies), focusing on the very popular $\mathcal{EL}$ language and $\mathcal{ALC}$ extended with boolean operators over the axioms. First, we extend existing negative results of incompatibility. Then, we place restrictions on the domains of eviction and reception functions on these logics that allows us to recover compatibility.

Keywords
Belief Change, Description Logics, Finite Bases

1. Introduction

In traditional paradigms of Belief Change, such as the AGM paradigm [1] for belief revision, and the KM paradigm [2] for belief update, the agent's epistemic state is represented as a set of formulae logically closed, called a theory, while the incoming information is represented as a single formula. The literature covering these paradigms often does not address the question of finite representation of the epistemic state. Moreover, one can see the representation of the incoming information as a formula as a restriction to the case when the incoming information is a set of models (since there can be sets of models that cannot be finitely represented as a formula).

To address these shortcomings, Guimarães et al. [3] recently proposed a Belief Change framework where the incoming information is a set of models. The authors focus on the question of finite representation, which is particularly relevant for formulas representing knowledge bases, since ontology reasoners are designed to deal with finite ontologies. The framework is proposed with two basic operations: eviction (removal of models) and reception (inclusion of models) [3]. It turns out that for many logics, in particular those in the field of Description Logic (DL), eviction and reception is not always possible (meaning that there are sets of models that cannot be finitely represented in a given DL language).

In this work, we generalize the framework by Guimarães et al. [3] by introducing the notion of 'compartments'. Intuitively, a compartment is a subset of formulae that can be expressed in a given DL and a subset of models taken from the whole set of DL models. This can be used to restrict the more general case of eviction and reception in a given DL, for example, by just considering finite models or by just considering a particular set of formulae where eviction and reception can be performed while preserving finiteness of the agent's epistemic state.

Our contribution We define the generalised framework using the notion of compartments, we prove some properties associated with the new framework, in particular, that eviction (and reception) compatibility with a satisfaction system implies eviction (and reception) compatibility of any compartment of that system. We then consider $\mathcal{EL}_\bot$, where we present two strategies for reception for the case in which the input is a single finite model. $\mathcal{ALC}_{bool}$ is neither eviction nor reception compatible (assuming an infinite signature for the reception case). We prove that it is not eviction compatible (the proof works with finite or infinite signature). Then, we build on our previous work [4] to establish eviction and reception compatibility w.r.t. compartments defined using quasimodels.

Related Work As related work, we mention traditional approaches in Belief Change which concern finiteness [5, 6]. Base-generated operations by Hansson [7] and the KM framework [2] belong to this category. These studies, however, only consider finitary propositional

21st International Workshop on Nonmonotonic Reasoning, September 2–4, 2023, Rhodes, Greece
ricardo.guimaraes@uib.no (R. Guimarães); ana.ozaki@uib.no (A. Ozaki); jandson.ribeiro@fernuni-hagen.de (J. S. Ribeiro)
https://jandsonribeiro.github.io/home/ (J. S. Ribeiro)
0000-0002-9622-4142 (R. Guimarães); 0000-0002-3889-6207 (A. Ozaki); 0000-0003-1614-6808 (J. S. Ribeiro)

logic. There are also works in Belief Change that replace formulas by models. Guerra and Wassermann [8], for example, propose a setting in which the epistemic state of an agent is given by a single Kripke model and the input is a formula in Linear Temporal Logic. In the field of Ontology Repair, Hieke et al. [9] use counter-models to implement contraction by a formula in the DL $\mathcal{EL}$. Additionally, neglecting syntax preservation to retain more motivated Pseudo-Contractions in Belief Change [10, 11, 12] and different forms of Ontology Repair such as via Axiom Weakening [13] and Gentle Repairs [14].

Organisation In the next section, we provide basic notions and notation relevant for this paper. In Section 3 we recall the framework by Guimarães et al. [3] and in Section 4 we present our generalised framework, with the already mentioned notion of compartments. In Sections 5 and 6, we consider the cases of the DLs $\mathcal{EL}_\perp$ and $\mathcal{ALC}_{bool}$ (and variants), respectively. Finally, we conclude in Section 7.

2. Preliminaries

We first provide basic general definitions and then we provide the necessary definitions for the DLs we consider.

2.1. Basic Definitions

The power set of a set A is denoted by $\mathcal{P}(A)$, while the set of all finite subsets of A is denoted by $\mathcal{P}_\mathrm{f}(A)$. We write $\mathcal{P}^*(A)$ to denote the non-empty subsets of A. Following Aiguier et al. [15], Delgrande et al. [16], and [17], we use satisfaction systems to define logics. A *satisfaction system* is a triple $\Lambda = (\mathcal{L}, \mathfrak{M}, \models)$, where $\mathcal{L}$ is a language, $\mathfrak{M}$ is a set of models, also called interpretations, and $\models$ is a satisfaction relation which contains all pairs $(M, \mathcal{B})$, where M is an interpretation and $\mathcal{B}$ is a *base* (that is, a subset of $\mathcal{L}$), such that M satisfies $\mathcal{B}$ (i.e., $M \models \mathcal{B}$). We denote by $\mathrm{Mod}_\Lambda(\mathcal{B})$ the set

$$\{M \in \mathfrak{M} \mid M \models \mathcal{B}\}.$$

We will write simply $\mathrm{Mod}(\mathcal{B})$ when the satisfaction system is clear from the context.

Satisfaction systems allow us to be more flexible and precise regarding the precise scope of the operations and constructions we define. This view also facilitates the generalisation of some results that do not depend on properties of the consequence relation of the logic.

A arbitrary set of models $\mathbb{M} \subseteq \mathfrak{M}$ within Λ is finitely representable iff there is $\mathcal{B} \in \mathcal{P}_\mathrm{f}(\mathcal{L})$ such that $\mathrm{Mod}(\mathcal{B}) = \mathbb{M}$. $\mathrm{FR}(\Lambda)$ denotes the collection of all *finitely representable sets of models in* Λ, that is, the set

$$\{\mathbb{M} \subseteq \mathfrak{M} \mid \exists \mathcal{B} \in \mathcal{P}_\mathrm{f}(\mathcal{L}) : \mathrm{Mod}(\mathcal{B}) = \mathbb{M}\}.$$

Also, we say that a set of formulae $\mathcal{B} \subseteq \mathcal{L}$ is finitely representable iff there is a $\mathcal{B}' \in \mathcal{P}_\mathrm{f}(\mathcal{L})$ with $\mathrm{Mod}(\mathcal{B}) = \mathrm{Mod}(\mathcal{B}')$. Additionally, we write $\times$ for the Cartesian product of two sets. Moreover, we denote the logical closure of a base $\mathcal{B}$ in a satisfaction system Λ by Cn_Λ, omitting the subscript when clear from the context.

2.2. Description Logic Definitions

Let $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_I}$ be countably infinite and pairwise disjoint sets of concept names, role names, and individual names, respectively. *$\mathcal{EL}$ concepts* are built according to the rule: $C ::= \top \mid A \mid (C \sqcap C) \mid \exists r.C$, where $A \in \mathsf{N_C}$. $\mathcal{EL}_\perp$ concepts extend $\mathcal{EL}$ by allowing $\perp$ (interpreted as the empty set). $\mathcal{ALC}$ concepts extend $\mathcal{EL}$ concepts with the rule $\neg C$ (recall that $C \sqcap \neg C$ is equivalent to $\perp$, so $\mathcal{ALC}$ extends $\mathcal{EL}_\perp$). $\mathcal{ALC}_{bool}$ formulae are expressions φ of the form

$$\alpha ::= C(a) \mid r(a, b) \mid (C = \top)$$

$$\varphi ::= \alpha \mid \neg(\varphi) \mid (\varphi \wedge \varphi)$$

where C is an $\mathcal{ALC}$ concept, $a, b \in \mathsf{N_I}$, and $r \in \mathsf{N_R}$. We may omit parentheses if there is no risk of confusion. The usual concept inclusions $C \sqsubseteq D$ can be expressed with $\top \sqsubseteq \neg C \sqcup D$ and $\neg C \sqcup D \sqsubseteq \top$, which is $(\neg C \sqcup D = \top)$. *Assertions* are expressions of the form $r(a, b)$ and $A(a)$, with $r \in \mathsf{N_R}$, $a, b \in \mathsf{N_I}$, and $A \in \mathsf{N_C}$. Whenever we speak of an $\mathcal{EL}_\perp$ finite base we mean a finite set of concept inclusions and assertions built from $\mathcal{EL}_\perp$ concepts. The same holds for $\mathcal{EL}$ and $\mathcal{ALC}$. The semantics of $\mathcal{EL}$, $\mathcal{EL}_\perp$, $\mathcal{ALC}$, and $\mathcal{ALC}_{bool}$ are defined using interpretations, as usual for DLs [18, 19].

3. Eviction and Reception

Guimarães et al. [17] defined two types of model change operations (functions from $\mathcal{P}_\mathrm{f}(\mathcal{L}) \times \mathcal{P}(\mathfrak{M})$ to $\mathcal{P}_\mathrm{f}(\mathcal{L})$) for modifying finite bases via a set of input models: eviction and reception. Eviction yields a base preserving as many models of the original as possible, while excluding the input models. Eviction operations are constructed using maximal subsets of the ideal resulting set of models, formalised in Definition 1.

Definition 1. Let $\Lambda = (\mathcal{L}, \mathfrak{M}, \models)$ be a satisfaction system. Also, let $\mathbb{M} \subseteq \mathfrak{M}$.

$$\mathrm{MaxFRSubs}(\mathbb{M}, \Lambda) := \{\mathbb{M}' \in \mathrm{FR}(\Lambda) \mid \mathbb{M}' \subseteq \mathbb{M} \text{ and } \nexists \mathbb{M}'' \in \mathrm{FR}(\Lambda) \text{ with } \mathbb{M}' \subset \mathbb{M}'' \subseteq \mathbb{M}\}.$$

Eviction can only be adequately defined in satisfaction systems that are *eviction-compatible*, that is, in those where MaxFRSubs is never empty. Hence, we can use a *FR selection function*, that is, a function sel :

$\mathcal{P}^*(\mathrm{FR}(\Lambda)) \to \mathrm{FR}(\Lambda)$, we can define eviction functions as follows.

Definition 2 ([17])**.** Let Λ be an eviction-compatible satisfaction system and sel a FR selection function on Λ. The *maxichoice eviction function on* Λ defined by sel is a map $\mathrm{evc}_{\mathrm{sel}} : \mathcal{P}_\mathrm{f}(\mathcal{L}) \times \mathcal{P}(\mathfrak{M}) \to \mathcal{P}_\mathrm{f}(\mathcal{L})$ such that:

$$\mathrm{Mod}(\mathrm{evc}_{\mathrm{sel}}(\mathcal{B}, \mathbb{M})) = \mathrm{sel}(\mathrm{MaxFRSubs}(\mathrm{Mod}(\mathcal{B}) \setminus \mathbb{M}, \Lambda)).$$

Theorem 3 characterises the maxichoice eviction defined before.

Theorem 3 ([17])**.** A model change operation evc, defined on an eviction-compatible satisfaction system Λ, is a maxichoice eviction function iff it satisfies the following postulates:

(success) $\mathbb{M} \cap \mathrm{Mod}(\mathrm{evc}(\mathcal{B}, \mathbb{M})) = \emptyset$.

(inclusion) $\mathrm{Mod}(\mathrm{evc}(\mathcal{B}, \mathbb{M})) \subseteq \mathrm{Mod}(\mathcal{B})$.

(finite retainment) If $\mathrm{Mod}(\mathrm{evc}(\mathcal{B}, \mathbb{M})) \subset \mathbb{M}' \subseteq \mathrm{Mod}(\mathcal{B}) \setminus \mathbb{M}$ then $\mathbb{M}' \notin \mathrm{FR}(\Lambda)$.

(uniformity) $\mathrm{MaxFRSubs}(\mathrm{Mod}(\mathcal{B}) \setminus \mathbb{M}, \Lambda) = \mathrm{MaxFRSubs}(\mathrm{Mod}(\mathcal{B}') \setminus \mathbb{M}', \Lambda)$ implies $\mathrm{Mod}(\mathrm{evc}(\mathcal{B}, \mathbb{M})) = \mathrm{Mod}(\mathrm{evc}(\mathcal{B}', \mathbb{M}'))$.

Reception produces a base that has all the models of the original and the input. Definition 4 details the construction employed for characterising and defining reception functions.

Definition 4. Let $\Lambda = (\mathcal{L}, \mathfrak{M}, \models)$ be a satisfaction system. Also, let $\mathbb{M} \subseteq \mathfrak{M}$.

$$\mathrm{MinFRSups}(\mathbb{M}, \Lambda) \coloneqq \{\mathbb{M}' \in \mathrm{FR}(\Lambda) \mid \mathbb{M} \subseteq \mathbb{M}' \text{ and } \nexists \mathbb{M}'' \in \mathrm{FR}(\Lambda) \text{ with } \mathbb{M} \subseteq \mathbb{M}'' \subset \mathbb{M}'\}.$$

As with eviction, reception can only be constructed in *reception-compatible* satisfaction systems. Reception-compatibility allows to define reception functions using FR selection function functions.

Definition 5 ([17])**.** Let $\Lambda = (\mathcal{L}, \mathfrak{M}, \models)$ be a reception-compatible satisfaction system and sel a FR selection function on Λ. The *maxichoice model reception function on* Λ defined by sel is a map $\mathrm{rcp}_{\mathrm{sel}} : \mathcal{P}_\mathrm{f}(\mathcal{L}) \times \mathcal{P}(\mathfrak{M}) \to \mathcal{P}_\mathrm{f}(\mathcal{L})$ such that:

$$\mathrm{Mod}(\mathrm{rcp}_{\mathrm{sel}}(\mathcal{B}, \mathbb{M})) = \mathrm{sel}(\mathrm{MinFRSups}(\mathrm{Mod}(\mathcal{B}) \cup \mathbb{M}, \Lambda)).$$

Maxichoice reception functions can also be characterised via a set of postulates.

Theorem 6 ([17])**.** A model change operation rcp, defined on a reception-compatible satisfaction system Λ, is a maxichoice reception function iff it satisfies the following postulates:

(success) $\mathbb{M} \subseteq \mathrm{Mod}(\mathrm{rcp}(\mathcal{B}, \mathbb{M}))$.

(persistence) $\mathrm{Mod}(\mathcal{B}) \subseteq \mathrm{Mod}(\mathrm{rcp}(\mathcal{B}, \mathbb{M}))$.

(finite temperance) If $\mathrm{Mod}(\mathcal{B}) \cup \mathbb{M} \subseteq \mathbb{M}' \subset \mathrm{Mod}(\mathrm{rcp}(\mathcal{B}, \mathbb{M}))$ then $\mathbb{M}' \notin \mathrm{FR}(\Lambda)$.

(uniformity) $\mathrm{MinFRSups}(\mathrm{Mod}(\mathcal{B}) \cup \mathbb{M}, \Lambda) = \mathrm{MinFRSups}(\mathrm{Mod}(\mathcal{B}') \cup \mathbb{M}', \Lambda)$ implies $\mathrm{Mod}(\mathrm{rcp}(\mathcal{B}, \mathbb{M})) = \mathrm{Mod}(\mathrm{rcp}(\mathcal{B}', \mathbb{M}'))$.

Next, in light of the negative results on compatibility of important satisfaction systems [17], we will modify the framework discussed in this section by restricting the input space of model change operations.

4. Generalising the Framework

While eviction and reception can be defined and characterised in some satisfaction systems, such as the usual for propositional logic and Kleene's three-valued logic, there are also important satisfaction systems that are neither eviction- nor reception-compatible, as it is the case of the DL $\mathcal{ALC}$ [17].

Here, we will attempt to circumvent the incompatibilities of a satisfaction system by placing restrictions on which bases and sets of models are allowed as input. We formalise these additional constraints with the notion of *compartment*.

Definition 7. A *compartment* of a satisfaction system $\Lambda = (\mathcal{L}, \mathfrak{M}, \models)$, is a pair $(\mathfrak{B}, \mathfrak{I})$ such that $\mathfrak{B} \subseteq \mathcal{P}_\mathrm{f}(\mathcal{L})$ and $\mathfrak{I} \subseteq \mathcal{P}(\mathfrak{M})$.

Formally, given a satisfaction system Λ, a model change operation modulo a compartment $(\mathfrak{B}, \mathfrak{I})$ is a function $f_\mathfrak{C} : \mathfrak{B} \times \mathfrak{I} \to \mathrm{FR}(\Lambda)$.

In the following, we generalise the notions of eviction- and reception-compatibility.

Definition 8. Let Λ be a satisfaction system,

1. A compartment $(\mathfrak{B}, \mathfrak{I})$ of Λ is eviction-compatible iff for all $\mathcal{B} \in \mathfrak{B}$ and $\mathbb{M} \in \mathfrak{I}$:
$$\mathrm{MaxFRSubs}(\mathrm{Mod}(\mathcal{B}) \setminus \mathbb{M}, \Lambda) \neq \emptyset.$$
2. A compartment $(\mathfrak{B}, \mathfrak{I})$ of Λ is reception-compatible iff for all $\mathcal{B} \in \mathfrak{B}$ and $\mathbb{M} \in \mathfrak{I}$:
$$\mathrm{MinFRSups}(\mathrm{Mod}(\mathcal{B}) \cup \mathbb{M}, \Lambda) \neq \emptyset.$$

If we consider an eviction-compatible compartment $(\mathfrak{B}, \mathfrak{I})$, we can adapt the notion of maxichoice eviction function from Definition 2, by defining it as a model change operator modulo $(\mathfrak{B}, \mathfrak{I})$. The eviction-compatibility of the compartment will ensure that the domain of the function is non-empty. Consequently, we obtain a version of Theorem 3 when the input is restricted to reception-compatible compartments.

Corollary 9. Let $(\mathfrak{B}, \mathfrak{I})$ be a eviction-compatible compartment of a satisfaction system Λ and evc be a model change operator modulo $(\mathfrak{B}, \mathfrak{I})$. The operator evc is a maxichoice eviction function modulo $(\mathfrak{B}, \mathfrak{I})$ iff it satisfies the postulates success, inclusion, finite retainment, and uniformity from Theorem 3, for all $\mathcal{B} \in \mathfrak{B}$ and $\mathbb{M} \in \mathfrak{I}$.

Proof sketch. The proof is analogous to the proof of Theorem 3 [17, Theorem 5]. □

Similarly, in a reception-compatible compartment $\mathfrak{C}$, we can define maxichoice reception functions modulo $\mathfrak{C}$ by constraining the domain of maxichoice reception functions from Definition 5. We get the following result as a consequence.

Corollary 10. Let $(\mathfrak{B}, \mathfrak{I})$ be a reception-compatible compartment of a satisfaction system Λ and rcp be a model change operator modulo $(\mathfrak{B}, \mathfrak{I})$. The operator rcp is a maxichoice reception function modulo $(\mathfrak{B}, \mathfrak{I})$ iff it satisfies success, persistence, finite temperance, and uniformity from Theorem 6, for all $\mathcal{B} \in \mathfrak{B}$ and $\mathbb{M} \in \mathfrak{I}$.

Proof sketch. The proof is analogous as the proof of Theorem 6 [17, Theorem 10]. □

The new definitions and results based on compartments generalise the original ones because considering the compartment $(\mathcal{P}_{\mathrm{f}}(\mathcal{L}), \mathrm{FR}(\Lambda))$ of $\Lambda = (\mathcal{L}, \mathfrak{M}, \models)$ yields the original constructions and properties defined for a satisfaction system Λ. Proposition 11 shows that the compatibility of a satisfaction system regarding eviction or reception is transferred to all of its compartments.

Proposition 11. Given a compartment $\mathfrak{C}$ of the satisfaction system Λ:

- If Λ is eviction-compatible then $\mathfrak{C}$ is eviction-compatible.
- If Λ is reception-compatible then $\mathfrak{C}$ is reception-compatible.

Proof. Let $\mathfrak{C} = (\mathfrak{B}, \mathfrak{I})$ be a compartment of $\Lambda = (\mathcal{L}, \mathfrak{M}, \models)$. For the first point, if $\Lambda = (\mathcal{L}, \mathfrak{M}, \models)$ is eviction-compatible, then for any $\mathcal{B} \in \mathfrak{B} \subseteq \mathcal{P}_{\mathrm{f}}\,\mathcal{L}$ and any $\mathbb{M} \in \mathfrak{I} \subseteq \mathcal{P}_{\mathrm{f}}(\mathfrak{M})$, it holds that $\mathrm{MaxFRSubs}(\mathcal{B} \setminus \mathfrak{M}, \Lambda) \neq \emptyset$. Hence $\mathfrak{C}$ is eviction-compatible.

The proof is analogous for the second point. □

In the next two sections we study the cases of the classical DLs $\mathcal{EL}$ and $\mathcal{ALC}$. Several DLs are neither eviction not reception-compatible [17]. These impossibility results are proved specifically for each of the investigated logics. We identify sufficient conditions for a DL to not be reception compatible.

Theorem 12. Let $\mathcal{L}$ be a monotonic DL with $\top$, that can represent inconsistencies and it is interpreted over an infinite signature. Then $\Lambda(\mathcal{L})$ is not reception-compatible.

Most of the expressive and interesting DLs can express inconsistencies (either with full negation or just $\bot$) and include the concept $\top$. Therefore, according to Theorem 12, if one wants to perform reception in these logics the only alternative is restricting to finite signatures. Eviction compatibility is lost if there is no way of representing inconsistencies [3] (since in this case one cannot remove all models). From now on, unless otherwise stated we consider only DLs over finite signatures and that can express inconsistencies. It is worth recalling that we are considering only compartments on finite models.

5. The Case of $\mathcal{EL}_\bot$

We investigate the case of the very popular $\mathcal{EL}$ ontology language. In particular, we devise two strategies for reception in $\mathcal{EL}_\bot$ for the case in which the input is a single finite model. Since these approaches are only defined for compartments in which the model class contains only singletons, we abuse the notation and write $\mathcal{I}$ instead of $\{\mathcal{I}\}$ whenever the meaning is clear.

5.1. A Model Product Approach for $\mathcal{EL}_\bot$

In this subsection, we consider an approach for performing reception in compartments of the DL $\mathcal{EL}_\bot$. We propose the following strategy to perform the reception of a base $\mathcal{B}$ with a single model (interpretation) $\mathcal{I}$: we turn $\mathcal{B}$ into one of its models, and we combine this model with $\mathcal{I}$ to produce the reception result. To combine these two models, we define a product operation which preserves exactly the information satisfied by both models. We then produce the finite base for reception from the model obtained by the product operation. If $\mathcal{B}$ is unsatisfiable the resut of the reception is a base that represents the input model exactly.

The main hurdle is to find a suitable model of the base. We need a model that satisfies exactly and only the information entailed by the knowledge base. Such models are called fit models:

Definition 13. A model $\mathcal{I}$ *fits* a base $\mathcal{B}$, iff (i) $\mathcal{I}$ is finite and (ii) for every $\mathcal{EL}_\bot$ formula α, $\mathcal{B} \models \alpha$ iff $\mathcal{I} \models \alpha$. Equivalently, we say that $\mathcal{B}$ fits $\mathcal{I}$.

Example 14 illustrates a base and one of its fit models.

Example 14. Let $\mathcal{B} = \{A \equiv \exists r.\top, A \equiv \exists r.A\}$. Also, let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ defined over $\{A, r\}$ and such that

$$\Delta^{\mathcal{I}} = \{x_1, x_2\} \quad A^{\mathcal{I}} \quad = \{x_1\} \quad r^{\mathcal{I}} = \{(x_1, x_1)\}.$$

Observation 15 shows that $\mathcal{I}$ fits $\mathcal{B}$.

Observation 15. The interpretation $\mathcal{I}$ from Example 14 fits the base $\mathcal{B}$ at that same example.

Indeed, in $\mathcal{EL}_\bot$, every finite model fits some finite set of concept inclusions (TBox). This result follows from the existence of algorithms that construct TBoxes that fit a given finite model [20, 21]. The existence of fit bases for any finite model also allows us to handle the case in which $\mathcal{B}$ is unsatisfiable. Therefore, in the remainder of this subsection, we consider only compartments $(\mathfrak{B}, \mathfrak{I})$ of $\Lambda(\mathcal{EL}_\bot)$ such that (1) all bases in $\mathfrak{B}$ present a fit model $\mathcal{I}$ such that $\{\mathcal{I}\} \in \mathfrak{I}$ and (2) $|\mathbb{M}| = 1$ for all $\mathbb{M} \in \mathfrak{I}$. Such compartments are called *fit compartments*. As usual, the product operation on models is as follows.

Definition 16. The *product* of two models $\mathcal{I}_1$ and $\mathcal{I}_2$ is the model $\mathcal{I} = \mathcal{I}_1 \times \mathcal{I}_2$ where

- $\Delta^{\mathcal{I}} := \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$;
- $A^{\mathcal{I}} := \{(d, e) \mid d \in A^{\mathcal{I}_1}, e \in A^{\mathcal{I}_2}\}$, for all $A \in \mathsf{N_C}$;
- $r^{\mathcal{I}} := \{((d, e), (d', e')) \mid (d, d') \in r^{\mathcal{I}_1}$ and $(e, e') \in r^{\mathcal{I}_2}\}$, for all $r \in \mathsf{N_R}$.
- The interpretation of complex concepts is defined as usual.

Proposition 17. Let $\mathcal{I} = \mathcal{I}_1 \times \mathcal{I}_2$. For all $\mathcal{EL}_\bot$ concepts C: $(u, v) \in C^{\mathcal{I}}$ iff $u \in C^{\mathcal{I}_1}$ and $v \in C^{\mathcal{I}_2}$.

The purpose of the product, say $\mathcal{I}_1 \times \mathcal{I}_2$, is to obtain a model that preserves precisely the information satisfied by both models $\mathcal{I}_1$ and $\mathcal{I}_2$. Although this is not true in general, there is a specific class of models in which the product satisfies such behaviour as long as the formulae of interest do not contain concepts logically equivalent to $\bot$ (Theorem 25).

Definition 18. For every two models $\mathcal{I}_1$ and $\mathcal{I}_2$ in such a class, and $\mathcal{EL}_\bot$ concepts C and D,

weak-preservation: if $\emptyset \not\models C \equiv D \equiv \bot$, then $\mathcal{I}_1 \times \mathcal{I}_2 \models C \sqsubseteq D$ iff $\mathcal{I}_1 \models C \sqsubseteq D$ and $\mathcal{I}_2 \models C \sqsubseteq D$.

Ideally, the property above should also cover concept inclusions involving concepts that are logically equivalent to $\bot$. This issue can be easily overcome by applying a rewriting function τ that swaps each $\bot$ symbol with an extra concept name $A_\bot$. As $A_\bot$ is a concept name, we can define τ ensuring the obtained concept is not logically equivalent to $\bot$, and for every model $\mathcal{I}$ in this desired class of models, we have

$$\mathcal{I} \models C \sqsubseteq D \text{ iff } \mathcal{I} \models \tau(C) \sqsubseteq \tau(D).$$

Definition 19. Let $\Sigma \subset \mathsf{N_C} \cup \mathsf{N_R} \cup \mathsf{N_I}$ be a signature and $A_\bot \in \mathsf{N_C} \setminus \Sigma$. We define $\tau : \mathcal{EL}_\bot(\Sigma) \to \mathcal{EL}_\bot(\Sigma \cup \{A_\bot\})$ inductively:

- $\tau(\bot) = A_\bot$,
- $\tau(B) = B$, if $B \in \mathsf{N_C} \setminus \{\bot\}$,
- $\tau(\exists r.C) = \exists r.\tau(C)$,
- $\tau(C \sqcap D) = \tau(C) \sqcap \tau(D)$,
- $\tau(C(a)) = (\tau(C)(a))$,
- $\tau(r(a, b)) = r(a, b)$.

The missing ingredient is to frame precisely the so well-behaved class of models mentioned above. In order to capture the *weak-preservation* condition, we consider models in which the extension of every concept that is not tautologically equivalent to $\bot$ must be non-empty. We start showing that every model can be turned into an equivalent model satisfying *weak-preservation*. By equivalent, we mean that the original model satisfies a formula α iff the new model satisfies its rewriting $\tau(\alpha)$. This model is called an ε-extension.

Definition 20. Let $A_\bot \in \mathsf{N_C}$, $\Sigma \subset ((\mathsf{N_C} \setminus \{A_\bot\}) \cup \mathsf{N_R} \cup \mathsf{N_I})$ be a signature, and $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a model defined over Σ. The ε-extension of $\mathcal{I}$ is the interpretation $\varepsilon(\mathcal{I}) = (\Delta^{\mathcal{I}} \cup \{\varepsilon\}, \cdot^{\varepsilon(\mathcal{I})})$ such that:

- $(A_\bot)^{\varepsilon(\mathcal{I})} = \{\varepsilon\}$.
- $B^{\varepsilon(\mathcal{I})} = B^{\mathcal{I}} \cup \{\varepsilon\}$ for all $B \in \mathsf{N_C} \setminus \{A_\bot\}$.
- $r^{\varepsilon(\mathcal{I})} = r^{\mathcal{I}} \cup \{(\varepsilon, \varepsilon)\}$ for all $r \in \Sigma \cup \mathsf{N_R}$.
- $a^{\varepsilon(\mathcal{I})} = a^{\mathcal{I}}$, for all $a \in \Sigma \cup \mathsf{N_I}$.

Where we assume w.l.o.g. that $\varepsilon \notin \Delta^{\mathcal{I}}$ and $A_\bot \in \mathsf{N_C} \setminus \Sigma$.

Proposition 21 states that the ε-extension of an interpretation $\mathcal{I}$ extends each concept C with a fixed sentinel symbol ε, as long as C is satisfiable.

Proposition 21. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a model defined over a finite signature Σ and $\varepsilon(\mathcal{I}) = (\Delta^{\mathcal{I}} \cup \{\varepsilon\}, \cdot^{\varepsilon(\mathcal{I})})$ its ε-extension. For all $\mathcal{EL}_\bot$ concepts C, either $\emptyset \models C \equiv \bot$ or $\varepsilon \in C^{\varepsilon(\mathcal{I})}$.

An ε-extension also preserves entailments over the original signature, as shown in Lemma 22.

Lemma 22. Let $A_\bot \in \mathsf{N_C}$, let $\Sigma \subset ((\mathsf{N_C} \setminus \{A_\bot\}) \cup \mathsf{N_R} \cup \mathsf{N_I})$ be a finite signature, and let $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ be a model over Σ, and α an $\mathcal{EL}_\bot$ formula over Σ. Then $\mathcal{I} \models \alpha$ iff $\varepsilon(\mathcal{I}) \models \alpha$.

Propositions 23 and 24 depict a convenient relationship between the translation function τ and ϵ-extensions.

Proposition 23. For all ε-extension $\varepsilon(\mathcal{I})$ and concept C, $(\tau(C))^{\varepsilon(\mathcal{I})} = C^\mathcal{I} \cup \{\varepsilon\}$.

Proposition 24. Let $\mathcal{I}$ be a model defined over the finite signature $\Sigma \subseteq (\mathsf{N_C} \setminus \{A_\bot\}) \cup \mathsf{N_R} \cup \mathsf{N_I}$ and C an $\mathcal{EL}_\bot$ concept over Σ. It holds that $(\tau(C))^{\varepsilon(\mathcal{I})} = C^{\varepsilon(\mathcal{I})} \cup \{\varepsilon\}$.

It is worth stressing that ε-extensions are models by definition, and therefore we can apply the product operation to them. This guarantees that the product of two ε-extensions satisfies weak-preservation (Definition 18), leading to Theorem 25.

Theorem 25. For all $\mathcal{EL}_\bot$ concepts C, D, we have that $\varepsilon(\mathcal{I}_1) \times \varepsilon(\mathcal{I}_2) \models \tau(C \sqsubseteq D)$ iff $\varepsilon(\mathcal{I}_1) \models \tau(C \sqsubseteq D)$ and $\varepsilon(\mathcal{I}_2) \models \tau(C \sqsubseteq D)$.

Distel [20], Guimarães et al. [21] have shown that in $\mathcal{EL}_\bot$, every finite model fits some finite base $\mathcal{B}$. Let $\mathrm{fit}(\cdot)$ be a function that maps each finite model $\mathcal{I}$ to some base that $\mathcal{I}$ fits. We say that fit is a fit assignment.

At this point, we have all the necessary ingredients to define our reception operation on products. The construction is suitable for all fit compartments in $\mathcal{EL}_\bot$.

Definition 26. Let $\mathfrak{C} = (\mathfrak{B}, \mathfrak{I})$ be a fit compartment of $\mathcal{EL}_\bot$, and fit a fit assignment. A product reception operation on $\mathfrak{C}$ is a function $\mathrm{rcp}_\times : \mathfrak{B} \times \mathfrak{I} \to \mathfrak{B}$, s.t

$$\mathrm{rcp}_\times(\mathcal{B}, \mathcal{I}) = \begin{cases} \mathrm{fit}(\mathcal{I}) & \text{if } \mathcal{B} \models \bot; \\ \mathrm{fit}(\varepsilon(\mathcal{I}_\mathcal{B}) \times \varepsilon(\mathcal{I})) & \text{otherwise.} \end{cases}$$

where $\mathcal{I}_\mathcal{B}$ is a finite model that fits $\mathcal{B}$.

Proposition 27. The ε-extensions are closed under the product operation.

Lemma 28. Let $\mathcal{I}$ be a model and φ an $\mathcal{EL}_\bot$ formula over the finite signature $\Sigma \subseteq (\mathsf{N_C} \setminus \{A_\bot\}) \cup \mathsf{N_R} \cup \mathsf{N_I}$. $\varepsilon(\mathcal{I}) \models \varphi$ iff $\varepsilon(\mathcal{I}) \models \tau(\varphi)$.

The product reception operation retains exactly the formulae entailed by the base and input model.

Corollary 29. Let $\mathfrak{C} = (\mathfrak{B}, \mathfrak{I})$ be a fit compartment of $\mathcal{EL}_\bot$, and $\mathrm{rcp}_\times$ a product operation on $\mathfrak{C}$. For all $\mathcal{EL}_\bot$ concept inclusions $C \sqsubseteq D$, $\mathrm{rcp}_\times(\mathcal{B}, \mathcal{I}) \models C \sqsubseteq D$ iff $\mathcal{B} \models C \sqsubseteq D$ and $\mathcal{I} \models C \sqsubseteq D$.

The $\mathcal{EL}_\bot$ satisfaction system presents some interesting behaviours regarding reception operations. One of such properties is the *reverse monotonic bijection property* (RMBP).

Definition 30 ([17]). A satisfaction system $\Lambda = (\mathcal{L}, \mathfrak{M}, \models)$ has the RMBP property iff for every $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{L}$, and $\mathcal{I} \in \mathfrak{M}$: $\mathcal{I} \in \mathrm{Mod}(\mathcal{B}_1 \cup \mathcal{B}_2)$ iff $\mathcal{I} \in \mathrm{Mod}(\mathcal{B}_1)$ and $\mathcal{I} \in \mathrm{Mod}(\mathcal{B}_2)$.

The RBMP induces a 'uniqueness' property of the functions $\mathrm{MinFRSups}$ and $\mathrm{MaxFRSubs}$.

Corollary 31 ([17]). Let Λ be the usual satisfaction system of $\mathcal{EL}_\bot$. For every set of models $\mathbb{M}$, $|\mathrm{MinFRSups}(\mathbb{M}, \Lambda)| \leq 1$.

In fit compartments of $\mathcal{EL}_\bot$, the product reception operations are exactly those reception operations that satisfy all four postulates from Theorem 6.

Theorem 32. Let $\mathfrak{C} = (\mathfrak{B}, \mathfrak{I})$ be a fit compartment of $\mathcal{EL}_\bot$. A model change operation rcp on $\mathfrak{C}$ satisfies all four postulates on Theorem 6 iff there is some product reception operation $\mathrm{rcp}_\times$ such that $\mathrm{Mod}(\mathrm{rcp}(\mathcal{B}, \mathcal{I})) = \mathrm{Mod}(\mathrm{rcp}_\times(\mathcal{B}, \mathcal{I}))$, for all bases $\mathcal{B}$ and all interpretations $\mathcal{I}$ in $\mathfrak{C}$.

While the restriction to fit compartments yields an elegant construction, it does not cover every $\mathcal{EL}_\bot$ base, not even when restricting ourselves only to concept inclusions (TBoxes) as Example 33.

Example 33. Let $\mathcal{B} = \{A \sqsubseteq \exists r.A\}$. For all $k \geq 0$ and $j \geq 1$ it holds that $\mathcal{B} \models \exists r^k.A \sqsubseteq \exists r^{(k+j)}.A$. Also, let $\mathcal{I}$ be a finite model of $\mathcal{B}$. Since $\Delta^\mathcal{I}$ is finite, for every $\mathcal{EL}_\bot$ concept C, there will be at most 2^n possible extensions under $\mathcal{I}$, where $n = \Delta^\mathcal{I}$. However, consider now the set of concepts $\{\exists r^i.A \mid 1 \leq i \leq 2^n + 1\}$. By the pigeonhole principle, there will be distinct concepts $\exists r^l.A$ and $\exists r^m.A$ in this set such that $\mathcal{I} \models \exists r^l.A \equiv \exists r^m.A$. W.l.o.g., let us assume $l < m$, then we have that $\mathcal{I} \models \exists r^m.A \sqsubseteq \exists r^l.A$ but $\mathcal{B} \not\models \exists r^m.A \sqsubseteq \exists r^l.A$. Since $\mathcal{I}$ is an arbitrary finite model of $\mathcal{I}$, $\mathcal{B}$ has no fit model.

In the next subsection, we explore another interesting approach which covers all reception-compatible compartments of $\Lambda(\mathcal{EL}_\bot)$.

5.2. A Saturation Strategy

The product operation we defined in the previous subsection works only for fit compartments. In this section, we propose a broader approach. We extract from the incoming model some specific formulae that it satisfies and we retain from such a set only the formulae that are entailed by the base. The obtained set corresponds to the reception result. This strategy works on every compartment $(\mathfrak{B}, \mathfrak{I})$ that is reception-compatible for DLs that are monotonic and idempotent and such that if $\mathbb{M} \in \mathfrak{I}$ then $|\mathbb{M}| = \{\mathcal{I}\}$ for some finite model $\mathcal{I}$.

As the result of the reception must be finite, we extract from the incoming model only formulae with size less or equal to a given number m.

The length of the greatest concept of an $\mathcal{EL}_\bot$ formula φ is given by

$$\mathrm{gcp}(\varphi) = \begin{cases} \max(|C|, |D|) & \text{if } \varphi = C \sqsubseteq D \\ |C| & \text{if } \varphi = C(a) \\ 1 & \text{if } \varphi = r(a,b) \end{cases}$$

We also extend this notion to finite bases: $\mathrm{gcp}(\mathcal{B}) = \max(\{\mathrm{gcp}(\varphi) \mid \varphi \in \mathcal{B}\})$.

Definition 34. Let m be a positive integer. The saturation of a model $\mathcal{I}$ bounded by m is the set $\mathcal{B}^m_\mathcal{I}$ such that $\varphi \in \mathcal{B}^m_\mathcal{I}$ iff $\mathrm{gcp}(\varphi) \leq m$ and $\mathcal{I} \models \varphi$.

Observation 35. For every positive integer m and interpretation $\mathcal{I}$, if $\varphi \in \mathcal{B}^m_\mathcal{I}$ then $\mathcal{I} \models \varphi$.

Lemma 36. If $\mathcal{B}$ is finite and $\mathcal{I} \models \mathcal{B}$ then $\mathcal{B} \subseteq \mathcal{B}^m_\mathcal{I}$, where $m = \mathrm{gcp}(\mathcal{B})$.

Proof. Let us assume that $\mathcal{B}$ is finite, $m = \mathrm{gcp}(\mathcal{B})$, and $\mathcal{I} \models \mathcal{B}$. Let $\mathcal{B}^m_\mathcal{I}$ be the saturation of $\mathcal{I}$ bounded to m. Let $\varphi \in \mathcal{B}$. As $m = \mathrm{gcp}(\mathcal{B})$, we get that $\mathrm{gcp}(\varphi) \leq m$. Therefore, as $\mathcal{I} \models \mathcal{B}$, we get that $\mathcal{I} \models \varphi$. This means that $\varphi \in \mathcal{B}^m_\mathcal{I}$. $\square$

In order to perform the reception of a knowledge base $\mathcal{B}$ with a model $\mathcal{I}$, we saturate $\mathcal{I}$ to an upper bound m, and we intersect such a set with the all information entailed by $\mathcal{B}$. Precisely, we define our reception operation as $Cn(\mathcal{B}) \cap \mathcal{B}^m_\mathcal{I}$.

Theorem 37. A compartment $\mathfrak{C} = (\mathfrak{B}, \mathfrak{I})$ is reception-compatible iff for all pairs $(\mathcal{B}, \mathcal{I}) \in \mathfrak{B} \times \mathfrak{I}$ there is some positive integer m such that

$$\mathrm{Mod}(\mathcal{B}') \in \mathrm{MaxFRSubs}(\mathrm{Mod}(\mathcal{B}) \cup \{\mathcal{I}\}, \Lambda),$$

where $\mathcal{B}' = \mathrm{Cn}(\mathcal{B}) \cap \mathcal{B}^m_\mathcal{I}$.

Theorem 37 characterises all reception-compatible compartments, including the maximal ones, via the saturation construction when the input is a single finite model. Although Theorem 37 focuses on $\mathcal{EL}_\bot$, it can be easily extended to other satisfaction systems, as the proof requires only the logic to be both monotonic and idempotent, as well as presenting a notion of length of formulae analogous to gcd.

6. The Case of $\mathcal{ALC}_{bool}$

The framework we presented in Section 4 is general enough to cover several satisfaction systems without imposing much constraints upon the logics being used to represent an agent's beliefs. However, there are interesting logics used for knowledge representation that are not reception-compatible, as it is the case of some DLs (Theorem 38). In this section, we investigate how to extend model change operations to one such logic as a study case. We look precisely at the logic $\mathcal{ALC}_{bool}$[1], which corresponds to the DL $\mathcal{ALC}$ enriched with boolean operators over $\mathcal{ALC}$ axioms. As $\mathcal{ALC}$ is a prototypical DL, it shares many similarities with other logics in the of DL family. Our results are built on proofs for the $\mathcal{ALC}$ case without boolean operators over the axioms [3].

We establish negative results for eviction compatibility. We denote by $\Lambda(\mathcal{ALC}_{bool})$ the satisfaction system with the entailment relation given by the standard semantics of $\mathcal{ALC}_{bool}$ [19].

Theorem 38. $\Lambda(\mathcal{ALC}_{bool})$ is neither eviction-compatible nor reception-compatible.

Proof. The fact that $\Lambda(\mathcal{ALC}_{bool})$ is not reception-compatible follows from Theorem 12 (the case for when the signature is finite is open). We then show that $\Lambda(\mathcal{ALC}_{bool})$ is not eviction-compatible (the proof works if the signature is finite or infinite). Let $\Lambda(\mathcal{ALC}_{bool}) = (\mathcal{L}_{\mathcal{ALC}_{\mathrm{bool}}}, \mathfrak{M}_{\mathcal{ALC}_{\mathrm{bool}}}, \models_{\mathcal{ALC}_{\mathrm{bool}}})$ be the usual satisfaction system for $\mathcal{ALC}_{\mathrm{bool}}$. For conciseness, we will write $\models$ instead of $\models_{\mathcal{ALC}_{\mathrm{bool}}}$ within this proof. Let $\mathcal{B}_\top = \{\bot \sqsubseteq \top\}$, that is, $\mathrm{Mod}(\mathcal{B}_\top) = \mathfrak{M}$. Also, given a fixed but arbitrary $a \in \mathsf{N_I}$ and $r \in \mathsf{N_R}$, we define models of the form $M^n = (\mathbb{N}, \cdot^{M^n})$ where

$$r^{M^n} = \{(i, i+1) \mid i \in \mathbb{N}, 0 \leq i < n\}$$

and $a^{M_n} = 0$, and similarly $M^\infty = (\mathbb{N}, \cdot^{M^\infty})$ where

$$r^{M^\infty} = \{(i, i+1) \mid i \in \mathbb{N}\}$$

and $a^{M^\infty} = 0$. Let $\mathbb{M}$ be the set of all models M such that for some $n \in \mathbb{N}$ we have that $a^M \in (\forall r^n.\bot)^M$. That is, there is no loop or infinite chain of elements connected via the role r starting from a^M. By definition of $\mathbb{M}$, we have that $M^\infty \notin \mathbb{M}$ since this model has an infinite chain of elements connected via the role r starting from a^M, while $M^n \in \mathbb{M}$ for all $n \in \mathbb{N}$.

To prove that $\Lambda(\mathcal{ALC}_{bool})$ is not eviction-compatible, we need to prove that there is no $\mathcal{B} \in \mathcal{P}_\mathrm{f}(\mathcal{L}_{\mathcal{ALC}_{\mathrm{bool}}})$ such that $\mathrm{Mod}(\mathcal{B}) \in \mathrm{MaxFRSubs}(\mathbb{M}, \Lambda(\mathcal{ALC}_{bool}))$, that is, $\mathrm{MaxFRSubs}(\mathbb{M}, \Lambda(\mathcal{ALC}_{bool})) = \emptyset$. Intuitively, we want to show that we cannot find a maximal $\mathcal{ALC}_{\mathrm{bool}}$ ontology that finitely represents the result of removing the models in $\mathfrak{M} \setminus \mathbb{M}$ from $\mathcal{B}_\top$. First, we recall the following claims.

Claim 39 ([3]). For every $\mathcal{ALC}$ concept C if there is $n \in \mathbb{N}$ such that for all $m \geq n$, with $m \in \mathbb{N}$, we have that $M^m \models C(a)$ then $M^\infty \models C(a)$.

[1] $\mathcal{ALC}_{bool}$ is also called $\mathcal{ALC}$-formula in [19], the former nomenclature facilitates the distinction between the logic and its formulae.

Claim 40 ([3]). For every $\mathcal{ALC}$ concept C if there is $n \in \mathbb{N}$ such that for all $m \geq n$, with $m \in \mathbb{N}$, we have that $M^m \models \top \sqsubseteq C$ then $M^\infty \models \top \sqsubseteq C$.

We are now ready to show that $\Lambda(\mathcal{ALC}_{bool})$ is not eviction-compatible. Suppose to the contrary that there is $\mathcal{B} \in \mathcal{P}_\mathrm{f}(\mathcal{L}_{\mathcal{ALC}_\mathrm{bool}})$ such that $\mathrm{Mod}(\mathcal{B}) \in \mathrm{MaxFRSubs}(\mathbb{M}, \Lambda(\mathcal{ALC}_{bool}))$. If there is $n \in \mathbb{N}$ such that $M^n \not\models \mathcal{B}$ then[2]

$$\mathcal{B}' := \mathcal{B} \vee (\bigsqcup_{k=0}^{n+1} (\exists r^k.\top \sqcap \neg\exists r^{k+1}.\top) = \top),$$

is such that $M^n \models \mathcal{B}'$. Moreover, $\mathrm{Mod}(\mathcal{B}) \subset \mathrm{Mod}(\mathcal{B}')$. By definition of $\mathcal{B}'$ and $\mathbb{M}$, we also have that $\mathrm{Mod}(\mathcal{B}') \in \mathrm{MaxFRSubs}(\mathbb{M}, \Lambda(\mathcal{ALC}_{bool}))$. This contradicts the assumption that $\mathrm{Mod}(\mathcal{B}) \in \mathrm{MaxFRSubs}(\mathbb{M}, \Lambda(\mathcal{ALC}_{bool}))$. So, for all $n \in \mathbb{N}$, we have that $M^n \models \mathcal{B}$.

Then, by Claims 39 and 40, it follows that $M^\infty \models \mathcal{B}$. Since, as already mentioned, $M^\infty \notin \mathbb{M}$, this contradicts the assumption that $\mathrm{Mod}(\mathcal{B}) \in \mathrm{MaxFRSubs}(\mathbb{M}, \Lambda(\mathcal{ALC}_{bool}))$. Thus, $\mathrm{MaxFRSubs}(\mathbb{M}, \Lambda(\mathcal{ALC}_{bool})) = \emptyset$. □

Quasimodels Now, we employ quasimodels in a new strategy for belief change in $\mathcal{ALC}_{bool}$. Our approach is based on the translation of formulae in $\mathcal{ALC}_{bool}$ into DNF. Let φ be an $\mathcal{ALC}_{bool}$ formula. Let $\mathsf{f}(\varphi)$ and $\mathsf{c}(\varphi)$ be the set of all subformulae and subconcepts of φ closed under single negation, respectively. A *concept type* for φ is a subset $\mathbf{c} \subseteq \mathsf{c}(\varphi)$ such that: $D \in \mathbf{c}$ iff $\neg D \notin \mathbf{c}$, for all $D \in \mathsf{c}(\varphi)$; and (2) $D \sqcap E \in \mathbf{c}$ iff $\{D, E\} \subseteq \mathbf{c}$, for all $D \sqcap E \in \mathsf{c}(\varphi)$. A *formula type* for φ is a subset $\mathbf{f} \subseteq \mathsf{f}(\varphi)$ such that: (1) $\phi \in \mathbf{f}$ iff $\neg\phi \notin \mathbf{f}$, for all $\phi \in \mathsf{f}(\varphi)$; and (2) $\phi \wedge \psi \in \mathbf{f}$ iff $\{\phi, \psi\} \subseteq \mathbf{f}$, for all $\phi \wedge \psi \in \mathsf{f}(\varphi)$. We may omit 'for φ' if this is clear from the context. A *model candidate* for φ is a triple $(T, o, \mathbf{f})$ such that T is a set of concept types, o is a function from $\mathsf{ind}(\varphi)$ to T, $\mathbf{f}$ a formula type, and $(T, o, \mathbf{f})$ satisfies the conditions: $\varphi \in \mathbf{f}$; $C(a) \in \mathbf{f}$ implies $C \in o(a)$; $r(a, b) \in \mathbf{f}$ implies $\{\neg C \mid \neg\exists r.C \in o(a)\} \subseteq o(b)$.

Definition 41 (Quasimodel). A model candidate $(T, o, \mathbf{f})$ for φ is a *quasimodel* for φ if the following holds

- for every concept type $\mathbf{c} \in T$ and every $\exists r.D \in \mathbf{c}$, there is $\mathbf{c}' \in T$ such that $\{D\} \cup \{\neg E \mid \neg\exists r.E \in \mathbf{c}\} \subseteq \mathbf{c}'$;
- for every concept type $\mathbf{c} \in T$ and every concept C, if $\neg C \in \mathbf{c}$ then this implies $(C = \top) \notin \mathbf{f}$;
- for every concept C, if $\neg(C = \top) \in \mathbf{f}$ then there is $\mathbf{c} \in T$ such that $C \notin \mathbf{c}$;
- T is not empty.

[2] Recall that M^n has a chain of $n + 1$ elements connected via the role r.

Theorem 42 establishes the connection between quasimodels and formulae in $\mathcal{ALC}_{bool}$.

Theorem 42 (Theorem 2.27 [19]). An $\mathcal{ALC}_{bool}$-formula φ is satisfiable iff φ has a quasimodel.

One can associate a model $\mathcal{I}_\mathcal{Q}$ to each quasimodel $\mathcal{Q}$ for φ such that $\mathcal{I}_\mathcal{Q} \models \varphi$ (see Definition 43).

Definition 43. Given a quasimodel $\mathcal{Q} = (T, o, \mathbf{f})$ for an $\mathcal{ALC}_{bool}$-formula φ, we define a model $\mathcal{I}_\mathcal{Q} = (\Delta^{\mathcal{I}_\mathcal{Q}}, \cdot^{\mathcal{I}_\mathcal{Q}})$ as follows:

- $\Delta^{\mathcal{I}_\mathcal{Q}} := T$
- $A^{\mathcal{I}_\mathcal{Q}} := \{t \in T \mid A \in c\}$ for all $A \in \mathsf{N_C}$;
- $r^{\mathcal{I}_\mathcal{Q}} := \{(t, t') \in T^2 \mid \neg\exists r.D \in t \Rightarrow \neg D \in t'\}$ for all $r \in \mathsf{N_R}$.

Given a class of models $\mathfrak{I}$, the class of $\mathcal{ALC}_{bool}$-formulae *induced by* φ is the class of $\mathcal{ALC}_{bool}$-formulae that contains φ and any $\mathcal{ALC}_{bool}$-formulae that is a boolean combination of atoms in φ.

Theorem 44. Consider the compartment $(\mathfrak{B}_\varphi, \mathfrak{I}_\varphi)$ where (1) $\mathfrak{I}_\varphi$ is the class of all subsets of models $\mathcal{I}_\mathcal{Q}$ with $\mathcal{Q}$ a quasimodel for an $\mathcal{ALC}_{bool}$-formula φ and (2) $\mathfrak{B}_\varphi$ is the class of $\mathcal{ALC}_{bool}$-formulae induced by φ. Then, $(\mathfrak{B}_\varphi, \mathfrak{I}_\varphi)$ is both eviction and reception-compatible.

Proof. This theorem follows from the results on model expansion and contraction presented earlier [4], where expansion corresponds to reception and contraction means eviction in our terminology. □

7. Conclusion

In this work, we investigate the model change operations of eviction and reception when applied to description logics such as $\mathcal{EL}$ and $\mathcal{ALC}$. We provide a more general negative result for reception in a large class of DLs. Moreover, generalise the framework proposed by Guimarães et al. [17] with the notion of compartments. These compartments allows us to define two reception functions in restricted domains, which satisfy the existing postulates for this operation. In particular, we restricted the input to singleton sets of finite models defined over finite signatures. While one of these reception functions applies specifically for fit $\mathcal{EL}_\bot$ compartments, the other can be employed in a broader class of compartments, including other DLs. Furthermore, we frame previous constructions for $\mathcal{ALC}_{bool}$ in the same framework, while extending the negative results for unrestricted eviction from $\mathcal{ALC}$ to $\mathcal{ALC}_{bool}$.

As future work, we include an investigation of general results of eviction-compatibility, similar to Theorem 12, in particular for $\mathcal{EL}_{\perp}$. We also aim at identifying eviction- and reception-compatible fragments for more satisfaction systems, in particular for Tarskian DLs. Finally, we will also explore the connections between model change operations and recent formula-based approaches for Belief Change [11, 12] and Ontology Repair [14, 9, 22].

Acknowledgments

The second author is supported by the NFR project "Learning Description Logic Ontologies", grant number 316022. The third author is supported by the German Research Association (DFG), project number 465447331.

References

[1] C. E. Alchourrón, P. Gärdenfors, D. Makinson, On the Logic of Theory Change: Partial Meet Contraction and Revision Functions, Journal of Symbolic Logic 50 (1985) 510–530.

[2] H. Katsuno, A. O. Mendelzon, Propositional knowledge base revision and minimal change, Artificial Intelligence 52 (1991) 263–294.

[3] R. Guimarães, A. Ozaki, J. S. Ribeiro, Finite based contraction and expansion via models, in: AAAI, 2023.

[4] J. S. Ribeiro, R. Guimarães, A. Ozaki, Revising ontologies via models: The ALC-formula case, in: DL, volume 2954 of *CEUR Workshop Proceedings*, CEUR-WS.org, 2021.

[5] B. Nebel, Belief Revision and Default Reasoning: Syntax-Based Approaches, in: KR 1991, Morgan Kaufmann, 1991, pp. 417–428.

[6] S. Dixon, W. Wobcke, The Implementation of a First-Order Logic AGM Belief Revision System, in: ICTAI 1993, IEEE Computer Society, 1993, pp. 40–47.

[7] S. O. Hansson, Knowledge-Level Analysis of Belief Base Operations, Artificial Intelligence 82 (1996) 215–235.

[8] P. T. Guerra, R. Wassermann, Two AGM-style characterizations of model repair, Ann. Math. Artif. Intell. 87 (2019) 233–257.

[9] W. Hieke, F. Kriegel, A. Nuradiansyah, Repairing $\mathcal{EL}$ TBoxes by means of countermodels obtained by model transformation, in: Proceedings of the 34th International Workshop on Description Logics (DL 2021), Bratislava, Slovakia, September 19-22, 2021, volume 2954 of *CEUR Workshop Proceedings*, CEUR-WS.org, 2021.

[10] Y. D. Santos, V. B. Matos, M. M. Ribeiro, R. Wassermann, Partial meet pseudo-contractions, International Journal of Approximate Reasoning 103 (2018) 11–27.

[11] V. B. Matos, R. Guimarães, Y. D. Santos, R. Wassermann, Pseudo-contractions as Gentle Repairs, in: Lecture Notes in Computer Science, Springer International Publishing, 2019, pp. 385–403.

[12] V. B. Matos, R. Wassermann, Repairing ontologies via kernel pseudo-contraction, in: Proceedings of the 20th International Workshop on Non-Monotonic Reasoning, NMR 2022, Part of the Federated Logic Conference (FLoC 2022), Haifa, Israel, August 7-9, 2022, volume 3197 of *CEUR Workshop Proceedings*, CEUR-WS.org, 2022, pp. 16–26. URL: https://ceur-ws.org/Vol-3197/paper2.pdf.

[13] N. Troquard, R. Confalonieri, P. Galliani, R. Peñaloza, D. Porello, O. Kutz, Repairing Ontologies via Axiom Weakening, in: AAAI 2018, AAAI Press, 2018, pp. 1981–1988.

[14] F. Baader, F. Kriegel, A. Nuradiansyah, R. Peñaloza, Making Repairs in Description Logics More Gentle, in: KR 2018, AAAI Press, 2018.

[15] M. Aiguier, J. Atif, I. Bloch, C. Hudelot, Belief revision, minimal change and relaxation: A general framework based on satisfaction systems, and applications to description logics, Artificial Intelligence 256 (2018) 160–180.

[16] J. P. Delgrande, P. Peppas, S. Woltran, General belief revision, J. ACM 65 (2018) 29:1–29:34.

[17] R. Guimarães, A. Ozaki, J. S. Ribeiro, Finite based contraction and expansion via models, CoRR abs/2303.03034 (2023).

[18] F. Baader, I. Horrocks, C. Lutz, U. Sattler, An Introduction to Description Logic, Cambridge University Press, 2017.

[19] K. Agi, F. Wolter, M. Zakharyaschev, D. Gabbay, Many-dimensional modal logics : theory and applications, Elsevier North Holland, Amsterdam Boston, 2003.

[20] F. Distel, Learning description logic knowledge bases from data using methods from formal concept analysis, Ph.D. thesis, Dresden University of Technology, 2011.

[21] R. Guimarães, A. Ozaki, C. Persia, B. Sertkaya, Mining EL bases with adaptable role depth, J. Artif. Intell. Res. 76 (2023) 883–924.

[22] F. Baader, P. Koopmann, F. Kriegel, A. Nuradiansyah, Optimal ABox Repair w.r.t. Static EL TBoxes: From Quantified ABoxes Back to ABoxes, in: The Semantic Web, Springer International Publishing, 2022, pp. 130–146.