Description Logics in the Calculus of Structures

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Abstract. We introduce a new proof system for the description logic $\mathcal{ALC}$ in the framework of the calculus of structures, a structural proof theory that employs *deep inference*. This new formal presentation introduces positive proofs for description logics. Moreover, this result makes possible the study of sub-structural refinements of description logics, for which a semantics can now be defined.

1 A calculus of structures for description logics

Proof systems in the calculus of structures are defined by a set of deep inference rules operating on *structures*[1]. The rules are said to be *deep* because unlike the sequent calculus for which rules must be applied at the root of sequents, the rules of the calculus of structures can be applied at any depth inside a structure.

As noted by Schild[2], $\mathcal{ALC}$ is a syntactic variant of propositional multi-modal logic $\mathsf{K}_{(m)}$. Therefore, since this logic involves no interaction between its modalities, its proof system in the calculus of structures can be straightforwardly extended from a proof system of unimodal K in the calculus of structures, such as the cut-free proof system $\mathsf{SKSg_K}$ described in [3].

Let $\mathcal{A}$ be a countable set equipped with a bijective function $\bar{\cdot} : \mathcal{A} \to \mathcal{A}$, such that $\bar{\bar{A}} = A$, and $\bar{A} \neq A$ for every $A \in \mathcal{A}$. The elements of $\mathcal{A}$ are called primitive concepts, and two of them are denoted by $\top$ and $\bot$ such that $\bar{\top} := \bot$ and $\bar{\bot} := \top$.

The set $\mathcal{R}$ of *prestructures* of $\mathcal{ALC}$ concepts is defined by the following grammar, where A is a primitive concept and R is a role name:

$$C, D ::= \top \mid \bot \mid A \mid \bar{C} \mid (C, D) \mid [C, D] \mid \exists R.C \mid \forall R.C \ .$$

On the set $\mathcal{R}$, the relation $=$ is defined to be the smallest congruence relation induced by the following equations.

Associativity	**Commutativity**
$(C,(D,E)) = ((C,D),E)$	$[C,D] = [D,C]$
$[C,[D,E]] = [[C,D],E]$	$(C,D) = (D,C)$

Units	**Negation**	**Roles**
$(C, \top) = C$	$\overline{(C, D)} = [\bar{C}, \bar{D}]$	$\overline{\forall R.C} = \exists R.\bar{C}$
$[C, \bot] = C$	$\overline{[C, D]} = (\bar{C}, \bar{D})$	$\overline{\exists R.C} = \forall R.\bar{C}$
$[\top, \top] = \top$	$\bar{\bar{C}} = C$	$\forall R.\top = \top$
$(\bot, \bot) = \bot$		$\exists R.\bot = \bot$

A *structure* is an element of $\mathcal{R}/=$, i.e. an equivalence class of prestructures. For a given structure C, the structure $\bar{C}$ is called its *negation*. *Contexts* are defined by the following syntax, where C stands for any structure: $S ::= \{\circ\} \mid [C, S] \mid (C, S)$. An *inference rule* is a scheme of the kind $\rho \frac{S\{C\}}{S\{D\}}$. This rule specifies a step of rewriting inside a generic context $S\{\circ\}$. A *proof* in a given system, is a finite chain of instances of inference rules in the system, whose uppermost structure is the unit $\top$.

2 System $\mathsf{SKSg}_{\mathcal{ALC}}$

The following set of rules defines the sound and complete cut-free proof system $\mathsf{SKSg}_{\mathcal{ALC}}$ for $\mathcal{ALC}$ in the calculus of structures :

$$\mathsf{i}{\downarrow}\frac{S\{\top\}}{S[C, \bar{C}]} \qquad \mathsf{s}\frac{S([C, D], E)}{S[C, (D, E)]} \qquad \mathsf{i}{\uparrow}\frac{S(C, \bar{C})}{S\{\bot\}}$$

$$\mathsf{w}{\downarrow}\frac{S\{\bot\}}{S\{C\}} \qquad \mathsf{w}{\uparrow}\frac{S\{C\}}{S\{\top\}}$$

$$\mathsf{c}{\downarrow}\frac{S[C, C]}{S\{C\}} \qquad \mathsf{c}{\uparrow}\frac{S\{C\}}{S(C, C)}$$

$$\mathsf{k}{\downarrow}\frac{S\{\forall R.[\bar{C}, D]\}}{S[\overline{\forall R.C}, \forall R.D]} \qquad \mathsf{k}{\uparrow}\frac{S(\overline{\exists R.C}, \exists R.D)}{S\{\exists R.(\bar{C}, D)\}}$$

References

1. Brünnler, K.: Deep Inference and Symmetry in Classical Proofs. Logos Verlag, Berlin (2004)
2. Schild, K.: A correspondence theory for terminological logics: Preliminary report. In: Proceedings of the International Joint Conference on Artificial Intelligence. (1991)
3. Stewart, C., Stouppa, P.: A systematic proof theory for several modal logics. In: Advances in Modal Logic. Volume 5., King's College (2005)