A Crisp Representation for Fuzzy $\mathcal{SHOIN}$ with Fuzzy Nominals and General Concept Inclusions

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Abstract. Fuzzy Description Logics are a family of logics which allow the representation of (and the reasoning within) structured knowledge affected by uncertainty and vagueness. They were born to overcome the limitations of classical Description Logics when dealing with such kind of knowledge, but they bring out some new challenges, requiring an appropriate fuzzy language to be agreed and needing practical and highly optimized implementations of the reasoning algorithms. In the current paper we face these problems by presenting a reasoning preserving procedure to obtain a crisp representation for a fuzzy extension of $\mathcal{SHOIN}$, which makes possible to reuse a crisp representation language as well as currently available reasoners, which have demonstrated a very good performance in practice. As an additional contribution, we define the syntax and semantics of a novel fuzzy version of the nominal construct and allow to reason within fuzzy general concept inclusions.

1 Introduction

Ontologies [1] are a core element in the layered architecture of the Semantic Web [2]. Description Logics (DLs for short) [3] are a family of logics for representing structured knowledge. The name of each logic is composed by some labels which identify the constructs of the logic. DLs have been proved to be very useful as ontology languages [4]. As it has been widely pointed out, classical ontologies and DLs are not appropriate to handle uncertain knowledge [5, 6] and since uncertainty is inherent to a lot of real-world application domains, the Semantic Web will not be fully operative as long as it does not provide means to manage it. A well studied solution is to extend DLs with fuzzy sets theory [7], producing fuzzy DLs (denoted with an f preceding the name of the corresponding DL and a subscript denoting the family of fuzzy operators considered e.g. $f_{KD}\mathcal{SHOIN}$ uses maximum t-conorm, minimum t-norm, and Kleene-Dienes implication).

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Nowadays, the World Wide Web Consortium (W3C) standard for ontology representation is OWL Web Ontology Language[1], a language comprising three sublanguages of increasing expressive power: OWL Lite, OWL DL and OWL Full (being OWL DL the most used level and nearly equivalent to $\mathcal{SHOIN}(\mathcal{D})$ [8] but without customised datatypes). In order to deal with uncertain knowledge, OWL may be extended to a fuzzy DL-based language e.g. FuzzyOWL [6], with the drawback that the large number of resources available (e.g. editors, reasoners or ontologies to be imported) should be adapted. Furthermore, reasoning within expressive DLs has a very high worst-case complexity (e.g. NPSPACE in $\mathcal{SHOIN}$) and, consequently, there exists a significant gap between the design of a decision procedure and the achievement of a practical implementation [9] (as a matter of fact, some of the OWL DL reasoners used in practice do not support full $\mathcal{SHOIN}(\mathcal{D})$ e.g. Racer [10] and FaCT [11]). Regarding fuzzy DLs, there does not exist any implemented reasoner for $f\mathcal{SHOIN}$. A reasoner for $f\mathcal{SHIN}(\mathcal{D})$ has been recently developed (fuzzyDL [2]), but its efficiency is still to be investigated. Moreover, the experience with crisp DLs ([9]) induces us to think that developing highly optimized implementations will be a hard task where ad-hoc mechanisms should be used for every particular fuzzy DL.

An alternative way to obtain fuzzy ontologies facing these two problems is *i)* to represent fuzzy DLs using crisp DLs and *ii)* to reduce reasoning within fuzzy DLs to reasoning within crisp DLs. This way it would be possible to translate them automatically into a crisp ontology language (e.g. OWL) and to use currently available reasoners (e.g. Pellet [12]). Unfortunately, there does not exist a lot of work following this line and the logics investigated are still far from OWL DL: [13] shows a reasoning preserving procedure for $f\mathcal{ALCH}$, [14] considers $f\mathcal{ALC}$ with truth values taken from an uncertainty lattice and [15], a restricted version of $f\mathcal{ALCQ}$ (e.g. they do not allow to reason within a TBox).

On the other hand, current fuzzy DLs still present some limitations which we think that should be overcome. Some works on fuzzy DLs deal with nominals (named individuals) but they choose not to fuzzify the nominal construct arguing that a fuzzy singleton set does not represent any real concept world [5, 6]. Hence, only crisp concepts can be defined extensively, as nominals either have to fully belong to them or not. Besides, although there have been proposed fuzzy general concept inclusions (which allow to constrain the truth value of a general concept inclusion or GCI) [5], current reasoning algorithms do not allow them.

Our work provides the following contributions. Firstly, we propose a different definition of $f\mathcal{SHOIN}$, including a fuzzy nominal construct and fuzzy GCIs. Secondly, we reduce reasoning in $f_{KD}\mathcal{SHOIN}$ to reasoning in $\mathcal{SHOIN}$, extending [13]. To the very best of our knowledge, there does not exist any reasoning algorithm dealing with such kind of fuzzy GCIs. The present paper is organized as follows. In the next section, we describe our fuzzy extension of $\mathcal{SHOIN}$. Then, Sect. 3 shows how to reduce it into crisp $\mathcal{SHOIN}$. Finally, in Sect. 4 we set out some conclusions and ideas for future work.

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[1] `http://www.w3.org/TR/owl-features`

[2] `http://gaia.isti.cnr.it/~straccia/software/fuzzyDL/fuzzyDL.html`

2 Fuzzy $\mathcal{SHOIN}$

In this section we define $f\mathcal{SHOIN}$, which extends $\mathcal{SHOIN}$ to the fuzzy case by letting (i) concepts denote fuzzy sets of individuals and (ii) roles denote fuzzy binary relations between individuals. Our logic is similar to [5, 6], adding fuzzy nominals and fuzzy GCIs. In fuzzy DLs most reasoning services are reducible to fKB satisfiability [16], so here in after we will only consider this task.

Syntax. $f\mathcal{SHOIN}$ assumes three alphabets of symbols, for concepts, roles and individuals. The concepts of the language (denoted C or D) can be built inductively from atomic concepts (A), atomic roles (R), top concept $\top$, bottom concept $\bot$, named individuals (o_i) and simple roles (S)[3] according to the following syntax rule (where n, m are natural numbers, $n \geq 0, m > 0$, $\alpha_i \in [0, 1]$):

$$
\begin{array}{lrl}
C, D \rightarrow & A \mid & \text{(atomic concept)} \\
& \top \mid & \text{(top concept)} \\
& \bot \mid & \text{(bottom concept)} \\
& C \sqcap D \mid & \text{(concept conjunction)} \\
& C \sqcup D \mid & \text{(concept disjunction)} \\
& \neg C \mid & \text{(concept negation)} \\
& \forall R.C \mid & \text{(universal quantification)} \\
& \exists R.C \mid & \text{(full existential quantification)} \\
& \{(o_1, \alpha_1), \ldots, (o_m, \alpha_m)\} \mid & \text{(nominals)} \\
& (\geq n\ S) \mid & \text{(at-least unqualified number restriction)} \\
& (\leq n\ S) & \text{(at-most unqualified number restriction)}
\end{array}
$$

If R_A is an atomic role, complex roles are built using this syntax rule:

$$R \rightarrow R_A \mid R^-$$

A fuzzy Knowledge Base (fKB) comprises two parts: the intensional knowledge, i.e. general knowledge about the application domain (a fuzzy Terminological Box or TBox K_T and a fuzzy Role Box or RBox K_R), and the extensional knowledge, i.e. particular knowledge about some specific situation (a fuzzy Assertional Box or ABox K_A with statements about individuals). A fuzzy ABox fK_A consists of a finite set of fuzzy assertions, which can be individual assertions or constraints on the truth value of a concept or role assertion. An individual assertion is either an inequality of individuals $\langle a \neq b \rangle$ or an equality of individuals $\langle a = b \rangle$ (they are necessary since we do not impose unique name assumption). A constraint on the truth value of a concept or role assertion is an expression of the form $\langle \Psi \geq \alpha \rangle$, $\langle \Psi > \beta \rangle$, $\langle \Phi \leq \beta \rangle$, $\langle \Phi < \alpha \rangle$, where Ψ is an assertion of the form $a : C$ or $(a, b) : R$, Φ is an assertion of the form $a : C$, $\alpha \in (0, 1]$ and $\beta \in [0, 1)$).

[3] A simple role is a non transitive role not having transitive sub-roles i.e. R is a sub-role of R' if $R \mathrel{\underline{\circledast}} R'$, where $\underline{\circledast}$ is the transitive-reflexive closure of $\sqsubseteq$

Note that fuzzy assertions of the form $\langle (a, b) : R \leq \beta\rangle, \langle (a, b) : R < \alpha\rangle$ are not allowed since they relate to negated roles, which are not part of $\mathcal{SHOIN}$. A fuzzy TBox fK_T consists of a finite set of fuzzy terminological axioms. A fuzzy terminological axiom is either a fuzzy GCI or a concept definition. A fuzzy GCI constrains the truth value of a GCI i.e. it is an expression of the form $\langle \Omega \geq \alpha\rangle$, $\langle \Omega > \beta\rangle$, $\langle \Omega \leq \beta\rangle$ or $\langle \Omega < \alpha\rangle$, where Ω is a GCI of the form $C \sqsubseteq$, $\alpha \in (0, 1]$ and $\beta \in [0, 1)$. We think that concept definitions should not be fuzzified, so $C \equiv D$ is an abbreviation of the pair of axioms $\langle C \sqsubseteq D \geq 1\rangle$ and $\langle D \sqsubseteq C \geq 1\rangle$. A fuzzy RBox fK_R consists of a finite set of fuzzy role axioms. A fuzzy role axiom is either a fuzzy role inclusion $R \sqsubseteq R'$, a fuzzy role definition $R \equiv R'$ (a short hand for both $R \sqsubseteq R'$ and $R' \sqsubseteq R$) or a transitive role axiom $trans(R)$.

Semantics. A fuzzy interpretation $\mathcal{I}$ is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a non empty set $\Delta^{\mathcal{I}}$ (the interpretation domain) and a fuzzy interpretation function $\cdot^{\mathcal{I}}$ mapping every individual onto an element of $\Delta^{\mathcal{I}}$, every concept C onto a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0, 1]$ and every role R onto a function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0, 1]$. $C^{\mathcal{I}}$ (resp. $R^{\mathcal{I}}$) is interpreted as the membership degree function of the fuzzy concept C (resp. fuzzy rol R) w.r.t. $\mathcal{I}$. $C^{\mathcal{I}}(a)$ (resp. $R^{\mathcal{I}}(a, b)$) gives us the degree of being the individual a an element of the fuzzy concept C (resp. the degree of being (a, b) an element of the fuzzy role R) under the fuzzy interpretation $\mathcal{I}$. The fuzzy interpretation function is extended to complex concepts and roles as:

$$
\begin{aligned}
\top^{\mathcal{I}}(a) &= 1 \\
\bot^{\mathcal{I}}(a) &= 0 \\
(C \sqcap D)^{\mathcal{I}}(a) &= C^{\mathcal{I}}(a) \wedge D^{\mathcal{I}}(a) \\
(C \sqcup D)^{\mathcal{I}}(a) &= C^{\mathcal{I}}(a) \vee D^{\mathcal{I}}(a) \\
(\neg C)^{\mathcal{I}}(a) &= \neg C^{\mathcal{I}}(a) \\
(\forall R.C)^{\mathcal{I}}(a) &= \inf_{b \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(a, b) \to C^{\mathcal{I}}(b)\} \\
(\exists R.C)^{\mathcal{I}}(a) &= \sup_{b \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(a, b) \wedge C^{\mathcal{I}}(b)\} \\
\{(o_1, \alpha_1), \ldots, (o_m, \alpha_m)\}^{\mathcal{I}}(a) &= \sup_{i \mid a \in \{o_i^{\mathcal{I}}\}} \alpha_i \\
(\geq 0)^{\mathcal{I}}(a) &= \top^{\mathcal{I}}(a) = 1 \\
(\geq m)^{\mathcal{I}}(a) &= \sup_{b_1, \ldots, b_m \in \Delta^{\mathcal{I}}} [\wedge_{i=1}^{m} S^{\mathcal{I}}(a, b_i) \bigwedge \wedge_{i<j} \{b_i \neq b_j\}] \\
(\leq n\ S)^{\mathcal{I}}(a) &= \neg(\geq n{+}1\ S)^{\mathcal{I}}(a) \\
(R^{-})^{\mathcal{I}}(a, b) &= R^{\mathcal{I}}(b, a)
\end{aligned}
$$

We will shortly justify our decision of fuzzifying the nominal construct by showing an example. Suppose we want to represent the concept of country where German is a widely spoken language as $C \equiv \{germany, austria, switzerland\}$. The classical semantics for the nominal construct is: $\{o_i\}^{\mathcal{I}}(a) = 1$ if $a \in \{o_i^{\mathcal{I}}\}$ or 0 otherwise. This semantics forces *switzerland* to fully belong to the concept or not, despite of German-speaking community of Switzerland represents only about two thirds of the total population of the country. On the contrary, our proposal allows to define $\{(germany, 1), (austria, 1), (switzerland, 0.67)\}$, which does represent a real-life concept. It is easy to see that our definition generalizes the previous definition for the nominal construct, as $\{o_1, \ldots, o_m\}$ is equivalent to $\{(o_1, 1), \ldots, (o_m, 1)\}$.

A fuzzy interpretation $\mathcal{I}$ satisfies (is a model of):

- A fuzzy assertion $\langle a : C \geq \alpha \rangle$ iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq \alpha$. Similar definitions can be given for $> \beta$, $\leq \beta$ and $< \alpha$.
- A fuzzy assertion $\langle (a, b) : R \geq \alpha \rangle$ iff $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq \alpha$. Similar definitions can be given for $> \beta$, $\leq \beta$ and $< \alpha$.
- An assertion $\langle a \neq b \rangle$ iff $a^{\mathcal{I}} \neq b^{\mathcal{I}}$ (resp. $\langle a = b \rangle$ iff $a^{\mathcal{I}} = b^{\mathcal{I}}$). Note that we consider individuals assertions to be crisp.
- A fuzzy GCI $\langle C \sqsubseteq D \geq \alpha \rangle$ iff $\inf_{a \in \Delta^{\mathcal{I}}} \{C^{\mathcal{I}}(a) \rightarrow D^{\mathcal{I}}(a)\} \geq \alpha$. Similar definitions can be given for $> \beta$, $\leq \beta$ and $< \alpha$.
- A concept definition $C \equiv D$ iff $C^{\mathcal{I}} = D^{\mathcal{I}}$.
- A role inclusion axiom $R \sqsubseteq R'$ iff $R^{\mathcal{I}} \subseteq R'^{\mathcal{I}}$.
- A role definition axiom $R \equiv R'$ iff $R^{\mathcal{I}} = R'^{\mathcal{I}}$.
- An axiom $trans(R)$ iff $\forall a, b \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(a, b) \geq \sup_{c \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(a, c) \wedge R^{\mathcal{I}}(c, b)$.
- A fKB $\langle fK_A, fK_T, fK_R \rangle$ iff it satisfies each element in K_A, K_T and K_R.

The definition of fuzzy GCIs allows concept subsumption to hold to a certain degree in $[0, 1]$. This does not hold for role inclusion axioms, which leads to a certain asymmetry in the expressivity. While this is not too elegant, it is a restriction imposed by the choice of the implication function, which would require the subjacent DL to have negated roles and role disjunction. However, for a higher practical utility, we have preferred to restrict ourselves to $\mathcal{SHOIN}$, closer to the DL underlying OWL DL.

The following lemma shows that our definition of $f\mathcal{SHOIN}$ is a sound extension of crisp $\mathcal{SHOIN}$:

Lemma 1. *Fuzzy interpretations coincide with crisp interpretations if we restrict to the membership degrees of 0 and 1 [6].*

Some properties. Here in after we will concentrate on $f_{KD}SHOIN$, restricting ourselves to the minimum t-norm $a \wedge b = \min\{a, b\}$, maximum t-conorm $a \vee b = \max\{a, b\}$, Łukasiewicz negation $\neg a = 1 - a$ and the Kleene-Dienes implication $a \rightarrow b = \max\{1 - a, b\}$. For instance, in the semantics of the at-least unqualified number restriction, $\wedge_{i<j}\{b_i \neq b_j\}$ means that there must exist n distinct elements of the domain. The choice of the t-norm and the t-conorm will be justified in Sect. 3. On the other hand, in fuzzy DLs it is very common to use the Kleene-Dienes implication in the semantics of universal quantification, so for the sake of coherence we have chosen to use it in the semantics of fuzzy GCIs as well. Similarly as in [17], $f_{KD}\mathcal{SHOIN}$ allows some sort of modus ponens over concepts and roles, even with the new semantics of fuzzy GCIs:

Lemma 2. *For $\alpha, \beta, \gamma \in [0, 1]$, $\alpha > 1 - \beta$ and $\bowtie = \{\geq, >\}$, the following properties are verified:*

(i) $\langle a : C \bowtie \alpha \rangle$ *and* $\langle C \sqsubseteq D \bowtie \beta \rangle$ *imply* $\langle a : D \geq \beta \rangle$.
(ii) $\langle (a, b) : R \bowtie \gamma \rangle$ *and* $\langle R \sqsubseteq R' \rangle$ *imply* $\langle (a, b) : R' \bowtie \gamma \rangle$.
(iii) $\langle (a, b) : R \bowtie \alpha \rangle$ *and* $\langle a : \forall R.C \bowtie \beta \rangle$ *imply* $\langle b : C \bowtie \beta \rangle$.

Unfortunately, the use of Kleene-Dienes implication in the semantics of fuzzy GCIs brings about two counter-intuitive effects. Firstly, a concept does not fully subsume itself i.e. $C \sqsubseteq C \Rightarrow \inf_{a \in \Delta^{\mathcal{I}}} \max\{1 - C^{\mathcal{I}}(a), C^{\mathcal{I}}(a)\} = 0.5$. Secondly, crisp concept subsumption forces fuzzy concepts to be crisp i.e. $\langle C \sqsubseteq D \geq 1 \rangle \Rightarrow \inf_{a \in \Delta^{\mathcal{I}}} \max\{1 - C^{\mathcal{I}}(a), D^{\mathcal{I}}(a)\} \geq 1$ which is true iff for each element of the domain $D^{\mathcal{I}}(a) = 1$ or $1 - C^{\mathcal{I}}(a) \geq 1 \Rightarrow C^{\mathcal{I}}(a) = 0$. These problems point out the need of further investigation involving alternative fuzzy operators. For example, using a residuum based implications (see [18] for a refresh on fuzzy operators) it is always true that $a \rightarrow b = 1$ if $a \leq b$, which would fix the first problem; while using Łukasiewicz implication ($a \rightarrow b = \min\{1, 1 - a + b\}$) would fix the second one.

3 A Crisp Representation for Fuzzy $\mathcal{SHOIN}$

In this section we show how to reduce a $f_{KD}\mathcal{SHOIN}$ fKB into a crisp Knowledge Base (KB). The procedure preserves reasoning, so existing $\mathcal{SHOIN}$ reasoners could be applied to the resulting KB. [13] presents a reasoning preserving transformation for $f_{KD}\mathcal{ALCH}$ into crisp $\mathcal{ALCH}$: firstly, some new atomic concepts and roles are defined, then some new axioms are added to preserve the semantics of the fKB and finally the ABox, the TBox and the RBox are mapped separately. Our reduction extends this work to $f_{KD}\mathcal{SHOIN}$. A slight difference is that our mapping of the TBox can introduce some new assertions about new individuals (not appearing in the initial fKB).

New Elements. Let A^{fK} and R^{fK} be the set of atomic concepts and atomic roles occurring in a fKB $fK = \langle fK_A, fK_T, fK_R \rangle$. In [13] it is shown that the set of the degrees which must be considered for any reasoning task is defined as $N^{fK} = X^{fK} \cup \{1 - \alpha | \alpha \in X^{fK}\}$, where X^{fK} is defined as follows:

$$\begin{aligned} X^{fK} = &\{0, 0.5, 1\} \cup \{\alpha | \langle \Psi \geq \alpha \rangle \in fK_A\} \cup \{\beta | \langle \Psi > \beta \rangle \in fK_A\} \\ &\cup \{1 - \beta | \langle \Phi \leq \beta \rangle \in fK_A\} \cup \{1 - \alpha | \langle \Phi < \alpha \rangle \in fK_A\} \\ &\cup \{\alpha | \langle \Omega \geq \alpha \rangle \in fK_T\} \cup \{\beta | \langle \Omega > \beta \rangle \in fK_T\} \\ &\cup \{1 - \beta | \langle \Omega \leq \beta \rangle \in fK_T\} \cup \{1 - \alpha | \langle \Omega < \alpha \rangle \in fK_T\} \end{aligned}$$

This also holds in $f_{KD}\mathcal{SHOIN}$, but note that it is no longer true when other fuzzy operators are considered. In that case, the process may calculate all possible degrees in $[0, 1]$ with a given precision, but further investigation is required. Without loss of generality, it can be assumed that $N^{fK} = \{\gamma_1, \ldots, \gamma_{|N^{fK}|}\}$ and $\gamma_i < \gamma_{i+1}, 1 \leq i \leq |N^{fK}| - 1$.

Now, for each $\alpha, \beta \in N^{fK}, \alpha \in (0, 1], \beta \in [0, 1)$, for each relation in $\{\geq, >, \leq, <\}$, for each $A \in A^{fK}$ and for each $R \in R^{fK}$, four new atomic concepts $A_{\geq\alpha}, A_{>\beta}, A_{\leq\beta}, A_{<\alpha}$ and two new atomic roles $R_{\geq\alpha}, R_{>\beta}$ are introduced. $A_{\geq\alpha}$ represents the crisp set of individuals which are instance of A with degree higher or equal than α i.e the α-cut of A. The other new elements are defined in a similar way. Neither $A_{<0}, A_{>1}, R_{>1}$ are considered (they are always empty sets) nor $A_{\leq 1}, A_{\geq 0}, R_{\geq 0}$ (they are always equivalent to the top concept).

The semantics of these newly introduced atomic concepts and roles is preserved by some terminological and role axioms. For each $1 \leq i \leq |N^{fK}| - 1$, for each $2 \leq j \leq |N^{fK}|$, for each $A \in A^{fK}$ and for each $R \in R^{fK}$, $T(N^{fK})$ is the smallest terminology containing the following axioms:

$$\begin{array}{rlrl} A_{\geq \gamma_{i+1}} & \sqsubseteq A_{>\gamma_i} & A_{>\gamma_i} & \sqsubseteq A_{\geq \gamma_i} \\ A_{<\gamma_j} & \sqsubseteq A_{\leq \gamma_j} & A_{\leq \gamma_i} & \sqsubseteq A_{<\gamma_{i+1}} \\ A_{\geq \gamma_j} \sqcap A_{<\gamma_j} & \sqsubseteq \bot & A_{>\gamma_i} \sqcap A_{\leq \gamma_i} & \sqsubseteq \bot \\ \top & \sqsubseteq A_{\geq \gamma_j} \sqcup A_{<\gamma_j} & \top & \sqsubseteq A_{>\gamma_i} \sqcup A_{\leq \gamma_i} \end{array}$$

Similarly, $R(N^{fK})$ is the smallest terminology containing these two axioms:

$$\begin{array}{rl} R_{\geq \gamma_{i+1}} & \sqsubseteq R_{>\gamma_i} \\ R_{>\gamma_i} & \sqsubseteq R_{\geq \gamma_i} \end{array}$$

It is easy to see that allowing expressions of the type $\langle (a,b) : R \leq \beta \rangle, \langle (a,b) : R < \alpha \rangle$ would need additional role constructs (role conjunction, role disjunction, bottom role and top role).

Mapping the ABox. Fuzzy assertions are mapped into $\mathcal{SHOIN}$ assertions using a mapping σ. Let $\gamma \in N^{fK}, \bowtie \in \{\geq, <, \leq, >\}$, $\sigma(fK_A) = \{\sigma(\Phi) | \Phi \in fK_A\}$, where $\sigma(\Phi)$ is defined as in the following table (where ρ is inductively defined on the structure of concepts and roles as in Table 1):

$$\begin{array}{c} \sigma(\langle a : C \bowtie \gamma \rangle) = a : \rho(C, \bowtie \gamma) \\ \sigma(\langle (a,b) : R \bowtie \gamma \rangle) = (a,b) : \rho(R, \bowtie \gamma) \\ \sigma(\langle a \neq b \rangle) = a \neq b \\ \sigma(\langle a = b \rangle) = a = b \end{array}$$

Mapping the TBox. $f\mathcal{SHOIN}$ fuzzy terminological axioms to either terminological axioms (for $\geq$ or $>$) or assertions (for $\leq$ and $<$). In the former case, we redefine $k(fK, fK_T)$ as $k(fK, fK_T) = \bigcup_{\Omega \in fK_T} k(\Omega)$, where $\Omega = \langle C \sqsubseteq D\{\geq, >\}\gamma \rangle$ and $k(\Omega)$ is defined as:

$$\begin{array}{c} k(\langle C \sqsubseteq D \geq \gamma \rangle) = \rho(C, > 1 - \gamma) \sqsubseteq \rho(D, \geq \gamma) \\ k(\langle C \sqsubseteq D > \gamma \rangle) = \rho(C, \geq 1 - \gamma) \sqsubseteq \rho(D, > \gamma) \end{array}$$

In the latter case, new assertions are necessary since negated terminological axioms are nor part of crisp $\mathcal{SHOIN}$. A new function $A(fK_T)$ adds these new assertions to the ABox. $A(fK_T) = \bigcup_{\Xi \in fK_T} A(\Xi)$, where $\Xi = \langle C \sqsubseteq D\{\leq, <\}\gamma \rangle$ and $A(\Xi)$ is defined as:

$$\begin{array}{c} A(\langle C \sqsubseteq D \leq \gamma \rangle) = x : \rho(C, \geq 1 - \gamma) \sqcap \rho(D, \leq \gamma) \\ A(\langle C \sqsubseteq D < \gamma \rangle) = x : \rho(C, > 1 - \gamma) \sqcap \rho(D, < \gamma) \end{array}$$

Note that how to modify the reduction process when alternative implication functions are used remains an open question.

Mapping the RBox. Role axioms are reduced using a function $k(fK, fK_R) = \bigcup_{\Omega \in fK_R} k(\Omega)$, where $k(\Omega)$ is defined as:

$$k(R \sqsubseteq R') = \bigcup_{\gamma \in N^{fK}, \bowtie \in \{\geq, >\}} \rho(R, \bowtie \gamma) \sqsubseteq \rho(R', \bowtie \gamma)$$
$$k(trans(R)) = \bigcup_{\gamma \in N^{fK}, \bowtie \in \{\geq, >\}} trans(\rho(R, \bowtie \gamma))$$

Table 1. Mapping ρ

x	y	$\rho(x, y)$
A	$\geq \gamma$	$A_{\geq \gamma}$ if $\gamma \neq 0, \top$ otherwise
A	$> \gamma$	$A_{> \gamma}$, if $\gamma \neq 1, \bot$ otherwise
A	$\leq \gamma$	$A_{\leq \gamma}$ if $\gamma \neq 0, \top$ otherwise
A	$< \gamma$	$A_{< \gamma}$, if $\gamma \neq 1, \bot$ otherwise
R	$\geq \gamma$	$R_{\geq \gamma}$ if $\gamma \neq 0, \top$ otherwise
R	$> \gamma$	$R_{> \gamma}$, if $\gamma \neq 1, \bot$ otherwise
$\top$	$\geq \gamma$	$\top$
$\top$	$> \gamma$	$\top$ if $\gamma \neq 1, \bot$ otherwise
$\top$	$\leq \gamma$	$\top$ if $\gamma = 1, \bot$ otherwise
$\top$	$< \gamma$	$\bot$
$\bot$	$\geq \gamma$	$\top$ if $\gamma = 0, \bot$ otherwise
$\bot$	$> \gamma$	$\bot$
$\bot$	$\leq \gamma$	$\top$
$\bot$	$< \gamma$	$\top$ if $\gamma \neq 0, \bot$ otherwise
$C \sqcap D$	$\{\geq, >\} \gamma$	$\rho(C, \{\geq, >\} \gamma) \sqcap \rho(D, \{\geq, >\} \gamma)$
$C \sqcap D$	$\{\leq, <\} \gamma$	$\rho(C, \{\leq, <\} \gamma) \sqcup \rho(D, \{\leq, <\} \gamma)$
$C \sqcup D$	$\{\geq, >\} \gamma$	$\rho(C, \{\geq, >\} \gamma) \sqcup \rho(D, \{\geq, >\} \gamma)$
$C \sqcup D$	$\{\leq, <\} \gamma$	$\rho(C, \{\leq, <\} \gamma \sqcap \rho(D, \{\leq, <\} \gamma)$
$\neg C$	$\{\geq, >\} \gamma$	$\rho(C, \{\leq, <\} 1 - \gamma)$
$\neg C$	$\{\leq, <\} \gamma$	$\rho(C, \{\geq, >\} 1 - \gamma)$
$\exists R.C$	$\{\geq, >\} \gamma$	$\exists \rho(R, \{\geq, >\} \gamma).\rho(C, \{\geq, >\} \gamma)$
$\exists R.C$	$\{\leq, <\} \gamma$	$\forall \rho(R, \{>, \geq\} \gamma).\rho(C, \{\leq, <\} \gamma)$
$\forall R.C$	$\{\geq, >\} \gamma$	$\forall \rho(R, \{>, \geq\} 1 - \gamma).\rho(C, \{\geq, >\} \gamma)$
$\forall R.C$	$\{\leq, <\} \gamma$	$\exists \rho(R, \{\geq, >\} 1 - \gamma).\rho(C, \{\leq, <\} \gamma)$
$\{(o_1, \alpha_1), \ldots, (o_m, \alpha_m)\}$	$\bowtie \gamma$	$\{o_i \mid \alpha_i \bowtie \gamma, 1 \leq i \leq n\}_{\bowtie \gamma}$
$\geq 0\ S$	$\bowtie \gamma$	$\rho(\top, \bowtie \gamma)$
$\geq m\ S$	$\{\geq, >\} \gamma$	$\geq m\ \rho(S, \{\geq, >\} \gamma)$
$\geq m\ S$	$\{\leq, <\} \gamma$	$\leq m{-}1\ \rho(S, \{>, \geq\} \gamma)$
$\leq n\ S$	$\{\geq, >\} \gamma$	$\leq n\ \rho(S, \{>, \geq\} 1 - \gamma)$
$\leq n\ S$	$\{\leq, <\} \gamma$	$\geq n{+}1\ \rho(S, \{\geq, >\} 1 - \gamma)$
R^-	$\bowtie \gamma$	$\rho(R, \bowtie \gamma)^-$

Discussion. A fKB $fK = \langle fK_A, fK_T, fK_R \rangle$ is reduced into a KB $K(fK) = \langle \sigma(fK_A) \cup A(fK_T), T(N^{fK}) \cup k(fK, fK_T), R(N^{fK}) \cup k(fK, fK_R) \rangle$. The complexity of our procedure is quadratic: the ABox is linear while the TBox and the RBox are quadratic. It is interesting to note that, while [13] reduces a fuzzy terminological axiom into a set of crisp terminological axioms, our semantics for fuzzy GCIs allows to reduce each axiom into either an axiom or an assertion. This reduction in the size of the TBox (although it is still quadratic) is very interesting since reasoning with GCIs is a source of computational complexity [19]. Finally, an important theorem can be shown:

Theorem 1. *A $f_{KD}\mathcal{SHOIN}$ fKB fK is satisfiable iff $K(fK)$ is satisfiable.*

Unfortunately, we cannot show the proof due to space limitations. Firstly, it has to be proved that the translation preserves the satisfiability of every single statement of the fKB. It can be shown that, if there exists a fuzzy interpretation satisfying a statement, then a crisp interpretation satisfying the result of its translation can be built. Secondly, it has to be proved that the translation preserves the satisfiability of the whole fKB. Then, it has be shown that the translation preserves the clashes. For example, the clash produced by the pair of conjugated axioms $\langle a : A \geq \gamma \rangle$ and $\langle a : A < \gamma \rangle$ is preserved, since the axiom $A_{\geq\gamma} \sqcap A_{<\gamma} \sqsubseteq \bot$ prevents any individual from belonging to A with degree $\geq \gamma$ and degree $< \gamma$.

4 Conclusions and Future Work

This paper has presented an alternative approach to achieve fuzzy ontologies, reusing currently existing crisp ontology languages and reasoners. In particular, after having presented a sound fuzzy extension of $\mathcal{SHOIN}$ including fuzzy nominals (enabling to define fuzzy sets extensively) and fuzzy GCIs (allowing to constrain the truth value of a GCI), we have presented a reasoning preserving procedure (quadratic in complexity) to reduce a $f_{KD}\mathcal{SHOIN}$ fKB into a crisp one. The semantics of fuzzy GCIs reduces the size of the resulting TBox w.r.t. [13], but imposes some counter-intuitive effects.

The main direction for future work is to perform an empirical evaluation in order to validate the theoretical results. From a theoretical point view, we are considering different fuzzy operators to avoid the counter-intuitive effects of the Kleene-Dienes implication. We also plan to include a crisp representation for fuzzy datatypes. Since OWL does not currently allow to define customised datatypes, it seems interesting to consider OWL Eu [20], a promising extension of OWL supporting them. Another interesting direction for future research is to consider the more expressive DL $\mathcal{SROIQ}$ [21] (providing some additional role constructs such as disjoint roles and negated role assertions) and which is the subjacent DL of OWL 1.1[4], an extension of OWL which has been recently proposed. We think that the additional expressivity may help to overcome the asymmetry in the definitions of fuzzy concept and role inclusion axioms.

[4] `http://www-db.research.bell-labs.com/user/pfps/owl/overview.html`

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