Ref.TH.1793-CERN

POSITIVITY RESTRICTIONS ON ANOMALOUS DIMENSIONS

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ABSTRACT

Arguments of positivity and conformal invariance in the Gell-Mann - Low limit imply positive anomalies γ_N (γ_N = scale dimension -2-N) for the operators $O_{\alpha_1,\ldots,\alpha_N}$ (N $\neq$ 0) which occur in the light cone expansion of two currents.

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Ref.TH.1793-CERN
9 January 1974

It is well known that the softness of the trace of the energy momentum tensor implies that the Gell-Mann low limit [1] of most renormalizable theories is not only scale but also conformal invariant [2]. It is generally accepted that such a limit corresponds to a bona fide field theory, the asymptotic theory [4]. We shall derive here, by a very simple procedure, general constraints on the scale dimensions of the (symmetric traceless) local tensor operators $O_{\alpha_1,\ldots\alpha_N}(x)$ (which in particular occur in the light cone expansion) of the asymptotic theory. Such scale dimensions are known to govern the short distance behaviour of the Green functions of the massive theory [5].

Let us consider the two point function

$$W_{\alpha_1\ldots\alpha_N;\beta_1\ldots\beta_N}(x) = \langle 0 | O_{\alpha_1\ldots\alpha_N}(x)\, O_{\beta_1\ldots\beta_N}(0) | 0 \rangle \tag{1}$$

where $O_{\alpha_1,\ldots\alpha_N}(x)$ are irreducible conformal tensors (i.e., Lorentz tensors $(\frac{N}{2}, \frac{N}{2})$ satisfying $[O_{\alpha_1,\ldots\alpha_N}(0), K_\lambda] = 0$ where K_λ generates special conformal transformations) of twist (= scale dimension minus order) $\tau_N = d_N - N$ $(N \geq 1$, d_N = scale dimension).

Conformal symmetry gives uniquely [5]

$$W_{\alpha_1\ldots\alpha_N;\beta_1\ldots\beta_N}(x) = (\text{constant})\,(x^2)^{-d_N} \left\{ \underset{\{\alpha\}\{\beta\}}{S}\, M_{\alpha_1\beta_1}(x) \ldots M_{\alpha_N\beta_N}(x) - \text{traces} \right\} \tag{2}$$

where S stands for symmetrization and

$$M_{\alpha\beta}(x) = 2\,\frac{x_\alpha x_\beta}{x^2} - g_{\alpha\beta}$$

We perform the (unique) decomposition

$$W_{\alpha_1\ldots\alpha_N;\beta_1\ldots\beta_N}(x) = \sum_{K=0}^{N} w_K(\tau_N)\, W^{N-K}_{\alpha_1\ldots\alpha_N;\beta_1\ldots\beta_N}(x) \tag{3}$$

where $W^{N-K}_{\alpha_1,\ldots\beta_N}(x)$ are homogeneous distributions satisfying

$$\partial_{\alpha_1}\ldots\partial_{\alpha_{K+1}} W^{N-K}_{\alpha_1\ldots\alpha_N;\beta_1\ldots\beta_N}(x) = 0 \tag{4}$$

(i.e., they contain spin values down to N-K) and $W_K(\tau_N)$ will be examined in the following. The spin structure of the homogeneous distributions $W^{N-K}_{\alpha_1,\dots\beta_N}(x)$ implies that they are positive distributions for $\tau_N \geq 1-K$. The necessary condition for positivity from Eq. (2) is then : $\tau_N \geq 1$. Only scale invariance is needed for this result [7]. To investigate further the restrictions from positivity we study the zeros of $W_K(\tau_N)$ in Eq. (3). At such zeros the K^{th} divergence of $O_{\alpha_1,\dots\alpha_N}(x)$ vanishes.

$$\partial_{\alpha_1}\dots\partial_{\alpha_K} O_{\alpha_1\dots\alpha_N}(x) = 0 \tag{5}$$

Eq. (5) is satisfied only at those τ_N such that [8]

$$[\partial_{\alpha_1}\dots\partial_{\alpha_K} O_{\alpha_1\dots\alpha_N}(0), K_\lambda] = 0 \tag{6}$$

A statement equivalent to [6] is that the quadratic and quartic Casimir operators C_I, C_{III}, on the conformal representations [8] of $O_{\alpha_1,\dots\alpha_N}$, satisfy

$$C_{I,III}(d_N, N) = C_{I,III}(d_N+K, N-K) \tag{7}$$

(we recall that the cubic Casimir C_{II} vanishes on tensor representations) Eq. (7) has the only solution

$$\tau_N = 3-K \tag{8}$$

We observe that the location of such "conformal zeros" is above that of the "kinematical" zeros (at $\tau_N \leq 1-K$) due to the divergence of the Fourier transforms of the distributions $W^{N-K}_{\alpha_1,\dots\beta_N}(x)$. For $W^{N-K}_{\alpha_1,\dots\beta_N}(x)$ such that its Fourier transform is a homogeneous distribution with an over-all normalization factor independent on τ_N, one has that $W_K(\tau_N)$ must be proportional to

$$\frac{\tau_N-2}{\tau_N-2+K}\,\frac{1}{\Gamma(\tau_N-1)} \tag{9}$$

For $\tau_N \geq 2$, $W_k(\tau_N) \geq 0$, whereas for $1 < \tau_N < 2$, $W_K(\tau_N) < 0$ $(K \geq 1)$. The necessary and sufficient condition for positivity is then $\tau_N \geq 2$ [10),11)]. Since none of the coefficients $W_K(\tau_N)$ change sign for $\tau_N \geq 2$ positivity is in fact ensured by positivity at the integer points $\tau_N = 2,3,4...$ which correspond to the case of free fields.

In conclusion we have shown that the operators $O_{\alpha_1,...\alpha_N}(x)$ $(N \geq 1)$, which can occur in the light cone expansion of two currents (weak and/ or electromagnetic), have scale dimensions [12),13)] $d_N \geq 2+N$ (provided conformal symmetry and positivity hold in the asymptotic theory) implying that [excluding the contribution of scalars $O(x)$] $J_\mu(x)\, J_\nu(0)$ is never more singular at $x^2 \sim 0$ than for the free theory, whereas the Green function, in Eq. (2), is never less singular.

ACKNOWLEDGMENTS

We have had discussions on this subject with R.J. Crewther, G. De Franceschi, G. Mack and G. Parisi, whom we would like to thank.

R E F E R E N C E S

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9) Eq. 8) can also be deduced, equally simply, from the manifestly conformal covariant formalism [see Eq. (A.41) in Ref. 8, p. 106].

10) A weaker result has been obtained by :
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11) Related conclusions are also obtained from different assumptions and procedures by :
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12) It is interesting to note that, as a consequence of our result, the contribution of the isovector second rank tensor to the Cottingham formula for the proton-neutron e.m. mass difference has an unambiguous sign which agrees with the choice of:
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13) We remark that our results do not imply nor are implied by the positivity restrictions which follow from positivity of the structure functions in deep inelastic e-p scattering; see for example in this connection :
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