(1969); R. F. Dashen and S. Y. Lee, Phys. Rev. 187, 2017 (1969); S. Okubo, Phys. Rev. D 3, 409 (1971). These articles contain additional references.

[27]Since we are only interested in the q-number part of the commutators here, we closely follow Okubo, Ref. 26.

[28]An added complexity occurs when considering nonconserved currents since the τ-ordered product of a current and a current divergence is not covariant.

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Continuation of Zero Contours from Weinberg's Low-Energy $\pi\pi$ Model to the ρ Region*

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We assume that the subthreshold zeros of Weinberg's $\pi\pi$ scattering amplitude smoothly continue into the physical region to become the Legendre zero of the ρ resonance. We show that this hypothesis of smooth zero contours is justified in the channel $\pi^+\pi^0 \to \pi^+\pi^0$. This enables us to give a new derivation of the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin relation and to predict the $I=2$ s wave correctly.

I. INTRODUCTION

Recent studies of the Veneziano model have served to emphasize that it is the zeros as well as the poles of the scattering amplitude that largely determine its structure. Wanders[1] has shown that Regge asymptotics with linear-trajectory functions follows from the distribution of zeros of certain meromorphic amplitudes. Phenomenologically Odorico[2] has traced, in both the high- and low-energy data, the zeros that must occur when poles in different channels cross.

This work is concerned with the role played by lines of zeros in low-energy $\pi\pi$ scattering. Since the current-algebra model of Weinberg[3] seems to be a good approximation to reality[4] in the neighborhood of the Mandelstam triangle, it is natural that several attempts have been made to unitarize Weinberg or other current-algebra models and so obtain an extrapolation of these models into the physical regions. Thereby one might hope to predict the phase shifts and in particular the mass and width of the ρ resonance. Attempts in this direction have been made for example by Kang and Lee.[5,6]

In contrast we want to show that it may be more useful and more reliable not to try to extrapolate the whole amplitude but just its zeros from one energy region to another. In Weinberg's linear model the on-mass-shell zeros of the invariant amplitudes, which are implied by the Adler self-consistency condition,[7] describe straight lines in the Mandelstam plane. We shall work with a quadratic model which preserves these Weinberg zeros for subthreshold energies, but whose zero contours are curved as unitarity requires.

What we mean by "zero contours" is unambiguous for an amplitude inside the Mandelstam triangle, where it is purely real. However, when an amplitude is complex we must define what we mean by its zero contours. In general, for real energies t, the complex amplitude

$$A[t, z_t = 1 + 2s/(t-4)]$$

will vanish at complex $z_t = z_0(t)$. We shall refer to the lines $z = \mathrm{Re}\, z_0(t)$ as the zero contours of this amplitude. Such contours correspond to the lines of minima of the differential cross section, $|A|^2$. We shall see that another definition (zeros of the real part of the amplitude) is not a useful concept.

We shall show under what circumstances zero contours can be useful. We recall that Barrelet[8] has shown that zero contours in $\pi^\pm p$ scattering are amazingly smooth. Odorico[2] has looked at minima of the differential cross sections in reactions like KN charge exchange and found them also to be smooth.

In Sec. II we construct a simple model based on current algebra for $z_0(t)$ inside the Mandelstam triangle. In Sec. III we determine the curvature of the zero contour through the Froissart-Gribov integral for the d-wave scattering length. In Sec. IV we discuss, in general, the smoothness of zero contours near resonances, and in Sec. V we give the connection between the zero contour in $\pi^+\pi^0 \to \pi^+\pi^0$ scattering and the ρ mass as well as the magnitude of the exotic s wave. Using the hypothesis of a smooth zero contour we give a new derivation of the KSRF (Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin) relation[9] (Sec. VI), and we predict the exotic s wave in Sec. VII. In Sec. VIII we explain why unitarity does not determine the absolute scale of our amplitude. In Sec. IX we show that the exotic d wave can be neglected in our example, but not in other cases. Section X contains our conclusions.

II. MODEL FOR THE ZERO CONTOUR

Our aim is to make the simplest reasonable ansatz for the zero contour in low-energy $\pi\pi$ scattering. We do this by constructing a model for the scattering amplitude inside the Mandelstam triangle which has essentially the features of Weinberg's current-algebra model. However, the zero contours in Weinberg's linear model are straight lines, while we shall see below that extrapolating into the ρ region requires zero contours to be curved. We make the simplicity hypothesis that it is sufficient to add terms quadratic in the Mandelstam variables to Weinberg's linear model in order to give such curvature. In doing so we shall require that certain aspects of Weinberg's subthreshold model be preserved.

For the larger part of this paper we concentrate on one particular zero contour, the one in the Chew-Mandelstam invariant amplitude $A(s,t,u)$.[10] This amplitude represents $\pi^+\pi^0$ elastic scattering in the t and u channels and the reaction $\pi^0\pi^0 \to \pi^+\pi^-$ in the s channel. It has the following isospin decomposition:

$$A(s,t,u) \equiv \tfrac{1}{3}(F^{s0} - F^{s2}) = \tfrac{1}{2}(F^{t1} + F^{t2}). \tag{2.1}$$

For this particular contour it is a good approximation to set the exotic $(I=2)$ d wave to zero (see Sec. IX). This approximation simplifies our illustration of the technique of smooth zero extrapolation.

The most general on-shell quadratic form for A (which is even under interchange of t and u) is

$$A(s,t,u) = a + bs + cs^2 + dtu, \tag{2.2}$$

which has just s, p, and d waves in the s, t, and u channels. Setting the exotic d wave to zero implies

$$c = 0. \tag{2.3}$$

We recall that the invariant amplitude in the Weinberg model[3] is given by[11]

$$A(s,t,u) = \tfrac{1}{4}L(s-1), \tag{2.4}$$

where

$$L = \frac{1}{8\pi f_\pi^{\,2}}, \tag{2.5}$$

with f_π = pion decay constant. We see that in Weinberg's model the amplitude A vanishes on the line $s=1$; this is just the on-shell appearance of the Adler zero.

We now ask which features of Weinberg's model, Eq. (2.4), we wish to preserve in our general expansion, Eqs. (2.2), (2.3). We recall that for the Weinberg amplitude the $I=0$ and 2 s-wave amplitudes have zeros at $s=\frac{1}{2}$ and $s=2$, respectively. These zeros are a general feature of models of $\pi\pi$ scattering.[4] However, the position of these s-wave zeros in the individual isospin amplitudes varies greatly from one model to another. In contrast the position of the zero in the s wave of the Chew-Mandelstam invariant amplitude, Eq. (2.1), is less model-dependent and generally appears fairly close to Weinberg's position of $s=1$, Eq. (2.4). Because of this we shall require that our amplitude also have an s-wave zero at $s=1$ and that the slope of the s-wave amplitude at this zero also be given by Weinberg. This reduces our amplitude, Eqs. (2.2), (2.3), to the following form:

$$A(s,t,u) = \tfrac{1}{4}L(s-1) + d(tu + s - \tfrac{5}{2}), \tag{2.6}$$

where d is an as yet unknown parameter, which will be determined in Sec. III. It should be stressed that if instead of taking Weinberg's model to be exact close to the s-wave zero in the amplitude A we had assumed, for example, that the s-wave scattering lengths were given exactly by Weinberg, our results would be insignificantly changed in the physical regions (see end of Sec. VI).

III. CURVATURE OF THE ZERO CONTOUR AND FROISSART-GRIBOV CALCULATION OF THE d-WAVE SCATTERING LENGTH

The free parameter d in our model for the zero contour, Eq. (2.6), is proportional to the d-wave scattering length, α_d, in the s channel:

$$\alpha_d \equiv \tfrac{1}{3}(a_2^0 - a_2^2) = -\tfrac{8}{15}d, \tag{3.1}$$

where the scattering lengths are defined by

$$a_l^I = \lim_{s\to 4} \frac{f_l^I(s)}{[\frac{1}{4}(s-4)]^l}. \tag{3.2}$$

The partial-wave amplitudes, $f_l^I(s)$, are related to the full isospin amplitudes by

$$F^{sI}(s,t) = \sum_{l=0}^{\infty} (2l+1) f_l^I(s) P_l\left(1 + \frac{2t}{s-4}\right), \tag{3.3}$$

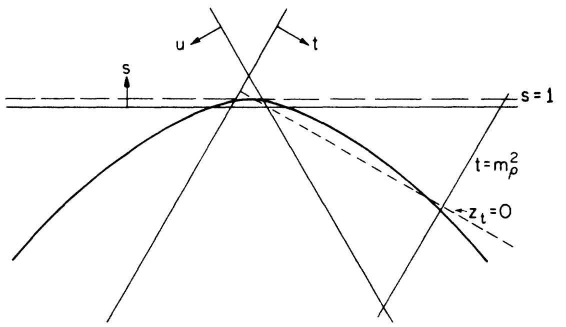

FIG. 1. The curve of $\mathrm{Re}z_0(t)$ in the Mandelstam plane, where $A[t, z_0(t)]=0$. A is the $\pi^+\pi^0 \to \pi^+\pi^0$ amplitude in the t and u channels. The zero contour in the Weinberg model is the straight line $s=1$.

and the phase shifts below inelastic threshold are defined by

$$f_l^I(s)=\left(\frac{s}{s-4}\right)^{1/2} e^{i\delta_l^I}\sin\delta_l^I . \tag{3.4}$$

We shall now determine the parameter d by calculating the d-wave scattering length α_d. This scattering length is given by the Froissart-Gribov representation

$$\alpha_d=\frac{16}{15\pi}\int_4^\infty \frac{dt}{t^3}\tfrac{1}{2}\left[\mathrm{Im}F^1(t,4)+\mathrm{Im}F^2(t,4)\right]. \tag{3.5}$$

Positivity of the t-channel absorptive parts implies that α_d must be positive. Therefore d is negative [see Eq. (3.1)] and the zero contour of the amplitude, Eq. (2.6), curves downwards into the physical-region t and u channels (see Fig. 1).

Because of the t^3 denominator in Eq. (3.5) the major contribution to this integral will come from the low-energy region, where we have just s and p waves. The s-wave contribution comes from the imaginary part of the exotic $I=2$ channel and is small compared to the dominant p-wave contribution from the ρ peak.[12] Evaluating the ρ contribution analytically in the "narrow-resonance approximation" (n.r.a.) we obtain

$$\alpha_d=\frac{8}{5}\frac{\Gamma_\rho}{m_\rho^4}\frac{m_\rho^2+4}{(m_\rho^2-4)^{3/2}}. \tag{3.6}$$

This determines the parameter d, Eq. (3.1), in terms of the mass, m_ρ, and width, Γ_ρ, of the ρ resonance.

IV. ZERO CONTOURS NEAR RESONANCES: THE SMOOTHNESS HYPOTHESIS

We now define more precisely what we mean by zero contours when the amplitude is complex, and we discuss the hypothesis that zero contours are smooth near resonances.

We work with the real part of the zeros of A and not with the zeros of the real part of A. More explicitly, for real values of the energy t we consider the complex zero $z_0(t)$ of the amplitude A, i.e., $A(t, z_0)=0$, and then we take the real part of this zero, $\mathrm{Re}z_0(t)$.

The basic hypothesis is that the real part of the zeros of A describe smooth lines even in the presence of resonances. In order to show when this hypothesis is reasonable, we consider a few typical examples:

(1) If $A(s,t)$ is the sum of two poles, one in s and the other in t with unequal strengths (g^2) and/or unequal widths (Γ), then the real part of the zero of A is a smooth line, while the zero of ReA gives a complicated pattern.

(2) If $A(t,z)$ in the scattering region of the t channel is built from a narrow p-wave resonance and a slowly varying s wave, then the real part of the zero of A is again a smooth line in the Mandelstam plane, while the zero of ReA gives a complicated pattern.

(3) On the other hand, with a narrow s-wave resonance and a constant p wave, both the real part of the zero of A and the zero of ReA show a violent variation.[13]

Barrelet[8] has shown that the real and imaginary parts of the zeros of the transversity amplitudes in $\pi^\pm p$ elastic scattering follow amazingly smooth lines. Odorico[2] has looked at minima (i.e., $\mathrm{Re}z_0$) of high- and low-energy differential cross sections. He has found that they follow smooth, often straight, lines, and associates them with the zeros which must occur when poles in different channels cross in the unphysical region.

Although there is no *a priori* reason why zeros should follow smooth paths, they are in fact smooth in many cases in nature and in simple Veneziano models.[14] In the amplitude we consider in this paper, $\pi^+\pi^0\to\pi^+\pi^0$, we have the ρ resonance in the p wave and a smooth exotic s-wave background. Therefore the zero contour is necessarily smooth in the ρ region.

V. ZERO CONTOUR IN THE ρ REGION

In the region of the ρ resonance the amplitude A is given by just the s and p waves:

$$A(t,z_t)=\tfrac{1}{2}S_2+\tfrac{3}{2}z_t\frac{m_\rho\Gamma_\rho}{m_\rho^2-t-im_\rho\Gamma_\rho}, \tag{5.1}$$

where $S_2=f_{l=0}^{I=2}(t)$ and where we have replaced $[t/(t-4)]^{1/2}$ by 1.

The zero contour near the ρ is approximately a straight line

$$\mathrm{Re}z_0(t) = -\tfrac{1}{3}\left[\mathrm{Re}S_2(t)\frac{m_\rho^2 - t}{m_\rho\Gamma_\rho} + \mathrm{Im}S_2(t)\right]. \tag{5.2}$$

In the narrow-resonance approximation [i.e., Γ in the resonance denominator, Eq. (5.1), is set equal to zero, while Γ in the numerator is kept at its normal value] the zero contour intersects the line $z=0$ at $t=m_\rho^2$. For a finite ρ width it is still true that the intersection $t=t_x$ of the zero contour with the line $z_t=0$ gives the ρ mass to a good approximation,

$$t_x - m_\rho^2 \ll m_\rho\Gamma_\rho, \tag{5.3}$$

because the exotic background has $\mathrm{Im}S_2 \ll \mathrm{Re}S_2$. The angle between the zero contour and the line $z_t=0$ gives the ratio between the strength of the s wave and the ρ width, thus:

$$\left[\frac{d}{dt}\mathrm{Re}z_0(t)\right]_{t=m_\rho^2} = \frac{\mathrm{Re}S_2(m_\rho^2)}{3m_\rho\Gamma_\rho}. \tag{5.4}$$

To summarize: If the zero contour is given, we can predict two quantities, m_ρ and $\mathrm{Re}S_2/\Gamma_\rho$, from the intersection point and the angle of intersection of the zero contour with the line $z_t=0$.

VI. THE KSRF RELATION REDERIVED

We now make the hypothesis that the zero contour of the s, p, d-wave model of Sec. II can be extrapolated to the ρ region, where it must be identified with the physical zero contour discussed in Sec. V (see Fig. 1).

The curvature of the contour in our modified Weinberg model is proportional to α_d/L. As discussed in Sec. V, this zero contour must cross the line $z_t=0$ close to the ρ mass. This determines α_d/L; using Eqs. (2.6) and (3.1) we obtain, in the n.r.a.,

$$\frac{\alpha_d}{L} = \frac{2}{15}\,\frac{m_\rho^2 - 2}{m_\rho^4 - 3m_\rho^2 + 1}. \tag{6.1}$$

On the other hand, α_d is proportional to the ρ width according to the Froissart-Gribov calculation of Eq. (3.6). Comparing the two expressions for α_d, Eq. (6.1) and Eq. (3.6), which were both obtained in the n.r.a., and using the definition [Eq. (2.5)], we find

$$\frac{1}{f_\pi^2} = \frac{96\pi\Gamma_\rho}{m_\rho^3}K, \tag{6.2}$$

with

$$K = \frac{1+4/m_\rho^2}{1-2/m_\rho^2}\,\frac{1-3/m_\rho^2+1/m_\rho^4}{(1-4/m_\rho^2)^{3/2}} = 1.35\,. \tag{6.3}$$

This relation is of the KSRF type[9] and relates the ρ width to the pion decay constant, giving

$$\begin{aligned} &f_\pi = 95 \text{ MeV (experimental value):}\\ &\quad\Gamma_\rho = 122 \text{ MeV},\\ &f_\pi = 83 \text{ MeV (Goldberger-Treiman value):}\\ &\quad\Gamma_\rho = 161 \text{ MeV}. \end{aligned} \tag{6.4}$$

The fact that our relation, Eqs. (6.2), (6.3), is empirically well satisfied gives support to our smooth-zero extrapolation hypothesis.[15]

The kinematic factor K reduces to unity in the limit $(m_\pi/m_\rho)^2 \to 0$, and Eq. (6.2) then takes exactly the form of the KSRF result.[9] In the simple B_4 model of Ref. 14, one obtains a KSRF-type relation with $K=\frac{1}{2}\pi=1.57$. We see that our factor K is about half way between the value in the old KSRF result and in the B_4 relation.

The explicit form of the kinematic factor, Eq. (6.3), depends on our assumption (discussed in Sec. II) that the s-channel s wave of our s, p, d-wave model and of Weinberg's linear model should agree exactly at the zero of this s wave. We now indicate how little the factor K in Eq. (6.3) is altered if we make different assumptions, for example, if we take instead (1) the s-wave scattering lengths, and (2) the whole amplitude near the point $s=1$, $t=u$, to be exactly as given by Weinberg. Writing the modified kinematic factor $K_i \equiv K\xi_i$, we obtain in case (1) $\xi_1 = 0.96$ and in case (2) $\xi_2 = 1.02$. Therefore our KSRF relation is changed by only ±4% if we modify our assumptions in Sec. II.

VII. PREDICTION OF THE $I=2$ s WAVE

The angle of intersection of the zero contour with the line $z=0$ determines $\mathrm{Re}S_2(m_\rho^2)/\Gamma_\rho$. Using Eqs. (2.6) and (3.1) we obtain the angle of intersection

$$\left[\frac{d}{dt}\mathrm{Re}z_0(t)\right]_{t=t_x} = \left[\frac{2t-3-2L/15\alpha_d}{(4-t)(t-1+2L/15\alpha_d)}\right]_{t=t_x}. \tag{7.1}$$

Using the value of α_d/L, Eq. (6.1), determined by the condition that the contour must intersect the line $z=0$ at the ρ mass in the n.r.a., we obtain

$$\left[\frac{d}{dt}\mathrm{Re}z_0(t)\right]_{t=m_\rho^2} = -\tfrac{1}{2}\left[\frac{t^2-4t+5}{(t-4)(t^2-3t+\frac{3}{2})}\right]_{t=m_\rho^2} = -\frac{1.11}{2m_\rho^2}, \tag{7.2}$$

with $m_\rho = 765$ MeV. Using Eq. (5.4) this gives

$$\mathrm{Re}S_2(m_\rho^2) = -\frac{3}{2}\,\frac{\Gamma_\rho}{m_\rho}1.11\,. \tag{7.3}$$

With $\Gamma_\rho = 135$ MeV we predict

$$\delta_{l=0}^{I=2}(m_\rho^2) = -18°, \tag{7.4}$$

evaluated using the n.r.a. for the ρ resonance. This is in reasonable agreement with the experimental results of, e.g., Baton *et al.* and Cohen *et al.*,[16] who give

$$\delta_0^2(750\text{ MeV}) = -12° \pm 6°,$$
$$\delta_0^2(790\text{ MeV}) = -15° \pm 3.5°, \tag{7.5}$$

respectively.

We now abandon the n.r.a. and use a nontrivial parametrization for the p wave from threshold up to the ρ resonance. This allows us to predict the $I=2$ s wave from threshold up to the ρ region, using the zero contour of Eq. (2.6). For the p-wave Breit-Wigner resonance tails we make the simplest ansatz which allows us to fix the p-wave scattering length at its Weinberg value, i.e.,

$$f_{l=1}^{I=1}(t) = \left(\frac{t}{t-4}\right)^{1/2} \frac{m_\rho \Gamma(t)}{m_\rho^2 - t - i m_\rho \Gamma_\rho(t)}, \tag{7.6}$$

with

$$\Gamma(t) = \Gamma_\rho \left(\frac{t-4}{t}\right)^{1/2} \left(\frac{t-4}{m_\rho^2 - 4}\right) \frac{1+\lambda}{1+\lambda(t-4)/(m_\rho^2-4)}. \tag{7.7}$$

The constant λ in the "barrier factor" is determined by the p-wave scattering length of the Weinberg model:

$$a_1^1 = \frac{4 m_\rho \Gamma_\rho}{(m_\rho^2 - 4)^2}(1+\lambda) = \tfrac{1}{3}L\,. \tag{7.8}$$

Using the definition (2.5) and our KSRF-type relation, Eqs. (6.2), (6.3), we obtain

$$1+\lambda = K\left(1 - \frac{4}{m_\rho^2}\right)^2 = 1.018\,. \tag{7.9}$$

Knowing the p wave and the zero contour we compute the $I=2$ s-wave phase shift; the result is shown in Fig. 2.[17] As can be seen, the resultant phase shift is in very reasonable agreement with all the recent data of Ref. 16. Near threshold the result agrees with the Weinberg model by construction. It is also in very good agreement with the values computed by Pennington and Protopopescu[4] using Roy's equations. This is the second piece of evidence in favor of our smooth zero extrapolation hypothesis.

We see that starting from a real amplitude, Eq. (2.6), which contains none of the correct analytic structure at threshold and pseudothreshold and which is therefore valid only inside the Mandelstam triangle, we have been able to compute both the real and imaginary parts of the $I=2$ s wave in the low-energy region. This has been achieved without recourse to a complicated unitarization technique but by assuming that zero contours extrapolate simply.

VIII. UNITARITY AND THE ABSOLUTE SCALE OF THE AMPLITUDES

One might have hoped that unitarity would determine the absolute scale of the amplitudes. This is not the case. Keeping the zero contour fixed and therefore the ρ mass fixed, we can multiply all scattering lengths and the exotic s wave by a factor λ while we keep the magnitude of the p wave at the resonance at its unitarity value and multiply the ρ width by λ. This scaling property is consistent with nonlinear unitarity, since with $(\text{Re}S_2)^2 \gg (\text{Im}S_2)^2$ the role played by unitarity is just to define the imaginary part of the partial-wave amplitude, S_2, from its real part, and not to determine the absolute scale of the amplitude.

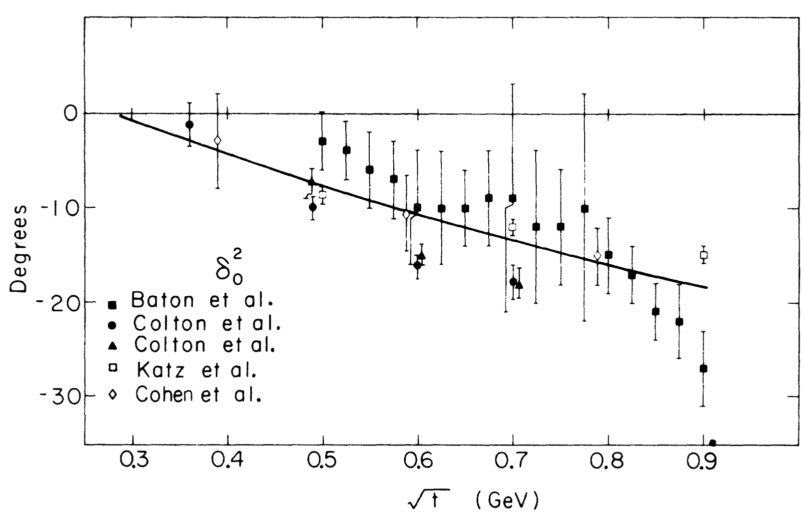

FIG. 2. The solid line shows the predicted values of $\delta_0^2(t)$. The data points are those of Ref. 16.

IX. THE EXOTIC d WAVE

We first show that it is a reasonable approximation to neglect the influence of the exotic d wave on that particular zero contour we have studied up to now. Afterwards we shall give an example where the influence of the exotic d wave is large.

In the general quadratic form, Eq. (2.2), we introduce the convenient parameter

$$\epsilon = \frac{c}{d} = -\frac{3}{2}\,\frac{a_2^2}{a_2^0 - a_2^2}\,. \tag{9.1}$$

We shall see below that a_2^2 is expected to be negative in our model, if not more generally (i.e., $\epsilon > 0$). Since $a_2^0 + 2a_2^2 > 0$ from positivity, we have

$$\tfrac{1}{2} > \epsilon > 0\,. \tag{9.2}$$

Using Eq. (2.2) we compute the shift of the zero contour at $t = m_\rho{}^2$ caused by $\epsilon \neq 0$:

$$\Delta z_0 = \frac{\Delta s}{\frac{1}{2}m^2} = \tfrac{1}{4}\epsilon < \tfrac{1}{8}\,. \tag{9.3}$$

We see that it is a reasonable approximation to neglect ϵ.

We now study the zero contour for the s-channel process $\pi^+\pi^+ \to \pi^+\pi^+$. If we neglect the exotic d wave, the s-channel amplitude has just an s wave and so the zero contour is a straight line at fixed s. In our model it is close to $s = 2$, while in Weinberg's model[3] and in the Lovelace-Veneziano model[14] it is exactly at $s = 2$. This is in conflict with unitarity. The contour $F^{s2} = 0$ corresponds to the vanishing of the t-channel amplitude

$$2F^{t0} - 3F^{t1} + F^{t2} = 0\,. \tag{9.4}$$

In the low-energy region we can approximate this by just the s and p waves and obtain

$$9\,\mathrm{Re}z_0(t) = 2\cos(\delta_0^0 - \delta_1^1)\frac{\sin\delta_0^0}{\sin\delta_1^1} + \cos(\delta_0^2 - \delta_1^1)\frac{\sin\delta_0^2}{\sin\delta_1^1}\,. \tag{9.5}$$

Now let us look at the $\mathrm{Re}z_0$ at $t = m_\rho{}^2$ (i.e., $\delta_1^1 = \frac{1}{2}\pi$); then

$$9\,\mathrm{Re}z_0 = 2\sin^2\delta_0^0 + \sin^2\delta_0^2 < 3\,. \tag{9.6}$$

We see that straightforward elastic unitarity demands

$$0 < \mathrm{Re}z_0(m_\rho{}^2) < \tfrac{1}{3}\,. \tag{9.7}$$

Putting in the value of δ_0^2 in Eq. (9.7) gives the even stronger restriction

$$0 < \mathrm{Re}z_0(m_\rho{}^2) < 0.23\,. \tag{9.8}$$

However, in our model without exotic d waves we have $F^{s2}(s \simeq 2, t) = 0$, i.e., $\mathrm{Re}z > 1$. As illustrated in Fig. 3, the violation of unitarity is rather dramatic. We see that exotic partial waves with $l \geq 2$ are clearly important.[18] The fact that unitarity in the low-energy $\pi\pi$ scattering does not allow this particular zero to follow the straight line, $s = 2$, predicted by Lovelace and Veneziano[14] is a warning against Odorico's idea[2] that the zeros of the Veneziano model are a universal feature little affected by unitarization.

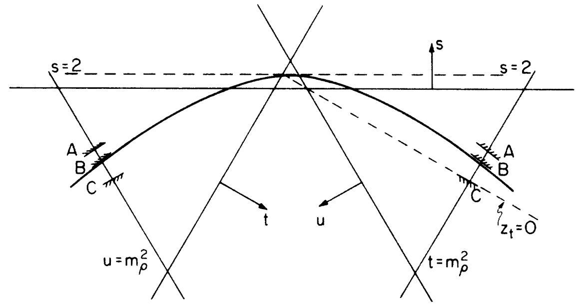

FIG. 3. A simple curve of $\mathrm{Re}z_0(t)$, where $F^{s2}[t, z_0(t)] = 0$, consistent with both Weinberg's amplitude inside the Mandelstam triangle and unitarity at $t = m_\rho{}^2$. F^{s2} is the $\pi^+\pi^+$ amplitude in the s channel and the $\pi^+\pi^- \to \pi^+\pi^-$ amplitude in the t and u channels. The zero contour in the Lovelace-Veneziano model and in the Weinberg model is the line $s = 2$. The bound A at $t = m_\rho{}^2$ is the elastic unitarity bound $\mathrm{Re}z_0 < \frac{1}{3}$ [Eq. (9.7)]; B is the stronger phenomenological bound $\mathrm{Re}z_0 < 0.23$ [Eq. (9.8)]; C is the bound $\mathrm{Re}z_0 > 0$ [Eq. (9.7)]. The zero contour must pass between B and C at $t = m_\rho{}^2$ to be consistent with unitarity.

X. SUMMARY AND CONCLUSIONS

We have extrapolated zero contours in $\pi\pi$ scattering from the region of the Mandelstam triangle where they coincide with the on-shell appearance of the Adler zero, as given by Weinberg's linear model, to the region of the ρ resonance, where they become its Legendre zero.[19] We have implemented the hypothesis of smooth zero contours by using an s, p, d-wave model for them. This hypothesis has enabled us to give a new derivation of the KSRF relation and to predict the $I = 2$ s wave in good agreement with experiment.

We have discussed under what assumptions the hypothesis of smooth zero contours is justified in low-energy scattering reactions; we have shown that it is justified in the specific channel we have considered and have referred to the work of both Odorico and Barrelet for empirical evidence for smoothness.

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[4]See, for example, M. R. Pennington and S. D. Protopopescu, Phys. Rev. D 7, 1429 (1973).

[5]L. S. Brown and R. L. Goble, Phys. Rev. Letters 20, 346 (1968); J. S. Kang and B. W. Lee, Phys. Rev. D 3, 284 (1971).

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[11]The pion mass is set equal to unity.

[12]The $I=2$ s-wave contribution is estimated to be less than 10% of the ρ contribution.

[13]An example of a nonsmooth zero contour for the case of a smooth p wave interfering with a threshold effect in the s wave has been discussed by M. R. Pennington and S. D. Protopopescu, Phys. Letters 40B, 105 (1972).

[14]C. Lovelace, Phys. Letters 28B, 264 (1968).

[15]From Eq. (6.1) with $f_\pi = 95$ MeV : $\alpha_d = 3.8\times10^{-4}$; with $f_\pi = 83$ MeV : $\alpha_d = 4.9\times10^{-4}$ evaluated in the n.r.a. This is to be compared with the experimental value in Ref. 4: $\alpha_d = (5.6\pm0.2)\times10^{-4}$.

[16]J. P. Baton *et al.*, Phys. Letters 33B, 525 (1970); 33B, 528 (1970); E. Colton *et al.*, Phys. Rev. D 3, 2028 (1971); W. M. Katz *et al.*, *Proceedings of the Conference on ππ and Kπ Interactions, Argonne National Laboratory, 1969*, edited by F. Loeffler and E. D. Malamud (Argonne National Laboratory, Argonne, Ill., 1969); D. Cohen *et al.*, Phys. Rev. D 7, 661 (1972).

[17]From Fig. 2 we see that $\delta_0^2(m_\rho^2) = -15.0°$. This is to be compared with the result of Eq. (7.4) which was obtained in the n.r.a. In calculating the phase shifts shown in Fig. 2 this approximation was not used. This accounts for the difference.

[18]In a model such as ours in which the only nonzero higher wave is the d wave and that is determined by just one parameter, its scattering length, we see that this scattering length has to be repulsive in the $I=2$ channel (i.e., $\epsilon > 0$) for such a change in curvature to result.

[19]As discussed in M. R. Pennington, thesis submitted to the University of London, 1971 (unpublished).