Note on the $\gamma\gamma$ contribution to $\pi^0 \to e^+e^-$ and $\eta \to \mu^+\mu^-$

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We review what is known on how the electromagnetic form factor for $\pi^0 \to e^+e^-$ and $\eta \to \mu^+\mu^-$ should behave, and show that a recent proposal which claims to resolve the discrepancy between theory and experiment for $\pi^0 \to e^+e^-$ is actually equivalent to assuming a form factor quite contrary to its known properties.

The rare decays $\pi^0 \to e^+e^-$ and $\eta \to \mu^+\mu^-$ have recently been studied with renewed interest, both theoretically[1–4] and experimentally.[5,6] The two-photon contribution to these decays is in general given by[1]

$$B^P \equiv \frac{\Gamma(P \to \gamma^*\gamma^* \to l^+l^-)}{\Gamma(P \to \gamma\gamma)} = 2\left(1 - \frac{4m_l^2}{m_P^2}\right)^{1/2}\left(\frac{\alpha m_l}{\pi m_P}\right)^2 |R|^2 \;, \tag{1}$$

where

$$R = \frac{i}{\pi^2 m_P^2} \times \int d^4k \frac{2[q^2k^2 - (k\cdot q)^2]F_P(k^2,(q-k)^2;m_P^2)}{(k^2+i\epsilon)[(q-k)^2+i\epsilon][(k-r)^2-m_l^2+i\epsilon]} \;, \tag{2}$$

with q, r, k, and $q-k$ as the four-momenta of the pseudoscalar meson P, the outgoing lepton l^-, and the two intermediate photons, respectively. The form factor $F_P(s_1,s_2;m_P^2)$ is normalized to unity at $s_1 = s_2 = 0$.

In an early calculation, Drell[7] noted that the integral in Eq. (2) is divergent if $F_P = 1$ for all s_1 and s_2. To obtain a convergent result, he used a dispersion relation in m_P^2 and assumed F_P to be a step function with an *ad hoc* cutoff, i.e., $F_P(0,0;m_P^2) = 1$ for $m_P^2 < \Lambda^2$ and zero otherwise. Unfortunately, the physical significance of such a model is not at all clear. In any case, more information on F_P has since become available, and three properties are now reasonably well established.

(a) The form factor with only one photon off the mass shell, $F_P(s,0;m_P^2)$, should have a resonant structure in the timelike region in accordance with vector-meson-dominance predictions.

(b) For large s, the unambiguous quantum-chromodynamics (QCD) prediction[8] is $F_P(s,0;m_P^2) \propto s^{-1}$.

(c) For large k^2, the form factor $F_P(k^2,(q-k)^2;m_P^2)$ should behave[9] as k^{-2}.

Given these constraints on the form factor, it is clear that the integral in Eq. (2) is now convergent. Moreover, these constraints are stringent enough that the predictions for B^P defined in Eq. (1) become more or less unique. Recently, two specific form factors satisfying these constraints were independently proposed:

$$F_P(s_1,s_2;m_P^2) = \frac{m_V^2}{m_V^2 - s_1 - s_2} \tag{3}$$

in Ref. 1; and

$$F_P(s_1,s_2;m_P^2) = \frac{1 - s_1s_2/3m_V^2Q^2}{(1-s_1/m_V^2)(1-s_2/m_V^2)} \;, \tag{4}$$

where $Q^2 = \frac{1}{2}(s_1 + s_2 - \frac{1}{2}m_P^2)$, in Ref. 2. Although these two form factors are quite different, the resulting predictions for B^P are very similar, namely, $B^\pi(e^+e^-) \simeq 0.6\times10^{-7}$ and $B^\eta(\mu^+\mu^-) \simeq 1.2\times10^{-5}$. The latter is consistent with the most recent experimental result[6] of $(1.66\pm0.54)\times10^{-5}$, whereas the former is considerably lower than the recently reported value[5] of $(1.8\pm0.6)\times10^{-7}$. This has led us to conclude that either the π^0 decay has other exotic contributions,[1] or perhaps, more likely, that a future measurement with smaller errors will come down in value. [It may be useful to point out that a previous measurement[10] of $\eta \to \mu^+\mu^-$ obtained $(5.9\pm2.2)\times10^{-5}$, a value much higher than the present value.[6]]

On the other hand, it has been claimed in a recent paper[4] that the measured $\pi^0 \to e^+e^-$ rate can be obtained theoretically by using a once-subtracted dispersion relation and, moreover, that those results are "form-factor-insensitive." In view of the above discussion, this appears very surprising indeed. On closer examination, it will turn out, as we show below, that this particular dispersion approach involves an implicit choice of a very peculiar form factor which fulfills none of the constraints (a) to (c) listed above. Hence it should not be considered as a *bona fide* resolution of the apparent discrepancy between theory and experiment.

The imaginary part of R in Eq. (2) can generally be written

$$\mathrm{Im}R(t) = \frac{\pi}{2\beta}\ln\left(\frac{1-\beta}{1+\beta}\right)f(t) \;, \tag{5}$$

where $\beta = (1 - 4m_l^2/t)^{1/2}$. For the on-shell amplitude for $\pi^0 \to e^+e^-$, no intermediate state other than the 2γ state can contribute to the imaginary part, to this order in α and to all orders in the strong interaction. By using the Cutkosky rules one then finds from Eq. (2) $f(m_\pi^2) = F_\pi(0,0;m_\pi^2) = 1$, due to the normalization of F_P.

The idea of Ref. 4 is to use Eq. (5) in a once-subtracted dispersion relation to calculate the real part of R. The first crucial assumption of that calculation (and a reason why it is unrealistic) is to make $f(t) = 1$ for *all t*. To see why this may not be a good assumption, consider, for example, $P \rightarrow V\gamma$ decay, where V is a vector meson. Since V undoubtedly couples to γ, the imaginary part of R in Eq. (5) must get an additional contribution whenever t exceeds m_V^2. This means that $f(t)$ as a function of t is, in general, quite complicated, and putting $f(t) \equiv 1$ is not necessarily a reasonable choice. [A general discussion of this and the relationship between calculating the integral in Eq. (2) directly and using dispersion relations is in preparation.[11]] Putting $f(t) \equiv 1$ unfortunately also makes the dispersion integral for $\mathrm{Re}R(t)$ divergent, calling for one subtraction which leaves a subtraction constant $R(0)$ to be determined. In Ref. 4 this $R(0)$ is arbitrarily put equal to zero. This is the second crucial assumption they make.

With $f(t) \equiv 1$, the once-subtracted dispersion relation reads

$$\mathrm{Re}[R(t)] - R(0) = \frac{t}{\pi} \int_0^\infty dt' \frac{\mathrm{Im}[R(t)]}{t'(t'-t)} \ , \quad (6)$$

where

$$\mathrm{Im}[R(t)] = \mathrm{Im}\left[\frac{i}{\pi^2 t} \int d^4k \frac{2[q^2k^2 - (q\cdot k)^2]}{k^2(q-k)^2[(k-r)^2 - m_l^2]}\right] . \quad (7)$$

Working in the center-of-mass frame of the pseudoscalar meson, and applying the mass-shell condition when using the Cutkosky rules, one finds

$$\frac{\mathrm{Im}[R(t)]}{t} = \mathrm{Im}\left[\frac{-i}{2\pi^2} \int \frac{d^4k}{k^2(q-k)^2[(k-r)^2 - m_l^2]}\right] . \quad (8)$$

With the choice of $R(0) \simeq 0$ as in Ref. 4, it is clear from Eqs. (6) and (8) that calculating a once-subtracted dispersion relation for R is equivalent to calculating an unsubtracted dispersion relation for the integral in Eq. (8). Indeed, by doing so, we recover the results of Ref. 4.

Now we compare Eq. (8) with Eq. (2), and see at once that it is equivalent to choosing a form factor

$$F_P(k^2, (q-k)^2; m_P^2) = -\frac{m_P^4}{4[q^2k^2 - (q\cdot k)^2]} \ , \quad (9)$$

which in terms of s_1 and s_2 becomes

$$F_P(s_1, s_2; m_P^2) = \frac{m_P^4}{m_P^4 + s_1^2 + s_2^2 - 2m_P^2(s_1+s_2) - 2s_1s_2} \ . \quad (10)$$

Unfortunately, this form factor is very unrealistic; it meets none of the criteria (a) to (c) listed above. In particular, it implies $F_P(s, 0; m_P^2) = m_P^4/(m_P^2 - s)^2$, which behaves like s^{-2} instead of s^{-1} as predicted by QCD for large s,[8] and, in addition, is not consistent with the experimental measurements of this form factor through the single-Dalitz-pair decays $\pi^0 \rightarrow e^+e^-\gamma$ and $\eta \rightarrow \mu^+\mu^-\gamma$.[12] (The data on $\pi^0 \rightarrow e^+e^-\gamma$ are actually not very conclusive[13] because of the extreme sensitivity of this result to the acceptance of the experimental apparatus for small invariant mass of the e^+e^- pair and to the proper treatment of radiative corrections.)

For the $\pi^0 \rightarrow e^+e^-$ decay one can see that of the two arbitrary choices $f(t) \equiv 1$ and $R(0) = 0$ made in Ref. 4, the latter is the most questionable one. In fact, calculating $R(0)$ explicitly using, e.g., the form factor Eq. (3) inserted in Eq. (2), one finds $R(0) \approx -22.2$. This is indeed very different from zero, since the relevant scale is set by $\mathrm{Im}R(m_\pi^2) \approx -17.5$. It is interesting to note that adding this value of $R(0)$ to the result in Ref. 4 would bring essential agreement between their value for B^π and the more standard one in Refs. 1 and 2. We finally remark that the discussion in Ref. 4 concerning the smallness of additional contributions from vector-meson intermediate states is not complete, since it does not treat their possible effect on the subtraction constant $R(0)$. In many models, this contribution can be explicitly calculated,[11] and one finds a typical behavior $R(0) \sim -3\ln(m_V/m_e)$.

In conclusion, we argue that the results of Ref. 4 are obtained by assumptions with little physical motivation, and whatever the cause of discrepancy between theory and experiment for $\pi^0 \rightarrow e^+e^-$, it is not resolved by their treatment.

Note added in proof. In the reply of Tupper and Samuel,[14] the weak and electromagnetic contributions to $p \rightarrow l^+l^-$ are unfortunately not separated. In this Comment, we have only treated the electromagnetic part, since the weak part is known to be negligible[1] for nonstrange mesons. Therefore, our $R(q^2)$ should be identified with $\frac{1}{2}[K(q^2) - K_{\mathrm{weak}}(q^2)]$ of Tupper and Samuel, or $K(0) = 2R(0) + K_{\mathrm{weak}}(0)$. The essence of our criticism is just that they put $K(0) = K_{\mathrm{weak}}(0)$ [Eq. (5) in Ref. 4], thereby enforcing $R(0) = 0$. Sample calculations for $\pi^0 \rightarrow e^+e^-$ (e.g., Ref. 11) tend to give $2R(0) \approx -45$; $K_{\mathrm{weak}}(0) \approx 0.08$.

As for the equivalence between taking the form factor in our Eq. (10) and inserting it into Eq. (2) compared to taking $f(t) = 1$, $R(0) = 0$ and inserting that into a once-subtracted dispersion relation, it is of course equivalence in the sense that this gives two different integral representations of the *same* analytic function of t [namely, the function in Eq. (6) of Ref. 4].

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