Neutral Kaon Decays into Lepton Pairs

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The decay modes $K_1^0 \to \mu^+\mu^-$ and $K_2^0 \to \mu^+\mu^-$ are discussed within the framework of first-order weak combined with fourth-order electromagnetic interactions. The possibility of obtaining lower bounds to the corresponding decay rates is examined, and estimates of the lower bounds are given in a perturbation-theory model. Possible enhancement effects due to strong interactions have been estimated through a simple model of final-state interactions. The assumption of CP invariance is made throughout.

I. INTRODUCTION

EXPERIMENTAL search for the decay modes $K_L \to \mu^+\mu^-$ and $K_S \to \mu^+\mu^-$ has been undertaken, primarily, to test the possible existence of neutral leptonic currents coupled to the strangeness-changing charge-conserving hadronic current. At present, the upper limits for the total branching ratios of these modes are[1] (at 90% confidence level)

$$\frac{\Gamma(K_L \to \mu^+\mu^-)}{\Gamma(K_L \to \text{all})} < 2.1\times10^{-7}, \qquad \frac{\Gamma(K_L \to e^+e^-)}{\Gamma(K_L \to \text{all})} < 1.5\times10^{-7}, \tag{1.1}$$

and[2]

$$\frac{\Gamma(K_S \to \mu^+\mu^-)}{\Gamma(K_S \to \text{all})} < 7.3\times10^{-6}. \tag{1.2}$$

Even in the absence of neutral leptonic currents, $K_L \to \mu^+\mu^-$ and $K_S \to \mu^+\mu^-$ are allowed decay modes through electromagnetic induction,[3] at first order in the Fermi coupling constant G ($G = 1.02\times10^{-5}/m_p^2$) and fourth order in the electric charge e. They are also allowed as "higher order" weak processes, thus providing an interesting possibility of probing weak interactions at a level beyond the effective first-order current-current Hamiltonian.[4] Possible tests of CP and CPT invariances involving these decay modes have also been discussed.[5,6] In view of future improvement of the experimental upper limits quoted above, it becomes of considerable interest to know the "expected" decay rates of these rare modes. In fact, various estimates of the decay rate of $K_L \to \mu^+\mu^-$ can already be found in the literature[7]; however, to our knowledge, nothing much is known about the decay rate of $K_S \to \mu^+\mu^-$.

This paper is primarily devoted to a study of the decay mode $K_S \to \mu^+\mu^-$ viewed as a first-order weak times fourth-order electromagnetic process. The possibility of obtaining a lower bound to the decay rate of this process from unitarity is examined, and estimates are given within the framework of a perturbation-theory model. Possible enhancement effects due to strong interactions have been taken into account only through a simple model of final-state interactions. The assumption of CP invariance is made throughout.

In Sec. II we examine the decay modes $K_L \to \mu^+\mu^-$ and $K_S \to \mu^+\mu^-$ from a phenomenological point of view. The implications of the experimental limits quoted above upon the possible existence of neutral leptonic

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[6] For a model of violation of CP invariance in which the decay $K_S \to \mu^+\mu^-$ plays an important role, see M. L. Good, L. Michel, and E. de Rafael, Phys. Rev. **151**, 1194 (1966).

[7] See Ref. 3; also L. M. Sehgal, Nuovo Cimento **45**, 785 (1966); C. Quigg and J. D. Jackson, UCRL Report No. 18487 (unpublished).

currents is briefly discussed, and order-of-magnitude estimates for the decay rates $K_L \to \mu^+\mu^-$ and $K_S \to \mu^+\mu^-$ are given. The use of the unitarity condition as a method to obtain lower bounds for the decay rates of these processes is discussed and the possibility of obtaining empirical bounds is considered.

In the light of the phenomenological discussion given in Sec. II, we review previous calculations of the decay rate $K_L \to \mu^+\mu^-$ in Sec. III.

Section IV is devoted to a calculation of a lower bound to the decay rate $K_S \to \mu^+\mu^-$ within the framework of a perturbation-theory model. The model consists in assuming a pointlike K-π-π weak coupling vertex and minimal electromagnetic interaction of pions and leptons. The contribution to the absorptive part of the $K_S \to \mu^+\mu^-$ amplitude from the intermediate states 2γ, 2π, and $2\pi\gamma$ is calculated within this model. The details of this calculation are given in Secs. IV A, IV C, and IV D, respectively. Possible enhancement effects due to the π-π strong interactions have been incorporated into the perturbation-theory model in an approximate way; i.e., only the π-π interaction in the state $J=0$, $I=0$ is taken into account. As a byproduct of this calculation, we obtain an estimate of the $K_S \to 2\gamma$ decay rate (always within the limitations of the model described above). Our estimate, however, disagrees with previous calculations which were made within a similar perturbation-theory framework. This is discussed in Sec. IV B.

The results obtained and conclusions are summarized in Sec. V. This is done in sufficient detail so that the reader who is not interested in the details of the calculation can omit Secs. IV A, IV C, and IV D.

Some technical details of the spin calculations have been relegated to Appendix A and the evaluation of certain integrals to Appendix B.

II. PHENOMENOLOGY

In $K_L \to \mu^+\mu^-$ and $K_S \to \mu^+\mu^-$ decays, the μ pair has total angular momentum $J=0$. There are two possible states for this system: ${}^3P_0$ and ${}^1S_0$, which are eigenstates of CP, with

$$CP|(\mu^+\mu^-)_{^3P_0}\rangle = +|(\mu^+\mu^-)_{^3P_0}\rangle ,$$
$$CP|(\mu^+\mu^-)_{^1S_0}\rangle = -|(\mu^+\mu^-)_{^1S_0}\rangle .$$

This means that, *at the limit of CP conservation* [i.e., $K_L \equiv K_2{}^0 = (K^0 + \bar{K}^0)/\sqrt{2}$ and $K_S \equiv K_1{}^0 = (K^0 - \bar{K}^0)/\sqrt{2}$],

$$K_2{}^0 \to (\mu^+\mu^-)_{^1S_0} \quad \text{and} \quad K_1{}^0 \to (\mu^+\mu^-)_{^3P_0}$$

are allowed transitions, whereas

$$K_2{}^0 \to (\mu^+\mu^-)_{^3P_0} \quad \text{and} \quad K_1{}^0 \to (\mu^+\mu^-)_{^1S_0}$$

are forbidden by CP invariance. If, furthermore, by analogy to the universal current-current interaction, we postulate a coupling of neutral lepton pairs to hadrons to the type[8]

$$g(J(x)_3{}^2)^{\mu}\bar{\psi}\gamma_\mu(1-\gamma_5)\psi + \text{H.c.}, \tag{2.1}$$

then, because of the $V-A$ structure of the leptonic current, only transitions to the $(\mu^+\mu^-)_{^1S_0}$ state are allowed. If the coupling constant g is real, we have that $K_2{}^0 \to \mu^+\mu^-$ is allowed and $K_1{}^0 \to \mu^+\mu^-$ is forbidden. From the experimental limits quoted above, and using the effective Hamiltonian defined in Eq. (2.1), one can set upper limits to the coupling constant g[9]:

$$\begin{aligned} g_\mu &< 1.3\times10^{-4} \sin\theta\, G/\sqrt{2}, \\ g_e &< 7.2\times10^{-2} \sin\theta\, G/\sqrt{2}. \end{aligned} \tag{2.2}$$

Here θ is the Cabibbo angle for the axial-vector current, and G is the Fermi coupling constant. The figures in these upper limits give the suppression of the couplings of neutral leptonic currents to hadrons as compared to the corresponding couplings of charged currents.

Regardless of the possible existence of neutral leptonic currents, the most general expression for the decay amplitude of the process $K^0 \to \mu^+\mu^-$ is as follows:

$$A[K^0 \to \mu^+(p')\mu^-(p)] = i\bar{u}(p)[F_1 + \gamma_5 F_2]v(p'). \tag{2.3a}$$

The scalar term (F_1) leads to transitions to the ${}^3P_0$ state; the pseudoscalar (F_2) to the ${}^1S_0$ state. Assuming CP invariance, the amplitude for the process $K^0 \to \mu^+\mu^-$ is then

$$A[\bar{K}^0 \to \mu^+(p')\mu^-(p)] = -i\bar{u}(p)[F_1 - \gamma_5 F_2]v(p'), \tag{2.3b}$$

and we have, with M the neutral kaon mass,

$$\Gamma(K_1{}^0 \to \mu^+\mu^-) = \frac{M}{4\pi}\left(1 - \frac{4m_\mu{}^2}{M^2}\right)^{3/2} |F_1|^2, \tag{2.4a}$$

$$\Gamma(K_2{}^0 \to \mu^+\mu^-) = \frac{M}{4\pi}\left(1 - \frac{4m_\mu{}^2}{M^2}\right)^{1/2} |F_2|^2. \tag{2.4b}$$

Next, we assume that these decays proceed via first-order weak times fourth-order electromagnetic interactions. The corresponding mechanisms are then described by the Feynman graphs shown in Fig. 1. Let us first write down the couplings corresponding to $K_1{}^0 \to \gamma\gamma$ and $K_2{}^0 \to \gamma\gamma$ decays. With

$$F_{\mu\nu} = \epsilon_\mu k_\nu - \epsilon_\nu k_\mu, \quad F_{\mu\nu}' = \epsilon_\mu' k_\nu' - \epsilon_\nu' k_\mu'$$

(k and k' are the energy-momenta of the γ's; ϵ and ϵ' are their polarizations), the decay amplitude for $K_1{}^0 \to \gamma\gamma$ is given by the scalar coupling

$$A(K_1{}^0 \to \gamma\gamma) = (H_1/M) F_{\mu\nu} F'^{\mu\nu}, \tag{2.5a}$$

and for $K_2{}^0 \to \gamma\gamma$ by the pseudoscalar coupling

$$A(K_2{}^0 \to \gamma\gamma) = (H_2/M) \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F'^{\rho\sigma}. \tag{2.5b}$$

The decay amplitudes for $K_1{}^0 \to \mu^+\mu^-$ and $K_2{}^0 \to \mu^+\mu^-$, as described in Fig. 1, are then given by the following

[8] See, e.g., Good, Michel, and de Rafael, Ref. 6.
[9] E. de Rafael, Phys. Rev. **157**, 1486 (1967).

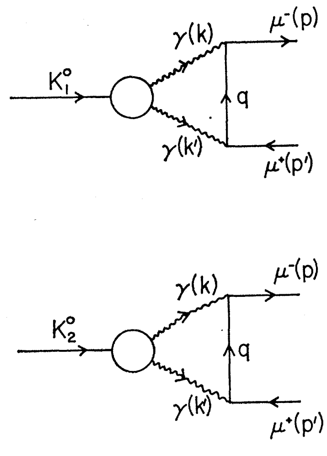

FIG. 1. Feynman diagrams describing $K_1^0 \to \mu^+\mu^-$ and $K_2^0 \to \mu^+\mu^-$ decays.

expressions:

$$A[K_{1,2}^0 \to \mu^+(p')\mu^-(p)] = \frac{1}{M}\int \frac{d^4q}{(2\pi)^4} H_{1,2}[M^2,(p-q)^2,(p'+q)^2] \times \frac{-i}{(p-q)^2+i\epsilon}\frac{-i}{(p'-q)^2+i\epsilon}(-ie)^2 T_{1,2}{}^{\mu\nu}\bar{u}(p)\gamma_\mu \times \frac{i(\not{q}+m_\mu)}{q^2-m^2+i\epsilon}\gamma_\nu v(p'), \quad (2.6)$$

with

$$T_1{}^{\mu\nu} = 2[(p-q)\cdot(p'+q)g^{\mu\nu}-(p-q)^\mu(p'+q)^\nu] \quad (2.7a)$$

and

$$T_2{}^{\mu\nu} = 4\epsilon^{\mu\nu\rho\sigma}(p-q)_\rho(p'+q)_\sigma. \quad (2.7b)$$

By comparison of these equations with the general expressions given in Eqs. (2.3a), and (2.3b), it can be seen that the form factors F_1 and F_2 are proportional to the lepton mass.[10] Thus, as an order-of-magnitude estimate, we expect

$$F_1 \sim F_2 \sim \frac{GM^2}{\sqrt{2}}\sin\theta \frac{\alpha^2}{\pi}\frac{m_\mu}{M}, \quad (2.8)$$

which corresponds to the following branching ratios:

$$\frac{\Gamma(K_1^0 \to \mu^+\mu^-)}{(K_1^0 \to \text{all})} \sim 3\times 10^{-11}, \qquad \frac{\Gamma(K_2^0 \to \mu^+\mu^-)}{(K_2^0 \to \text{all})} \sim 2\times 10^{-8}. \quad (2.9)$$

[10] This can be readily seen from Eq. (2.6) by inspection of the possible terms which are left after the d^4q integration and application of the Dirac equation. Another way is by use of the projectors on the triplet and singlet states of the $\mu^+\mu^-$ system given in Eqs. (A15) and (A16) of Appendix A. The amplitudes $A(K_1^0 \to \mu^+\mu^-)$ and $A(K_2^0 \to \mu^+\mu^-)$ are proportional to respectively, $\mathrm{tr}P_{\mathrm{out}}{}^{(1)}(p,p')\gamma_\mu i(\not{q}+m_\mu)\gamma_\nu$ and $\mathrm{tr}P_{\mathrm{out}}{}^{(0)}(p,p')\gamma_\mu i(\not{q}+m_\mu)\gamma_\nu$. Clearly, only terms proportional to m_μ survive after the trace operation.

Since $|F_i| \geq |\mathrm{Abs}F_i|$, $i=1, 2$, it is clear from Eqs. (2.4a) and (2.4b) that knowledge of the absorptive parts of the form factors F_1 and F_2 will give us lower bounds to the $K_1^0 \to \mu^+\mu^-$ and $K_2^0 \to \mu^+\mu^-$ decay rates. By use of the unitarity condition, the quantities $\mathrm{Abs}F_i$ can be expressed as follows[11]:

$$\mathrm{Abs}F_1 = \frac{-i}{2[2(M^2-4m_\mu^2)]^{1/2}}(2\pi)^4 \sum_\lambda \int d\rho_\lambda \times \delta^{(4)}(p+p'-\sum p_\lambda)\langle\lambda|T|(\mu^+\mu^-)_{^3P_0}\rangle^* \times A(K^0 \to \lambda), \quad (2.10a)$$

$$\mathrm{Abs}F_2 = \frac{-i}{2[2(M^2-4m_\mu^2)]^{1/2}}(2\pi)^4 \sum_\lambda \int d\rho_\lambda \times \delta^{(4)}(p+p'-\sum p_\lambda)\langle\lambda|T|(\mu^+\mu^-)_{^1S_0}\rangle^* \times A(K^0 \to \lambda), \quad (2.10b)$$

where the summations are extended over all possible intermediate states $|\lambda\rangle$, allowed by phase space, which are in the same invariant subspace of the strong and electromagnetic interactions S matrix as $|(\mu^+\mu^-)_{^3P_0}\rangle$ in the case of $\mathrm{Abs}F_1$ and as $|(\mu^+\mu^-)_{^1S_0}\rangle$ in the case of $\mathrm{Abs}F_2$; $d\rho_\lambda$ denotes the phase-space volume element corresponding to $|\lambda\rangle$. By inspection it can be seen that, to order Ge^4, the possible intermediate states in Eq. (2.10a) are: 2γ in a $CP=+1$ state, 2π; $2\pi\gamma$, where the γ can be a bremsstrahlung γ ray; and $3\pi\gamma$, where the γ has to be emitted directly at the K-3π interaction box. Correspondingly in Eq. (2.10b), the possible intermediate states to order Ge^4 are: 2γ in a $CP=-1$ state; $2\pi\gamma$, where the γ is emitted directly at the K-2π vertex; and 3π. Clearly, there is not much hope of getting a rigorous estimate of all these contributions since their calculation involves a detailed knowledge of weak and electromagnetic interactions of hadrons. One can, however, try to separate those terms which can be calculated and combine the rest into observables which in principle can be obtained from experiment. More specifically, we suggest the following procedure.

[11] Equations (2.10) depend critically on the assumption of CP invariance in $K \to \mu^+\mu^-$ decays. In general, with

$$A[K^0 \to \mu^+(p')\mu^-(p)] = i\bar{u}(p)[F_1+\gamma_5 F_2]v(p')$$

and $A[\bar{K}^0 \to \mu^+(p')\mu^-(p)] = i\bar{u}(p)[G_1+\gamma_5 G_2]v(p')$ and *under the assumption of CPT invariance alone*, we have

$$G_1^* + F_1 = \frac{1}{2[2(M^2-4m_\mu^2)]^{1/2}}(2\pi)^4 \sum_\lambda \int d\rho_\lambda\, \delta^{(4)}(p+p'-\sum p_\lambda) \times \langle\lambda|T|(\mu^+\mu^-)_{3P_0}\rangle^* A(K^0 \to \lambda),$$

and correspondingly for the combination $-G_2^*+F_2$. Under the assumption of CP invariance, $G_1=-F_1$, $G_2=F_2$, and Eqs. (2.10) follow. For a general discussion of CPT invariance and unitarity in the K^0-$\bar{K}^0$ system, see J. S. Bell and J. Steinberger, in *Proceedings of the Oxford International Conference on Elementary Particles, 1965* (Rutherford High-Energy Laboratory, Chilton, Berkshire, England, 1966), pp. 195–222. See also N. Beyers, S. W. MacDowell, and C. N. Yang in *Proceedings of the Seminar on High Energy Physics and Elementary Particles* (IAEA, Vienna, 1965), pp. 955–980.

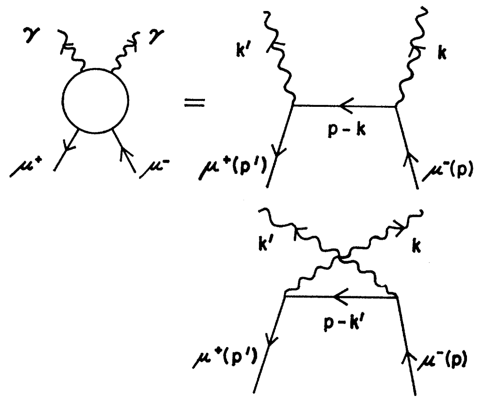

FIG. 2. Feynman diagrams describing $\mu^+\mu^-\to 2\gamma$ transitions contributing to the amplitude $\langle(2\gamma)_1|T|(\mu^+\mu^-)_{^3P_0}\rangle$.

Let us call $\Omega_{1,2}$ the contributions to the right-hand side in Eqs. (2.10) from all possible intermediate states other than the 2γ states. The contribution from the 2γ state can be explicitly calculated and is proportional to $\mathrm{Re}H_{1,2}(M^2)$. Notice that the terms $\langle 2\gamma|T|(\mu^+\mu^-)_{^3P_0}\rangle$ and $\langle 2\gamma|T|(\mu^+\mu^-)_{^1S_0}\rangle$ in Eqs. (2.10) are only needed in the Born approximation (see Fig. 2). Thus, with

$$\beta_\mu=(1-4m_\mu^2/M^2)^{1/2}, \tag{2.11}$$

we get

$$\mathrm{Abs}F_{1,2}=\mathrm{Re}H_{1,2}(M^2)\frac{1}{\sqrt{2}}\alpha\frac{m_\mu}{M}\frac{1}{\beta_\mu}\ln\frac{1+\beta_\mu}{1-\beta_\mu}+\Omega_{1,2}. \tag{2.12}$$

The quantities $\Omega_{1,2}$ can be bounded in terms of cross sections and decay rates by use of the Schwartz inequality:

$$|\Omega_{1,2}|\leq\frac{1}{2\beta_\mu}[\beta_\mu M^2\sigma_{1,2}(M^2)]^{1/2}\left(\frac{1}{M}\Gamma_{1,2}\right)^{1/2}\equiv\tilde{\Omega}_{1,2}. \tag{2.13}$$

Here, $\sigma_1(M^2)$ and $\sigma_2(M^2)$ denote the total cross sections for $(\mu^+\mu^-)_{^3P_0}$ and $(\mu^+\mu^-)_{^1S_0}$, respectively, to all final states other than 2γ at a c.m. energy equal to the K^0 mass; and Γ_1 and Γ_2 are the K_1^0 and K_2^0 decay rates into all final states except 2γ. Then, *provided that*

$$\tilde{\Omega}_{1,2}<\frac{1}{\sqrt{2}}\alpha\frac{m_\mu}{M}\frac{1}{\beta_\mu}\ln\left(\frac{1+\beta_\mu}{1-\beta_\mu}\right)|\mathrm{Re}H_{1,2}(M^2)|, \tag{2.14}$$

we have a lower bound for the decay rates $K_1^0\to\mu^+\mu^-$ and $K_2^0\to\mu^+\mu^-$, since then

$$|\mathrm{Abs}F_{1,2}|\geq\frac{1}{\sqrt{2}}\alpha\frac{m_\mu}{M}\frac{1}{\beta_\mu}\ln\left(\frac{1+\beta_\mu}{1-\beta_\mu}\right)\times|\mathrm{Re}H_{1,2}(M^2)|-\tilde{\Omega}_1. \tag{2.15}$$

The quantities $\mathrm{Re}H_{1,2}(M^2)$ in Eq. (2.15) are related to the $K_1^0\to 2\gamma$ and $K_2^0\to 2\gamma$ decay rates as follows:

$$\Gamma(K_1^0\to 2\gamma)=(M/16\pi)\{[\mathrm{Re}H_1(M^2)]^2+[\mathrm{Im}H_1(M^2)]^2\}, \tag{2.16a}$$

$$\Gamma(K_2^0\to 2\gamma)=(M/4\pi)\{[\mathrm{Re}H_2(M^2)]^2+[\mathrm{Im}H_2(M^2)]^2\}. \tag{2.16b}$$

Again, we can obtain information on $\mathrm{Im}H_{1,2}(M^2)$ from their corresponding unitarity conditions. To order Ge^2, $\mathrm{Im}H_1(M^2)$ can be estimated by saturation of the corresponding unitarity sum with the contribution from the 2π intermediate state (see Figs. 3 and 4). The result of this calculation, which is reported in detail in Sec. IV A, is

$$\mathrm{Im}H_1(M^2)=\alpha\frac{1}{2\sqrt{6}}\frac{1}{M}[\sqrt{2}\eta_0A_0+\mathrm{Re}A_2]\times(1-\beta_\pi^2)\ln\frac{1+\beta_\pi}{1-\beta_\pi}, \tag{2.17}$$

where A_0 and A_2 are the transition amplitudes between K^0 and two pions in an isospin state $I=0$, and 2, respectively; η_0 is an enhancement factor due to the strong interactions of the 2π system in the isospin state $I=0$; and β_π is the pion velocity in the c.m. system, i.e.,

$$\beta_\pi=(1-4m_\pi^2/M^2)^{1/2}. \tag{2.18}$$

It is harder to make a similiar estimate of $\mathrm{Im}H_2(M^2)$. This quantity, however, can be bounded applying again the Schwartz inequality to the corresponding unitarity sum. Thus we have

$$|\mathrm{Im}H_2(M^2)|^2\leq\tfrac{1}{8}[\Gamma(K_2^0\to\mathrm{all})/M]\times M^2\sigma_{J=0,P=-1}{}^{(\gamma\gamma)}\equiv\tilde{H}_2, \tag{2.19}$$

where $\sigma_{J=0,P=-1}{}^{(\gamma\gamma)}(M^2)$ denotes the total $\gamma\gamma$ cross section to states with total angular momentum $J=0$ and negative parity. In $K_2^0\to 2\gamma$ decays, we expect the dominant contribution to the decay rate to come from $\mathrm{Re}H_2(M^2)$ (in particular, from the π^0 and η pole diagrams). Then, *provided that*

$$\tilde{H}_2<(4\pi/M)\Gamma(K_2^0\to 2\gamma), \tag{2.20}$$

we have

$$|\mathrm{Re}H_2(M^2)|^2\geq 4\pi\Gamma(K_2^0\to 2\gamma)/M-\tilde{H}_2. \tag{2.21}$$

To summarize, let us state the conditions for the existence of semiempirical lower bounds to the $K_{1,2}^0\to\mu^+\mu^-$ decay rates.

(i) For $K_1^0\to\mu^+\mu^-$, *provided that*

$$\tilde{\Omega}_1<\frac{1}{\sqrt{2}}\alpha\frac{m_\mu}{M}\frac{1}{\beta_\mu}\ln\frac{1+\beta_\mu}{1-\beta_\mu}\times\left(16\pi\frac{1}{M}\Gamma(K_1^0\to 2\gamma)-[\mathrm{Im}H_1(M^2)]^2\right)^{1/2}, \tag{2.22}$$

where $\tilde{\Omega}_1$ is defined in Eq. (2.13) and $\mathrm{Im}H_1(M^2)$ in Eq.

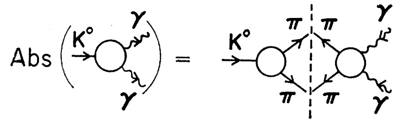

FIG. 3. Unitarity diagram corresponding to the 2π contribution to $\mathrm{Im}H_1(t)$ in Eq. (4.8).

(2.17), then

$$\Gamma(K_1^0\to\mu^+\mu^-)\geq\frac{M}{4\pi}\beta_\mu^3\times\left\{\frac{1}{\sqrt{2}}\alpha\frac{m_\mu}{M}\frac{1}{\beta_\mu}\ln\frac{1+\beta_\mu}{1-\beta_\mu}\left[16\pi\frac{1}{M}\Gamma(K_1^0\to 2\gamma)-\frac{\alpha^2}{24}\frac{1}{M^2}[\sqrt{2}\eta_0A_0+\mathrm{Re}A_2]^2(1-\beta_\pi^2)^2\ln^2\left(\frac{1+\beta_\pi}{1-\beta_\pi}\right)\right]^{1/2}-\frac{1}{2\beta_\mu}[\beta_\mu M^2\sigma_1(M^2)]^{1/2}\left(\frac{1}{M}\Gamma_1\right)^{1/2}\right\}^2. \quad (2.23)$$

(ii) For $K_2^0\to\mu^+\mu^-$, *provided that*

$$M^2\sigma_{J=0,P=-1}{}^{(\gamma\gamma)}(M^2)<32\pi\frac{\Gamma(K_2^0\to 2\gamma)}{\Gamma(K_2^0\to \mathrm{all})}, \quad (2.24)$$

and

$$\tilde{\Omega}_2<\frac{1}{\sqrt{2}}\alpha\frac{m_\mu}{M}\frac{1}{\beta_\mu}\ln\frac{1+\beta_\mu}{1-\beta_\mu}\times\left(4\pi\frac{1}{M}\Gamma(K_2^0\to 2\gamma)-\tilde{H}_2\right)^{1/2}, \quad (2.25)$$

where $\tilde{\Omega}_2$ is defined in Eq. (2.13) and $\tilde{H}_2$ in Eq. (2.19), then

$$\Gamma(K_2^0\to\mu^+\mu^-)\geq\frac{M}{4\pi}\beta_\mu\times\left[\frac{1}{\sqrt{2}}\alpha\frac{m_\mu}{M}\frac{1}{\beta_\mu}\ln\frac{1+\beta_\mu}{1-\beta_\mu}\left(4\pi\frac{1}{M}\Gamma(K_2^0\to 2\gamma)-\frac{1}{8}\frac{\Gamma(K_2^0\to\mathrm{all})}{M}M^2\sigma_{J=0,P=-1}{}^{(\gamma\gamma)}(M^2)\right)^{1/2}-\frac{1}{2\beta_\mu}[\beta_\mu M^2\sigma_2(M^2)]^{1/2}\left(\frac{1}{M}\Gamma_2\right)^{1/2}\right]^2. \quad (2.26)$$

Equations (2.23) and (2.26) are inequalities between observable quantities. Their usefulness is obviously limited by the fact that they involve quantities like $\sigma_1(M^2)$, $\sigma_2(M^2)$, and $\sigma_{J=0,P=-1}{}^{(\gamma\gamma)}(M^2)$, which are far from being measurable at present. However, they provide consistency conditions which might become interesting in the future.

In order to obtain a "numerical" estimate of the right-hand sides of Eqs. (2.23) and (2.26), one is clearly forced to make approximations and then find their justification within some model. In Sec. III we discuss such a possibility for the $K_2^0\to\mu^+\mu^-$ case, and in Secs. IV and V for the $K_1^0\to\mu^+\mu^-$ case.

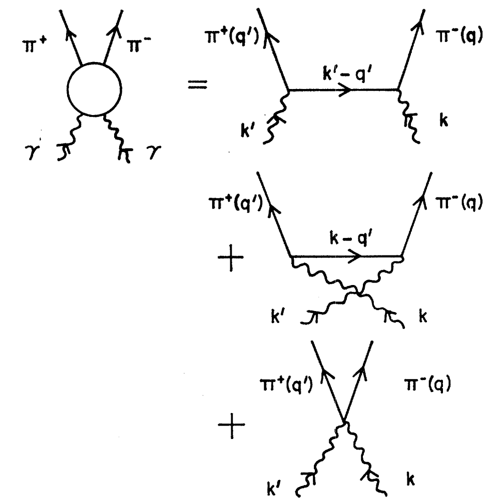

FIG. 4. Feynman diagrams describing $2\gamma\to\pi^+\pi^-$ transitions contributing to the amplitude $\langle(\pi\pi)_I|T|(2\gamma)_1\rangle$.

III. $K_2^0\to\mu^+\mu^-$ DECAY

Neglecting the last two terms in the right-hand side of Eq. (2.26), one gets the following prediction:

$$\frac{\Gamma(K_2^0\to\mu^+\mu^-)}{\Gamma(K_2^0\to\gamma\gamma)}\geq\alpha^2\left(\frac{m_\mu}{M}\right)^2\frac{1}{2\beta_\mu}\ln^2\left(\frac{1+\beta_\mu}{1-\beta_\mu}\right)=1.2\times10^{-5}. \quad (3.1)$$

This result has been previously obtained by Quigg and Jackson,[12,13] who refer to it as the unitarity bound.

There are two recent measurements[14] of the total branching ratio for the mode $K_L\to 2\gamma$:

$$\frac{\Gamma(K_L\to 2\gamma)}{\Gamma(K_L\to\mathrm{all})}=(4.68\pm0.65)\times10^{-4} \quad \text{(Banner *et al.*)} \quad (3.2a)$$

$$=(5.3\pm1.5)\times10^{-4} \quad \text{(Arnold *et al.*)}. \quad (3.2b)$$

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[12] See Quigg and Jackson, Ref. 7. A similar prediction for η decay was first obtained by Geffen and Young (D. A. Geffen and B. L. Young, Phys. Rev. Letters **15**, 316 (1965)] and rediscovered by Callan and Treiman [C. G. Callen, Jr., and S. B. Treiman, *ibid.* **18**, 1083 (1967); **19**, 57(E) (1967)]. We should like to emphasize that the approximations involved in the case of η decay are far more rigorous than in K_2^0 decay. Equation (3.1) is also implicitly contained in a paper by Sehgal (see L. M. Sehgal, Ref. 7), and is explicitly contained in another paper by Sehgal (see L. M. Sehgal, Ref. 13).

[13] L. M. Sehgal, Phys. Rev. **183**, 1511 (1969).

[14] M. Banner, J. W. Cronin, J. K. Liu, and J. E. Pilcher, Phys. Rev. Letters **21**, 1103 (1968); R. Arnold, I. A. Budakov, D. C. Cundy, G. Myatt, F. Nezrick, G. H. Trilling, W. Venus, H. Yoshiki, B. Aubert, P. Heusse, E. Nagy, and C. Pascaud, Phys. Letters **28B**, 56 (1968).

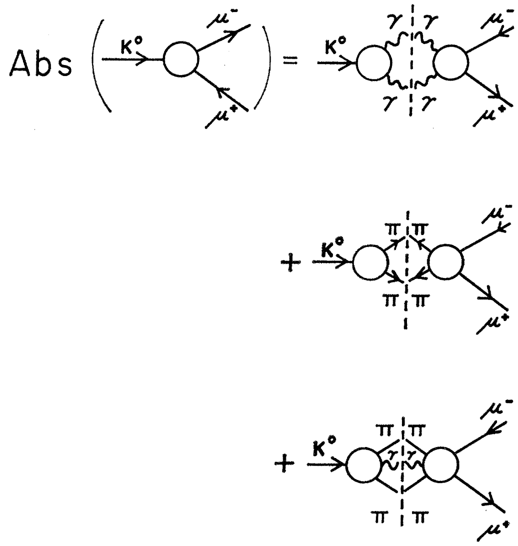

FIG. 5. Unitarity diagrams corresponding to the 2γ, 2π, and $2\pi\gamma$ contributions to Abs F_1 in Eq. (2.10a).

Taking a weighted average of these numbers, we obtain from Eq. (3.1) the following lower limit:

$$\frac{\Gamma(K_2{}^0\to\mu^+\mu^-)}{\Gamma(K_2{}^0\to\text{all})}\geq 6\times10^{-9}, \tag{3.3}$$

to be compared with the order-of-magnitude estimate given in Eq. (2.9).

Equation (3.1) has been obtained assuming that to order Ge^4, only the 2γ intermediate state gives a significant contribution to the unitarity sum in Eq. (2.10b), Also, it assumes that $\mathrm{Im}H_2(M^2)$ is negligible, i.e.,

$$\Gamma(K_2{}^0\to2\gamma)\simeq(M/16\pi)[\mathrm{Re}H_2(M^2)]^2.$$

Here, we would like to present some qualitative arguments which add credibility to these assumptions. We shall discuss separately the contributions to $\mathrm{Abs}F_2$ in Eq. (2.10b) from the $2\pi\gamma$, and the 3π intermediate states; and comment on the $K_2{}^0\to2\gamma$ decay rate.

A. $2\pi\gamma$ Intermediate State

Let us call $\Omega_2(2\pi\gamma)$ the corresponding contribution to $\mathrm{Abs}F_2$. We have [see Eq. (2.13)]

$$|\Omega_2(2\pi\gamma)|\leq\frac{1}{2\beta_\mu}\{\beta_\mu M^2[\sigma(\mu^+\mu^-)_{^1S_0}\to\pi\pi\gamma]\}^{1/2}\times\left(\frac{1}{M}\Gamma(K_2{}^0\to\pi\pi\gamma)\right)^{1/2}. \tag{3.4}$$

Experimentally,[15]

$$\frac{\Gamma(K_2{}^0\to\pi\pi\gamma)}{\Gamma(K_2{}^0\to\text{all})}<4\times10^{-4}, \tag{3.5}$$

which is a smaller branching ratio than the 2γ mode. A very crude order-of-magnitude estimate of $\sigma[(\mu^+\mu^-)_{^1S_0}\to\pi\pi\gamma]$ gives

$$\sigma[(\mu^+\mu^-)_{^1S_0}\to\pi\pi\gamma]\sim(\alpha/\pi)\sigma[(\mu^+\mu^-)_{^1S_0}\to2\gamma]; \tag{3.6}$$

in fact, this might very well be an overestimate, since the phase space for $\pi\pi\gamma$ is smaller than for $\gamma\gamma$. The cross section for $(\mu^+\mu^-)_{^1S_0}\to2\gamma$ can be calculated from quantum electrodynamics:

$$\sigma[(\mu^+\mu^-)_{^1S_0}\to2\gamma]=16\pi\alpha^2\frac{1}{M^2\beta_\mu}\left(\frac{m_\mu}{M}\right)^2\ln^2\left(\frac{1+\beta_\mu}{1-\beta_\mu}\right). \tag{3.7}$$

Then, our estimate of $\Omega_2(\pi\pi\gamma)$ is

$$|\Omega_2(\pi\pi\gamma)|\lesssim9\times10^{-14}. \tag{3.8}$$

B. 3π Intermediate State

We call $\Omega_2(3\pi)$ the 3π contribution to $\mathrm{Abs}F_2$ in Eq. (2.10b). Again, we have

$$\Omega_2(3\pi)\leq\frac{1}{2\beta_\mu}\{\beta_\mu M^2\sigma[(\mu^+\mu^-)_{^1S_0}\to3\pi]\}^{1/2}\times\left(\frac{1}{M}\Gamma(K_2{}^0\to3\pi)\right)^{1/2}. \tag{3.9}$$

Experimentally,[16]

$$\frac{\Gamma(K_L\to3\pi)}{\Gamma(K_L\to\text{all})}\sim34\%. \tag{3.10}$$

Unless there are particular enhancements in the process $(\mu^+\mu^-)_{^1S_0}\to3\pi$, we expect from an order-of-magnitude estimate

$$\sigma[(\mu^+\mu^-)_{^1S_0}\to3\pi]\sim(\alpha/\pi)^2\times0.10\sigma[(\mu^+\mu^-)_{^1S_0}\to2\gamma], \tag{3.11}$$

where the factor 0.10 is the $3\pi/2\gamma$ dimensionless phase-space ratio.[17] Thus, we have as a crude estimate of $\Omega_2(3\pi)$

$$|\Omega_2(3\pi)|\lesssim4\times10^{-14}. \tag{3.12}$$

C. Comment on $K_2{}^0\to2\gamma$ Rate

With the coupling defined in Eq. (2.5b) we have that [see Eq. (2.16b)]

$$\Gamma(K_2{}^0\to2\gamma)=(M/4\pi)|H_2(M^2)|^2.$$

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[15] R. C. Thatcher, A. Abashian, R. J. Adams, D. W. Carpenter, R. E. Mischke, B. M. K. Nefkens, J. H. Smith, L. J. Verhey, and A. Wattenburg, Phys. Rev. **174**, 1674 (1968).

[16] See. e.g., A. H. Rosenfeld, N. Barash-Schmidt, A. Barbaro-Galtieri, L. R. Price, P. Söding, C. G. Wohl, M. Roos, and W. W. Willis, Rev. Mod. Phys. **41**, 109 (1969).

[17] That is, the ratio

$$\frac{1}{M^2}\int\prod_{i=1}^{3}dp_i\,\theta(p_i)\delta(p_i{}^2-m_\pi{}^2)\delta^{(4)}(p-\sum p_i)$$

over

$$\int dp_1dp_2\,\theta(p_1)\delta(p_1{}^2)\theta(p_2)\delta(p_2{}^2)\delta^{(4)}(p-p_1-p_2).$$

Here we want to bound the imaginary part of $H_2(M^2)$. The expression for this bound has already been given in Eq. (2.19). It involves the unknown factor $\sigma_{J=0,P=-1}{}^{(\gamma\gamma)}(M^2)$. We expect that the major contribution to this cross section comes from the $\gamma\gamma\to 3\pi$ transition. Also, it seems reasonable to assume that

$$\sigma_{J=0}{}^{\gamma\gamma\to 3\pi}(M^2)\lesssim\sigma_{J=0}{}^{\gamma\gamma\to 2\pi}(M^2)\,; \qquad (3.13)$$

we estimate the latter cross section assuming minimal electromagnetic coupling for pions. Within this model,

$$\sigma_{J=0}{}^{\gamma\gamma\to 2\pi}(M^2)=32\pi\alpha^2\frac{1}{M^2\beta_\pi}\left(\frac{m_\pi}{M}\right)^4\ln^2\left(\frac{1+\beta_\pi}{1-\beta_\pi}\right), \qquad (3.14)$$

and the condition stated in Eq. (2.24) is largely satisfied. Also the condition stated in Eq. (2.25) is satisfied, i.e., from Eqs. (3.8) and (3.12) we have the $\tilde{\Omega}_2\sim 1.3\times 10^{-13}$, while the right-hand side of Eq. (2.25) amounts to 1.4×10^{-12}.

We can now reconsider Eq. (2.26). Using the order-of-magnitude estimate which we have made above, we find

$$\frac{\Gamma(K_2{}^0\to\mu^+\mu^-)}{\Gamma(K_2{}^0\to 2\gamma)}\geq\alpha^2\left(\frac{m_\mu}{M}\right)^2\frac{1}{2\beta_\mu}\times\ln^2\left(\frac{1+\beta_\mu}{1-\beta_\mu}\right)(1-0.2)\,; \qquad (3.15)$$

i.e., we estimate that the corrections to the lower bound given in Eq. (3.1) could be as large as 20%.

IV. $K_1{}^0\to\mu^+\mu^-$ DECAY CALCULATIONS

In this case, the lower bound analogous to the one derived in Eq. (3.1) for $K_2{}^0\to\mu^+\mu^-$ would be obtained neglecting the last two terms of Eq. (2.23). Thus,

$$\frac{\Gamma(K_1{}^0\to\mu^+\mu^-)}{\Gamma(K_1{}^0\to 2\gamma)}\geq\alpha^2\left(\frac{m_\mu}{M}\right)^2 2\beta_\mu\ln^2\left(\frac{1+\beta_\mu}{1-\beta_\mu}\right). \qquad (4.1)$$

This "bound" has been obtained assuming that to order Ge^4 the unitarity sum in Eq. (2.10a) is dominated by the 2γ intermediate state.[18] Then, one simply has[19]

$$\mathrm{Abs}F_1(M^2)=\frac{\alpha}{\sqrt{2}}\frac{m_\mu}{M}\,\mathrm{Re}H_1(M^2)\frac{1}{\beta_\mu}\ln\frac{1+\beta_\mu}{1-\beta_\mu}.$$

In deriving Eq. (4.1) it is assumed, furthermore, that

$$\Gamma(K_1{}^0\to 2\gamma)\simeq\frac{M}{16\pi}[\mathrm{Re}H_1(M^2)]^2,$$

i.e., that $\mathrm{Im}H_1(M^2)$ is negligible compared to $\mathrm{Re}H_1(M^2)$. It is known, however, that the 2π intermediate state gives contributions of order Ge^2 to $\mathrm{Im}H_1(M^2)$, so this assumption is not really justified.[20] The same 2π intermediate state also gives a contribution of order Ge^4 to $\mathrm{Abs}F_1(M^2)$ which might be comparable to that from the 2γ state. It seems to us that Eq. (4.1) can be a very misleading bound, and more work on the possible contributions from other intermediate states than 2γ is needed in this case.

This section consists of the details of a perturbation-theory calculation of $\mathrm{Abs}F_1(M^2)$ in Eq. (2.10a). We shall assume a pointlike K-π-π weak interaction and minimal electromagnetic interaction of pions and leptons. The details of calculations corresponding to contributions to $\mathrm{Abs}F_1(M^2)$ from the intermediate states 2γ, 2π, and $2\pi\gamma$ (see Fig. 5) are reported in different subsections. The connection with the phenomenological discussion given in Sec. II is made in Sec. V.

A. Contribution from Two-Photon Intermediate State

The $|2\gamma\rangle$ intermediate state is in an eigenstate of CP

$$CP|(2\gamma)_1\rangle=+|(2\gamma)_1\rangle,$$

where, in terms of helicity states,

$$|(2\gamma)_1\rangle=(1/\sqrt{2})(|++\rangle+|--\rangle).$$

The T matrix $\langle(2\gamma)_1|T|(\mu^+\mu^-)_{^3P_0}\rangle$, which appears on the right-hand side in Eq. (2.10a), can easily be obtained from the Feynman diagrams corresponding to $\mu^+\mu^-$ annihilation into 2γ's (see Fig. 2). More precisely,

$$\langle(2\gamma_1)|T|(\mu^+\mu^-)_{^3P_0}\rangle^*=(1/\sqrt{2})\times[T_0{}^{(1)}(++)+T_0{}^{(1)}(--)]_{J=0}{}^* \qquad (4.2)$$

and

$$T_0{}^{(1)}(\pm\pm)=(-ie)^2\times\mathrm{Tr}\left\{P^{(1)}(p,p')_{\text{``in''}}\left[\frac{\gamma\cdot\epsilon_\pm{}'i(p-k+m_\mu)\gamma\cdot\epsilon_\pm}{(p-k)^2-m_\mu{}^2+i\epsilon}+\frac{\gamma\cdot\epsilon_\pm i(p-k'+m_\mu)\gamma\cdot\epsilon_\pm{}'}{(p'-k)^2-m_\mu{}^2+i\epsilon}\right]\right\}. \qquad (4.3)$$

$\epsilon_\pm$ and $\epsilon_\pm{}'$ are the polarization vectors of the γ's with momenta k and k', corresponding to helicities + and −. In Eq. (4.2) the subscript $J=0$ denotes that only the components $T_0{}^{(1)}(\pm\pm)$ with total angular momentum $J=0$ contribute to the $(\mu^+\mu^-)\to 2\gamma$ transition. $P^{(1)}(p,p')_{\text{``in''}}$ in Eq. (4.3) is the projector on the $^3P_0$ state of the incoming $\mu^+\mu^-$ system and is evaluated in Appendix A. We find

$$T_0{}^{(1)}(\pm\pm)=\frac{-8}{\sqrt{2}}ie^2\frac{m_\mu}{\sqrt{t}}\frac{\beta_\mu}{1-\beta_\mu{}^2\cos^2\theta}, \qquad (4.4)$$

[18] See Sehgal, Ref. 13. The result obtained by Sehgal does not agree, however, with our Eq. (4.1). The error can be traced to a mistake in Eq. (9), p. 1512, of his paper. One of the authors (JS) would like to thank Dr. Sehgal for correspondence regarding his paper.

[19] See the derivation in Sec. IV A below.

[20] V. Barger, Nuovo Cimento **32**, 128 (1964); B. R. Minart and E. de Rafael, Nucl. Phys. **B8**, 131 (1968).

where $\frac{1}{2}\sqrt{t}$ is the energy of the μ in the c.m. system, θ is the c.m. scattering angle, and β_μ is the velocity of the μ in this system, i.e.,

$$\beta_\mu=(1-4m_\mu^2/t)^{1/2}. \tag{4.5}$$

Notice that in our case $t=M^2$, where M is the K^0 mass. The projection upon the $J=0$ amplitude leads to the result

$$\frac{1}{\sqrt{2}}[T_0^{(1)}(++)+T_0^{(1)}(--)]_{J=0}{}^* = ie^2\frac{4m_\mu}{\sqrt{t}}\ln\frac{1+\beta_\mu}{1-\beta_\mu}. \tag{4.6}$$

The contribution to AbsF_1 in Eq. (2.10a) from the 2γ intermediate states is therefore

$$\text{Abs}F_1^{(2\gamma)}(t)=\frac{\alpha}{\sqrt{2}}\frac{m_\mu}{\sqrt{t}}H_1(t)\frac{1}{\beta_\mu}\ln\frac{1+\beta_\mu}{1-\beta_\mu}. \tag{4.7}$$

Only Re$H_1(t)$ gives a net contribution to Abs$F_1^{(2\gamma)}(t)$. The term $i\,\text{Im}H_1(t)$, which in principle gives an imaginary contribution to Abs$F_1^{(2\gamma)}(t)$, cancels with a corresponding term coming from the 2π intermediate state, as we shall see later. However, to estimate Re$H_1(t)$ we shall need Im$H_1(t)$; the latter is obtained by writing the unitarity condition for the process $K_1^0\to\gamma\gamma$ and saturating the sum over intermediate states with the 2π system only. This is justified at the approximation $[O(Ge^2)]$ that we want to know $H_1(t)$. Thus we have, assuming CP invariance,

$$2\,\text{Im}H_1(t)=\frac{1}{\sqrt{t}}\sum_I(2\pi)^4\int d\rho_I\,\delta(q+q'-p-p') \times\langle(2\pi)_I|T|(2\gamma)_1\rangle^* A(K^0\to(2\pi)_I), \tag{4.8}$$

where the summation is over the 2π system with isospin $I=0$ and 2. The corresponding unitarity diagram is shown in Fig. 3.

If only elastic unitarity is taken into account, the amplitudes $A[K^0\to(2\pi)_I]$ for $I=0$, 2 have the following structure[21]:

$$A[K^0\to(2\pi)_I]=iA_Ie^{i\delta_I},\quad I=0,2 \tag{4.9}$$

where δ_I is the s-wave π-π phase shift for scattering in an isospin state $I=0$, 2 at a total c.m. energy equal to $\sqrt{t}$, i.e., the K^0 mass for on-shell kaons.

The T-matrix term in Eq. (4.8) can be written in the following way:

$$\langle(2\pi)_I|T|(2\gamma)_1\rangle=(1/\sqrt{2})[B_I(++)+B_I(--)]_{J=0}\times\eta_Ie^{i\delta_I}. \tag{4.10}$$

Here $B_I(\pm\pm)$ are the amplitudes for the annihilation of γ's with helicities ±1 and ±1 into two pions in an isospin state I. The subscript $J=0$ means that only the s-wave projections of these amplitudes are taken; and η_I is an enhancement factor due to the strong interactions of the 2π system in the isospin state I.

When the two pions are in a relative s state, we have

$$\eta_0(t)=\exp[\sigma_0(t)],$$

where

$$\sigma_0(t)=\frac{t}{\pi}P\int_{4m_\pi^2}^\infty dt'\frac{\delta_0(t')}{t'(t'-t)} =\frac{t}{\pi}\int_{4m_\pi^2}^\infty dt'\frac{\delta_0(t')-\delta_0(t)}{t'(t'-t)}-\frac{\delta_0(t)}{\pi}\ln\left(\frac{t-4m_\pi^2}{4m_\pi^2}\right).$$

For definiteness we use $\delta_0(t)$ corresponding to the "broad σ" model and take the numerical values for the forward dispersion relation solution of Morgan and Shaw.[22] We set $\delta_0(t)=\frac{1}{2}\pi$ above 1 BeV. Thus we get

$$\eta_0(t=M^2)=1.56. \tag{4.11}$$

Since the s-wave $I=2$, $\pi\pi$ scattering has no known structure, we shall set $\eta_2(t)=1$.

The amplitudes $B_I(\pm\pm)$ are obtained from the Feynman diagrams shown in Fig. 4. Assuming minimal coupling for the pion field, we finally obtain

$$\text{Im}H_1(t)=\frac{\alpha}{2\sqrt{6}}\frac{1}{\sqrt{t}}[\sqrt{2}\eta_0A_0+\text{Re}A_2] \times\frac{4m_\pi^2}{t}\ln\frac{1+\beta_\pi}{1-\beta_\pi}, \tag{4.12}$$

where β_π is the π velocity in the c.m. system,

$$\beta_\pi=(1-4m_\pi^2/t)^{1/2}.$$

Numerically (using $A_0=5.09\times10^{17}$ sec^{-1} and, because $|A_2|\ll A_0$, setting Re$A_2=0$), the value of Eq. (4.12) at $t=M^2$ is

$$\text{Im}H_1(M^2)=1.65\times10^{-9}. \tag{4.13}$$

To estimate Re$H_1(t)$, we write an unsubtracted dispersion relation for the quantity $H_1(t)/\sqrt{t}$, which is the coefficient of the factor $F_{\mu\nu}F'^{\mu\nu}$ in Eq. (2.5a), and we assume that $A_0(t)$ and Re$A_2(t)$ in Eq. (4.12) may be approximated by their on-mass-shell values at $t=M^2$ throughout the integration range. However, the specific t dependence for η_0 given above is used. With these assumptions, we find at $t=M^2$

$$\text{Re}H_1(M^2)=-1.02\times10^{-9}. \tag{4.14}$$

Inserting this result into Eq. (4.7), we conclude that the 2γ intermediate state in Eq. (2.10a) leads to an over-

[21] See, e.g., Bell and Steinberger, Ref. 11.

[22] D. Morgan and G. Shaw, Nucl. Phys. **B10**, 261 (1969).

all contribution

$$\mathrm{Abs}F_1{}^{(2\gamma)}(M^2) = -3.71\times10^{-12} + i\alpha^2\frac{1}{\sqrt{3}}(\sqrt{2}\eta_0 A_0 + \mathrm{Re}A_2)\frac{m_\mu}{M^2}\left(\frac{m_\pi}{M}\right)^2 \times\frac{1}{\beta_\mu}\ln\frac{1+\beta_\mu}{1-\beta_\mu}\ln\frac{1+\beta_\pi}{1-\beta_\pi}. \quad (4.15)$$

As we have already mentioned, the imaginary part in the right-hand side of Eq. (4.15) cancels with a corresponding contribution from the 2π intermediate states.

It is of interest to compare the result obtained in Eq. (4.15) with the purely perturbation-theoretical prediction. The latter is obtained by setting $\eta_0=1$ in Eq. (4.12) and then performing the dispersion integral, i.e.,

$$\mathrm{Re}H_1(M^2) = \frac{\alpha}{\pi}\frac{1}{2M\sqrt{6}}(\sqrt{2}A_0+\mathrm{Re}A_2)P\int_{4m_\pi^2}^{\infty}\frac{dt}{t-M^2} \times\frac{M^2}{t}\frac{4m_\pi^2}{t}\ln\frac{1+(1-4m_\pi^2/t)^{1/2}}{1-(1-4m_\pi^2/t)^{1/2}}$$

$$=\frac{\alpha}{\pi}\frac{1}{2M\sqrt{6}}(\sqrt{2}A_0+\mathrm{Re}A_2) \times\left\{-2+\tfrac{1}{2}(1-\beta_\pi{}^2)\left[\pi^2-\ln^2\left(\frac{1-\beta_\pi}{1+\beta_\pi}\right)\right]\right\}, \quad (4.16)$$

where

$$\beta_\pi=(1-4m_\pi{}^2/M^2)^{1/2}.$$

Numerically,

$$\mathrm{Re}H_1(M^2) = -0.65\times10^{-9} \quad \text{(pert. th.)}. \quad (4.17)$$

From Eqs. (4.7) and (4.16), we get the perturbation-theory result

$$\mathrm{Abs}F_1{}^{(2\gamma)}(M^2) =\frac{\alpha^2}{\pi}\frac{1}{\sqrt{3}}(\sqrt{2}A_0+\mathrm{Re}A_2)\frac{m_\mu}{M^2}\frac{1}{2\beta_\mu}\ln\left(\frac{1+\beta_\mu}{1-\beta_\mu}\right) \times\left\{-1+\tfrac{1}{4}(1-\beta_\pi{}^2)\left[\pi^2-\ln^2\left(\frac{1-\beta_\pi}{1+\beta_\pi}\right)\right]\right\} + i\alpha^2\frac{1}{\sqrt{3}}(\sqrt{2}A_0+\mathrm{Re}A_2)\frac{m_\mu}{M^2}\left(\frac{m_\pi}{M}\right)^2\frac{1}{\beta_\mu} +\ln\left(\frac{1+\beta_\mu}{1-\beta_\mu}\right)\ln\left(\frac{1+\beta_\pi}{1-\beta_\pi}\right). \quad (4.18)$$

Numerically, we get for the real part of the right-hand side of Eq. (4.18)

$$\mathrm{Abs}F_1{}^{(2\gamma)}(M^2) = -2.18\times10^{-12} \quad \text{(pert. th.)}. \quad (4.19)$$

B. $K_1{}^0\to\gamma\gamma$ Decay Rate

It is clear from Eqs. (2.16a), (4.13), and (4.14) that as a byproduct of the calculations discussed in the preceding subsections we also have a prediction of the $K_1{}^0\to\gamma\gamma$ decay rate:

$$\Gamma(K_1{}^0\to2\gamma) = 5.7\times10^4\ \mathrm{sec}^{-1}. \quad (4.20)$$

If instead of Eq. (4.14) we use the perturbation-theory result for $\mathrm{Re}H_1(M^2)$ [i.e., Eq. (4.17)], then

$$\Gamma(K_1{}^0\to2\gamma) = 2.2\times10^4\ \mathrm{sec}^{-1} \quad \text{(pert. th.)}. \quad (4.21)$$

In view of possible future experiments in the 2γ decay mode of the K^0-$\bar{K}^0$ system,[23] it is of interest to compare these rates to the $K_2{}^0\to2\gamma$ rate. Using the experimental values quoted in Eqs. (3.2) we have

$$\frac{\Gamma(K_1{}^0\to2\gamma)}{\Gamma(K_2{}^0\to2\gamma)} = 6.0 \quad (4.22)$$

and

$$\frac{\Gamma(K_1{}^0\to2\gamma)}{\Gamma(K_2{}^0\to2\gamma)} = 2.3 \quad \text{(pert. th.)}. \quad (4.23)$$

It must be emphasized that Eq. (4.22) is a *model-dependent* prediction, and it should be taken more as an indication of how the strong-interaction effects can alter the simple perturbation-theory result given in Eq. (4.23) than as a rigorous prediction.

Our result for $\Gamma(K_1{}^0\to2\gamma)$ disagrees with previous estimates made within a similar perturbation-theory framework.[20,24] In Barger[20] and in Martin and de Rafael[20] the incorrect assumption that the helicity amplitude [i.e., $H_1(t)\times\sqrt{t}$] obeys an unsubtracted dispersion relation was made.[25] The correct dispersion relation is written down in a recent book by Nishijima[24]; however, there appears to be an unfortunate error in the explicit integration.[26]

C. Contribution from Two-Pion Intermediate State

The 2π intermediate states in Eq. (2.10a) can be in isospin states $I=0$ and 2. Their over-all contribution to $\mathrm{Abs}F_1(t)$ is as follows:

$$\mathrm{Abs}F_1{}^{(2\pi)}(t) = \frac{-i}{[2(t-4m_\mu{}^2)]^{1/2}}\frac{\beta_\pi}{16\pi} \times\sum_I\langle(2\pi)_I|T|(\mu^+\mu^-)_{{}^3P_0}\rangle^* A[K^0\to(2\pi)_I], \quad (4.24)$$

[23] For a discussion of possible measurements which have bearing on the question of CP noninvariance in K^0-$\bar{K}^0\to2\gamma$ decays, see L. M. Sehgal and L. Wolfenstein, Phys. Rev. **162**, 1362 (1967); and B. R. Martin and E. de Rafael, Ref. 20.

[24] K. Nishijima, *Fields and Particles* (Benjamin, New York, 1969), pp. 351–359.

[25] Furthermore, in Martin and de Rafael (Ref. 20) the rates for $K_1{}^0\to2\gamma$ should be divided by a factor of 2.

[26] One of the authors (EdeR) wishes to thank Professor F. Yndurain for an enlightening discussion on this point.

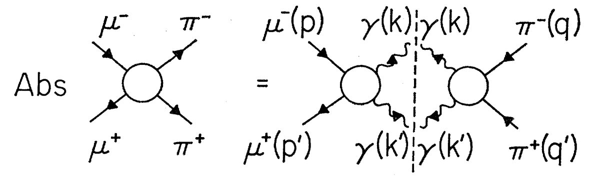

FIG. 6. Unitarity diagram corresponding to the 2γ contribution to $\mathrm{Abs}\langle(2\pi)_I|T|(\mu^+\mu^-)_{^3P_0}\rangle$ in Eq. (4.25).

where $\langle(2\pi)_I|T|(\mu^+\mu^-)_{^3P_0}\rangle$ is the transition amplitude for $\mu^+\mu^-$ annihilation in a $^3P_0$ state to 2π in an isospin state I. The absorptive part of this amplitude can easily be obtained from unitarity by saturation with the 2γ intermediate state only (see Fig. 6), i.e.,

$$\mathrm{Abs}\langle(2\pi)_I|T|(\mu^+\mu^-)_{^3P_0}\rangle=(1/16\pi)\langle(2\gamma)_1|T|(2\pi)_I\rangle^* \times\langle(2\gamma)_1|T|(\mu^+\mu^-)_{^3P_0}\rangle. \quad (4.25)$$

Both amplitudes on the right-hand side have been estimated in Sec. IV A [see Eqs. (4.2), (4.6), and (4.10)]. They were obtained from the Feynman diagrams shown in Figs. 2 and 4. We shall recall that

$$\langle(2\gamma)_1|T|(\mu^+\mu^-)_{^3P_0}\rangle=-ie^2\frac{4m_\mu}{\sqrt{t}}\ln\left(\frac{1+\beta_\mu}{1-\beta_\mu}\right) \quad (4.26)$$

and

$$\langle(2\gamma)_1|T|(2\pi)_I\rangle =-ie^2\frac{1}{\sqrt{2}}\frac{8m_\pi^2}{t}\frac{1}{\beta_\pi}\ln\left(\frac{1+\beta_\pi}{1-\beta_\pi}\right)C_I\eta_I e^{i\delta_I}, \quad (4.27)$$

with

$$C_0=\sqrt{\tfrac{2}{3}} \quad\text{and}\quad C_2=\sqrt{\tfrac{1}{3}}.$$

η_I are the same enhancement factors as in Eq. (4.10). Therefore,

$$\mathrm{Abs}\langle(2\pi)_I|T|(\mu^+\mu^-)_{^3P_0}\rangle=\alpha^2\frac{8\pi}{\sqrt{2}}C_I\eta_I e^{-i\delta_I} \times\frac{m_\pi^2}{t}\frac{m_\mu}{\sqrt{t}}\frac{4}{\beta_\pi}\ln\left(\frac{1+\beta_\pi}{1-\beta_\pi}\right)\ln\left(\frac{1+\beta_\mu}{1-\beta_\mu}\right), \quad (4.28)$$

and, inserting this result into Eq. (4.24), we have

$$\mathrm{Abs}F_1^{(2\pi)}(t)=\frac{-1}{16\pi}\frac{\beta_\pi}{\beta_\mu}\mathrm{Re}\langle\pi^+\pi^-|T|(\mu^+\mu^-)_{^3P_0}\rangle\frac{1}{\sqrt{t}}\frac{1}{\sqrt{6}} \times[\sqrt{2}\eta_0A_0+\mathrm{Re}A_2]-i\alpha^2\frac{1}{\sqrt{t}}\frac{1}{\sqrt{3}}(\sqrt{2}\eta_0A_0+\mathrm{Re}A_2) \times\frac{m_\mu}{\sqrt{t}}\left(\frac{m_\pi}{\sqrt{t}}\right)^2\frac{1}{2\beta_\mu}\ln\left(\frac{1+\beta_\mu}{1-\beta_\mu}\right)\ln\left(\frac{1+\beta_\pi}{1-\beta_\pi}\right). \quad (4.29)$$

Here we get again an imaginary part which is precisely the opposite of the one encountered in the calculation of $\mathrm{Abs}F_1^{(2\gamma)}(M^2)$ in Eq. (4.15). Therefore, these terms cancel when substituted in the general expression giving

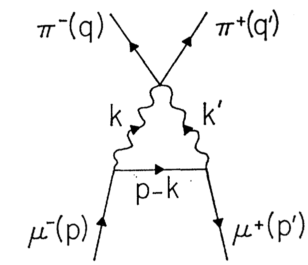

FIG. 7. Seagull-type diagram contributing to $\mathrm{Re}\langle\pi^+\pi^-|T|(\mu^+\mu^-)_{^3P_0}\rangle$.

$\mathrm{Abs}F_1(t)$ [see Eq. (2.10a)]. From Eq. (4.29), we see that the problem now reduces to a calculation of $\mathrm{Re}\langle\pi^+\pi^-|T|(\mu^+\mu^-)_{^3P_0}\rangle$. Altogether, there are three Feynman diagrams, at fourth order, contributing to this amplitude. We shall refer to that drawn in Fig. 7 as the "seagull-type diagram" and to those drawn in Fig. 8 as the "box-type diagrams" and proceed to their calculation separately.

1. Calculation of Seagull-Type Diagram

We write the corresponding T matrix in the following way:

$$\langle\pi^+\pi^-|T|\mu^+\mu^-\rangle_{\mathrm{seagull}}=\bar{v}(p')A(t)u(p).$$

Then,

$$\langle\pi^+\pi^-|T|(\mu^+\mu^-)_{^3P_0}\rangle_{\mathrm{seagull}} =\mathrm{Tr}P^{(1)}(p,p')_{\text{``in''}}A(t), \quad (4.30)$$

and, from the Feynman diagram of Fig. 7, we get

$$\mathrm{Im}A(t)=(-ie)^2(-2ie^2)g^{\mu\nu}\frac{(2\pi^2)}{(2\pi)^4} \times\int d^4k\,\delta(k^2)\delta(k'^2)\frac{1}{\mathrm{Tr}P^{(1)}(p,p')_{\text{``in''}}} \times\mathrm{Tr}\left\{P^{(1)}(p,p')_{\text{``in''}}\left[\frac{\gamma^\mu i(\not{p}-\not{k}+m_\mu)\gamma^\nu}{(p-k)^2-m_\mu^2}\right]\right\}. \quad (4.31)$$

The calculation of the right-hand side yields

$$\mathrm{Im}A(t)=4\pi\alpha^2\frac{m_\mu}{\beta_\mu}\frac{1}{t}Y_2(t), \quad (4.32)$$

with

$$Y_2(t)=\frac{1+\beta_\mu^2}{\beta_\mu^2}\ln\frac{1-\beta_\mu}{1+\beta_\mu}+\frac{2}{\beta_\mu}, \quad t\geq 4m_\mu^2. \quad (4.33)$$

From this, $\mathrm{Re}A(t)$ can be obtained from the dispersion integral

$$\frac{P}{\pi}\int_0^\infty\frac{dt'}{t'-t}\frac{Y_2(t')}{\beta_\mu't'}=\frac{Y_1(t)}{\pi t\beta_\mu},$$

where,[27] after analytic continuation of $Y_2(t')$ to the region $0\leq t'\leq 4m_\mu^2$,

$$Y_1(t)=\frac{1+\beta_\mu^2}{\beta_\mu^2}\left[\mathrm{Li}_2\left(-\frac{1-\beta_\mu}{1+\beta_\mu}\right)-\mathrm{Li}_2\left(-\frac{1+\beta_\mu}{1-\beta_\mu}\right)\right]-\frac{2}{\beta_\mu}\ln\left(\frac{4}{1-\beta_\mu^2}\right). \quad (4.34)$$

Therefore, we have that

$$\mathrm{Re}\langle\pi^+\pi^-|T|(\mu^+\mu^-)_{^3P_0}\rangle_{\mathrm{seagull}}=\frac{8}{\sqrt{2}}\alpha^2\frac{m_\mu}{\sqrt{t}}Y_1(t), \quad (4.35)$$

and the real contribution to $\mathrm{Abs}F_1^{(2\pi)}(t)$ from the seagull term is

$$\mathrm{Abs}F_1^{(2\pi)}(t)_{\mathrm{seagull}}=-\frac{\alpha^2}{4\pi}\frac{1}{\sqrt{t}}\frac{1}{\sqrt{3}}(\sqrt{2}\eta_0A_0+\mathrm{Re}A_2)\times\frac{m_\mu}{\sqrt{t}}\frac{\beta_\pi}{\beta_\mu}Y_1(t), \quad (4.36)$$

with $\eta_0=1$ in the case of perturbation theory.

2. *Calculation of Box-Type Diagrams*

The relevant diagrams are shown in Fig. 8. By analogy with the previous calculation of the seagull-type diagram, we write the corresponding T matrix

$$\langle\pi^+\pi^-|T|\mu^+\mu^-\rangle_{\mathrm{box}}=\bar{v}(p')W(s,t)u(p),$$

from which

$$\langle\pi^+\pi^-|T|(\mu^+\mu^-)_{^3P_0}\rangle_{\mathrm{box}}=\frac{1}{2}\int_{-1}^{+1}d\cos\phi\times\mathrm{Tr}P^{(1)}(p,p')_{\text{``in''}}W(s,t), \quad (4.37)$$

where ϕ is the c.m. scattering angle. We see from Eq. (4.29) that only the real part of the above amplitude is needed to calculate $\mathrm{Abs}F_1^{(2\pi)}(t)$.

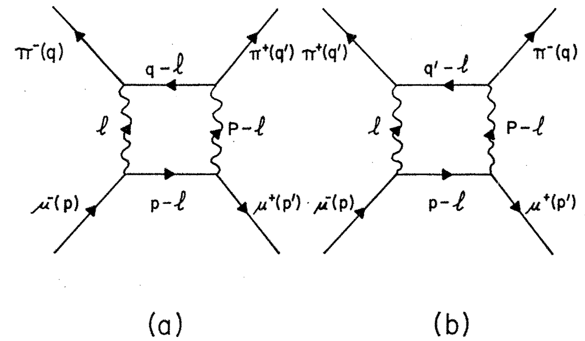

FIG. 8. Box-type diagrams contributing to $\mathrm{Re}\langle\pi^+\pi^-|T|(\mu^+\mu^-)_{^3P_0}\rangle$.

In terms of the usual variables

$$t=(p+p')^2=P^2,$$
$$s=(p-q^2)=m_\pi^2+m_\mu^2-\tfrac{1}{2}t+\tfrac{1}{2}t\beta_\pi\beta_\mu\cos\phi,$$
$$u=(p-q')^2=m_\pi^2+m_\mu^2-\tfrac{1}{2}t-\tfrac{1}{2}t\beta_\pi\beta_\mu\cos\phi,$$

it is convenient to transform to the variable θ, where

$$s=(m_\pi+m_\mu)^2-m_\pi m_\mu(1+\theta)^2/\theta, \quad (4.38)$$

so that the partial-wave projection becomes

$$\frac{1}{2}\int_{-1}^{+1}d\cos\phi=\frac{m_\pi m_\mu}{t\beta_\pi\beta_\mu}\int_{x''}^{x'}\frac{(1-\theta^2)}{\theta^2}d\theta, \quad (4.39)$$

where

$$x'^2=\frac{(1-\beta_\mu)(1+\beta_\pi)}{(1+\beta_\mu)(1-\beta_\pi)},\quad x''^2=\frac{(1-\beta_\mu)(1-\beta_\pi)}{(1+\beta_\mu)(1+\beta_\pi)}. \quad (4.40)$$

We consider first the contribution from Fig. 8(a). The matrix element associated with this diagram has an infrared divergence in the photon mass, and the corresponding integrals must be treated with care. Evaluation by the usual Feynman parametrization is very tedious and we prefer to use the Mandelstam representation for the square box.[28] From Fig. 8(a) we have (with minimal electromagnetic coupling)

$$W^{(a)}(s,t)=-i\frac{e^4}{(2\pi)^4}\int d^4l\frac{(2\boldsymbol{q}-\boldsymbol{l}-\boldsymbol{P})[(\boldsymbol{p}-\boldsymbol{l})+m_\mu](2\boldsymbol{q}-\boldsymbol{l})}{(l^2-\lambda^2)[(P-l)^2-\lambda^2][(p-l)^2-m_\mu^2][(q-l)^2-m_\pi^2]}, \quad (4.41)$$

where λ denotes the small photon mass. Application of the Dirac equation reduces the numerator to

$$N^{(a)}(s,t)=8p\cdot q\boldsymbol{q}-8l\cdot q\boldsymbol{q}-8p\cdot q\boldsymbol{l}+4m_\pi^2\boldsymbol{l}+2l\cdot p\boldsymbol{l}, \quad (4.42)$$

where we have dropped terms proportional to the photon mass.

[27] We use the dilogarithmic function as defined by L. Lewin, *Dilogarithms and Associated Functions* (MacDonald, London, 1958), i.e.,

$$\mathrm{Li}_2(x)\equiv-\int_0^x\frac{\ln(1-t)dt}{t}.$$

[28] S. Mandelstam, Phys. Rev. **115**, 1742 (1959); R. E. Cutkosky, J. Math. Phys. **1**, 429 (1960); A. C. T. Wu, Kgl. Danske Videnskab Selskab, Mat.-Fys. Medd. **33**, No. 3 (1961); W. B. Rolnick, Phys. Rev. Letters **16**, 544 (1966).

We will consider first the following integral:

$$L(s,t)=\mathrm{Re}\left[\frac{-i}{2\pi^2}\int d^4l\frac{1}{(l^2-\lambda^2)(P^2-2P\cdot l+l^2-\lambda^2)(l^2-2p\cdot l)(l^2-2q\cdot l)}\right],$$

which we will calculate by use of the Mandelstam representation; i.e., we write

$$L(s,t)=\int\frac{ds'}{(s'-s)}\int\frac{dt'}{(t'-t)}\rho(s',t'), \qquad (4.43)$$

where only the t' integration is a principal-value integration, and

$$\rho^{-2}(s,t)=t(t-4\lambda^2)[(m_\pi+m_\mu)^2-s][(m_\pi-m_\mu)^2-s]-4\lambda^4st. \qquad (4.44)$$

The region of integration is bounded by $\rho(s',t')\geq 0$. When $\lambda\to 0$, we can obviously drop the term in λ^4 and the integrals separate. The t' integration, in the limit $\lambda\to 0$, gives $t^{-1}(\ln\lambda^2/t+i\pi)$ and the s' integration (see Appendix B) gives $(-\theta\ln\theta)[m_\pi m_\mu(1-\theta^2)]^{-1}$, where θ is defined by Eq. (4.38); i.e.,

$$\theta=\frac{[(m_\pi+m_\mu)^2-s]^{1/2}-[(m_\pi-m_\mu)^2-s]^{1/2}}{[(m_\pi+m_\mu)^2-s]^{1/2}+[(m_\pi-m_\mu)^2-s]^{1/2}}. \qquad (4.45)$$

The final answer for L is

$$L(s,t)=\frac{-\theta\ln\theta}{tm_\pi m_\mu(1-\theta^2)}\ln\frac{\lambda^2}{t}. \qquad (4.46)$$

We consider next the real part of the integral

$$\frac{-i}{2\pi^2}\int d^4l \times\frac{l_\mu}{(l^2-\lambda^2)(P^2-2P\cdot l+l^2-\lambda^2)(l^2-2p\cdot l)(l^2-2q\cdot l)}=A(s,t)P_\mu+B(s,t)p_\mu+C(s,t)q_\mu. \qquad (4.47)$$

Setting all the lines on the mass shell we can again find double spectral functions corresponding to A, B, and C. These spectral functions are found by solving

$$\int d^4l\, l_\mu\delta(l^2-\lambda^2)\delta(P^2-2P\cdot l)\delta(l\cdot p)\delta(l\cdot q)=A'P_\mu+B'p_\mu+C'q_\mu.$$

Contracting both sides with P, p, and q gives three equations whose solution is

$$\begin{aligned}A'(s,t)&=\rho(s,t)[(m_\pi+m_\mu)^2-s][(m_\pi-m_\mu)^2-s]\Delta^{-1},\\ B'(s,t)&=t\rho(s,t)(s+m_\pi^2-m_\mu^2)\Delta^{-1},\\ C'(s,t)&=t\rho(s,t)(s+m_\mu^2-m_\pi^2)\Delta^{-1},\end{aligned} \qquad (4.48)$$

where

$$\Delta=2\{[(m_\pi+m_\mu)^2-s][(m_\pi-m_\mu)^2-s]+st\}. \qquad (4.49)$$

The coefficients A, B, and C are now given by double spectral integrals over the spectral functions A', B', and C', respectively. In fact, $\gamma\cdot P$ is zero when taken between the spinors, so we only require B (C can be obtained from B by a simple permutation of the masses). Note that the term in $l\cdot q$ of Eq. (4.42) does not contribute to $W^{(a)}(s,t)$, and the term in $(l\cdot p)\boldsymbol{l}$ gives rise to integrals which vanish in the limit $\lambda^2\to 0$.

We have for $B(s,t)$

$$B(s,t)=\int\frac{ds'}{s'-s}P\int\frac{dt'}{t'-t}\frac{t'(s'+m_\pi^2-m_\mu^2)}{\Delta(s',t')}\rho(s',t'), \qquad (4.50)$$

and we integrate first over t' dropping the term in λ^4. Taking the limit $\lambda\to 0$ after integration yields a result independent of the photon mass. [The integral in Eq. (4.50) is not infrared-divergent.] We are left with

$$B(s,t)=\int_{(m_\pi+m_\mu)^2}^{\infty}\frac{ds'}{s'-s}\frac{s'+m_\pi^2-m_\mu^2}{\Delta(s',t)}\times\frac{\ln|[(m_\pi+m_\mu)^2-s'][(m_\pi-m_\mu)^2-s']/s't|}{[((m_\pi+m_\mu)^2-s')((m_\pi-m_\mu)^2-s')]^{1/2}}.$$

We now use partial fractions and reduce the integration over s' to several basic integrals which we calculate in Appendix B. The real part gives

$$\begin{aligned}B(s,t)=\frac{1}{m_\mu m_\mu}\Bigg\{&\frac{-\theta}{(1-\theta^2)}\frac{s+m_\pi^2-m_\mu^2}{\Delta(s,t)}\chi(\theta)\\ &+\frac{x'}{(1-x'^2)}\frac{-\frac{1}{4}t(\beta_\pi-\beta_\mu)^2+m_\pi^2-m_\mu^2}{2t\beta_\pi\beta_\mu[s+\frac{1}{4}t(\beta_\pi-\beta_\mu)^2]}\chi(x')\\ &-\frac{x''}{(1-x''^2)}\frac{-\frac{1}{4}t(\beta_\pi+\beta_\mu)^2+m_\pi^2-m_\mu^2}{2t\beta_\pi\beta_\mu[s+\frac{1}{4}t(\beta_\pi+\beta_\mu)^2]}\chi(x'')\Bigg\}, \end{aligned}\qquad (4.51)$$

where x' and x'' are given in Eq. (4.40) and $\chi(\theta)$ is defined in Appendix B, Eq. (B6). The function $C(s,t)$ only differs from $B(s,t)$ by the interchange of m_π^2 and m_μ^2 in the numerator terms in Eq. (4.51).

The factors in the denominators of the above equation appear to give singularities when we try to integrate over θ. We see, however, that they cancel with similar terms in the numerator when we apply the projection operator $P^{(1)}$. Consider the form of the numerator

$$N^{(a)}=8q\cdot p\boldsymbol{q}+(4m_\pi^2-8q\cdot p)\boldsymbol{l}.$$

After the integration over l, the second term becomes proportional to

$$B(s,t)\boldsymbol{p}+C(s,t)\boldsymbol{q}.$$

Applying the projection operator $P^{(1)}$, for the incoming $\mu^+\mu^-$ system, we find

$$\mathrm{Tr}P^{(1)}(p,p')\text{``in''}\boldsymbol{p}=(2t)^{1/2}\beta_\mu m_\mu, \tag{4.52}$$

$$\mathrm{Tr}P^{(1)}(p,p')\text{``in''}\boldsymbol{q}=\frac{4m_\mu(p'\cdot q-p\cdot q)}{(2t)^{1/2}\beta_\mu}. \tag{4.53}$$

Thus the result of the application of the projection operator on the real part of $W^{(a)}(s,t)$ is

$$\begin{aligned}\mathrm{Tr}[P^{(1)}(p,p')\text{``in''}W^{(a)}]&\\ =\alpha^2\frac{16m_\mu}{\beta_\mu(2t)^{1/2}}&(m_\pi^2+m_\mu^2-s)(s-u)L(s,t)\\ -\alpha^2\frac{8(s-m_\mu^2)}{m_\pi\beta_\mu(2t)^{1/2}}&\left[\frac{2\theta}{1-\theta^2}\chi(\theta)+\frac{(\beta_\mu-\beta_\pi)}{\beta_\pi}\frac{x'}{(1-x'^2)}\chi(x')\right.\\ &\left.-\frac{(\beta_\mu+\beta_\pi)}{\beta_\pi}\frac{x''}{(1-x''^2)}\chi(x'')\right].\end{aligned} \tag{4.54}$$

Expressing the variables in terms of θ and substituting in the unitarity equation (2.10a) finally gives

$$\begin{aligned}\mathrm{Abs}F_1^{(2\pi,a)}=&\frac{\alpha^2}{2\pi}A(+\,-)\frac{m_\pi m_\mu^2}{t^2\beta_\mu^3}\int_{x'}^{x''}d\theta\,\frac{(1+\theta^2)}{\theta^2}\left[t-2m_\pi m_\mu\frac{(1+\theta^2)}{\theta}\right]\ln\theta\,\ln\frac{\lambda^2}{\tau}+\frac{\alpha^2}{\pi}\frac{A(+\,-)}{2}\frac{m_\mu}{t^2\beta_\mu^3}\\ &\times\int_{x'}^{x''}\frac{d\theta}{\theta}\left[m_\pi^2-m_\pi m_\mu\frac{(1+\theta^2)}{\theta}\right]\chi(\theta)+\frac{\alpha^2}{2\pi}\frac{A(+\,-)}{2}\frac{m_\mu}{t^2\beta^2}\int_{-1}^{+1}d\cos\phi\,(s-m_\mu^2)[\chi(x'')-\chi(x')],\end{aligned} \tag{4.55}$$

where

$$A(+\,-)=(1/\sqrt{3})(\sqrt{2}\eta_0A_0+\mathrm{Re}A_2).$$

After a considerable amount of algebra, we find

$$\begin{aligned}\frac{2m_\pi m_\mu}{t^2}\int_{x'}^{x''}\frac{(1+\theta^2)}{\theta^2}\left[t-2m_\pi m_\mu\frac{(1+\theta^2)}{\theta}\right]\ln\theta\,d\theta&=Q(t)\\ &=-\left(\beta_\mu-\tfrac{1}{2}(1-\beta_\mu^2)\ln\left|\frac{1+\beta_\mu}{1-\beta_\mu}\right|\right)\left(\beta_\pi-\tfrac{1}{2}(1-\beta_\pi^2)\ln\left|\frac{1+\beta_\pi}{1-\beta_\pi}\right|\right)\end{aligned} \tag{4.56}$$

for the first integral. The second integral can be integrated analytically in terms of trilogarithmic functions, but the algebra involved would be so long that we computed it numerically. In terms of $\cos\phi$, the third integral is trivial, so the final result is

$$\begin{aligned}\mathrm{Abs}F_1^{(2\pi,a)}(t)=&\frac{\alpha^2}{4\pi}A(+\,-)\frac{m_\mu}{t\beta_\mu^3}Q(t)\ln\frac{\lambda^2}{t}+\frac{\alpha^2}{2\pi}A(+\,-)\frac{m_\mu m_\pi^2}{t^2\beta_\mu^3}\int_{x'}^{x''}\frac{d\theta}{\theta}\left(1-\frac{m_\mu}{m_\pi}\frac{(1+\theta^2)}{\theta}\right)\chi(\theta)\\ &+\frac{\alpha^2}{4\pi}A(+\,-)\frac{m_\mu}{t^2\beta_\mu^2}(2m_\pi^2-t)[\chi(x'')-\chi(x')].\end{aligned} \tag{4.57}$$

Now we turn to Fig. 8(b). The matrix element can be derived from Eq. (4.41) by the substitution $q\rightarrow q'$, which changes s into u. Hence this graph does not have to be calculated independently and we can immediately write down the answer for $\mathrm{Tr}(P^{(1)}W_{2\pi}^{(b)}(t,u))$ if we define a variable ψ by

$$u=(m_\pi+m_\mu)^2-m_\pi m_\mu[(1+\psi)^2/\psi]. \tag{4.58}$$

In terms of this variable, the partial-wave projection becomes

$$\tfrac{1}{2}\int_{-1}^{+1}d\cos\phi=\frac{m_\pi m_\mu}{t\beta_\pi\beta_\mu}\int_{x''}^{x'}\frac{(1-\psi)^2}{\psi^2}d\psi, \tag{4.59}$$

and there is no difference between the answer for the crossed and direct graphs. Multiplying by a factor of 2, the total contribution can now be split up into two separate terms, one of which is infrared divergent, viz.,

$$\mathrm{Abs}F_{1,1}^{(2\pi,a+b)}=\frac{\alpha^2}{2\pi}A(+\,-)\frac{m_\mu}{t\beta_\mu^3}Q(t)\ln\left(\frac{\lambda^2}{t}\right), \tag{4.60}$$

and a finite part, viz.,

$$\begin{aligned}\mathrm{Abs}F_{1,2}^{(2\pi,a+b)}=&\frac{\alpha^2}{4\pi}A(+\,-)\frac{m_\mu}{t\beta_\mu^2}(1+\beta_\pi^2)\\ &\times[\chi(x')-\chi(x'')]-\frac{\alpha^2}{\pi}A(+\,-)\frac{m_\mu}{t^2\beta_\mu^3}\\ &\times\int_{x''}^{x'}\frac{d\theta}{\theta}\left(m_\pi^2-m_\pi m_\mu\frac{(1+\theta^2)}{\theta}\right)\chi(\theta).\end{aligned} \tag{4.61}$$

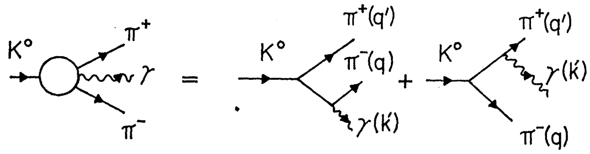

FIG. 9. Feynman diagram contributing to $A(K^0 \to 2\pi\gamma)$ in Eq. (4.64).

We shall show in Sec. IV D that the term containing the infrared divergence is exactly canceled by a similar term coming from the $\pi\pi\gamma$ intermediate state. The total contribution from the $\pi\pi$ intermediate state is thus the sum of Eq. (4.61) and the term from the seagull graph, Eq. (4.36). Evaluating the integrals numerically, we find

$$\mathrm{Abs}F_1{}^{(2\pi)}(M^2)_{\text{seagull}} = -4.65\times 10^{-12} \qquad (4.62)$$

and

$$\mathrm{Abs}F_{1,2}{}^{(2\pi,a+b)}(M^2) = 4.88\times 10^{-12}. \qquad (4.63)$$

D. Contribution from Pion-Photon Intermediate State

The contribution to $\mathrm{Abs}F_1(t)$ in Eq. (2.10a) from the $2\pi\gamma$ intermediate states is

$$\mathrm{Abs}F_1{}^{(2\pi\gamma)}(t) = \frac{-i}{2[2(t-4m_\mu^2)]^{1/2}}\frac{1}{(2\pi)^5}\int dq dq' dk' \times\theta(q)\delta(q^2-m_\pi^2)\theta(q')\delta(q'^2-m_\pi^2)\theta(k')\delta(k'^2-\lambda^2) \times\delta^{(4)}(p+p'-q-q'-k')\langle 2\pi\gamma|T|(\mu^+\mu^-)_{{}^3P_0}\rangle^* \times A(K^0\to 2\pi\gamma). \qquad (4.64)$$

To estimate the decay amplitude $A(K^0\to 2\pi\gamma)$ we shall limit ourselves to the consideration of bremsstrahlung photons only; i.e., possible direct transition terms are neglected.[29] The relevant Feynman diagrams are shown in Fig. 9. From these we have

$$A(K^0\to 2\pi\gamma) = (-ie)A(+\,-)2i \times\left(\frac{q\cdot\epsilon'}{\lambda^2+2q\cdot k'} - \frac{q'\cdot\epsilon'}{\lambda^2+2q'\cdot k'}\right), \qquad (4.65)$$

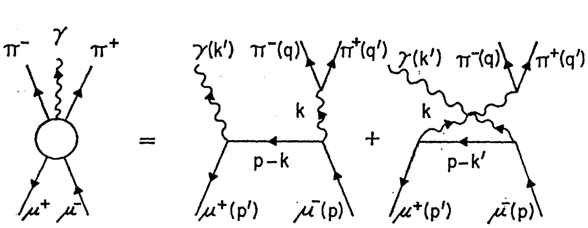

FIG. 10. Feynman diagrams contributing to $\langle 2\pi\gamma|T|(\mu^+\mu^-)_{{}^3P_0}\rangle$ in Eq. (4.64).

[29] This is in agreement with the scope of the model which we have adopted [K-π-π pointlike vertex and minimal electromagnetic interactions for pions]. The pions are still in an S-wave state so our enhancement factor η_0 can be taken into account. If we were to consider hard photons rather than bremstrahlung photons, then the pions would be in states of higher angular momentum. These would correspond to the so-called "structure-dependent" amplitudes for the decay $K\to\pi\pi\gamma$, and lead to transitions with $CP=-1$.

FIG. 11. The c.m. frame of the $\mu^+\mu^-$ system. θ is the angle between the momentum of the virtual photon **k** and the μ^+ momentum **p'** [see Eq. (4.67) and Fig. 10].

where $A(+\,-)$ denotes the amplitude $A(K^0\to\pi^+\pi^-)$ and ϵ' is the polarization of the photon.

The transition amplitude $\langle 2\pi\gamma|T|(\mu^+\mu^-)_{{}^3P_0}\rangle$ is obtained from the Feynman diagrams shown in Fig. 10. Thus we have

$$\langle 2\pi\gamma|T|(\mu^+\mu^-)_{{}^3P_0}\rangle = (-ie)^3\,\mathrm{Tr}P^{(1)}(p,p')\text{``in''} \times\left\{\frac{\gamma\cdot\epsilon' i(\not p-\not k+m_\mu)}{(p-k)^2-m_\mu^2}\gamma_\mu + \gamma_\mu\frac{i(\not p-\not k'+m_\mu)\gamma\cdot\epsilon'}{(p-k')^2-m_\mu^2}\right\} \times\frac{-ig^{\mu\nu}}{k^2-\lambda^2}(q-q')_\nu. \qquad (4.66)$$

In Eqs. (4.65) and (4.66), λ is a small mass given to the photon in order to deal with the infrared divergences.

The rest of this section is essentially devoted to technical details concerning the integrations in Eq. (4.64). They form the contents of the following subsections. The final expression of $\mathrm{Abs}F_1{}^{(2\pi\gamma)}$ as a function of t only is given in Sec. IV D 4.

1. *Kinematics*

We have, according to Fig. 10,

$$p+p'=k'+q+q', \quad k=q+q'$$

and

$$p^2=p'^2=m_\mu^2,\quad q^2=q'^2=m_\pi^2,\quad k'^2=\lambda^2,\quad k^2=s. \qquad (4.67)$$

It is convenient to choose as integration variables in Eq. (4.64) s, i.e., the invariant mass squared of the π^+-π^- system, and the angles θ, θ', and ϕ' defined as follows: In the c.m. frame of the μ^+-μ^- system (see Fig. 11), θ is the angle between the momentum of the virtual photon **k** and the μ^+ momentum **p'**; in the c.m. frame of the π^+-π^- system (see Fig. 12), θ' is the angle between the π^- momentum **q** and the direction of **k** as viewed in that system; ϕ' is the corresponding azimuthal angle

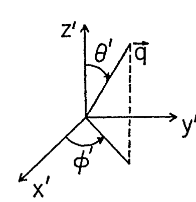

FIG. 12. The c.m. frame of the $\pi^+\pi^-$ system. θ' is the angle between the μ^- momentum **q** and the direction of **k** as viewed in that system. ϕ' is the corresponding azimuthal angle [see Eq. (4.67) and Fig. 11].

(see Fig. 12). In terms of these variables, the phase-space integral in Eq. (4.64) becomes

$$\int dq dq' dk'\ \theta(q)\delta(q^2-m_\pi^2)\theta(q')\delta(q'^2-m_\pi^2)\theta(k')\delta(k'^2-\lambda^2)\delta^{(4)}(p+p'-q-q'-k')$$

$$=\int_{4m_\pi^2}^{(\sqrt{t}-\lambda)^2} ds\ 2\pi\frac{\lambda^{1/2}(t,\lambda^2,s)}{4t}2\pi\frac{\lambda^{1/2}(s,m_\pi^2,m_\pi^2)}{4s}\frac{1}{2}\int_{-1}^{+1}d(\cos\theta)\frac{1}{2}\int_{-1}^{+1}d(\cos\theta')\frac{1}{2\pi}\int_0^{2\pi}d\phi', \quad (4.68)$$

where

$$\lambda(a,b,c)=a^2+b^2+c^2-2ab-2ac-2bc.$$

Notice that, e.g., $\lambda^{1/2}(s,m_\pi^2,m_\pi^2)/2\sqrt{s}$ is the momentum $|\mathbf{q}|$ of one of the pions in the π-π c.m. system. In fact, we shall find it useful to write several expressions in terms of the variables

$$\beta_\pi'=\frac{\lambda^{1/2}(s,m_\pi^2,m_\pi^2)}{s}=\left(1-\frac{4m_\pi^2}{s}\right)^{1/2}, \quad (4.69)$$

which is the velocity of one of the pions in the π-π c.m. system in units of the velocity of light; and

$$\beta_\mu=\frac{\lambda^{1/2}(t,m_\mu^2,m_\mu^2)}{t}=\left(1-\frac{4m_\mu^2}{t}\right)^{1/2}, \quad (4.70)$$

which is the velocity of one of the muons in the over-all c.m. system. Thus we have for the muon propagators which appear in Eq. (4.66)

$$\frac{1}{(p-k)^2-m_\mu^2}=\frac{-2}{(t-s-\lambda^2)}\frac{1}{1-\beta_\mu\gamma\cos\theta}, \quad (4.71a)$$

$$\frac{1}{(p-k')^2-m_\mu^2}=\frac{-2}{(t-s-\lambda^2)}\frac{1}{1+\beta_\mu\gamma\cos\theta}, \quad (4.71b)$$

with

$$\gamma\equiv\frac{\lambda^{1/2}(t,s,\lambda^2)}{t-s-\lambda^2}\xrightarrow[\lambda\to 0]{}1, \quad (4.72)$$

and

$$\frac{1}{\lambda^2+2q\cdot k'}=\frac{2}{t-s+\lambda^2}\frac{1}{1+\beta_\pi'\gamma'\cos\theta'}, \quad (4.73a)$$

$$\frac{1}{\lambda^2+2q'\cdot k'}=\frac{2}{t-s+\lambda^2}\frac{1}{1-\beta_\pi'\gamma'\cos\theta'}, \quad (4.73b)$$

with

$$\gamma'\equiv\frac{\lambda^{1/2}(t,s,\lambda^2)}{t-s+\lambda^2}\xrightarrow[\lambda\to 0]{}1. \quad (4.74)$$

2. *Integration over Phase Space and Separation of Infrared Divergences*

Substituting the expressions given in Sec. IV D 1 into Eqs. (4.65) and (4.66) and performing the sum over the polarization of the outgoing photon, we get

$$\sum_{\text{pol}}\langle 2\pi\gamma|T|(\mu^+\mu^-)_{^3P_0}\rangle^* A(K^0\to 2\pi\gamma)=(-ie)^4A(+\ -)\frac{2i}{s-\lambda^2}\frac{2}{t-s+\lambda^2}\frac{-2}{t-s-\lambda^2}\frac{1}{1-\beta_\pi'^2\gamma'^2\cos^2\theta'}$$

$$\times\text{Tr}P^{(1)}(p,p')_{\text{``in''}}\left\{[Q(\not p-\not k+m_\mu)Q-\beta_\pi'\gamma'\cos\theta'\not k(\not p-\not k+m_\mu)Q]\frac{1}{1-\beta_\mu\gamma\cos\theta}\right.$$

$$\left.+[Q(\not p-\not k+m_\mu)Q-\beta_\pi'\gamma'\cos\theta'\ Q(\not p-\not k'+m_\mu)\not k]\frac{1}{1+\beta_\mu\gamma\cos\theta}\right\}, \quad (4.75)$$

where

$$Q=q-q'.$$

After performing the trace operation and the trivial integration upon ϕ', we arrive at the following expression:

$$\text{Abs}F_1^{(2\pi\gamma)}(t)=\frac{-1}{(2\pi)^3}(-ie)^4A(+\ -)\frac{1}{2}\frac{m_\mu}{t^2}\int_{4m_\pi^2}^{(\sqrt{t}-\lambda)^2}\frac{ds}{s-\lambda^2}\left(1-\frac{4m_\pi^2}{s}\right)\frac{\gamma'\beta_\pi'}{t-s-\lambda^2}\frac{1}{2}\int_{-1}^{+1}d(\cos\theta)\frac{1}{2}\int_{-1}^{+1}d(\cos\theta')$$

$$\times\frac{1}{1-\beta_\mu^2\gamma^2\cos^2\theta}\frac{1}{1-\beta_\pi'^2\gamma'^2\cos^2\theta'}\{ts+[\gamma\gamma's(t-s+\lambda^2)-ts]\cos^2\theta-ts\cos^2\theta'+[ts+\tfrac{1}{2}(t+s-\lambda^2)^2-\tfrac{1}{2}\gamma\gamma'$$

$$\times[(t+\lambda^2)(t+s-\lambda^2)+s(t-s+\lambda^2)+\tfrac{1}{2}(t+s-\lambda^2)^2]]\cos^2\theta\cos^2\theta'\}. \quad (4.76)$$

Notice that for $\lambda \to 0$, the expression inside the bracket $\{\ \}$ becomes

$$\{\ \} \xrightarrow[\lambda\to 0]{} ts(1-\cos^2\theta-\cos^2\theta'+\cos^2\theta\cos^{2'})+\tfrac{1}{4}(t-s)(4s\cos^2\theta-3s\cos^2\theta\cos^2\theta'-t\cos^2\theta\cos^2\theta'), \quad (4.77)$$

and only the first term, i.e., $ts(\cdots)$ leads to an infrared divergence.

The angular integrations which appear in Eq. (4.76) are

$$\frac{1}{2}\int_{-1}^{+1} d(\cos\theta')\frac{1}{1-\beta_\pi'^2\gamma'^2\cos^2\theta'}=\frac{1}{2\beta_\pi'\gamma'}\ln\frac{1+\beta_\pi'\gamma'}{1-\beta_\pi'\gamma'}, \quad (4.78a)$$

$$\frac{1}{2}\int_{-1}^{+1} d(\cos\theta)'\frac{\cos^2\theta'}{1-\beta_\pi'^2\gamma'^2\cos^2\theta'}=\frac{1}{\beta_\pi'^2\gamma'^2}\left(\frac{1}{2\beta_\pi'\gamma'}\ln\frac{1+\beta_\pi'\gamma'}{1-\beta_\pi'\gamma'}-1\right), \quad (4.78b)$$

and similar integrals for the terms depending on $\cos\theta$. Thus, we are left with the problem of separating the infrared divergences in integrals of the type

$$I_{1,2}=\int_{4m_\pi^2}^{(\sqrt{t}-\lambda)^2}\frac{ds}{s-\lambda^2}\left(1-\frac{4m_\pi^2}{s}\right)\frac{s}{t-s-\lambda^2}\times Z_{1,2}(\gamma'\beta_\pi'), \quad (4.79)$$

where

$$Z_1(\gamma'\beta_\pi')=\tfrac{1}{2}\ln\frac{1+\beta_\pi'\gamma'}{1-\beta_\pi'\gamma'}$$

and

$$Z_2(\gamma'\beta_\pi')=\frac{1}{2\beta_\pi'^2\gamma'^2}\ln\frac{1+\beta_\pi'\gamma'}{1-\beta_\pi'\gamma'}-\frac{1}{\beta_\pi'\gamma'}.$$

This can be done using the following change of variables:

$$z=(t-s-\lambda^2)/2m_\pi\sqrt{t}.$$

Then, with

$$\beta_\pi\equiv(1-4m_\pi^2/t)^{1/2} \quad (4.80)$$

we obtain

$$I_1=\tfrac{1}{2}\beta_\pi^2\ln\frac{1-\beta_\pi}{1+\beta_\pi}\ln\frac{\lambda}{m_\pi}-\tfrac{1}{2}\beta_\pi^2\ln\frac{1-\beta_\pi}{1+\beta_\pi}\times\ln\left(\frac{\sqrt{t}}{2m_\pi}\beta_\pi^2\right)+V_1, \quad (4.81a)$$

where V_1 is a convergent integral

$$V_1=\tfrac{1}{2}\int_0^{[(\sqrt{t})/2m_\pi]\beta_\pi^2}\frac{dz}{z}\left[\left(1-\frac{4m_\pi^2}{t-2m_\pi(\sqrt{t})z}\right)\times\ln\frac{1+\{1-4m_\pi^2/[t-2m_\pi(\sqrt{t})z]\}^{1/2}}{1-\{1-4m_\pi^2/[t-2m_\pi(\sqrt{t})z]\}^{1/2}}-\beta_\pi^2\ln\frac{1+\beta_\pi}{1-\beta_\pi}\right], \quad (4.81b)$$

and

$$I_2=\left(\tfrac{1}{2}\ln\frac{1-\beta_\pi}{1+\beta_\pi}+\beta_\pi\right)\ln\frac{\lambda}{m_\pi}-\left(\tfrac{1}{2}\ln\frac{1-\beta_\pi}{1+\beta_\pi}+\beta_\pi\right)\times\ln\left(\frac{\sqrt{t}}{2m_\pi}\beta_\pi^2\right)+V_2, \quad (4.82a)$$

where V_2 is also a convergent integral

$$V_2=\tfrac{1}{2}\int_0^{[(\sqrt{t})/2m_\pi]\beta_\pi^2}\frac{dz}{z}\times\left[\ln\frac{1+\{1-4m_\pi^2/[t-2m_\pi(\sqrt{t})z]\}^{1/2}}{1-\{1-4m_\pi^2/[t-2m_\pi(\sqrt{t})z]\}^{1/2}}-\ln\frac{1+\beta_\pi}{1-\beta_\pi}+2\{1-4m_\pi^2/[t-2m_\pi(\sqrt{t})z]\}^{1/2}-2\beta_\pi\right]. \quad (4.82b)$$

From the results given in Eqs. (4.81a) and (4.82a) and using Eq. (4.77) it is easy now to get the over-all contribution to $\mathrm{Abs}F_1{}^{(2\pi\gamma)}$ in Eq. (4.76) from the infrared-divergent terms. We find

$$\mathrm{Abs}F_1{}^{(2\pi\gamma)}(t)=\frac{\alpha^2}{\pi}A(+\,-)\frac{m_\mu}{t}\frac{1}{\beta_\mu^3}\times\left[(1-\beta_\mu^2)\tfrac{1}{2}\ln\frac{1-\beta_\mu}{1+\beta_\mu}+\beta_\mu\right]\times\left[(1-\beta_\pi^2)\tfrac{1}{2}\ln\frac{1-\beta_\pi}{1+\beta_\pi}+\beta_\pi\right]\times\ln\frac{\lambda}{m_\pi}+\text{finite terms}. \quad (4.83)$$

Notice that this infrared-divergent term cancels with the one obtained in the evaluation of the contribution to $\mathrm{Abs}F_1(t)$ from the 2π intermediate states [see Eq. (4.60)]. We discuss the evaluation of the finite terms contributing to $\mathrm{Abs}F_1{}^{(2\pi\gamma)}(t)$ in Sec. IV D 3.

3. *Evaluation of Finite Terms Contributing to* $\mathrm{Abs}F_1^{(2\pi\gamma)}(t)$

From the results of the preceding subsection, it can be seen that there are three types of finite terms contributing to $\mathrm{Abs}F_1^{(2\pi\gamma)}(t)$:

(1) Terms which have already appeared in the evaluation of the infrared contribution [see Eqs. (4.81a) and (4.82a)]. They lead to a net contribution

$$-\frac{\alpha^2}{\pi}A(+\,-)\frac{m_\mu}{t}\frac{1}{\beta_\mu^3}\left[(1-\beta_\mu^2)^{\frac{1}{2}}\ln\frac{1-\beta_\mu}{1+\beta_\mu}+\beta_\mu\right]\times\left[(1-\beta_\pi^2)^{\frac{1}{2}}\ln\frac{1-\beta_\pi}{1+\beta_\pi}+\beta_\pi\right]\frac{1}{2}\ln\frac{\beta_\pi^4}{1-\beta_\pi^2}. \quad (4.84)$$

(2) Contributions from the integrals V_1 and V_2 [see Eqs. (4.81b) and (4.82b)]. These lead to a net contribution

$$-\frac{\alpha^2}{\pi}A(+\,-)\frac{m_\mu}{t}\frac{1}{\beta_\mu^3}\left[(1-\beta_\mu^2)^{\frac{1}{2}}\ln\frac{1-\beta_\mu}{1+\beta_\mu}+\beta_\mu\right]\times(V_1-V_2). \quad (4.85)$$

(3) Terms from the convergent part of the integral in Eq. (4.76). According to Eq. (4.77), these lead to a net contribution

$$\frac{\alpha^2}{\pi}A(+\,-)\frac{m_\mu}{t}\frac{1}{\beta_\mu^3}\left[\frac{1}{2}\ln\frac{1-\beta_\mu}{1+\beta_\mu}+\beta_\mu\right]\frac{1}{4t}\times\int_{4m_\pi^2}^{t}\frac{ds}{s}\left[-2s\beta_\pi'^2\ln\frac{1-\beta_\pi'}{1+\beta_\pi'}+(3s+t)\beta_\pi'+(3s+t)\frac{1}{2}\ln\frac{1-\beta_\pi'}{1+\beta_\pi'}\right]. \quad (4.86)$$

To evaluate the integral V_1-V_2 [see Eqs. (4.81b) and (4.82b)] it is convenient to make the following change of variable:

$$1-4m_\pi^2/[t-2m_\pi(\sqrt{t})z]=x^2.$$

Thus, we have

$$V_1-V_2=\frac{4m_\pi^2}{t}\int_0^{\beta_\pi}dx\frac{x}{x^2-1}\frac{1}{x^2-\beta_\pi^2}\times\left[(x^2-1)\ln\frac{1+x}{1-x}-(\beta_\pi^2-1)\ln\frac{1+\beta_\pi}{1-\beta_\pi}\right]-\frac{4m_\pi^2}{t}\int_0^{\beta_\pi}dx\frac{2x}{x^2-1}\frac{1}{x+\beta_\pi}.$$

The second integral can be expressed in terms of logarithmic functions; the first involves logarithmic and dilogarithmic functions. We find

$$V_1-V_2=(1-\beta_\pi^2)\tfrac{1}{6}\pi^2+(1-\beta_\pi)\ln\left(\frac{1+\beta_\pi}{1-\beta_\pi}\right)+2\beta_\pi\ln\left(\frac{1+\beta_\pi}{2}\right)+(1-\beta_\pi^2)\left\{\frac{1}{4}\ln\left[\frac{1-\beta_\pi}{1+\beta_\pi}\beta_\pi^4\right]\times\ln\left(\frac{1+\beta_\pi}{1-\beta_\pi}\right)-\mathrm{Li}_2\left(\frac{1-\beta_\pi}{1+\beta_\pi}\right)\right\}. \quad (4.87)$$

The integral appearing in Eq. (4.86) can also be evaluated by using the change of variables

$$1-4m_\pi^2/s=x^2,$$

and is expressible in terms of logarithmic and dilogarithmic functions as well. The result we find is

$$\frac{1}{4m_\pi^2}\int_{4m_\pi^2}^{t}\frac{ds}{s}\left[-2s\beta_\pi'^2\ln\frac{1-\beta_\pi'}{1+\beta_\pi'}+(3s+t)\frac{1}{2}\ln\frac{1-\beta_\pi'}{1+\beta_\pi'}+(3s+t)\beta_\pi'\right]$$
$$=-\tfrac{1}{3}\pi^2+\frac{\beta_\pi}{2(1-\beta_\pi^2)}-\frac{\pi^2}{12(1-\beta_\pi^2)}+\frac{7}{4}\ln\left(\frac{1-\beta_\pi}{1+\beta_\pi}\right)-\frac{3}{2(1-\beta_\pi^2)}\ln\left(\frac{1-\beta_\pi}{1+\beta_\pi}\right)+\left(2+\frac{1}{2(1-\beta_\pi^2)}\right)\times\left[\frac{1}{2}\ln\left(\frac{1-\beta_\pi^2}{4}\right)\ln\left(\frac{1+\beta_\pi}{1-\beta_\pi}\right)+\ln\left(\frac{1-\beta_\pi}{2}\right)\ln\left(\frac{1+\beta_\pi}{2}\right)+2\mathrm{Li}_2\left(\frac{1+\beta_\pi}{2}\right)\right]. \quad (4.88)$$

4. *Expression for* $\mathrm{Abs}F_1^{(2\pi\gamma)}$

The final result for $\mathrm{Abs}F_1^{(2\pi\gamma)}$ consists of an infrared part, as given by Eq. (4.83), and a finite part, which is the sum of the expressions given in Eqs. (4.84)–(4.86). Using the results quoted in Eqs. (4.87) and (4.88), the over-all expression for $\mathrm{Abs}F_1^{(2\pi\gamma)}$ can be cast into the following form:

$$\mathrm{Abs}F_1^{(2\pi\gamma)}(t)=-\frac{\alpha^2}{\pi}A(+\,-)\frac{m_\mu}{t}\left\{\frac{1}{\beta_\mu^3}\left[\beta_\mu+(1-\beta_\mu^2)^{\frac{1}{2}}\ln\frac{1-\beta_\mu}{1+\beta_\mu}\right]\left[\beta_\pi+(1-\beta_\pi^2)^{\frac{1}{2}}\ln\frac{1-\beta_\pi}{1+\beta_\pi}\right]\ln\frac{m_\pi}{\lambda}+\frac{1}{2\beta_\mu}\ln\frac{1+\beta_\mu}{1-\beta_\mu}\Phi^{(1)}(\beta_\pi)+\frac{1}{\beta_\mu^2}\left[\frac{1}{2\beta_\mu}\ln\frac{1+\beta_\mu}{1-\beta_\mu}-1\right]\Phi^{(2)}(\beta_\pi)\right\}, \quad (4.89)$$

where

$$\Phi^{(1)}(\beta_\pi)=(1-\beta_\pi^2)\tfrac{1}{6}\pi^2+\tfrac{1}{2}\beta_\pi \ln\left(\frac{\beta_\pi^4}{1-\beta_\pi^2}\right)+2\beta_\pi \ln\left(\frac{1+\beta_\pi}{2}\right)-(1-\beta_\pi)\ln\frac{1-\beta_\pi}{1+\beta_\pi}$$
$$-\tfrac{1}{2}(1-\beta_\pi^2)\ln(1-\beta_\pi)\ln\frac{1-\beta_\pi}{1+\beta_\pi}-(1-\beta_\pi^2)\mathrm{Li}_2\left(\frac{1-\beta_\pi}{1+\beta_\pi}\right) \quad (4.90)$$

and

$$\Phi^{(2)}(\beta_\pi)=\frac{\pi^2}{48}+\tfrac{1}{8}\beta_\pi-(1-\beta_\pi^2)\tfrac{1}{12}\pi^2+\tfrac{5}{8}\ln\frac{1-\beta_\pi}{1+\beta_\pi}-\tfrac{1}{2}\beta_\pi\ln[\tfrac{1}{16}\beta_\pi^4(1-\beta_\pi^2)]+(7/16)(1-\beta_\pi^2)\ln\frac{1-\beta_\pi}{1+\beta_\pi}$$
$$-\tfrac{1}{16}\ln\left[\frac{(1+\beta_\pi)^2}{16}(1-\beta_\pi^2)\right]\ln\frac{1-\beta_\pi}{1+\beta_\pi}+\tfrac{1}{4}(1-\beta_\pi^2)\ln\left[\frac{16}{(1+\beta_\pi)^2}\frac{1-\beta_\pi}{1+\beta_\pi}\right]\ln\frac{1-\beta_\pi}{1+\beta_\pi}$$
$$+\tfrac{1}{4}\mathrm{Li}_2\left[-\left(\frac{1-\beta_\pi}{1+\beta_\pi}\right)\right]+\tfrac{1}{2}(1-\beta_\pi^2)\mathrm{Li}_2\left[\left(\frac{1-\beta_\pi}{1+\beta_\pi}\right)^2\right]. \quad (4.91)$$

Some remarks are in order:

(1) We note that the infrared term in Eq. (4.89) cancels with the corresponding infrared divergence encountered in the calculation of $\mathrm{Abs}F_1^{(2\pi)}(t)$ [see Eq. (4.60)]. From this cancellation one is left, however, with a finite quantity

$$-\frac{\alpha^2}{\pi}A(+\,-)\frac{m_\mu}{t}\frac{1}{\beta_\mu^3}\left[\beta_\mu+(1-\beta_\mu^2)\tfrac{1}{2}\ln\frac{1-\beta_\mu}{1+\beta_\mu}\right]$$
$$\times\left[\beta_\pi+(1-\beta_\pi^2)\tfrac{1}{2}\ln\frac{1-\beta_\pi}{1+\beta_\pi}\right]\tfrac{1}{2}\ln\frac{1-\beta_\pi^2}{4}, \quad (4.92)$$

which certainly gives a net contribution to the total $\mathrm{Abs}F_1(t)$.

(2) For $t=M^2$, i.e., for the on-shell $K\to\mu^+\mu^-$ decay, we find numerically that

$$\Phi^{(1)}(\beta_\pi)=0.257 \quad (4.93)$$

and

$$\Phi^{(2)}(\beta_\pi)=-0.088. \quad (4.94)$$

Then, with

$$A(+\,-)=(1/\sqrt{3})(\sqrt{2}\eta_0A_0+\mathrm{Re}A_2),$$

we find that

$$\mathrm{Abs}F_1^{(2\pi\gamma)}(M^2)_{\mathrm{finite}}$$
$$=-\frac{\alpha^2}{\pi}A(+\,-)\frac{m_\mu}{M^2}\left\{\frac{1}{2\beta_\mu}\ln\frac{1+\beta_\mu}{1-\beta_\mu}\Phi^{(1)}(\beta_\pi)\right.$$
$$\left.+\frac{1}{\beta_\mu^2}\left[\frac{1}{2\beta_\mu}\ln\frac{1+\beta_\mu}{1-\beta_\mu}-1\right]\Phi^{(2)}(\beta_\pi)\right\}=-1.10\times10^{-12}. \quad (4.95)$$

(3) For $t=M^2$, the numerical value of the expression given in Eq. (4.92) is 1.53×10^{-12}.

(4) Adding the results obtained in Eqs. (4.62), (4.63), (4.95), and (4.92), we get the net real contribution to $\mathrm{Abs}F_1(M^2)$ from the 2π and $2\pi\gamma$ intermediate states:

$$\mathrm{Abs}F_1^{(2\pi)}(M^2)+\mathrm{Abs}F_1^{(2\pi\gamma)}(M^2)=0.68\times10^{-12}. \quad (4.96)$$

V. $K_1^0\to\mu^+\mu^-$ DECAY: SUMMARY AND CONCLUSIONS

1. We have estimated a lower bound for the decay rate of the process $K_s^0\to\mu^+\mu^-$. This has been done in the following way: We assume *CP* invariance to hold, and write the decay rate as

$$\Gamma(K_1^0\to\mu^+\mu^-)=\frac{M}{4\pi}\left(1-\frac{4m_\mu^2}{M^2}\right)^{3/2}|F_1|^2. \quad (2.4a)$$

Since $|F_1|\geq|\mathrm{Abs}F_1|$, it is clear that a calculation of the absorptive part of the form factor F_1 gives us a lower bound to the rate $\Gamma(K_1^0\to\mu^+\mu^-)$.

The quantity $\mathrm{Abs}F_1$ has been estimated by saturating the unitary condition

$$\mathrm{Abs}F_1=\frac{-i(2\pi)^4}{2[2(M^2-4m_\mu^2)]^{1/2}}\sum_\lambda\int d\rho_\lambda\,\delta^{(4)}(p+p'-\textstyle\sum p_\lambda)$$
$$\times\langle\lambda|T|(\mu^+\mu^-)_{^3P_0}\rangle^*A(K^0\to\lambda) \quad (2.10a)$$

to order Ge^4 (G is the Fermi constant, e the electric charge).

2. We have calculated assuming a pointlike weak coupling at the K-π-π vertex and minimal electromagnetic interaction for pions and muons. This implies a summation in Eq. (2.10a) over 2γ, 2π, and $2\pi\gamma$ intermediate states (see the diagrams of Fig. 2).

(i) From the 2γ contribution to Eq. (2.10a), we get the result

$$\mathrm{Abs}F_1^{(2\gamma)}=\frac{\alpha^2}{\pi}\frac{1}{\sqrt{3}}(\sqrt{2}A_0+\mathrm{Re}A_2)\frac{1}{M}\frac{m_\mu}{M}\frac{1}{2\beta_\mu}\ln\frac{1+\beta_\mu}{1-\beta_\mu}$$
$$\times\left\{-1+\left(\frac{1-\beta_\pi^2}{2}\right)\left[\pi^2-\ln^2\left(\frac{1+\beta_\pi}{1-\beta_\pi}\right)\right]\right\}, \quad (4.18)$$

where $\beta_\mu=(1-4m_\mu^2/M^2)^{1/2}$, $\beta_\pi=(1-4m_\pi^2/M^2)^{1/2}$, and A_0,A_2 are the amplitudes for the transition of K^0 to two pions in isospin states $I=0$ and $I=2$, respectively.

Numerically,

$$\mathrm{Abs}F_1^{(2\gamma)} = -2.18\times 10^{-12} \quad \text{(pert. th.)}. \qquad (4.19)$$

(ii) From the 2π contribution to Eq. (2.10a), we find for the seagull term (see Fig. 7)

$$\mathrm{Abs}F_1^{(2\pi)}(\text{seagull}) = -\frac{\alpha^2}{\pi}\frac{1}{\sqrt{3}}(\sqrt{2}A_0+\mathrm{Re}A_2) \times\frac{1}{M}\frac{m_\mu}{M}\frac{\beta_\pi}{4\beta_\mu}Y_1(t), \qquad (4.36)$$

with[27]

$$Y_1(t) = \frac{1+\beta_\mu^2}{\beta_\mu^2}\left[\mathrm{Li}_2\left(-\frac{1-\beta_\mu}{1+\beta_\mu}\right)-\mathrm{Li}_2\left(-\frac{1+\beta_\mu}{1-\beta_\mu}\right)\right] -\frac{2}{\beta_\mu}\ln\left(\frac{4}{1-\beta_\mu^2}\right), \qquad (4.34)$$

and, for the box diagrams contribution [see Fig. 8 and Eqs. (4.60) and (4.61)],

$$\mathrm{Abs}F_1^{(2\pi)}(\text{box}) = \frac{\alpha^2}{\pi}\frac{1}{\sqrt{3}}(\sqrt{2}A_0+\mathrm{Re}A_2)\frac{1}{M}\frac{m_\mu}{M}\left\{\frac{1}{\beta_\mu^3}Q(M^2)\ln\frac{\lambda}{M}+\frac{1}{4\beta_\mu^2}(1+\beta_\pi^2)[\chi(x')-\chi(x'')] -\frac{1}{\beta_\mu^3}\int_{x''}^{x'}\frac{d\theta}{\theta}\left(m_\pi^2-m_\pi m_\mu\frac{(1+\theta^2)}{\theta}\right)\chi(\theta)\right\}, \qquad (5.1)$$

with

$$Q(M^2) = -\left[\beta_\mu-\tfrac{1}{2}(1-\beta_\mu^2)\ln\left(\frac{1+\beta_\mu}{1-\beta_\mu}\right)\right]\left[\beta_\pi-\tfrac{1}{2}(1-\beta_\pi^2)\ln\left(\frac{1+\beta_\pi}{1-\beta_\pi}\right)\right], \qquad (4.56)$$

$$x' = \left[\frac{(1-\beta_\mu)(1+\beta_\pi)}{(1+\beta_\mu)(1-\beta_\pi)}\right]^{1/2}, \quad x'' = \left[\frac{(1-\beta_\mu)(1-\beta_\pi)}{(1+\beta_\mu)(1+\beta_\pi)}\right]^{1/2}, \qquad (4.40)$$

and $\chi(\theta)$ defined in Appendix B, Eq. (B6). Numerically,

$$\mathrm{Abs}F_1^{(2\pi)}(\text{seagull}) = -2.98\times 10^{-12} \quad \text{(pert. th.)} \qquad (5.2)$$

and

$$\mathrm{Abs}F_1^{(2\pi)}(\text{box}) = \frac{\alpha^2}{\pi}\frac{1}{\sqrt{3}}(\sqrt{2}A_0+\mathrm{Re}A_2)\frac{1}{M}\frac{m_\mu}{M}\frac{1}{\beta_\mu^3}Q(M^2)\ln\frac{\lambda}{M}+3.14\times 10^{-12} \quad \text{(pert. th.)}. \qquad (5.3)$$

(iii) From the $2\pi\gamma$ contribution to Eq. (2.10a), we find [See Figs. 9 and 10, and Eq. (4.89)]

$$\mathrm{Abs}F_1^{(2\pi\gamma)} = \frac{\alpha^2}{\pi}\frac{1}{\sqrt{3}}(\sqrt{2}A_0+\mathrm{Re}A_2)\frac{1}{M}\frac{m_\mu}{M}\left\{-\frac{1}{\beta_\mu^3}Q(M^2)\ln\frac{\lambda}{m_\pi}-\frac{1}{2\beta_\mu}\ln\frac{1+\beta_\mu}{1-\beta_\mu}\Phi^{(1)}(\beta_\pi) -\frac{1}{\beta_\mu^2}\left[\frac{1}{2\beta_\mu}\ln\frac{1+\beta_\mu}{1-\beta_\mu}-1\right]\Phi^{(2)}(\beta_\pi)\right\}, \qquad (5.4)$$

where the functions $\Phi^{(1)}(\beta_\pi)$ and $\Phi^{(2)}(\beta_\pi)$ are given in Eqs. (4.90) and (4.91), respectively. Numerically,

$$\mathrm{Abs}F_1^{(2\pi\gamma)} = -\frac{\alpha^2}{\pi}\frac{1}{\sqrt{3}}(\sqrt{2}A_0+\mathrm{Re}A_2)\frac{1}{M}\frac{m_\mu}{M}\frac{1}{\beta_\mu^3}Q(M^2)\ln\frac{\lambda}{m_\pi}-0.70\times 10^{-12} \quad \text{(pert. th.)}. \qquad (5.5)$$

Notice that the infrared divergence in $\mathrm{Abs}F_1^{(2\pi)}$(box) [Eq. (5.3)] cancels with the corresponding infrared divergence in $\mathrm{Abs}F_1^{(2\pi\gamma)}$ [Eq. (5.4)]. Thus we have

$$\mathrm{Abs}F_1^{(2\pi)}(\text{box})+\mathrm{Abs}F_1^{(2\pi\gamma)} = -\frac{\alpha^2}{\pi}\frac{1}{\sqrt{3}}(\sqrt{2}A_0+\mathrm{Re}A_2)\frac{1}{M}\frac{m_\mu}{M}\frac{1}{\beta_\mu^3}Q(M^2)\ln\frac{m_\pi}{M}+2.44\times 10^{-12} = 3.42\times 10^{-12}. \qquad (5.6)$$

It turns out that the contribution from $\mathrm{Abs}F_1^{(2\pi)}$ (seagull) [Eq. (5.2)] almost cancels the infrared divergence-free combination $\mathrm{Abs}F_1^{(2\pi)}(\text{box})+\mathrm{Abs}F_1^{(2\pi\gamma)}$. Thus the quantity

$$\mathrm{Abs}F_1^{(2\pi)}(\text{seagull})+\mathrm{Abs}F_1^{(2\pi)}(\text{box})+\mathrm{Abs}F_1^{(2\pi\gamma)} = 0.44\times 10^{-12}, \qquad (5.7)$$

which is gauge invariant and infrared convergent, gives a contribution to AbsF_1 which is smaller than the contribution from the 2γ intermediate state [Eq. (4.19)] and opposite in sign. The over-all result is

$$\begin{aligned}\mathrm{Abs}F_1 = \mathrm{Abs}F_1^{(2\gamma)} + \mathrm{Abs}F_1^{(2\pi)}(\text{seagull}) \\ + \mathrm{Abs}F_1^{(2\pi)}(\text{box}) + \mathrm{Abs}F_1^{(2\pi\gamma)} \\ = -1.74\times10^{-12} \quad (\text{pert. th.}),\end{aligned} \tag{5.8}$$

and from Eq. (1.5) we have, therefore, that

$$\Gamma(K_1^0 \to \mu^+\mu^-) \geq 1.35\times10^{-1}\ \text{sec}^{-1} \quad (\text{pert. th.}), \tag{5.9}$$

which corresponds to a branching ratio

$$\frac{\Gamma(K_1^0 \to \mu^+\mu^-)}{\Gamma(K_1^0 \to \text{all})} \geq 1.6\times10^{-11} \quad (\text{pert. th.}), \tag{5.10}$$

not much smaller than the "first-guess" estimate given in Eq. (2.9).

3. We have also made a very simple model to estimate the possible enhancement effects of the π-π strong interactions in our calculations. This is described in Sec. IV A. Practically, it amounts to the substitution

$$(1/\sqrt{3})(\sqrt{2}A_0 + \mathrm{Re}A_2) \to (1/\sqrt{3})(\sqrt{2}\eta_0 A_0 + \eta_2\,\mathrm{Re}A_2)$$

(where η_I is an enhancement factor due to the strong interactions of the 2π system in the isospin state I) in all the perturbation-theory results except in the evaluation of Re$H_1(M^2)$. Here the explicit dependence of η_0 on the dispersion variable has been taken into account. As Re$A_2 \ll |A_0|$, we have always set the term $\eta_2\,\mathrm{Re}A_2 = 0$, which means that η_0 is purely a multiplicative factor in the 2π and $2\pi\gamma$ contributions.

The corresponding value for AbsF_1, where enhancement factors are taken into account, is given by the sum of Eqs. (4.15) and (4.96):

$$\mathrm{Abs}F_1 = -3.03\times10^{-12}, \tag{5.11}$$

to be compared with the perturbation-theory prediction of Eq. (5.8). From Eq. (5.11), we obtain that

$$\Gamma(K_1^0 \to \mu^+\mu^-) \geq 4.37\times10^{-1}\ \text{sec}^{-1}, \tag{5.12}$$

and, therefore,

$$\frac{\Gamma(K_1^0 \to \mu^+\mu^-)}{\Gamma(K_L^0 \to \text{all})} \geq 5.1\times10^{-11}, \tag{5.13}$$

which is our final prediction.

4. Our results so far have been entirely for the decay of K_1^0 into a muon pair. However, we can easily obtain the result for K_1^0 decay into an electron pair by changing m_μ into m_e in our equations. We quote without further comment the electron equivalents of Eqs. (5.12) and (5.13), i.e.,

$$\Gamma(K_1^0 \to e^+e^-) \geq 1.26\times10^{-3}\ \text{sec}^{-1}, \tag{5.14}$$

and, therefore,

$$\frac{\Gamma(K_1^0 \to e^+e^-)}{\Gamma(K_1^0 \to \text{all})} \geq 1.5\times10^{-13}. \tag{5.15}$$

5. We should like to close this section with two comments on the reliability of the bound given above.

(i) In principle, first-order weak times fourth-order electromagnetic interactions might be competitive with second-order weak interactions [$GM_K^2(\alpha/\pi)^2 = 1.5\times10^{-11}$ and $(GM_K^2)^2 = 0.8\times10^{-11}$]. However, second-order weak interactions only contribute to the real part of F_1; hence, they can only increase the lower bound given in Eq. (5.12).

(ii) One source of uncertainty in our calculation comes from the estimate of the real part of the $K_1^0 \to 2\gamma$ amplitude, i.e., the quantity we call Re$H_1(M^2)$, which has been estimated in Sec. IV A. It is clear that other channels than the 2π are open in the dispersion integral from $4m^2$ to ∞, viz., 3π, $N\bar{N}$, etc., which are not taken into account in our estimate [see Eq. (4.16)]. There exists, however, the empirical possibility of improving this; clearly, one expects to observe $K_1^0 \to 2\gamma$ decays before $K_1^0 \to \mu^+\mu^-$ decays will be detected. In that case $|\mathrm{Re}H_1(M^2)|$ can be taken from experiment, using the estimated value for Im$H_1(M^2)$ [see Eq. (4.13)]. If $|\mathrm{Re}H_1(M^2)| < 1.02\times10^{-9}$, then the 2π and 2γ and perhaps $3\pi\gamma$ channels might become competitive with the 2γ. In this case it is clear that a more elaborate estimate than our perturbation model is needed. If $|\mathrm{Re}H_1(M^2)| > 1.02\times10^{-9}$, then one can treat other channels than the 2γ as a small correction. In this case there is room for improvement of the $\Gamma(K_1^0 \to \mu^+\mu^-)$ bound.

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APPENDIX A

Throughout the calculation described in the text, we have used a covariant expression for the projector on the triplet state of the $\mu^+\mu^-$ system. In this appendix, we give the details of how to construct this projector, as well as the projector on the singlet state.

With our normalization of Dirac spinors, the projector on a particle state with energy momentum p and polarization s is

$$u(p,s)\otimes\bar{u}(p,s) = \tfrac{1}{2}(\not{p}+m)(1+\gamma_5 \not{s}), \tag{A1}$$

where

$$p^2 = m^2, \quad s^2 = -1, \quad \text{and} \quad s\cdot p = 0.$$

This is a well-known expression due to Michel and Wightman.[30]

Next, let us denote by $s^{(i)}$ ($i = 1, 2, 3$) three four-vectors such that

$$p\cdot s^{(i)} = 0, \quad s^{(i)}\cdot s^{(j)} = -\delta_{ij}, \quad i, j = 1, 2, 3$$

and

$$\epsilon^{\mu\nu\rho\sigma} p_\mu s_\nu^{(1)} s_\rho^{(2)} s_\sigma^{(3)} = m.$$

[30] L. Michel and A. S. Wightman, Phys. Rev. 98, 1190 (1955).

We choose the third component $s_\mu^{(3)}$ as the quantization axis, and call λ the magnetic quantum number, or helicity, along this axis. It was shown by Bouchiat and Michel[31] that the projector given in Eq. (A1) can then be written as follows:

$$u(p,\lambda')\otimes\bar{u}(p,\lambda)=\tfrac{1}{2}(\not p+m)(\delta_{\lambda\lambda'}+\gamma_5\mathbf{s}\cdot\boldsymbol{\tau}_{\lambda\lambda'})\,, \quad (A2)$$

where $\mathbf{s}\cdot\boldsymbol{\tau}_{\lambda\lambda'}$ is a shorthand notation for $(\mathbf{s}^{(1)}\tau_1+\mathbf{s}^{(2)}\tau_2+\mathbf{s}^{(3)}\tau_3)_{\lambda\lambda'}$; τ_1, τ_2, and τ_3 are the usual Pauli matrices. Equation (A2) can be viewed as a 2×2 matrix acting on the helicity indices λ and λ'.

The generalization to the case where two energy-momenta are involved, i.e., expressions of the type $u(p',\lambda')\otimes\bar{u}(p,\lambda)$, was made by Nuyts.[32] It reads

$$u(p',\lambda')\otimes\bar{u}(p,\lambda)=\frac{1}{[2(mm'+p\cdot p')]^{1/2}}\times\tfrac{1}{2}(\not p'+m')(\not p+m)(\delta_{\lambda\lambda'}+\gamma_5\mathbf{s}\cdot\boldsymbol{\tau}_{\lambda\lambda'})\,, \quad (A3)$$

where $p^2=m^2$ and $p'^2=m'^2$. We note that in this case, λ' is the helicity along an axis s' such that

$$s'=\Lambda_{p/m\to p'/m'}s\,,$$

where $\Lambda_{p/m\to p'/m'}$ is the pure Lorentz transformation (or boost) which takes p/m to p'/m' and leaves invariant the two-plane orthogonal to the one defined by p and p'.

A general method to construct all the possible projectors for spin-$\frac{1}{2}$ particles has been given by Michel.[33] In our case, we are interested in expressions for $u(p',\lambda')\otimes\bar{v}(p,\lambda)$ and $v(p',\lambda')\otimes\bar{u}(p,\lambda)$. These can easily be obtained from Eq. (A3) in the following way: We choose s to be the helicity axis corresponding to p, i.e.,

$$s_0=\frac{|\mathbf{p}|}{m},\quad \mathbf{s}=\frac{E}{m}\frac{\mathbf{p}}{|\mathbf{p}|},\quad E^2-|\mathbf{p}|^2=m^2.$$

Then we simply have $v(p,-\lambda)=\gamma_5 u(p,\lambda)$, and $\bar{v}(p,\lambda)=-\bar{u}(p,-\lambda)\gamma_5$. All we have to notice then is that

$$\gamma_5(\delta_{\lambda,-\lambda'}+\gamma_5\mathbf{s}\cdot\boldsymbol{\tau}_{\lambda,-\lambda'})\to\gamma_5(\tau_1)_{\lambda\lambda'}+\mathbf{s}\cdot(\boldsymbol{\tau}\tau_1)_{\lambda\lambda'}$$
$$-(\delta_{-\lambda\lambda'}+\gamma_5\mathbf{s}\cdot\boldsymbol{\tau}_{-\lambda\lambda'})\gamma_5\to-[\gamma_5(\tau_1)_{\lambda\lambda'}-\mathbf{s}(\tau_1\boldsymbol{\tau})_{\lambda\lambda'}].$$

Thus we have

$$u(p',\lambda')\otimes\bar{v}(p,\lambda)=\frac{-1}{[2(mm'+p\cdot p')]^{1/2}}\tfrac{1}{2}(\not p'+m')(\not p+m)\times[\gamma_5(\tau_1)_{\lambda\lambda'}-\mathbf{s}\cdot(\tau_1\boldsymbol{\tau})_{\lambda\lambda'}], \quad (A4)$$

$$v(p',\lambda')\otimes\bar{u}(p,\lambda)=\frac{1}{[2(mm'+p\cdot p')]^{1/2}}\times\tfrac{1}{2}(-\not p'+m')(-\not p+m)[\gamma_5(\tau_1)_{\lambda\lambda'}+\mathbf{s}\cdot(\boldsymbol{\tau}\tau_1)_{\lambda\lambda'}]. \quad (A5)$$

[31] C. Bouchiat and L. Michel, Nucl. Phys. **5**, 416 (1958).
[32] J. Nuyts, Bull. Acad. Roy. Belg. **47**, 566 (1961).
[33] L. Michel, Institut des Hautes Etudes Scientifiques Report, 1963 (unpublished).

It is useful to expand the right-hand side of Eqs. (A4) and (A5) in the basis of 16 γ matrices:

$$\begin{pmatrix}u(p',\lambda')\otimes\bar{v}(p,\lambda)\\ v(p',\lambda')\otimes\bar{u}(p,\lambda)\end{pmatrix}=\frac{\pm1}{2[2(mm'+p\cdot p')]^{1/2}}\times\Big[(\pm A_\mu\gamma^\mu\gamma^5+T_{\mu\nu}\sigma^{\mu\nu}+P\gamma_5)(\tau_1)_{\lambda\lambda'}+(\mathbf{S}1\pm\mathbf{V}_\mu\gamma^\mu\pm\mathbf{A}_\mu\gamma^\mu\gamma_5+\mathbf{T}_{\mu\nu}\sigma^{\mu\nu})\cdot\begin{pmatrix}\tau_1\boldsymbol{\tau}\\ \boldsymbol{\tau}\tau_1\end{pmatrix}_{\lambda\lambda'}\Big], \quad (A6)$$

with

$$\mathbf{S}=-m(p'\cdot\mathbf{s})\,, \quad (A7)$$

$$\mathbf{V}_\mu=(p'\cdot\mathbf{s})p_\mu-(mm'+p\cdot p')\mathbf{s}_\mu\,, \quad (A8)$$

$$A_\mu=mp_\mu'+m'p_\mu\,,\quad \mathbf{A}_\mu=i\epsilon_{\mu\nu\rho\sigma}p^\nu p'^\rho\mathbf{s}^\sigma\,, \quad (A9)$$

$$T_{\mu\nu}=\tfrac{1}{4}\epsilon_{\mu\nu\rho\sigma}(p^\rho p'^\sigma-p'^\rho p^\sigma)\,,$$
$$\mathbf{T}_{\mu\nu}=\tfrac{1}{2}i[(mp_\mu'+m'p_\mu)\mathbf{s}_\nu-\mathbf{s}_\mu(mp_\nu'+mp_\nu)]\,, \quad (A10)$$

$$P=mm'+p\cdot p'\,. \quad (A11)$$

The projector on the triplet state of the outgoing $\mu^+\mu^-$ system has three components:

$$P^{(1)}(p',p)=(1/\sqrt{2})[v(p',+)\otimes\bar{u}(p,-)-v(p',-)\otimes\bar{u}(p,+)],$$
$$P_{\pm1}^{(1)}(p',p)=v(p',\pm)\otimes\bar{u}(p,\pm)\,.$$

The projector on the singlet state is given by

$$P^{(0)}(p',p)=(1/\sqrt{2})[v(p',+)\otimes\bar{u}(p,-)+v(p',-)\otimes\bar{u}(p,+)].$$

Using Eq. (A6), it can be seen that

$$P^{(1)}=\frac{-1}{2(mm'+p\cdot p')^{1/2}}[S^{(3)}1-V_\mu^{(3)}\gamma^\mu-A_\mu^{(3)}\gamma^\mu\gamma_5+T_{\mu\nu}^{(3)}\sigma^{\mu\nu}], \quad (A12)$$

$$P_{\pm1}^{(1)}=\frac{1}{2(mm'+p\cdot p')^{1/2}}[S^{(\mp)}1-V_\mu^{(\mp)}\gamma^\mu-A_\mu^{(\mp)}\gamma^\mu\gamma_5+T_{\mu\nu}^{(\mp)}\sigma^{\mu\nu}], \quad (A13)$$

where, for example, $S^{(\pm)}$ means $(S^{(1)}\pm iS^{(2)})/\sqrt{2}$. For the singlet projector we have

$$P^{(0)}=\frac{1}{2(mm'+p\cdot p')^{1/2}}(-A_\mu\gamma^\mu\gamma_5+T_{\mu\nu}\sigma^{\mu\nu}+P\gamma_5)\,. \quad (A14)$$

The explicit form of $P^{(1)}(p',p)$ needed in the text is the following: With

$$p^2=p'^2=m^2,\quad t=(p+p')^2,$$

and

$$s_\mu^{(3)}=\frac{1}{m}\frac{1}{[t(t-4m^2)]^{1/2}}[tp_\mu-2m^2(p+p')_\mu],$$

we have

$$P^{\text{``out''}(1)}=\frac{1}{2[2(t-4m^2)]^{1/2}}[(t-4m^2)1+2m(p'-p)_\mu\gamma^\mu+i(p_\mu p_\nu'-p_\mu'p_\nu)\sigma^{\mu\nu}]. \quad \text{(A15)}$$

Correspondingly, we have for the singlet projector

$$P^{\text{``out''}(0)}=\frac{1}{2(2t)^{1/2}}[-2m(p+p')_\mu\gamma^\mu\gamma_5+\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}(p^\rho p'^\sigma-p'^\rho p^\sigma)\sigma^{\mu\nu}+t\gamma_5]. \quad \text{(A16)}$$

Sometimes we shall also need the projectors on the incoming $\mu^+\mu^-$ system. If p is the energy momentum of the incoming μ^- and p' of the incoming μ^+, we have

$$P^{\text{``in''}(1)}=\frac{1}{2[2(t-4m^2)]^{1/2}}[(t-4m^2)1+2m(p'-p)_\mu\gamma^\mu+i(p_\mu'p_\nu-p_\mu p_\nu')\sigma^{\mu\nu}], \quad \text{(A17)}$$

$$P^{\text{``in''}(0)}=\frac{-1}{2(2t)^{1/2}}[2m(p+p')_\mu\gamma^\mu\gamma_5+\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}(p'^\sigma p^\sigma-p^\rho p'^\sigma)\sigma^{\mu\nu}+t\gamma_5]. \quad \text{(A18)}$$

APPENDIX B

The integrals over s' in Eqs. (4.43) and (4.50) are best carried out by the following changes of variable:

$$s'=(m_\pi-m_\mu)^2+m_\pi m_\mu(1+y)^2/y,$$
$$s=(m_\pi-m_\mu)^2-m_\pi m_\mu(1-\theta)^2/\theta.$$

Hence the integral over s' in Eq. (4.43) becomes

$$\int_{(m_\pi+m_\mu)^2}^{\infty}\frac{ds'}{s'-s}\{[(m_\pi+m_\mu)^2-s'][(m_\pi-m_\mu)^2-s']\}^{-1/2}=\frac{1}{m_\pi m_\mu}\int_0^1\frac{dy}{(y+\theta)(y+1/\theta)}=\frac{-\theta}{m_\pi m_\mu(1-\theta)^2}\ln\theta \quad \text{(B1)}$$

and the integration of $B'(s',t)$ and $C'(s',t)$ over s' yields

$$\int_{(m_\pi+m_\mu)^2}^{\infty}\frac{ds'}{s'-s}\{[(m_\pi+m_\mu)^2-s'][(m_\pi-m_\mu)^2-s']\}^{-1/2}\ln\left|\frac{[(m_\pi+m_\mu)^2-s'][(m_\pi-m_\mu)^2-s']}{s't}\right|$$
$$=\frac{1}{m_\pi m_\mu}\int_0^1\frac{dy}{(y+\theta)(y+1/\theta)}\left[\ln\left(\frac{m_\pi m_\mu}{t}\right)+\ln\left(\frac{(1-y)^2}{y}\right)+\ln\left(\frac{(1+y)^2}{(a+y)(1/a+y)}\right)\right], \quad \text{(B2)}$$

where $a=m_\mu/m_\pi$. The third integral in Eq. (B2) would be zero if the masses were equal. If we call these integrals R_1, R_2, and R_3, respectively, then, after some algebra we find

$$R_1=\frac{-\theta}{m_\pi m_\mu(1-\theta^2)}\left(\ln\theta\ln\frac{m_\pi m_\mu}{t}\right), \quad \text{(B3)}$$

$$R_2=\frac{-\theta}{m_\pi m_\mu(1-\theta^2)}\left[\tfrac{1}{6}\pi^2+\tfrac{1}{2}\ln^2\theta+\ln\theta\ln\frac{(1+\theta)^2}{\theta}+2\text{Li}_2(-\theta)\right], \quad \text{(B4)}$$

$$R_3=\frac{-\theta}{m_\pi m_\mu(1-\theta^2)}\left[\ln\theta\ln\frac{(1-\theta)^2}{(a-\theta)(1/a-\theta)}+2\text{Li}_2\left(-\frac{1+\theta}{1-\theta}\right)-\text{Li}_2\left(-\frac{1+\theta}{a-\theta}\right)-\text{Li}_2\left(-\frac{1+\theta}{1/a-\theta}\right)\right.$$
$$-2\text{Li}_2\left(-\frac{\theta}{1-\theta}\right)+\text{Li}_2\left(\frac{-\theta}{a-\theta}\right)+\text{Li}_2\left(\frac{-\theta}{1/a-\theta}\right)+2\text{Li}_2\left(\frac{1-\theta}{1+\theta}\right)-\text{Li}_2\left(\frac{1-a\theta}{1+\theta}\right)-\text{Li}_2\left(\frac{1-\theta/a}{1+\theta}\right)-2\text{Li}_2(1-\theta)$$
$$+\text{Li}_2(1-a\theta)+\text{Li}_2(1-\theta/a)+\ln^2\left(\frac{1+\theta}{1-\theta}\right)-\tfrac{1}{2}\ln^2\left(\frac{1+\theta}{1-a\theta}\right)-\tfrac{1}{2}\ln^2\left(\frac{1+\theta}{1-\theta/a}\right)$$
$$\left.-\ln^2(1-\theta)+\tfrac{1}{2}\ln^2(1-a\theta)+\tfrac{1}{2}\ln^2(1-\theta/a)\right]. \quad \text{(B5)}$$

Note that R_3 has been written so that all the dilogarithms and logarithms are real for θ in the range of integration between x' and x''. Finally, we define $\chi(\theta)$ by

$$\chi(\theta)=-m_\pi m_\mu[(1-\theta^2)/\theta](R_1+R_2+R_3). \quad \text{(B6)}$$