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Field-theory approach to the quantum Hall effect

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Fradkin's formulation of statistical field theory is applied to the Coulomb interacting electron gas in a magnetic field. The electrons are confined to a plane in normal three-dimensional space and also interact with the physical three-dimensional electromagnetic field. The magnetic-translation-group Ward identities are derived. By using them, it is shown that the exact electron propagator is diagonalized in the basis of the wave functions of the free electron in a magnetic field whenever the magnetic-translation-group symmetry is unbroken. The general tensor structure of the polarization operator is obtained and used to show that the Chern-Simons action always describes the Hall-effect properties of the system. A general proof of the Strêda formula for the Hall conductivity is presented. It follows that the coefficient of the Chern-Simons terms in the long-wavelength approximation is exactly given by this relation. Such a formula, expressing the Hall conductivity as a simple derivative, in combination with a diagonal form of the full propagator, allows us to obtain a simple expression for the filling factor and the Hall conductivity. Indeed, these results, after assuming that the chemical potential lies in a gap of the density of states, lead to the conclusion that the Hall conductivity is given without corrections by $\sigma_{xy}=\nu e^2/h$, where ν is the filling factor. In addition, it follows that the filling factor is independent of the magnetic field if the value of the chemical potential remains in the gap.

I. INTRODUCTION

The integer quantum-Hall effect (IQHE) and the fractional one (FQHE) have been a subject of very active research in the field of condensed matter physics in this decade.[1] This interest is also shared by quantum-field-theory (QFT) theorists.[2] In this connection it has been stressed that many advances of this research can be further stimulated by a closer collaboration between field-theory and condensed-matter theorists.[1] Most of the theoretical activity in this field was developed in the framework of the many-particle quantum mechanics (QM).[3,4] Relatively few works have been devoted to developing an interacting-field-theory treatment.[5,6]

The present work intends to apply Fradkin's functional approach to quantum statistics to the study of these effects.[7] The general aim is to exploit the generality of those methods to investigate some exact properties of the nonrelativistic Coulomb interacting electron gas confined to a plane in the physical three-dimensional (3D) space.[8]

The main conclusions of the work are organized in order to show that the Hall conductivity is exactly given by the product of the filling factor and e^2/h whenever the Fermi level lies in a gap of the density of states. The effects of impurities are completely disregarded in the present approach.

The diagonalization property of the exact one-particle propagator, shown in a recent paper,[9] and rederived here, helps in simplifying the discussion. The Strêda formula for the Hall conductivity and its equivalence with the coefficient of the Chern-Simons action is obtained in the context of the statistical QFT for the interacting electron gas.[10,11] As the Hall conductivity is given by that relation as a simple derivative of the density with respect to the magnetic field, and the density is also expressed in a simple way thanks to the diagonalization property, closed expressions for the filling factor and Hall conductivity are obtained. Finally, it is argued that when the Fermi level lies in a gap of the density of states, the Hall-conductivity formula

$$\sigma_{xy}=\nu\frac{e^2}{h} ,$$

where ν is the filling factor, is an exact one. In addition, the filling ratio ν is independent of the magnetic field if the Fermi level remains inside the gap.

In the second section the functional approach is presented. Section III is devoted to giving a sketched derivation of the diagonalization property of the propagator in the functional formalism. The tensor structure of the polarization operator is obtained in Sec. IV. It serves for the derivation of the Strêda formula from the finite-temperature QFT in Sec. V. The filling factor and Hall-conductivity expression are obtained in Sec. VI. The proof of the proportionality with the filling factor of the Hall conductivity is given in Sec. VII, and the independence of the filling factor on B when the Fermi energy lies in a gap is shown.

II. FUNCTIONAL APPROACH

As mentioned in the Introduction, in this paper the analysis of the QHE is performed by using Fradkin's functional formulation for statistics as restricted to nonrelativistic systems.[7] We take the quantum 2D-electron plasma embedded in a real 3D plane as described by the following temperature Green's-function generating functional

$$Z=\int D\psi^* D\psi\, e^{(S)}\ , \tag{1}$$

where the action S is given by

$$\begin{aligned} S=\frac{1}{\hbar c}\int dx_4 \Bigg\{ \int \psi^* \left[-c\hbar\partial_4 - \left[\mathbf{p}_\sigma - \frac{e}{c}\mathbf{A}_\sigma^t \right]^2 \Big/ (2m) + \mu + ieA_4^t \right] \psi\, d^2x \\ -\frac{\lambda}{2}\int [\psi^*(x)\psi^*(x')U(\mathbf{x}-\mathbf{x}')\psi(x')\psi(x)]d^2x\, d^2x' + \lambda \int \psi^*(x)\psi(x)U(\mathbf{x}-\mathbf{x}')n_0 d^2x\, d^2x' \Bigg\} \\ +\frac{1}{\hbar c}\int d^3x\, dx_4 \left[-\frac{1}{16\pi}\left[(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{2}{\alpha}(\partial_\mu A_\mu)^2 \right] \right] + \int [\psi^*(x)\eta(x)+\eta^*(x)\psi(x)]d^2x\, dx_4\ . \end{aligned} \tag{2}$$

In (2) the fermion fields ψ and ψ^* are functions of the coordinates x_1 and x_2 of the points in the electron-gas-confinement plane $x_3=0$. The electromagnetic fields A_μ are functions of all the three-space coordinates $\mathbf{x}$. The x_4 arguments for all the fields are real numbers corresponding to the Matsubara theory. The external fermion sources η,η^* are Grassmannian functions of the same kind as their corresponding fields.[7] The parameters e, m, and μ are the electron charge, mass, and the chemical potential, respectively. The Gaussian units are used throughout the work. Other definitions that are needed in (2) and will be required are the following:

$$\begin{aligned} &d^2x=dx_1dx_2\ ,\\ &d^3x=dx_1dx_2dx_3,\quad x_4\in(0,\beta)\ ,\\ &p_\mu=-i\hbar\frac{\partial}{\partial x_\mu}=-i\hbar\partial_\mu\ ,\\ &\mathbf{p}_\sigma=(p_1,p_2,0,0)\ ,\\ &\mathbf{A}_\sigma^t=(A_1^t,A_2^t,0,0),\quad \mathbf{n}=(0,0,1)\ , \end{aligned} \tag{3}$$

where $\beta=c\hbar/(kT)$ (k is the Boltzmann constant and T is the temperature). The total electromagnetic field A_μ^t is defined by

$$A_\mu^t=A_\mu^e(x)+A_\mu(x)\ , \tag{4}$$

in which A_μ^e is the vector potential of the homogeneous magnetic field in the symmetrical gauge

$$\mathbf{A}^e=\tfrac{1}{2}B\mathbf{n}\times\mathbf{x},\quad A_4^e=0\ . \tag{5}$$

The $A_\mu(x)$ field in (4) is the mean value of the quantum electromagnetic field. It needs to be stated that in (1) the electromagnetic field functional integral is absent because it was exactly performed after the static approximation was assumed. This approximation results in the Coulomb interaction four-fermion term in (2). However, the mean field A_μ remains as a dynamical quantity, having its own equations of motion.[7] This field will play an important role in the following discussion. Finally, the term in (2) containing the arbitrary parameter α corresponds to the fixing of the Lorentz-gauge condition in the quantization procedure.[7] Here the value $\alpha=1$ will be selected that simplifies the photon propagator to be the inverse of the d'Alembertian operator. The parameter λ is the coupling constant of the Coulomb interaction.

In (2) the factor n_0 represents the homogeneous background of charges (jellium) that compensates the electron charge density at equilibrium. As usual, the Coulomb interaction potential is damped at large distances in the way

$$U(\mathbf{r})=\frac{\lambda}{|\mathbf{r}|}\exp(-\mu|\mathbf{r}|)\ ,$$

in order to have convergence in the calculations. At the end, the limit $\mu\to 0$ must be taken. The coupling constant value $\lambda=e^2$ must be fixed for concrete calculations. It is worthwhile to remark that some of the conclusions in this paper correspond only to the zero-temperature limit. In each case the validity conditions will be stated explicitly.

III. MAGNETIC TRANSLATION GROUP AND DIAGONALIZATION OF THE MASS OPERATOR

It is well known that the translational-invariance properties of electron in a homogeneous magnetic field are mathematically described by the so-called magnetic translation group (MTG).[12] In Ref. 9 it was also argued that the generators of this group, when represented in the space of states, commute with the Coulombic interaction Hamiltonian. Then it follows that the theory described by the generating functional (1) should retain this symmetry if the ground state does not break it. In this section the Ward identities stemming from the magnetic translation group will be obtained in their functional formulation. By using these relations the exact diagonalization of the mass operator will also be shown. This property occurs in the representation determined by a complete set of solutions of the free-electron problem in the magnetic field. Such a result was also derived in Ref. 9. It should be stated here that the diagonality of the mass operator in QED was proven by Ritus.[13] Then our result constitutes the extension of this conclusion to the nonrelativistic context. It should be mentioned that the study of the Ward identities associated with charge conservation [U(1) symmetry] had been presented in Refs. 14–16. As stressed above, the Ward identities we have presented here are the ones corresponding to the magnetic translational symmetry.

The infinitesimal magnetic-translation-group transformation, leaving the action S in (2) invariant when all the sources vanish, is given as[9]

$$\psi(x)\rightarrow\psi(x)+\frac{1}{i\hbar}b_j\left[p_j+\frac{e}{c}A_j^e\right]\psi(x)\ , \tag{6}$$

$$\psi^*(x)\rightarrow\psi^*(x)+\frac{i}{i\hbar}b_j\left[p_j-\frac{e}{c}A^e\right]\psi^*(x)\ , \tag{7}$$

$$A_\mu(x)\rightarrow A_\mu(x)\ , \tag{8}$$

with b_j, $j=1,2$ being infinitesimal parameters.

After performing the change of variables (6)–(8) in (1), the desired Ward indentities may be obtained as

$$\int d^2x\,dx_4\left[\frac{1}{i\hbar}\left[p_j-\frac{e}{c}A_j^e\right]\frac{\delta Z}{\delta\eta_s(x)}\eta_s(x)+\eta_s^*(x)\frac{1}{i\hbar}\left[p_j+\frac{e}{c}A_j^e\right]\frac{\delta Z}{\delta\eta_s^*(x)}\right]-\int d^3x\,dx_4\left[\frac{\delta Z}{\delta A_\mu(x)}\partial_j A_\mu(x)\right]=0\ . \tag{9}$$

In order to arrive at (9) the translational invariance of the free electromagnetic action term in (2) was employed. The Grassmann functional derivatives over η and η^* in (9) are of the "right" and "left" types, respectively.[17] In terms of the generating functional of the connected Green functions

$$W=\ln Z\ , \tag{10}$$

the relation (9) is transformed into

$$\int d^2x\,dx_4\left[-\frac{1}{i\hbar}\left[G_j^*(x)\frac{\delta W}{\delta\eta_s(x)}\right]\eta_s(x)+\frac{1}{i\hbar}\eta^*(x)\left[G_{j(x)}\frac{\delta W}{\delta\eta_s^*(x)}\right]\right]-\int d^3x\,dx_4\left[\frac{\delta W}{\delta A_\mu(x)}\partial_j A_\mu(x)\right]=0\ , \tag{11}$$

where the following notation for the generators of the MTG has been introduced:

$$G_j(x)=p_j+\frac{e}{c}A_j^e,\quad G_j^*(x)=-p_j+\frac{e}{c}A_j^e\ . \tag{12}$$

Let us now apply the Ward identities (11) to the proof of the diagonalization of the mass operator in the basis of the free-electron eigenfunctions in the magnetic field.[9]

After taking the derivatives of (11) with respect to two functional arguments $\eta^*(x),\eta(x')$ and making all the sources and the field A_μ vanish, the Ward identity for the one-particle propagator is obtained in the form

$$G_j(x)G_{rs}(x,x')=G_{rs}(x,x')\overleftarrow{G}_j^*(x')\ , \tag{13}$$

where the arrow means that the derivative is acting on the left.

Relation (13) expresses that the generator of the magnetic translation group commutes with the exact Green's function. It may be also shown by acting with $G_j(x)$ on (13) that the following relation is valid:

$$G^2(x)G_{rs}(x,x')=G_{rs}(x,x')[\overleftarrow{G}^2(x')]^*\ , \tag{14}$$

where it has been defined as

$$G^2(x)=G_i(x)G_i(x)\ . \tag{15}$$

Let us introduce the operator

$$H=\left[p_i-\frac{e}{c}A_i^e\right]\left[p_i-\frac{e}{c}A_i^e\right]\ , \tag{16}$$

which is proportional to the one-particle Hamiltonian and also defines the expression for the free-electron Green propagator. By using the definition of G_j in (12) and the explicit expression (5) for the vector potential A_i^e, the following equation can be obtained:

$$H=G^2-\frac{2eB}{c}L_3\ , \tag{17}$$

in which L_3 is the third component of the angular momentum operator for a free particle

$$L_3=\varepsilon^{ij3}x_ip_j\ . \tag{18}$$

Among the three operators H, G^2, and L_3, the following commutation relations can be obtained:

$$[G^2,H]=0\ , \tag{19}$$

$$[G^2,L_3]=0\ , \tag{20}$$

$$[H,L_3]=0\ . \tag{21}$$

It also follows that the rotational invariance of system allows us to show, through the use of its corresponding Ward identities, the additional relation[9]

$$L_3(x)G_{rs}(x,x')=G_{rs}(x,x')\overleftarrow{L}_3(x')\ . \tag{22}$$

Therefore, after using (19)–(22) and (14), the following relation is obtained:

$$H(x)G_{rs}(x,x')=G_{rs}(x,x')\overleftarrow{H}(x')\ . \tag{23}$$

The identities (22) and (23) imply that the eigenfunctions of the Green's-function kernel can be selected as the common set of eigenfunctions of the free Hamiltonian and the angular momentum operators. More details about this result can be found in Ref. 9. This property of the exact one-particle Green's function, as mentioned before, becomes the extension to the nonrelativistic (and statistical) framework of the analogous result derived by Ritus for QED.[13] It should be mentioned that this conclusion is in no way restricted to the 2D-electron gas. The argumentation works equally well for the 3D-electron system.

From the commutativity of the inverse of the free propagator of the system with H and L_3 and (23), the diagonalization of the exact mass operator in the basis of common eigenfunctions of H and L_3 immediately follows. The explicit form for the propagator in the temporal Fourier representation takes the form[9]

$$G_{\alpha\beta}(\mathbf{x},\mathbf{x}',k_4)=\delta_{\alpha\beta}\sum_{n=0}^{\infty}G_n(k_4)\varphi_n^0(0)\varphi_n^{0*}(\mathbf{x}'-\mathbf{x})\times\exp\left[\frac{ie\,\mathbf{A}^e(\mathbf{x})}{\hbar c}\cdot(\mathbf{x}-\mathbf{x}')\right], \quad (24)$$

where $\varphi_n^m(\mathbf{x})$ are the normalized eigenfunctions of the free-electron problem. In arriving at (24) the sum over the angular momentum eigenvalues was explicitly calculated by means of the formula[18]

$$\sum_{m=-\infty}^{n}\varphi_n^m(\mathbf{x})\varphi_n^{m*}(\mathbf{x}')=\varphi_n^0(0)\varphi_n^{0*}(\mathbf{x}'-\mathbf{x})\times\exp\left[\frac{ie}{\hbar c}\mathbf{A}^e(\mathbf{x})\cdot(\mathbf{x}-\mathbf{x}')\right], \quad (25)$$

where the sum runs down to $-\infty$ because we have considered the magnetic field in the positive x_3 axis direction and the electric charge $e<0$.

The diagonal form (24) is a greatly simplifying result. It expresses the fact that the spatial dependence of the propagator is kinematically fixed. That is, similar to the way in which the translational invariance in the absence of magnetic field allows the Fourier decomposition of the propagator, the MTG, when the field is present, determines the spatial dependence of the propagator in terms of the Laguerre functions. Formula (24) is also a generalization of the results of Girvin and McDonald for one-particle density matrices.[19,20] The generalization of (24) to the case of crossed electric and magnetic fields has been presented in Ref. 21.

IV. LINEAR RESPONSE AND THE CHERN-SIMONS TERM

In Ref. 8 the general tensor structure of the polarization operator characterizing the linear-response properties of the electromagnetic field was calculated. This result allowed the authors to point out the relevance of the Chern-Simons action for the description of QHE. The argumentation was performed in the one-loop approximation. It may be considered that one of the central aims of the present work is to present the generalization of the above conclusions of Ref. 8 to all orders in perturbation theory for IQHE and FQHE.

In this section the expression of the polarization tensor Π in the functional approach is presented. After passing to the Fourier representations and using the transversality property, the general tensor structure is obtained in terms of the characteristic vectors of the problem. Then when the long-wavelength approximation is considered, it is shown that the Chern-Simons (CS) action always describes the Hall-effect properties of the system. A formula for the Hall conductivity (or what is the same as the coefficient of the Chern-Simons terms) is also obtained. It serves in the next section to obtain a derivation of the Strěda formula for the Chern-Simons term coefficient in the context of statistical QFT.[11] The equation of motion for the mean electromagnetic field $A_\mu(x)$ is given in Fradkin's approach by[7]

$$\left.\frac{\delta W[\eta^*,\eta,A]}{\delta A_\mu(x)}\right|_{\eta^*=\eta=0}=\frac{ien_0}{c\hbar}u_\mu\,,\qquad W=\ln Z,\quad u_\mu=(0,0,0,1)\,, \quad (26)$$

where the fermion external sources vanish. The relation (26) is a highly nonlinear one. The corresponding equations for the small perturbations of the background magnetic field are obtained by performing a functional expansion in A_μ and retaining the linear terms. Then for the expansion of W up to quadratic terms we have

$$W[0,0,A]=W[0,0,0]+\int\left.\frac{\delta W[0,0,A]}{\delta A_\mu}\right|_{A=0}A_\mu(x)d^4x+\tfrac{1}{2}\int A_\mu(x)\left.\frac{\delta^2 W}{\delta A_\mu(x)\delta A_\nu(x)}\right|_{A=0}\times A_\nu(y)d^4x d^4y+O(A^3)\,. \quad (27)$$

After substituting into (26) it follows that

$$\left[\frac{\delta W[0,0,A]}{\delta A_\mu(x)}\right]_{A=0}+\int\left[\frac{\delta^2 W[0,0,A]}{\delta A_\mu(x)\delta A_\nu(y)}\right]_{A=0}\times A_\nu(y)d^4y+O(A^2)=\frac{ien_0}{c\hbar}u_\mu\,. \quad (28)$$

Thus under the assumption of no spontaneous breaking of the symmetry of the external magnetic field it follows that

$$\left[\frac{\delta W}{\delta A_\mu(x)}\right]_{\eta,\eta^*,A=0}=\frac{ien_0}{c\hbar}u_\mu\,. \quad (29)$$

In physical words, this condition expresses the assumption that the system does not develop any internal electromagnetic field in addition to the constant magnetic field. That is, the zero field $A_\mu=0$ must be a solution of the quantum equation of motion (26).

The linear Maxwell equations coming from (28) after performing the functional derivatives of W take the form

$$\frac{1}{4\pi\hbar c}\partial^2 A_\mu(x)+\int\Pi_{\mu\nu}(x,x')A_\nu(x')d^4x'=0\,. \quad (30)$$

In arriving at (30) the value $\alpha=1$ was substituted for the gauge parameter. The polarization tensor $\Pi_{\mu\nu}$ in (30) takes the explicit form

$$\Pi_{\mu\nu}(x,x')=\frac{e^2}{\hbar mc^3}\delta(x_3)\delta(x_3')\delta^{(3)}(x-x')\frac{P_{\mu\nu}\delta^2 Z}{Z\delta\eta(x^+)\delta\eta^*(x)}\Bigg|_{\eta,\eta^*,A=0}$$
$$-\delta(x_3)\left[\frac{ieu_\nu}{\hbar c}+\frac{eP_{\mu\alpha}}{2\hbar mc^2}\left[p_\alpha(x)-p_\alpha(x^+)-\frac{2e}{c}A^e_\alpha(x)\right]\right]\frac{1}{Z}\frac{\delta^3 Z}{\delta\eta_s(x^+)\delta\eta_s^*(x)\delta A_\nu(x')}\Bigg|_{\eta,\eta^*,A=0}, \quad (31)$$

where the electron-gas four-velocity[7] [as given in (26)],

$$u_\mu=(0,0,0,1)=\delta_{\mu 4}, \quad (32)$$

has been introduced in addition, with the projection on the gas plane Lorentz tensor

$$P_{\mu\nu}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}=\delta_{\mu 1}\delta_{\nu 1}+\delta_{\mu 2}\delta_{\nu 2}. \quad (33)$$

In (31) the three-dimensional Dirac δ function is defined by

$$\delta^{(3)}(x-x')=\delta(x_1-x_1')\delta(x_2-x_2')\delta(x_4-x_4'). \quad (34)$$

The special δ functions in the x_3 coordinates reflect the loss of translational invariance implied by the confinement of the gas to the plane $x_3=0$. Finally, the x^+ four-vector is defined as

$$x_\mu^+=x_\mu+\delta u_\mu, \quad \delta>0, \quad (35)$$

where the limit $\delta\to 0$ is implicitly understood in (31). This variable takes care of the correct ordering of the operators to which the functional derivatives are associated. It may be useful to remember that the spatial dependence of the fermion variables is always on the variables x_1, x_2, and x_4, with x_3 excluded.

The tensor $\Pi_{\mu\nu}$ obeys the so-called transversality condition.[7] It can be directly deduced from the Ward identity associated with the gauge invariance and has a close relation with charge conservation. In the coordinate representation it reads as

$$\partial_\mu\Pi_{\mu\nu}(x,x')=0. \quad (36)$$

Relation (36) will play an important role in fixing a close form for the tensor structure of $\Pi_{\mu\nu}$.

Let us now introduce the momentum representation in the variables of the (x_1,x_2) plane and the Matsubara variable x_4 in the following way:

$$\Pi_{\mu\nu}(k,x_3,x_3')=\int d^2x\,dx_4 e^{-ik_\alpha(x_\alpha-x_\alpha')}\Pi_{\mu\nu}(x,x'), \quad (37)$$

$$\Pi_{\mu\nu}(x,x')=\int\frac{d^2k\,dk_4}{(2\pi)^3}e^{ik_\alpha x_\alpha}\Pi_{\mu\nu}(k,x_3,x_3'), \quad (38)$$

where the four-momentum k is defined as $k_\mu=(k_1,k_2,0,k_4)$, and the translational invariance in the plane of the gas has been considered in (37) by assuming that Π only depends on the difference of the variables $x_\alpha-x_\alpha'$, $\alpha=1$, 2, and 4. In the x_3 variable there is no such invariance.

The explicit dependence of $\Pi_{\mu\nu}$ on the x_3 variable may be exactly obtained. For this purpose it should be noted that performing the functional derivative over A in (31) gives rise to x' and ν dependences, which are symmetrical with respect to the x and μ ones. Thus a global factor $\delta(x_3)\delta(x_3')$ appears that completely defines the dependence of Π on x_3 and x_3'. Furthermore, it also follows that

$$\Pi_{3\alpha}(x,x')=\Pi_{\alpha 3}(x,x')=0, \quad \alpha=1,2,3,4. \quad (39)$$

Thus the polarization tensor takes the form

$$\Pi_{\mu\nu}(k,x_3,x_3')=\Pi_{\mu\nu}(k)\delta(x_3)\delta(x_3'), \quad (40)$$

with

$$n_\alpha\Pi_{\alpha\mu}(k)=\Pi_{\mu\alpha}(k)n_\alpha=0 \quad (41)$$

expressing the vanishing of all the components with an index equal to 3 in terms of the four-vector normal to the gas plane

$$n_\mu=(0,0,1,0). \quad (42)$$

The transversality property (36) takes the form

$$k_\alpha\Pi_{\alpha\mu}(k)=\Pi_{\mu\alpha}(k)k_\alpha=0. \quad (43)$$

The conclusion arises that the linear-response properties of this problem are described by a special tensor $\Pi_{\mu\nu}(k)$. It has the same basic properties as the one corresponding to a purely two-dimensional electron gas interacting with an also 2D electromagnetic field.[22,23]

The remaining part of tensor structure of $\Pi_{\mu\nu}(k)$ may be expressed in terms of three scalar functions by using the transversality property (43). After performing some algebraic operations, the following result may be derived:

$$\Pi_{\mu\nu}(k)=\frac{[\pi_1(k)+\pi_2(k)]}{k_4^2}l_\mu^{(1)}l_\nu^{(1)}+\pi_1(k)l_\mu^{(2)}l_\nu^{(2)}+\frac{\pi_3(k)}{k_4}\varepsilon^{\mu\alpha\nu\beta}n_\alpha k_\beta, \quad (44)$$

where the newly defined four-vectors are given in the rest frame of the gas by the expressions

$$l_\mu^{(1)}=\left[\frac{k_4\mathbf{k}}{|\mathbf{k}|},-|\mathbf{k}|\right], \quad (45)$$

$$l_\mu^{(2)}=\left[\frac{\mathbf{n}\times\mathbf{k}}{|\mathbf{k}|},0\right], \quad (46)$$

where

$$\mathbf{n}=(0,0,1)\ , \tag{47}$$

$$\mathbf{k}=(k_1,k_2,0)\ . \tag{48}$$

The scalar functions in (44) satisfy the following relations:

$$\pi_1(k)=\pi_1(-k)\ , \tag{49}$$

$$\pi_2(k)=\pi_2(-k)\ , \tag{50}$$

$$\pi_3(k)=-\pi_3(-k)\ . \tag{51}$$

The main information in (44) is that the last contribution in the sum that breaks the space-time inversion as implied by (51) is exactly the Chern-Simons term when π_3/k_4 is taken in the zero-momenta limit. Therefore, the conclusion arises that for 2D electron system the Chern-Simons action describes the Hall-effect properties, no matter whether at the quantized or normal regimes. It is also interesting that the Chern-Simons appearance of the parity breaking term in (44) is a direct consequence of the gauge invariance as expressed by the transversality relations (43). The connection of the Chern-Simons action with the QHE was argued for the first time in Refs. 22–24. Here it is also argued that the Hall conductivity is always described by the Chern-Simons terms as a simple consequence of gauge invariance and two dimensionality. In addition, it follows that the embedding of the 2D electron gas into the real three-dimensional space does not destroy this result.

V. STRÊDA FORMULA FOR INTERACTING ELECTRONS AS THE COEFFICIENT OF THE CS ACTION

The main objective of this section will be the derivation of the Strêda formula for the static value of the Hall conductivity for an interacting electron gas in the framework of statistical QFT. The formula for the conductivity tensor in terms of the polarization operator is given by[7]

$$\sigma_{ij}(k)=\frac{\hbar c^2}{k_4}\Pi_{ij}(k)\ , \tag{52}$$

in which the δ-function structure in the x_3 variables is not considered as being common to all the $\Pi_{\mu\nu}$ components. Such a dependence only expresses the physical fact that all the internal currents and charges are confined to the plane $x_3=0$.

The interest here is in the static value of the Hall conductivity tensor, which is determined in the zero-momenta limit (after the analytical continuation in the k_4 variable[7]) by

$$\begin{aligned}\sigma_{ij}^{(H)}&=\lim_{|\mathbf{k}|\to 0}\left[\lim_{k_4\to 0}\left[\frac{\pi_3(k)}{k_4^2}\varepsilon^{i\alpha j\beta}n_\alpha k_\beta\right]\right]\hbar c^2\\&=\lim_{|\mathbf{k}|\to 0}\left[\lim_{k_4\to 0}\left[\frac{\pi_3(k)}{k_4}\right]\right]\varepsilon^{i3j}\hbar c^2\\&=\sigma_{xy}\varepsilon^{ij3}\ .\end{aligned} \tag{53}$$

After expressing π_3 in terms of the Π_{ij} components we have

$$\begin{aligned}\frac{\pi_3(k)}{k_4}&=-\frac{1}{|\mathbf{k}|k_4}l_i^{(2)}\Pi_{ij}k_j\\&=-\frac{1}{\mathbf{k}^2k_4}\varepsilon^{ilm}n_lk_m\Pi_{ij}k_j\\&=\frac{1}{\mathbf{k}^2}\varepsilon^{ilm}n_lk_m\Pi_{i4}\ ,\end{aligned} \tag{54}$$

where relation (46) for $l_i^{(2)}$ and the transversality condition have been used. After derivating the transversality relation over the spatial momenta and taking the limit $k_4\to 0$, the following formula is obtained:

$$\Pi_{i4}=-k_l\frac{\partial\Pi_{l4}}{\partial k_i}\Bigg|_{k_4\to 0}\ , \tag{55}$$

which when substituted in (53) and using

$$\Pi_{i4}(k)=\int d^2r\,dr_4e^{-i\mathbf{k}\cdot\mathbf{r}}\Pi_{i4}(r)\ , \tag{56}$$

allows us to write the relation

$$\sigma_{xy}=-\hbar c^2\lim_{|\mathbf{k}|\to 0}\left[\int d^2r\,dr_4e^{-i\mathbf{k}\cdot\mathbf{r}}\frac{n_lk_mk_j}{\mathbf{k}^2}\times(-ir_i)\Pi_{j4}(r)\varepsilon^{ilm}\right]\ , \tag{57}$$

in which, according to (31) and the translation invariance,

$$\begin{aligned}\Pi_{j4}(r)&=-\frac{e}{2\hbar mc^2}\left[p_j(r)-p_j(r^+)-2\frac{e}{c}A_j^e(r)\right]\\&\quad\times\int\frac{1}{Z}\frac{\delta^3Z}{\delta\eta_s(r^+)\delta\eta_s^*(r)\delta A_4(0,x_3')}dx_3'\\&=\left[\int\frac{\delta}{\delta A_j(r,x_3)}\left[\frac{1}{Z}\frac{\delta Z}{\delta A_4(0,x_3')}\right]\times dx_3dx_3'\right]_{\eta,\eta^*,A=0}\ .\end{aligned} \tag{58}$$

Before continuing, let us note that the limit in (57) may be taken by fixing an arbitrary direction for the vector $\mathbf{k}$ in the plane. Then we may define two orthogonal unit vectors $t_i^{(1)}$ and $t_i^{(2)}$, which consequently obey

$$t_i^{(1)}t_j^{(1)}+t_i^{(2)}t_j^{(2)}=\delta_{ij}\ . \tag{59}$$

By considering the limit (57) for the $\mathbf{k}$ direction along each of the unit vectors $t^{(i)}$, $i=1,2$, and performing the semisum of both expressions, it follows that

$$\sigma_{xy}=-\frac{\hbar c^2}{2}\lim_{|\mathbf{k}|\to 0}\left[\int d^2r\,dr_4e^{-i\mathbf{k}\cdot\mathbf{r}}(-i)\varepsilon^{ilj}\times n_lr_i\Pi_{j4}(r)\right]\ . \tag{60}$$

The substitution of (58) into (60) gives

$$\sigma_{xy}=-\frac{\hbar c^2}{2}\lim_{|\mathbf{k}|\to 0}\int d^2r\, dr_4 e^{-i\mathbf{k}\cdot\mathbf{r}}(-i)\varepsilon^{ilj}n_l r_i \times\left[\int \frac{\delta}{\delta A_j(r,x_3)}\left[\frac{1}{Z}\frac{\delta Z}{\delta A_4(0,x_3')}\right]\times dx_3 dx_3'\right|_{\eta,\eta^*,A=0}, \quad (61)$$

which after considering

$$\frac{dA_i^e(r)}{dB}=\tfrac{1}{2}\varepsilon^{ilj}n_l r_j \quad (62)$$

takes the form

$$\sigma_{xy}=-\hbar c^2 i\int d^2r\, dr_4\frac{dA_i^e(r)}{dB}\left[\frac{\delta}{\delta A_i(r,x_3)}\left[\frac{1}{Z}\frac{\delta Z}{\delta A_4(0,x_3')}\right]\right]\Bigg|_{\eta,\eta^*,A=0} =-\hbar c^2 i\int\frac{d}{dB}\left[\frac{1}{Z}\frac{\delta Z}{\delta A_4(0,x_3')}\right]\Bigg|_{\eta,\eta^*,A=0}dx_3' . \quad (63)$$

But

$$\frac{1}{Z}\frac{\delta Z}{\delta A_4(0,x_3')}=-\frac{ie}{\hbar c}\frac{1}{Z}\frac{\delta^2 Z\delta(x_3')}{\delta\eta_s(0^+)\delta\eta_s^*(0)} =i\frac{e}{\hbar c}n(0)\delta(x_3') , \quad (64)$$

where $n(0)$ is the density of particles. Thus, the substitution of (64) into (63) gives the Strêda formula

$$\sigma_{xy}=ec\frac{dn}{dB}\bigg|_{\mu,T=\text{const}} . \quad (65)$$

The relation (65) expresses the Hall conductivity as a simple derivative of the density of particles, but the density also has a simple expression in terms of the fermion Green's function. Thus the obtained diagonalization property of the exact propagator can further simplify the discussion of the Hall conductivity. The result (65) is valid at $T\neq 0$.

VI. THE FILLING-FACTOR FORMULA AND OTHER RELATIONS

According to (24) we have for the fermion Green's function

$$G_{\alpha\beta}(x,x')=\frac{1}{Z}\frac{\delta^2 Z}{\delta\eta_\alpha^*(x)\delta\eta_\beta(x')} =\delta_{\alpha\beta}\sum_{n=0}^{\infty}G_n(x_4-x_4')\varphi_n^0(0)\varphi_n^{0*}(\mathbf{x}-\mathbf{x}') \times\exp\left[i\frac{e\,\mathbf{A}^e(\mathbf{x})}{\hbar c}(\mathbf{x}-\mathbf{x}')\right] . \quad (66)$$

Therefore for the density of particles it results to

$$n=-\frac{1}{Z}\frac{\delta^2 Z}{\delta\eta_s(0^+)\delta\eta_s^*(0)} =-2\sum_{n=0}^{\infty}G_n(-\delta)|\varphi_n^0(0)|^2\bigg|_{\delta\to 0^+} , \quad (67)$$

where in the zero-temperature limit

$$\lim_{\delta\to 0^+}G_n(-\delta)=\lim_{\delta\to 0^+}\int\frac{dk_4}{2\pi}\frac{e^{-ik_4\delta}\hbar c}{ic\hbar k_4+\varepsilon_n-\mu+\sigma_n(k_4)} . \quad (68)$$

In (68) the $\sigma_n(k_4)$ are the eigenvalues of the mass operator associated with the particular eigenfunction φ_n^m. The independence of σ_n of m can be shown by using the results of the Sec. III.[9]

After considering that

$$|\varphi_n^0(0)|^2=\frac{1}{2\pi r_0^2}=\frac{|eB|}{hc} ,$$

the density n can be written as follows:

$$n=-2\frac{|eB|}{hc}\sum_{n=0}^{\infty}\lim_{\delta\to 0^+}\int_{-\infty}^{\infty}\frac{dk_4}{2\pi}\frac{\hbar c\, e^{-ik_4\delta}}{ic\hbar k_4+\varepsilon_n-\mu+\sigma_n(k_4)} \equiv\frac{|eB|}{hc}\nu , \quad (69)$$

in which the filling factor ν is defined.

As a matter of checking, it is possible to disregard the $\sigma_n(k_4)$ in (69). In such a case the integral in it may be readily calculated to give (if μ lies in the gap between two Landau levels)

$$n=\frac{2|eB|}{hc}\sum_{n=0}^{\infty}\Theta(\mu-\varepsilon_n) , \quad (70)$$

in which $\Theta(x)$ is the Heaviside function. The result (70) is the expected one in the tree approximation for the Green's function.

In continuing, let us examine the value of the conductivity predicted by the use of the Strêda formula (65) and the expression (69) for the density. Substituting (69) into (65),

$$\sigma_{xy}=ec\frac{dn}{dB}\bigg|_{\mu,T=\text{const}}=ec\frac{d}{dB}\left[\nu\frac{eB}{hc}\right]$$
$$=\frac{e^2}{h}\frac{d}{dB}(\nu B)=\frac{e^2}{h}\left[\nu+B\frac{d\nu}{dB}\right]_{\mu,T=\text{const}} . \quad (71)$$

Hence the Hall conductivity is exactly proportional to the filling factor ν if this magnitude does not depend on B.

Below, a relation expressing $d\nu/dB$ in terms of the derivatives over μ and λ will be obtained. Note that under the changes of variables

$$q=k_4\frac{c}{\omega_0},\quad \mu'=\mu/(\hbar\omega_0),\quad \omega_0=\frac{|eB|}{mc} , \quad (72)$$

the filling factor expresses as

$$\nu=-\sum_{n=0}^{\infty}\lim_{\delta'\to 0}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{2e^{-iq\delta'}}{iq+n+\frac{1}{2}-\mu'+\sigma_n/(\hbar\omega_0)} , \quad (73)$$

where the mass-operator eigenvalue satisfies with independence of m (Ref. 9) the relation

$$\sigma_n(k_4)=\int\int\varphi_n^{m*}(x)\Sigma(\mathbf{x},\mathbf{x}',k_4)\varphi_n^m(\mathbf{x}')d^2x\,d^2x' . \quad (74)$$

In (74) the trivial spinor structure of the wave function and Σ are not considered. That is,

$$\Sigma_{\alpha\beta}(\mathbf{x},\mathbf{x}',k_4)=\delta_{\alpha\beta}\Sigma(\mathbf{x},\mathbf{x}',k_4) . \quad (75)$$

Let us consider now the generating functional (1) evaluated at zero electromagnetic field. In this case it generates the fermionic Green's functions of the problem, in particular the one-electron one. Now it is worthwhile to introduce a new set of dimensionless integration fields and space-time variables according to

$$\mathbf{x}\to r_0\mathbf{z} , \quad (76)$$

$$x_4\to\frac{c}{\omega_0}z_4 , \quad (77)$$

$$\psi(x)\to\psi(z)/r_0 , \quad (78)$$

$$\psi^*(x)\to\psi^*(z)/r_0 . \quad (79)$$

After that, the generating functional in the new variables takes the form

$$Z[\xi,\xi^*]=\int D\psi^*(z)D\psi(z)e^S$$

where

$$S=\int dz_4\left\{\int\psi^*(z)\left[-\frac{\partial}{\partial z_4}-\frac{1}{2}\left[-i\nabla+\frac{\mathbf{n}\times\mathbf{z}}{2}\right]^2+\mu'\right]\psi(z)d^2z\right.$$
$$-\frac{1}{2}\int\psi^*(z)\psi^*(z')\frac{\lambda'}{|z-z'|}\psi(z')\psi(z)d^2z\,d^2z'$$
$$\left.+\int\psi^*(z)\psi(z)\frac{\lambda'}{|z-z'|}r_0^2n_0d^2z\,d^2z'\right\}+\int[\xi^*(z)\psi(z)+\psi^*(z)\xi(z)]d^2z\,dz_4 , \quad (80)$$

where, taking into account (29) and (67),

$$r_0^2n_0=\left[\frac{1}{Z}\int D\psi^*(z)D\psi(z)\psi^*(0^+)\psi(0)e^S\right]_{\xi,\xi^*=0} , \quad (81)$$

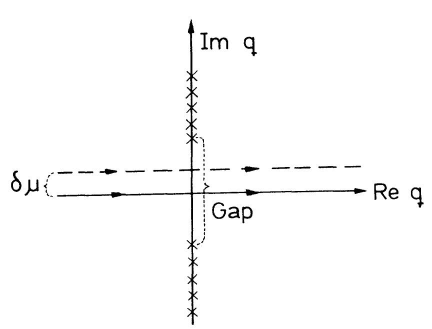

FIG. 1. Chemical-potential increment $\delta\mu$ as a shift of the integration contour in (85).

and new auxiliary sources ξ and ξ^* have been introduced. The parameters μ' and λ' are given by

$$\mu'=\frac{\mu}{\hbar\omega_0},\quad \lambda'=\frac{\lambda}{r_0\hbar\omega_0} . \quad (82)$$

Thus, at $T=0$, any fermionic Green's function with p legs, after being multiplied by r_0^p is a function of the parameters B, μ, and λ only through the adimensional constants λ' and μ'. It also follows that in (73) $\sigma_n/(\hbar\omega_0)$ is the eigenvalue of the mass operator calculated within the transformed generating functional description. Then $\sigma_n/(\hbar\omega_0)$ is a function of B only through μ' and λ'. This fact leads to the following relation:

$$\frac{d\nu}{dB}\bigg|_{\mu,\lambda=\text{const}}=-\frac{1}{B}\left[\mu\frac{\partial\nu}{\partial\mu}+\frac{1}{2}\lambda\frac{\partial\nu}{\partial\lambda}\right]$$
$$=-\frac{\hbar}{2B^2|e|}\left[\mu\frac{\partial n}{\partial\mu}+\frac{1}{2}\lambda\frac{\partial n}{\partial\lambda}\right] . \quad (83)$$

Thus the B derivative of ν expresses linearly in terms of the derivatives of n with respect to μ and λ. It is necessary to stress that such a result assumes that the system satisfies the equilibrium equation (29) when calculating the derivatives. In physical terms, this means that the background of compensating charges maintains the system's neutrality upon variation of the parameters. It is apparent that this condition is strongly connected with the plateau's stability and the role of impurities as a heat bath.

In this work we have discussed the consequences of the symmetries for the clean samples. However, the generalization of the results along the lines of the one presented in Refs. 14–16 for the electric [U(1) symmetry] charge in the presence of impurities seems feasible and is worthwhile to investigate.

VII. THE EXACT FORMULA $\sigma=\nu(e^2/h)$ WHEN μ LIES IN A GAP

In this section it will be argued that the both derivatives

$$\left.\frac{\partial n}{\partial \mu}\right|_{B,\lambda=\text{const}}$$

and

$$\left.\frac{\partial n}{\partial \lambda}\right|_{B,\mu=\text{const}}$$

vanish under the condition that the Fermi level lies in a gap of the density of states. If such is the case, then from (71) arises the exact result

$$\sigma_{xy}=\frac{e^2}{h}\nu=\frac{e^2}{h}\frac{n}{2\pi r_0^2} . \tag{84}$$

Let us analyze the expression (69) for the density at $T=0$,

$$n=-2\frac{|eB|}{hc}\sum_{n=0}^{\infty}\lim_{\delta\to 0^+}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{e^{-iq\delta}}{iq-\mu'+n+\frac{1}{2}+\sigma_n^*} , \tag{85}$$

where σ_n^* depends on q and μ' in the way

$$\sigma_n^*(q,\mu',\lambda')=\sigma_n^*(iq-\mu',\lambda') , \tag{86}$$

as may be seen from the modified generating functional (80).

It can be noticed that, as the Lehmann representation implies, the singularities of the integrand in (85) must correspond to a branch cut or a set of poles along the imaginary axis. In statistical QFT, however, the energy of the excitations is measured from the Fermi energy. Therefore, the condition for a gap is that a sufficiently small open interval of the imaginary axis including the origin does not contain any pole or branch cut. However, as may be seen from (85) and (86), a small change of μ (or μ') is equivalent to a parallel shift (because the integrand depends on $iq-\mu'$) of the integration contour. Thus, under the validity of the above condition for a gap, that deformation does not alter the result of the integral. In this way it follows that

$$\left.\frac{dn}{d\mu}\right|_{B,\lambda=\text{const}}=0 . \tag{87}$$

Figure 1 graphically shows the above description.

Now, let us consider the $\partial n/\partial\lambda$ derivative. For this purpose the following expression for the density in terms of the thermodynamical potential will be used:

$$n=-\frac{1}{V}\left.\frac{\partial\Omega}{\partial\mu}\right|_{T,V,B=\text{const}} , \tag{88}$$

where Ω is defined by

$$\Omega=-kT\ln Z . \tag{89}$$

Thus $\partial n/\partial\lambda$ may be written as follows:

$$\frac{\partial n}{\partial\lambda}=-\frac{1}{V}\frac{\partial}{\partial\mu}\left[\frac{\partial\Omega}{\partial\lambda}\right]_{T,V,B=\text{const}} . \tag{90}$$

For the λ derivative of Ω the following expression can be obtained by standard methods:[25]

$$\frac{\partial\Omega}{\partial\lambda}=\frac{1}{2\lambda}\int d^2x\lim_{\mathbf{x}'\to\mathbf{x}}\lim_{x_4'\to x_4^+}\left[-c\hbar\frac{\partial}{\partial x_4}-\left[\mathbf{p}-\frac{e}{c}\mathbf{A}^e(x)\right]^2\Big/(2m)+\mu\right]G_{\alpha\alpha}(x,x') . \tag{91}$$

After substituting the diagonal form (24) for the exact propagator, the following expression can be obtained in the zero-temperature limit:

$$\frac{\partial n}{\partial\lambda}=c\frac{\partial}{\partial\mu}\left[\sum_{n=0}^{\infty}\lim_{\delta\to 0^+}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{(-iq+\mu'-n-\frac{1}{2})e^{-iq\delta}}{-iq+\mu'-n-\frac{1}{2}-\sigma_n^*(q)}\right] , \tag{92}$$

where c is a constant. Relation (92) expresses $\partial n/\partial\lambda$ as a derivative over μ of an integration of the kind similar to (85). Therefore, the small changes of μ are again equivalent to parallel shifts of the integration axis in the complex q plane. Since in this process, by assumption, the axis is not passing over any singularity, the result of the integration is unchanged, and thus $\partial n/\partial\lambda$ vanishes. Then the independence of the filling factor from the magnetic field follows, as well as the exactness of the formula $\sigma_{xy}=\nu e^2/h$.

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