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Resolvent approach to the Volterra equation as a tool for EAS modeling

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Transport equations for cascading particles in extensive air shower of cosmic rays can be transformed to Volterra equations of the second kind. The numerical resolvent for equation is constructed on the two-dimensional lattice in the case of longitudinal development of shower. The method can be used as an efficient alternative to Monte Carlo technique becoming cumbersome at the highest energies.

1. Introduction

Energy spectrum of primary cosmic radiation (PCR) extends beyond 10^{20} eV where the only observational method is detection of extensive air showers (EAS) produced by cosmic rays (CRs) in atmosphere. Due to faint primary CR intensity the giant arrays are needed to detect PCR of the highest energy. The properties of primary CRs have to be deduced from the measured (at the ground level) parameters of secondary components of EAS. Quantitative results rely on the shower simulation based on model assumptions about particle interactions. At present the most commonly used method of the shower modeling is Monte Carlo simulation technique which traces individual particles [1, 2]. However, at the highest energies computational burden limits the applicability of the method. The thinning algorithm [3] is destined to solve this problem, but above $E_0 = 10^{19}$ eV it is also powerless. An alternative is the numerical solution of cascade equations which is considerably faster than Monte Carlo approach [4, 5]. In this paper the common method is revisited - the solution of integral Volterra equation which results in powerful algorithms of numerical shower modeling.

The first application of the Volterra equation resolvent to the problem of cosmic ray cascades was attempted 50 years ago in the form of 'successive generations method' [6]. At that time the solution of hadron transport equation was possible only in the case of the delta function approximation of the production spectrum of secondaries in nuclear interactions.

Today, the numerical resolvent constructed on a lattice gives the solution for any production spectrum of hadrons using personal computer or notebook [7].

2. Particle transport equations

The primary particle generates secondary hadrons in inelastic nuclear collision in air, which in turn produce the next generation and so on, forming hadronic cascade. Integro-differential equation for the density of the given kind of hadrons $f(x, E)$ is

$$\frac{\partial f(x,E)}{\partial x} = -(1+\frac{B}{xE})f(x,E) + \int_E^{E_{max}} f(x,u)w(E,u)du + F(x,E), \qquad (1)$$

where $F(x, E)$ is a source function, the depth x is in free path length units, $w(E, u)$ is production spectrum and B is a decay constant of hadrons. For the simplicity, free path length (and some other parameters hereinafter) is supposed constant, otherwise, the energy interval can be chosen where it is approximately constant. Substituting $f(x, E) \to e^{-x}x^{-B/E}f(x, E)$ one can transform it to Volterra equation

$$f(x,E) = e^{-x}x^{-B/E}[\int_0^x e^t t^{B/E} \int_E^{E_{max}} f(t,u)w(E,u)dudt + \int_0^x e^t t^{B/E} F(t,E)dt]. \qquad (2)$$

To model longitudinal development of the cascade we have a system of Volterra equations for every kind of hadrons, chained step by step due to derivative shower components. The solution of Volterra's equation may be obtained with the aid of resolvent in the form of contraction operator [8]. If $\hat{R}$ denotes the linear integral operator in equation (2), then one has successive terms

$$\begin{aligned} f_1 &= \hat{R}F \\ f_2 &= \hat{R}f_1 \\ &\ldots\ldots\ldots\ldots\ldots \\ f_{n+1} &= \hat{R}f_n \\ &\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{3}$$

A sum of all these terms is the solution of equation (2). The physical meaning of n-th term is a successive generation of hadrons born in nuclear interaction of $(n-1)$-th generation.

3. Numerical resolvent on the lattice

In order to approximate the resolvent it is convenient to choose rectangular lattice $\{h_i, y_k\}$, where $h_i = i\delta h, i = 0, .., N$ is atmospheric altitude in $h_0 = 6.9$ km units; $y_k = k\delta y, k = 0, .., M$ is the rapidity. The longitudinal rapidity $y = 0.5\log((1+v_{\|})/(1-v_{\|}))$ is preferable value rather than energy in quadrature formula due to its logarithmic increase with energy $E = \sqrt{m^2 + \langle p_\perp \rangle^2} cosh(y)$ and rather simple Lorentz transform of rapidity distributions - a shift along y.

Quadrature approximation of integrals in equation (2) gives an iteration procedure to calculate the lattice representation of a solution as a partial sum limit

$$f_{ik} = e^{-x_i} x_i^{-\frac{B}{E_k}} \delta h \sum_{j=0}^{i} \alpha_j x_j e^{x_j} x_j^{\frac{B}{E_k}} [\delta y \sum_{l=k}^{M} \beta_l w_{kl} f_{jl} + F_{jk}], \tag{4}$$

where α_j, β_l are quadrature coefficients ensuring the accuracy of the integral approximation. The convergence to the exact solution of equation (2) is provided by a theorem of which the necessary and sufficient conditions are: i) quadrature coefficients are bounded above; ii) quadrature errors are uniformly convergent to zero; iii) the lattice scaling down forces calculation errors to zero uniformly in N, M [8].

Computer program is composed of hook modules which calculate the particular component parameters. Quadrature is implemented using cubic spline approximation. All integral equation kernels, desired solution etc are given as matrices at the lattice node locus $\{h_i, y_k\}$. Boundary conditions are given at the top of atmosphere and at y_{max} as the attenuation curve for the primary particle.

The integral over atmospheric height becomes insurmountable at $E \leq B$ due to strong variation of the integrand multiplied by $x^{B/E}$. The way out was found using additional virtual h_i layers inserted on the lattice at low energies where the spline interpolation is applied to slowly varying part of the integrand.

In-situ error checking of the program is performed using energy balance of the shower components. On every height layer the fraction of the primary energy is calculated carried by electromagnetic, muonic+neutrino and hadronic components. The accuracy of energy calculation was checked changing the rapidity interval δy twice. Energy imbalance is possible where the production spectrum has the width in rapidity comparable to that allowed kinematically. In this case the control of kernel matrix is needed on the lattice. The identity

$$\int_0^{y_{max}} f(x,y) m_\perp \cosh y dy = \int_0^{y_{max}} m_\perp \cosh y \int_y^{y_{max}} f(x,u) w(y,u) du dy$$

that is the corollary of sum rules suits the purpose.

4. Algorithm validation

Correctness of the program and the accuracy achievable were established using simple cascade models with analytically tractable solutions [9]. The first one ('delta model') is convenient to check the height integrals and the meson decay tracking.

Let the cascade consisting of charged pions only be initiated by the primary nucleon of energy E_0. All free path lengths are assumed equal 1, as the inelasticity coefficients, too; production spectrum of pions is delta function: $w(E,U) = n\delta(E - U/n)$, where $n(U)$ is the multiplicity of secondaries. The nucleon density $N(x,E) = \exp(-x)\delta(E - E_0)$ is a source function for charged pions. A solution for pions is given as a sum of successive generations of energy $E_{i+1} = E_i/n(E_i)$, $i = 0, ..., \infty$:

$$f(x, E > 0) = e^{-x} \sum_{i=1}^{\infty} x^i \prod_{j=1}^{i} \frac{n(E_{j-1})}{j + B/E_j}. \tag{5}$$

Comparison of the numerical solution and formula (5) is given in Figure 1.

The second ('scaling') model is characterized by scaling behavior in the form of $w(y,u) = w(u-y)$ rapidity distribution; the simplest version is assumed here - the kernel $w(u-y) = 1$ in $y \in (0,u)$. This leads to the multiplicity of secondaries $n \sim \log E$. The decay constant is 0 in this model while other parameters are the same as in delta model. Resultant solution is

$$f(x,y) = e^{-x} x J_0(\sqrt{x(y_{max} - y)}), \tag{6}$$

where J_0 is Bessel function of zero order; $E_0 = \sqrt{m^2 + \langle p_\perp \rangle^2} cosh(y_{max})$. Series expansion of the Bessel function gives successive generations variant of the solution

$$f(x,y) = e^{-x} \sum_{k=0}^{\infty} \frac{x^{k+1}(y_{max} - y)^k}{(k+1)!k!}.$$

In Figure 2 the exact solution is compared to numerical one obtained at different iterations. Equivalent number of pion generations sufficient for the solution accuracy better than 1% is $m \geq 15$. A series of tests conducted using both models have proved applicability of the algorithm and revealed an optimal choice of the lattice spacing needed to achieve the solution accuracy $\sim 1\%$: $\delta h = 0.05$, $\delta y = 0.5$.

5. Conclusions

The resolvent for Volterra equation provides a numerical solution algorithm of EAS cascade equations adaptable to any model of nuclear interactions. The software implementation for PC manifests the program as effective and reliable tool for EAS modeling.

Two analytically tractable models give the test feasibility of the program in use. They are mutually complementary in checking the accuracy of quadrature approximation of integrals over x and y. Broadening the rapidity distribution width of the production spectrum implies an increase in the number of iterations to achieve the adequate accuracy of the solution: from single iteration for delta model up to $m = 15$ in the case of uniform rapidity distribution.

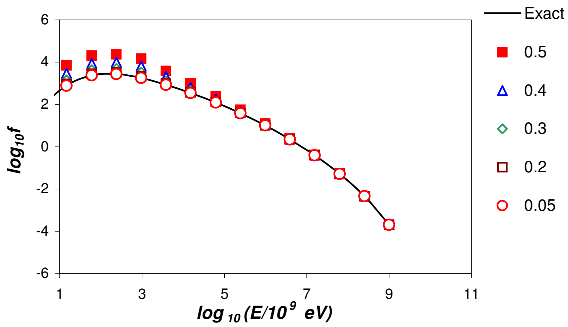

Figure 1. Pions density $f(x_0, E)$ calculated at the sea level x_0 with different steps in height $\delta h = 0.05, \ldots, 0.5$ (marks are indicated on the right) and exact solution in the delta model.

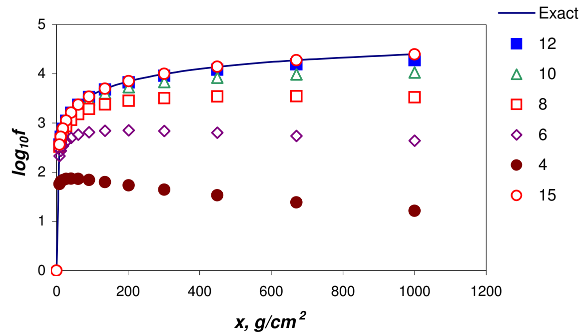

Figure 2. Cascade curve of pions with fixed rapidity $y = 2$ calculated with $m = 4, \ldots, 15$ iterations in the program (m is indicated on the right) and exact solution in the scaling model.

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