CERN-PH-TH/2006-090

Finiteness Properties of the $\mathcal{N}=4$ Super-Yang–Mills Theory in Supersymmetric Gauge

L. Baulieu*, G. Bossard†, S. P. Sorella‡

*Theory Division, CERN, Switzerland§

*† LPTHE, CNRS and Universités Paris VI - Paris VII, Paris, France¶

‡Departamento de Física Teórica Instituto de Física, UERJ, Universidade do Estado do Rio de Janeiro, Brazil ‖

Abstract

With the introduction of shadow fields, we demonstrate the renormalizability of the $\mathcal{N}=4$ super-Yang–Mills theory in component formalism, independently of the choice of UV regularization. Remarkably, by using twisted representations, one finds that the structure of the theory and its renormalization is determined by a subalgebra of supersymmetry that closes off-shell. Starting from this subalgebra of symmetry, we prove some features of the superconformal invariance of the theory. We give a new algebraic proof of the cancellation of the β function and we show the ultraviolet finiteness of the 1/2 BPS operators at all orders in perturbation theory. In fact, using the shadow field as a Maurer–Cartan form, the invariant polynomials in the scalar fields in traceless symmetric representations of the internal R-symmetry group are simply related to characteristic classes. Their UV finiteness is a consequence of the Chern–Simons formula.

*email address: baulieu@lpthe.jussieu.fr

†email address: bossard@lpthe.jussieu.fr

‡email address: sorella@uerj.br

§1211-Geneve 23, Switzerland

¶4 place Jussieu, F-75252 Paris Cedex 05, France.

‖Rua São Francisco Xavier 524, 20550-013 Maracanã Rio de Janeiro, Brazil.

1 Introduction

In a recent paper, new fields have been introduced for supersymmetric gauge theories, which we called shadow fields. These fields are elements of BRST doublets. They determine a BRST like operator for the supersymmetry invariance, which is fully compatible with the gauge symmetry [1]. Shadow fields also allow for the construction of new classes of gauges that interpolate between the usual Faddeev–Popov gauges and new ones, which are explicitly supersymmetric. These gauges give an additional Slavnov–Taylor identity, for controlling supersymmetry at the quantum level. The observables are determined from the cohomology of the BRST operator for gauge invariance, and their supersymmetry covariance can be established from the new Slavnov–Taylor identity.

It has been conjectured for many years that the $\mathcal{N} = 4$ super-Yang–Mills theory is a superconformal theory. Its β function vanishes at all orders in perturbation theory [2, 3]. The superconformal invariance is a basic feature of the AdS/CFT Maldacena's conjecture [4]. In this paper we use the new supersymmetric gauges to directly prove in perturbation theory some of the results that are implied by the conformal invariance of the $\mathcal{N} = 4$ super-Yang–Mills theory, with the only symmetry hypothesis of super Poincaré supersymmetry.

In [1], the renormalization of supersymmetric theories, using shadow fields, was detailed when the supersymmetry algebra closes off-shell, that is, for cases where auxiliary fields exist for closing the whole supersymmetry algebra. In the $\mathcal{N} = 4$ super-Yang–Mills theory, no such auxiliary fields exist. Here, we will bypass this point by combining the methodology of [1] and the existence of a supersymmetry subalgebra with 9 generators, which is large enough to constrain the classical $\mathcal{N} = 4$ super-Yang–Mills action, and small enough to be closed without the use of equations of motion. This subalgebra was introduced in [5, 6], using methods that are specific to topological field theory and involve twisted variables.

The Slavnov–Taylor identities that are allowed by the introduction of the shadow fields enable us to prove the renormalizability of the $\mathcal{N} = 4$ theory, without any assumption on the choice of the ultraviolet regularization. In this paper, we show the absence of a possible anomaly for the $\mathcal{N} = 4$ theory. Algebraic methods have been used to show that invariant polynomials depending on one of the scalar fields of the supersymmetry multiplet are protected from renormalization [7]. We will extend this result to the whole set of local observables of the $\mathcal{N} = 4$ super-Yang–Mills theory, which are invariant polynomials in scalar fields, taking their values in any given traceless symmetric representation of the

R-symmetry group. These operators are the 1/2 BPS primary operators. We then obtain the finiteness of the whole 1/2 BPS multiplets, by supersymmetry covariance. We will give a new proof of the cancellation of the β function at all orders in perturbation theory. Remarkably, all these results are obtained by using small sectors of the supersymmetry algebra. The latter exist, thanks to the possibility of twisting the supersymmetry algebra, by combining the R-symmetry and the Lorentz symmetry. In order to prove the stability of the action under renormalization, it is in fact sufficient to use a sector of the supersymmetry algebra with 6 generators. And, to prove that all 1/2 BPS primary operators are protected operators, one uses a sector of the supersymmetry algebra with 5 generators. A key feature of this proof is the Chern–Simons formula, where the shadow field can be identified as a Maurer–Cartan form. It appears that the scalar observables of topological field theories determine the 1/2 BPS primary operators by covariance under the R-symmetry of the supersymmetric theory. In fact, their finiteness property is closely related to the existence of characteristic classes.

We will suppose everywhere that the gauge group is simple.

2 $\mathcal{N}=4$ super-Yang–Mills theory in the twisted variables

2.1 Fields and symmetries

Consider the $\mathcal{N}=4$ multiplet $(A, \lambda^\alpha, \phi^i)$ in a flat euclidean space $Spin(4) \cong SU(2)_+ \times SU(2)_-$, where α, i are indices in the **4** and the **6** of the internal symmetry $SL(2,\mathbb{H})$, which is the euclidean version of the $SU(4)$ R-symmetry in Minkowski space. The components of spinor and scalar fields λ^α and ϕ^i can be twisted, i.e., decomposed on irreducible representations of the following subgroup[1]

$$SU(2)_+ \times \text{diag}\big(SU(2)_- \times SU(2)_R\big) \times U(1) \subset SU(2)_+ \times SU(2)_- \times SL(2,\mathbb{H}) \qquad (1)$$

We redefine $SU(2) \cong \text{diag}\big(SU(2)_- \times SU(2)_R\big)$. The $\mathcal{N}=4$ multiplet is decomposed as follows

$$(A_\mu, \Psi_\mu, \eta, \chi^I, \Phi, \bar\Phi) \qquad (L, h_I, \bar\Psi_\mu, \bar\eta, \bar\chi_I) \qquad (2)$$

In this equation, the vector index μ is a "twisted world index", which stands for the $(\frac{1}{2}, \frac{1}{2})$ representation of $SU(2)_+ \times SU(2)$. The index I is for the adjoint representation of

[1] Usually, one means by twist a redefinition of the energy momentum tensor that we do not consider here.

the diagonal $SU(2)$. In fact, any given field X^I can be identified as a twisted antiselfdual 2-form $X_{\mu\nu^-}$, by using the flat hyperKähler structure $J^I_{\mu\nu}$.

The twisted "matter" multiplet $(L, h_I, \bar{\Psi}_\mu, \bar{\eta}, \bar{\chi}_I)$ involves therefore four scalars, which are now assembled as a scalar L and an antiselfdual 2-form h_I. For the sake of clarity, we will shortly display a table with the representations of the twisted fields under the various symmetries, as well as their commutation properties.

The ten-dimensional super-Yang–Mills theory determines by dimensional reduction the untwisted $\mathcal{N}=4$ super-Yang–Mills theory. Analogously, the twisted eight-dimensional $\mathcal{N}=2$ theory determines the twisted formulation of the $\mathcal{N}=4$ super-Yang–Mills theory in four dimensions [8]. An horizontality conditions is given in [5], which determines the maximal supersymmetry subalgebra of the twisted $\mathcal{N}=2,\ d=8$ theory, which can be closed without the equations of motion. We will consider this horizontality condition with minor modifications, which are convenient for perturbation theory. It defines the BRST operator s associated to gauge invariance and the graded differential operator Q, which generates the maximal off-shell closed supersymmetry subalgebra. The latter depends on nine twisted supersymmetry parameters, which are one scalar ϖ and one eight-dimensional vector ε. s and Q are thus obtained by expanding over all possible gradings the following horizontality condition

$$\begin{aligned}(d+s+Q-\varpi i_\varepsilon)\big(A+\Omega+c\big)+\big(A+\Omega+c\big)^2\\ =F+\varpi\Psi+g(\varepsilon)\eta+i_\varepsilon\chi+\varpi^2\Phi+|\varepsilon|^2\bar{\Phi}\end{aligned} \tag{3}$$

and its associated Bianchi relation

$$\begin{aligned}(d+s+Q-\varpi i_\varepsilon)\big(F+\varpi\Psi+g(\varepsilon)\eta+i_\varepsilon\chi+\varpi^2\Phi+|\varepsilon|^2\bar{\Phi}\big)\\ +[A+\Omega+c\,,\,F+\varpi\Psi+g(\varepsilon)\eta+i_\varepsilon\chi+\varpi^2\Phi+|\varepsilon|^2\bar{\Phi}]=0\end{aligned} \tag{4}$$

Ω is the Faddeev–Popov ghost, while c is the shadow field [1]. The action of Q on the physical fields decomposes as a gauge transformation with parameter c and a supersymmetry transformation δ^{Susy}, as follows

$$Q=\delta^{Susy}-\delta^{\text{gauge}}(c) \tag{5}$$

By dimensional reduction of these formula one obtains the maximal subalgebra of the $\mathcal{N}=4$ supersymmetry algebra which can be closed off-shell [6]. By dimensional reduction in four dimensions, the eight components of the vector ε decomposes into one scalar ω, one antiselfdual 2-form written as an $SU(2)$ triplet υ_I and one four-dimensional vector

ε^μ. The corresponding supersymmetry algebra is

$$
\begin{aligned}
\delta^{\mathcal{S}usy} A &= \varpi\Psi + \omega\bar\Psi + g(\varepsilon)\eta + g(J_I\varepsilon)\chi^I + \upsilon_I J^I(\bar\Psi)\\
\delta^{\mathcal{S}usy} \Psi &= -\varpi d_A\Phi - \omega\big(d_A L + T\big) + i_\varepsilon F + g(J_I\varepsilon)H^I + g(\varepsilon)[\Phi,\bar\Phi] - \upsilon_I\big(d_A h^I + J^I(T)\big)\\
\delta^{\mathcal{S}usy} \Phi &= -\omega\bar\eta + i_\varepsilon\Psi - \upsilon_I\bar\chi^I\\
\delta^{\mathcal{S}usy} \bar\Phi &= \varpi\eta\\
\delta^{\mathcal{S}usy} \eta &= \varpi[\Phi,\bar\Phi] - \omega[\bar\Phi, L] + \mathscr{L}_\varepsilon\bar\Phi - \upsilon_I[\bar\Phi, h^I]\\
\delta^{\mathcal{S}usy} \chi^I &= \varpi H^I + \omega[\bar\Phi, h^I] + \mathscr{L}_{J^I\varepsilon}\bar\Phi - \upsilon_I[\bar\Phi, L] + \varepsilon^I{}_{JK}\upsilon^J[\bar\Phi, h^K]\\
\delta^{\mathcal{S}usy} H^I &= \varpi[\Phi,\chi^I] + \omega\big([L,\chi^I] - [\eta, h^I] - [\bar\Phi,\bar\chi^I]\big) - \mathscr{L}_{J^I\varepsilon}\eta - [\bar\Phi, i_{J^I\varepsilon}\Psi] + \mathscr{L}_\varepsilon\chi^I\\
&\qquad + \upsilon_J[h^J,\chi^I] + \upsilon^I\big([\eta, L] + [\bar\Phi,\bar\eta]\big) - \varepsilon^I{}_{JK}\upsilon^J\big([\eta, h^K] + [\bar\Phi,\bar\chi^K]\big)\\
\delta^{\mathcal{S}usy} L &= \varpi\bar\eta - \omega\eta + i_\varepsilon\bar\Psi - \upsilon_I\chi^I\\
\delta^{\mathcal{S}usy} \bar\eta &= \varpi[\Phi, L] + \omega[\Phi,\bar\Phi] + \mathscr{L}_\varepsilon L + i_\varepsilon T + \upsilon_I\big(H^I + [h^I, L]\big)\\
\delta^{\mathcal{S}usy} \bar\Psi &= \varpi T - \omega d_A\bar\Phi - g(\varepsilon)[\bar\Phi, L] + g(J_I\varepsilon)[\bar\Phi, h^I] + \upsilon_I J^I(d_A\bar\Phi)\\
\delta^{\mathcal{S}usy} T &= \varpi[\Phi,\bar\Psi] + \omega\big(-d_A\eta - [\bar\Phi,\Psi] + [L,\bar\Psi]\big) - g(\varepsilon)\big([\eta, L] + [\bar\Phi,\bar\eta]\big)\\
&\qquad + g(J_I\varepsilon)\big([\eta, h^I] + [\bar\Phi,\bar\chi^I]\big) + \mathscr{L}_\varepsilon\bar\Psi + \upsilon_I\big([h^I,\bar\Psi] + J^I(d_A\eta + [\bar\Phi,\bar\Psi])\big)\\
\delta^{\mathcal{S}usy} h^I &= \varpi\bar\chi^I + \omega\chi^I - i_{J^I\varepsilon}\bar\Psi - \upsilon^I\eta - \varepsilon^I{}_{JK}\upsilon^J\chi^K \qquad (6)\\
\delta^{\mathcal{S}usy} \bar\chi^I &= \varpi[\Phi, h^I] + \omega\big([L, h^I] - H^I\big) + \mathscr{L}_\varepsilon h^I - i_{J^I\varepsilon}T + \upsilon^I[\Phi,\bar\Phi] + \upsilon_J[h^J, h^I] + \varepsilon^I{}_{JK}\upsilon^J H^K
\end{aligned}
$$

Notice the presence of auxiliary fields H_I and T_μ, for a total of $7 = 3+4$ degrees of freedom. They have been introduced to lift some degeneracy when solving the horizontality condition, while ensuring that the supersymmetry algebra closes, according to

$$(\delta^{\mathcal{S}usy})^2 = \delta^{\text{gauge}}(\omega(\varphi) + \varpi i_\varepsilon A) + \varpi\mathcal{L}_\varepsilon \qquad (7)$$

with

$$\omega(\varphi) \equiv \varpi^2\Phi + \varpi\omega L + \varpi\upsilon_I h^I + (\omega^2 + \upsilon_I\upsilon^I + |\varepsilon|^2)\bar\Phi \qquad (8)$$

As explained in [1], the field dependent gauge transformation that appears in the commutator of two supersymmetries (7) justifies the introduction of the shadow field c, with the following Q transformation

$$Qc = \omega(\varphi) + \varpi i_\varepsilon A - c^2 \qquad (9)$$

When all parameters, but ϖ, vanish, $\omega(\varphi)$ can be identified as a topological ghost of ghost [12].

The s transformations of physical fields are their gauge transformations with parameter Ω.

In order to solve the degeneracy in the horizontality condition $s\, c + Q\Omega + [c, \Omega] = 0$, one introduces the field μ, with

$$
\begin{aligned}
&s\,\Omega = -\Omega^2 && Q\Omega = -\mu - [c, \Omega] \\
&s\, c = \mu && \\
&s\,\mu = 0 && Q\mu = -\big[\omega(\varphi), \Omega\big] - \varpi \mathscr{L}_\varepsilon \Omega - [c, \mu]
\end{aligned}
\tag{10}
$$

As explained in [1], the use of c and Ω allows one to disentangle the gauge symmetry and supersymmetry in the gauge-fixing process. In fact, antighosts and antishadows must be introduced, in order to concretely perform a gauge-fixing, which we will choose to be Q-invariant. The new fields come as a BRST quartet, and their transformation laws are as follows:

$$
\begin{aligned}
&s\,\bar{\mu} = \bar{c} && s\,\bar{c} = 0 && s\,\bar{\Omega} = b && s\, b = 0 \\
&Q\bar{\mu} = \bar{\Omega} && Q\bar{c} = -b && Q\bar{\Omega} = \varpi \mathcal{L}_\varepsilon \bar{\mu} && Qb = -\varpi \mathcal{L}_\varepsilon \bar{c}
\end{aligned}
\tag{11}
$$

On all the fields of the theory one has $(d + s + Q - \varpi i_\varepsilon)^2 = 0$, that is:

$$
\begin{aligned}
s^2 = 0 \qquad & Q^2 = \varpi \mathcal{L}_\varepsilon \\
\{ s, Q\} &= 0
\end{aligned}
\tag{12}
$$

Q depends on 9 parameters. We see that if we restrict to the subalgebra with five parameters, by taking $\varepsilon = 0$, we have $Q^2 = 0$. This observation will be shortly used.

The grading of the fields is determined from the assignments of the Faddeev–Popov ghost number and the shadow number [1]. The other quantum numbers are those for the global symmetry $SU(2)_+ \times SU(2) \times U(1)$. Together with δ^{Susy}, the latter invariance gives rise to a well-defined graded subalgebra of the whole $\mathcal{N} = 4$ symmetry, which is big enough to completely determine the $\mathcal{N} = 4$ action [6]. As we will see shortly, the Ward identities associated to the invariance under this subalgebra also determine the theory at the quantum level, in such a way that one recovers eventually the whole symmetry of the $\mathcal{N} = 4$ super-Yang–Mills theory, including its $Spin(4) \times SL(2, \mathbb{H})$ R-symmetry, after untwisting. All relevant quantum numbers are summarized in the following tables

	A	h_I	Ψ	η	χ_I	$\bar\Psi$	$\bar\eta$	$\bar\chi_I$	Φ	L	$\bar\Phi$	H_I	T
canonical dimension	1	1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	1	1	1	2	2
$U(1)$	0	0	1	-1	-1	-1	1	1	2	0	-2	0	0
$SU(2)_+$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0	0	0	0	0	$\frac{1}{2}$
$SU(2)$	$\frac{1}{2}$	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	1	0	0	0	1	$\frac{1}{2}$
commutation property	$-$	$+$	$+$	$-$	$-$	$+$	$-$	$-$	$+$	$+$	$+$	$+$	$-$

Table I : Quantum numbers of physical fields (with ghost and shadow number zero)

	Ω	$\bar\Omega$	b	$\bar\mu$	$\bar c$	c	μ	ϖ	ω	ε	υ_I	χ	$\chi^{(s)}$	$\chi^{(Q)}$	$\chi^{(Q_s)}$
canonical dimension	0	2	2	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	d	$4-d$	$\frac{7}{2}-d$	$\frac{7}{2}-d$
ghost number	1	-1	0	-1	0	0	1	0	0	0	0	g	$-1-g$	$-g$	$-1-g$
shadow number	0	0	0	-1	-1	1	1	1	1	1	1	s	$-s$	$-1-s$	$-1-s$
$U(1)$	0	0	0	0	0	0	0	-1	1	1	1	u	$-u$	$-u$	$-u$
$SU(2)_+$	0	0	0	0	0	0	0	0	0	$\frac{1}{2}$	0	j_+	j_+	j_+	j_+
$SU(2)$	0	0	0	0	0	0	0	0	0	$\frac{1}{2}$	1	j	j	j	j
commutation property	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$\pm$	$\mp$	$\mp$	$\pm$

Table II : Quantum numbers of ghosts, shadows and sources for s , Q, sQ transformations where χ stands for any field of the theory.

2.2 Classical action and gauge-fixing

The classical action S is defined as a gauge-invariant local functional in the physical fields, which has canonical dimension four, is invariant under the global symmetry $SU(2)_+ \times SU(2) \times U(1)$ and under the Q symmetry with its nine supersymmetry generators. The corresponding lagrangian density is any given linear combination of

$$\begin{aligned}\mathcal{L}_4^0 \equiv \operatorname{Tr} \Big(& \frac{1}{2} F_\wedge F + H_I J^I \star F - \star H_I H^I + \varepsilon_{IJK} \star H^I [h^J, h^K] + \star H_I [L, h^I] + T \star T \\ & + T \star d_A L + J_I \star T d_A h^I + d_A \Phi \star d_A \bar\Phi + \chi_I J^I \star d_A \Psi + \Psi \star d_A \eta - \bar\chi_I J^I \star d_A \bar\Psi - \bar\Psi \star d_A \bar\eta \\ & + \star\eta[\Phi, \eta] + \Psi[\bar\phi, \star\Psi] + \star\chi_I[\Phi, \chi^I] + \star\bar\eta[\bar\Phi, \bar\eta] + \bar\Psi[\Phi, \star\bar\Psi] + \star\bar\chi_I[\bar\Phi, \bar\chi^I] \\ & - \star\eta[L, \bar\eta] - \Psi[L, \star\bar\Psi] - \star\chi_I[L, \bar\chi^I] - \star\eta[h_I, \bar\chi^I] + \star\bar\eta[h_I, \chi^I] \\ & + J_I \star \Psi[h^I, \bar\Psi] - \star\varepsilon_{IJK}\chi^I[h^J, \bar\chi^K] - \star[\Phi, \bar\Phi]^2 - \star[\Phi, h_I][\bar\Phi, h^I] - \star[\Phi, L][\bar\Phi, L] \Big) \end{aligned} \quad (13)$$

and the of topological term

$$Ch_4^0 \equiv \frac{1}{2} \operatorname{Tr} F_\wedge F \quad (14)$$

Here, we are only interested in the sector of zero instanton number $\int Ch^0_4 = 0$, in such a way that

$$S = \frac{1}{g^2}\int \mathcal{L}^0_4 \tag{15}$$

The only physical parameter of the theory is thus the coupling constant g. Modulo the elimination of the auxiliary fields H and T, and the addition of a topological term, $\mathcal{L}^0_4$ can be identified as the untwisted $\mathcal{N}=4$ supersymmetric lagrangian.

One can check that the action S is in fact invariant under the sixteen generators of supersymmetry. Remarkably, we found that it is already completely determined by the δ^{Susy} invariance, when the supersymmetry generators are reduced to the six ones associated to ϖ, ω and ε [6].

As shown in [1], the introduction of the trivial BRST quartet $(\bar\mu, \bar c, \bar\Omega, b)$ allows a renormalizable supersymmetric (i.e. Q invariant) gauge-fixing $s\,\mathbf{\Psi}$ for S, where $\mathbf{\Psi}$ is a Q-exact gauge fermion that depends on the shadow fields and on the supersymmetry parameters

$$\mathbf{\Psi} = Q\int \mathrm{Tr}\ \bar\mu(d\star A - \alpha b) \tag{16}$$

Here, we will restrict to the shadow-Landau gauge $\alpha = 0$. More Ward identities exist in this gauge, which is stable under renormalization. They greatly simplify the renormalization problems. When $\alpha = 0$, the supersymmetric gauge-fixing action is

$$\begin{aligned} s\,\mathbf{\Psi} = \int \mathrm{Tr}\ \Big(& bd\star A - \bar\Omega d\star d_A\Omega + \bar c d\star d_A c + \bar\mu d\star d_A\mu \\ &- \bar c d\star\big(\varpi\Psi + \omega\bar\Psi + g(\varepsilon)\eta + g(J_I\varepsilon)\chi^I + \upsilon_I J^I(\bar\Psi)\big) \\ &+ \bar\mu d\star\big([d_A\Omega, c] + [\Omega, \varpi\Psi + \omega\bar\Psi + g(\varepsilon)\eta + g(J_I\varepsilon)\chi^I + \upsilon_I J^I(\bar\Psi)]\big)\Big) \end{aligned} \tag{17}$$

Let $\varphi^{(s)}$, $\varphi^{(Q)}$ and $\varphi^{(Q_s)}$ be respectively the sources of the s, Q and $s\,Q$ transformations of the fields. The gauge-fixed complete action Σ, including the insertions of these operators, is

$$\begin{aligned} \Sigma = S + s\,\mathbf{\Psi} + \sum_a \int (-1)^a\Big(&\varphi^{(s)}{}_a\, s\,\varphi^a + \varphi^{(Q)}{}_a Q\varphi^a + \varphi^{(Q_s)}{}_a\, s\,Q\varphi^a\Big) \\ &+ \int \mathrm{Tr}\ \Big(\Omega^{(s)}\Omega^2 - \Omega^{(Q)}Q\Omega - \Omega^{(Q_s)}\, s\,Q\Omega + \mu^{(Q)}Q\mu - c^{(Q)}Qc\Big) \end{aligned} \tag{18}$$

Owing to the source dependence, the s and Q invariance can be expressed as functional identities, namely the Slavnov–Taylor identities, defined in [1]

$$\mathcal{S}_{(s)}(\Sigma) = 0 \qquad \mathcal{S}_{(Q)}(\Sigma) = 0 \tag{19}$$

Choosing the class of "linear gauges" (16), one has equations of motion for the quartet $(\bar{\mu}, \bar{c}, \bar{\Omega}, b)$ that imply functional identities $\overline{\mathcal{G}_\bullet}(\Sigma) = 0$, which can be used as Ward identities. Moreover, the shadow-Landau gauge allows for further functional identities, associated to the equations of motion of Ω, c and μ, $\mathcal{G}^\bullet(\Sigma) = 0$, which constitute a BRST quartet with the global gauge transformations. We refer to [1] for the detailed expressions of these antighost and ghost Ward identities. All Ward identities verify consistency conditions and their solutions determine a Lie algebra of linear functional operators. The linearized Slavnov–Taylor operators associated to s and Q satisfy the algebra

$$\mathcal{S}_{(s)|\Sigma}{}^2 = 0 \qquad \mathcal{S}_{(Q)|\Sigma}{}^2 = \varpi \mathcal{P}_\varepsilon$$
$$\left\{\mathcal{S}_{(s)|\Sigma}, \mathcal{S}_{(Q)|\Sigma}\right\} = 0 \tag{20}$$

$\mathcal{P}_\varepsilon$ is the differential operator which acts as the Lie derivative along ε on all fields and external sources. It must be noted that the Green functions depend on the supersymmetry parameters generated by the Q-exact gauge-fixing term, but not the physical observables [1].

3 Renormalization of the action

The problem of the renormalization of supersymmetric theories is strongly simplified in the case where the supersymmetry algebra closes without the use of the equations of motion, provided one uses shadow fields [1]. To treat the renormalization of the $\mathcal{N} = 4$ model, for which no set of auxiliary fields exists, we will adapt the results of [1], using the maximal off-shell closed Q symmetry of the $\mathcal{N} = 4$ model, with nine generators. This reduces the question of the anomalies and of the stability of the theory to algebraic problems, which involve only the physical fields and the differential δ^{Susy}.

3.1 Anomalies

The possible anomalies associated to the Ward identities

$$\mathcal{S}_{(s)}(\Gamma) = 0 \qquad \mathcal{S}_{(Q)}(\Gamma) = 0 \qquad \overline{\mathcal{G}_\bullet}(\Gamma) = 0 \qquad \mathcal{G}^\bullet(\Gamma) = 0 \tag{21}$$

are related to the cohomology $\mathcal{H}^* = \oplus_{s\in\mathbb{N}}\mathcal{H}^s$ of the differential complex of gauge-invariant functionals in the physical fields and the supersymmetry parameters (ϖ, ω, υ_I and ε^μ) with differential δ^{Susy}. The shadow number s defines the grading of this complex. The non-trivial anomalies are either elements of $\mathcal{H}^1$ or a doublet made out of the Adler–Bardeen anomaly and of a supersymmetric counterpart, which exists if and only if the cocycle

$$\int \mathrm{Tr}\ \big(F_\wedge \delta^{Susy} A_\wedge \delta^{Susy} A + \omega(\varphi) F_\wedge F\big) \tag{22}$$

is δ^{Susy}-exact [1]. If this cocycle were δ^{Susy}-exact in the $\mathcal{N} = 4$ theory, its restriction to the value ($\varpi = 1$, $\omega = \upsilon = \varepsilon = 0$) of the supersymmetry parameters would also be δ_1^{Susy}-exact. This restricted operator δ_1^{Susy} can be identified with the equivariant form of the topological BRST operator defined in the generalized Donaldson–Witten theory associated to twisted $\mathcal{N} = 4$ super-Yang–Mills theory [6] and the cocycle (22) can be identified with the Donaldson–Witten invariant

$$\int \mathrm{Tr}\ \big(\Phi F_\wedge F + \Psi_\wedge \Psi_\wedge F\big) \tag{23}$$

The latter expression is a non-trivial cohomology class. Thus the consistency equations for both s and Q symmetries forbid the possibility of an Adler–Bardeen anomaly in $\mathcal{N} = 4$ super-Yang–Mills theory. In fact, one obtains an analogous result for the cases $\mathcal{N} = 2, 3$.

This very simple demonstration gives an algebraic proof of the absence of Adler–Bardeen anomaly in Yang–Mills theories with extended supersymmetry.

As for the absence of purely supersymmetric anomaly, one can straightforwardly compute that $\mathcal{H}^1$ is empty. Indeed, the invariance under only six supersymmetry generators is sufficient to determine the classical action [6]. Then, one finds that the possible elements of $\mathcal{H}^1$ associated to transformations linear in ϖ, ω, and ε must be trivial. After their elimination, power counting forbids the possibility of functionals, which are linear in the parameter υ_I and satisfy all the invariances required by the consistency conditions.

As a corollary, one finds that, when one renormalizes the theory and adjusts the 1PI generating functional at a given order of perturbation theory by adding non-invariant counter-terms, it is sufficient to consider the Slavnov–Taylor identity with the six generators. Once this is done, it is automatic that one has also restored the Slavnov–Taylor identity with nine generators.

Therefore, in the absence of a solution to the consistency conditions of the functional operators associated to the Ward identities, it is by definition possible to renormalize the $\mathcal{N} = 4$ super-Yang–Mills theory, while maintaining all the Ward identities of the shadow-Landau gauge. The process is straightforward, and independent of the choice of the regularization.

3.2 Stability

In order to ensure that the renormalized action does not depend on more parameters than the classical lagrangian that we are starting from, one must prove its stability property. Basically, this amounts to prove that the most general local solution of the Ward identities, which can be imposed in perturbation theory, has the same form as the local gauge-fixed effective action that one starts from to define perturbation theory.

In [1], within the framework of our class of renormalizable gauges, we reduced the question of the stability of the action to that of finding the most general supersymmetry algebra acting on the set of physical fields of the theory. Using power counting, we checked by inspection that, for the $\mathcal{N}=4$ theory, the solution of this problem is unique, modulo a rescaling of each physical field and modulo a redefinition of the auxiliary fields H_I and T

$$\begin{aligned} H_I^{\mathcal{R}} &= z_{10} H_I + z_{11} J_I^{\mu\nu} F_{\mu\nu} + z_{12}\varepsilon_{IJK}[h^I, h^K] + z_{13}[L, h_I] \\ T_\mu^{\mathcal{R}} &= z_{20} T_\mu + z_{21} D_\mu L + z_{22} J_{I\,\mu}{}^{\nu} D_\nu h^I \end{aligned} \tag{24}$$

Here the z's are arbitrary coefficients. Such a non linear renormalizations can in fact be avoided. For this, one defines H^I and T in such way that

$$\frac{\delta^L S}{\delta H^I} = -2 \star H_I \qquad \frac{\delta^L S}{\delta T} = 2 \star T \tag{25}$$

The auxiliary fields then decouple and are not renormalized. This property can be checked by using Ward identities associated to the equations of motion of these auxiliary fields, which are consistent with the whole set of Ward identities of the theory. To define such Ward identities, one adds to the action sources that are tensors of rank two in the adjoint representation of the gauge group, for the local operator $c\otimes\Omega$ (where the tensor product is for the adjoint representation of the gauge group) and for its s, Q and sQ variations.

It follows that the source independent part, $S^{\mathcal{R}} + s^{\mathcal{R}}\boldsymbol{\Psi}^{\mathcal{R}}$, of the most general local solution of Ward identities is determined by its invariance under both renormalized symmetries $s^{\mathcal{R}}$ and $Q^{\mathcal{R}}$. The graded differential operators $s^{\mathcal{R}}$ and $Q^{\mathcal{R}}$ have the same expression as s and Q, with a mere substitution of the bare fields and the coupling constant into renormalized ones. Modulo these substitutions, $S^{\mathcal{R}}$, defined as the most general local functional of ghost and shadow number zero, power counting four, and invariant under all the global symmetries, which is invariant under $\delta^{Susy\,\mathcal{R}}$ and belongs to the cohomology of $\mathcal{S}_{(s)|\Sigma}$, is the same as S in Eq. (15). In our class of gauge, the gauge-fixing term keeps the same form, due to the ghost and antighost Ward identities. One can thus write, for the most general possible local solution of Ward identities for the $\mathcal{N}=4$ theory

$$\Sigma^{\mathcal{R}} = \frac{1}{{g_{\mathcal{R}}}^2}\int \mathcal{L}_4^{0\,\mathcal{R}} + s^{\mathcal{R}} Q^{\mathcal{R}} \int \mathrm{Tr}\ \bar\mu d\star A + \sum_a \int (-1)^a \Big({\varphi^{(s)}}_a s^{\mathcal{R}} \varphi^a + {\varphi^{(Q)}}_a Q^{\mathcal{R}} \varphi^a + {\varphi^{(Q_s)}}_a s^{\mathcal{R}} Q^{\mathcal{R}} \varphi^a \Big) + \int \mathrm{Tr}\ \Big(-\Omega^{(s)} s^{\mathcal{R}}\Omega - \Omega^{(Q)} Q^{\mathcal{R}}\Omega - \Omega^{(Q_s)} s^{\mathcal{R}} Q^{\mathcal{R}}\Omega + \mu^{(Q)} Q^{\mathcal{R}}\mu - c^{(Q)} Q^{\mathcal{R}} c \Big) \quad (26)$$

3.3 Callan–Symanzik equation

We define m as the subtraction point. The renormalized generating functional Γ of 1PI vertices of fields and insertion of s, Q and sQ transformations of all fields verifies by construction the Callan–Symanzik equation

$$\mathcal{C}\,\Gamma = 0 \quad (27)$$

With our choice of gauge, the supersymmetry parameters do not get renormalized, because of the Ward identities. Thus, in the shadow-Landau gauge, the unique parameter of the theory that can be possibly renormalized is the coupling constant g.

Because of the quantum action principle, $m\frac{\partial\Gamma}{\partial m}$ is equal to the insertion of a local operator in the 1PI generating functional satisfying all the linearized functional identities associated to the Ward identities. Using furthermore the stability property of the effective action, one obtains that the anomalous dimensions of the fields can be adjusted, order by order in perturbation theory, in such a way that the Callan–Symanzik operator takes the following form

$$\mathcal{C}\,\mathscr{F} \equiv m\frac{\partial\mathscr{F}}{\partial m} + \beta\frac{\partial\mathscr{F}}{\partial g} + \mathcal{S}_{(s)|\mathscr{F}}\mathcal{S}_{(Q)|\mathscr{F}} \int \Big(\sum_a \gamma^a\, \varphi^a {\varphi^{(Q_s)}}_a + \gamma^{(A)}\,\mathrm{Tr}\ \bar\mu \frac{\delta^L \mathscr{F}}{\delta b} \Big) = m\frac{\partial\mathscr{F}}{\partial m} + \beta\frac{\partial\mathscr{F}}{\partial g} - \sum_a \gamma^a \int \Big(\varphi^a \frac{\delta^L\mathscr{F}}{\delta\varphi^a} - {\varphi^{(s)}}_a \frac{\delta^L\mathscr{F}}{\delta{\varphi^{(s)}}_a} - {\varphi^{(Q)}}_a \frac{\delta^L\mathscr{F}}{\delta{\varphi^{(Q)}}_a} - {\varphi^{(Q_s)}}_a \frac{\delta^L\mathscr{F}}{\delta{\varphi^{(Q_s)}}_a} \Big) + \gamma^{(A)} \int \mathrm{Tr}\ \Big(\bar\mu\frac{\delta^L\mathscr{F}}{\delta\bar\mu} + \bar c\frac{\delta^L\mathscr{F}}{\delta\bar c} + \bar\Omega\frac{\delta^L\mathscr{F}}{\delta\bar\Omega} + b\frac{\delta^L\mathscr{F}}{\delta b} \Big) \quad (28)$$

Any given local operator $\mathcal{O}_A$ generally mixes under renormalization with all other operators with equal or lower canonical dimensions, except if a symmetry forbids this phenomenon.

To generate insertions of any observable $\mathcal{O}_A$ in the 1PI generating functional Γ, one couples them to external sources u^A and redefine

$$\Sigma \to \Sigma[u] = \Sigma + \sum_A \int \langle u^A, \mathcal{O}_A \rangle \tag{29}$$

Renormalization can only mix a finite number of local operators, because of power counting. To control renormalization, one must generically introduce new sources v^X for other operators and extend $\Sigma[u]$ into $\Sigma[u, v]$ in such a way that one can define the s and Q transformations of sources u^A and v^X so that $\Sigma[u, v]$ satisfies all the Ward identities of the theory. By doing so, the Slavnov–Taylor, ghost and antighost operators get modified by source dependent terms. Then, for any given observable with a given canonical dimension, the theory generated by $\Sigma[u, v]$ can be renormalized in such a way that it satisfies the same Ward identities as the theory generated by Σ, provided one has introduced the large enough but finite set of sources v^X and that the introduction of these new sources does not generate anomalies.

The quantum action principle implies that the Callan–Symanzik equation for the 1PI generating functional $\Gamma[u, v]$ can be written as follows

$$\mathcal{C}\,\Gamma[u, v] = \big[L[u, v] \cdot \Gamma[u, v]\big] \tag{30}$$

The right hand side stands for the insertion of a local functional $L[u, v]$ of canonical dimension four in $\Gamma[u, v]$. Because of the commutation property between the Callan–Symanzik operator and the functional operators associated to the Ward identities of the theory, this insertion must satisfy all the modified linearized functional identities associated to the Ward identities including the source dependence

$$\begin{aligned} \mathcal{S}_{(s)|\Gamma}\big[L \cdot \Gamma\big] &= 0 \qquad \mathcal{S}_{(Q)|\Gamma}\big[L \cdot \Gamma\big] = 0 \\ \overline{L\mathcal{G}_\bullet}\,\big[L \cdot \Gamma\big] &= 0 \qquad L\mathcal{G}^\bullet\,\big[L \cdot \Gamma\big] = 0 \end{aligned} \tag{31}$$

Therefore, the local functional L must be invariant under all the global symmetries of the theory and verify

$$\mathcal{S}_{(s)|\Sigma} L = \mathcal{S}_{(Q)|\Sigma} L = \overline{L\mathcal{G}_\bullet}\, L = L\mathcal{G}^\bullet\, L = 0 \tag{32}$$

where the linearized operators are assumed to contain the source dependent modifications associated to $\Sigma[u, v]$. The most general form of L thus corresponds to the most general u, v dependent invariant counterterm, solution of Ward identities [9].

4 Physical observables

Physical observables are defined as the correlation functions of gauge-invariant functionals $\mathcal{O}^{\text{inv}}_A$ of physical fields,

$$\Big\langle \mathcal{O}^{\text{inv}}_A\,\mathcal{O}^{\text{inv}}_B\,\mathcal{O}^{\text{inv}}_C\cdots\Big\rangle \tag{33}$$

They belong to the cohomology of s . Thus, they do not depend on the gauge parameters, including the supersymmetry parameters [1]. One can study them in the shadow-Landau gauge without loss of generality. We mentioned in section 3.3 that the supersymmetry parameters are not renormalized in this gauge. Thus, for any set of functions of the supersymmetry parameter $f^A(\varpi,\omega,\upsilon_I,\varepsilon)$, one has

$$\Big\langle\Big(\sum_A f^A\mathcal{O}^{\text{inv}}_A\Big)\,\mathcal{O}^{\text{inv}}_B\,\mathcal{O}^{\text{inv}}_C\cdots\Big\rangle=\sum_A f^A\Big\langle\mathcal{O}^{\text{inv}}_A\,\mathcal{O}^{\text{inv}}_B\,\mathcal{O}^{\text{inv}}_C\cdots\Big\rangle \tag{34}$$

The Slavnov–Taylor identities imply that the insertion of any given gauge-invariant functional in the physical fields $\mathcal{O}^{\text{inv}}_A$ are renormalized such that

$$\Big[\delta^{Susy}\mathcal{O}^{\text{inv}}_A\cdot\Gamma\Big]=\mathcal{S}_{(Q)|\Gamma}\Big[\mathcal{O}^{\text{inv}}_A\cdot\Gamma\Big] \tag{35}$$

This equation and the factorization property (34) imply that physical observables fall into supersymmetry multiplets, when they are sandwiched between physical states.

Eq. (34) is a useful property, since it is often convenient to introduce field functionals under the form $\mathcal{O}=\sum_A f^A\mathcal{O}^{\text{inv}}_A$. One can study the observables $\big\langle\mathcal{O}^{\text{inv}}_A\cdots\big\rangle$ through correlations functions involving the functional $\mathcal{O}$, as long as each $\mathcal{O}^{\text{inv}}_A$ is unambiguously defined by $\mathcal{O}$ at the classical level.

In the shadow-Landau gauge, it is therefore meaningful to define the physical observables as the field functionals in the cohomology of s , including the ones with a dependence on the supersymmetry parameters. Observables are allowed to have an arbitrary positive shadow number.

5 Protected and 1/2 BPS operators

Some of the local operators of the $\mathcal{N}=4$ super-Yang–Mills theory are protected from renormalization. A strong definition of this property is

$$\mathcal{C}\,\Big[\mathcal{O}\cdot\Gamma\Big]=0 \tag{36}$$

expressing the vanishing of the corresponding anomalous dimension, $\gamma_{\mathcal{O}}=0$. However, the form of this equation must be slightly relaxed, since we are interested in physical operators, and their finiteness is only meaningful for their values between physical states. We thus define a protected physical operator by the request that it satisfies the previous condition, up to an unphysical s-exact term, namely

$$\mathcal{C}\left[\mathcal{O}^{\rm inv}\cdot\Gamma\right] = \mathcal{S}_{(s)|\Gamma}\left[\Upsilon^{(\mathcal{O})}\cdot\Gamma\right] \tag{37}$$

Here $\Upsilon^{(\mathcal{O})}$ is a local functional of ghost number -1. Its expression can be gauge-dependent.

Well-known protected local operators of the $\mathcal{N}=4$ super Yang–Mills theory are those belonging to BPS multiplets. In superconformal theory it is natural to classify the physical observables in irreducible superconformal multiplets. In each superconformal multiplet, there is a superconformal primary operator that is annihilated by the so-called special supersymmetry generators at the point $x^\mu=0$. Moreover, the action of supersymmetry generators on a superconformal primary operator generates all operators of its superconformal multiplet. When at least one of the supercharges commutes with the superconformal primary operator of a superconformal multiplet, the latter is called BPS. Such irreducible multiplets are short. They play an important role in the AdS/CFT correspondence. Superconformal invariance implies that the dimension of any operators belonging to such multiplets do not receive radiative corrections.

The 1/2 BPS primary operators are the primary operators that are annihilated by half of the supersymmetry generators. They are the gauge-invariant polynomials in the scalar fields of the theory in a traceless symmetric representation of the $SO(5,1)$ R-symmetry group. In this section we will prove that all the 1/2 BPS primary operators, and thus all their descendants, are protected operators, without assuming that the theory is conformal, using only Ward identities associated to gauge and supersymmetry invariance.

In the gauge $\varepsilon=0$ the operator Q is nilpotent. The Lie algebra valued function of the scalar fields $\omega(\varphi)$ that characterizes the field dependent gauge transformations that appear in the commutators of the supersymmetries depends in this case on five parameters,

$$\omega(\varphi) = \varpi^2\Phi + \varpi\omega L + \varpi\upsilon_I h^I + (\omega^2+\upsilon_I\upsilon^I)\bar\Phi \tag{38}$$

If we had considered all the supersymmetry generators, $\omega(\varphi)$ would take the form

$$\omega(\varphi)_{16} = \left(\bar\epsilon\tau^i\epsilon\right)\phi_i \tag{39}$$

where τ^i are the six-dimensional gamma matrices for the $SL(2,\mathbb{H})$ spinor representations. We have not indicated the $SL(2,\mathbb{H})$ index for the spinor ϵ. The quantity $\omega(\varphi)$ is a particular case of $\omega(\varphi)_{16}$, when ϵ satisfies, among other conditions, that $(\bar\epsilon\gamma^\mu\epsilon)=0$.

Thus, the expansion of any invariant polynomial in $\omega(\varphi)$ in powers of the supersymmetry parameters gives operators that belong to symmetric representations of $SO(5,1)$. Moreover, because of the original ten-dimensional Fiertz identity

$$\big(\bar{\boldsymbol{\varepsilon}}\Gamma_m\boldsymbol{\varepsilon}\big)\Gamma^m\boldsymbol{\varepsilon}=0 \tag{40}$$

for any given commuting Majorana–Weyl spinor $\boldsymbol{\varepsilon}$, any given $\mathcal{N}=4$ Majorana spinor that satisfies $(\bar\epsilon\gamma^\mu\epsilon)=0$, is such that

$$\big(\bar\epsilon\tau^i\epsilon\big)\big(\bar\epsilon\tau_i\epsilon\big)=0 \tag{41}$$

Therefore, all operators obtained from the expansion of $\omega(\varphi)$ belong to traceless symmetric representations of $SO(5,1)$.

In fact, the invariant polynomials $\mathcal{P}\big(\omega(\varphi)\big)$ give by expansion in the supersymmetry parameters the whole traceless symmetric representations of $SO(5,1)$. To obtain this result, it is sufficient to show that this expansion provides an equal number of operators than there are components in the representations. It is convenient to use a four dimensional notation, with $\daleth=(0,I)$, $\upsilon^\daleth\equiv(\omega,\upsilon^I)$ and $h_\daleth\equiv(L,h_I)$. Call $X(n_+,n_-)$ the sum of the monomials, of degree n_+ in ϖ and n_- in $\upsilon^\daleth$, which may stand in the expansion of $\mathcal{P}\big(\omega(\varphi)\big)$. For $n_+\geqslant n_-$, $X(n_+,n_-)$ takes the following form[2]

$$X(n_+,n_-)\propto \text{sTr }\Phi^{\frac{n_+-n_-}{2}}\bigg((\upsilon^\daleth h_\daleth)^{n_-}+\sum_{p=1}^{\frac{n_-}{2}}C^p_{n_+\,n_-}(\upsilon^\daleth h_\daleth)^{n_--2p}(\upsilon_\daleth\upsilon^\daleth\Phi\bar\Phi)^p\bigg) \tag{42}$$

where sTr is the symmetrized trace and $C^p_{n_+\,n_-}=\frac{n_-!(\frac{n_+-n_-}{2})!}{p!(n_--2p)!(p+\frac{n_+-n_-}{2})!}$. By defining $S^n_d=\frac{(d+n-1)!}{(d-1)!n!}$ as the dimension of the symmetric representation of rank n in $SO(d)$, $X(n_+,n_-)$ gives $S_4^{n_-}$ operators. For $n_+<n_-$, $X(n_+,n_-)$ takes the form

$$X(n_+,n_-)\propto \text{sTr }(\upsilon_\daleth\upsilon^\daleth\bar\Phi)^{\frac{n_--n_+}{2}}\bigg((\upsilon^\daleth h_\daleth)^{n_+}+\sum_{p=1}^{\frac{n_+}{2}}C^p_{n_-\,n_+}(\upsilon^\daleth h_\daleth)^{n_+-2p}(\upsilon_\daleth\upsilon^\daleth\Phi\bar\Phi)^p\bigg) \tag{43}$$

and gives $S_4^{n_+}$ operators. By expanding an invariant polynomial of degree n as a power series in the supersymmetry parameters, one thus obtains

$$\sum_{n_-=0}^{n}S_4^{n_-}+\sum_{n_+=0}^{n-1}S_4^{n_+}=S_4^n+2\sum_{p=0}^{n-1}S_4^p \tag{44}$$

[2]For simplicity we have written $X(n_+,n_-)$ in the simplest case where $\mathcal{P}\big(\omega(\varphi)\big)=\text{Tr }\omega(\varphi)^{\frac{n_++n_-}{2}}$. The demonstration extends trivially to any invariant polynomials.

independent operators in the traceless symmetric representation of $SO(5,1)$. The traceless symmetric representation of $SO(5,1)$ of rank n is of dimension $S_6^n - S_6^{n-2}$. One can then compute by recurrence that

$$S_{d-2}^n + 2\sum_{p=0}^{n-1} S_{d-2}^p = S_d^n - S_d^{n-2} \tag{45}$$

One has

$$S_{d-2}^2 + 2(S_{d-2}^0 + S_{d-2}^1) = S_d^2 - S_d^0 = \frac{(d-1)(d+2)}{2} \tag{46}$$

and

$$\begin{aligned} S_{d-2}^n + 2\sum_{p=0}^{n-1} S_{d-2}^p - \big(S_{d-2}^{n-1} + 2\sum_{p=0}^{n-2} S_{d-2}^p\big) &= S_{d-2}^n + S_{d-2}^{n-1} \\ S_d^n - S_d^{n-2} - \big(S_d^{n-1} - S_d^{n-3}\big) &= \frac{(d+n-4)!}{(d-1)!n!}(2n+d-3) \end{aligned} \tag{47}$$

Thus, we finally have the result that any gauge invariant polynomial in the scalar fields that belongs to a traceless symmetric representations of $SO(5,1)$ can be represented by an invariant polynomial $\mathcal{P}$ in $\omega(\varphi)$.

Since $Q^2 = 0$ with the restricted symmetry with the five parameters ω, ϖ, v^I, we can use the horizontality condition (2.1) and the Chern–Simons formula. It implies that, for any given invariant symmetrical polynomial $\mathcal{P}$, one has

$$\mathcal{P}\big(\omega(\varphi)\big) = Q\,\Delta\big(c, \omega(\varphi)\big) \tag{48}$$

with

$$\Delta\big(c, \omega(\varphi)\big) = \int_0^1 dt\, \mathcal{P}\big(c \,|\, t\omega(\varphi) + (t^2 - t)c^2\,\big) \tag{49}$$

where $\mathcal{P}(X) \equiv \mathcal{P}(X, X, X, \cdots)$ and

$$\mathcal{P}(Y|X) \equiv \mathcal{P}(Y, X, X, \cdots) + \mathcal{P}(X, Y, X, \cdots) + \mathcal{P}(X, X, Y, \cdots) + \cdots \tag{50}$$

Any given polynomial in the scalar fields belonging to a traceless symmetric representation of $SO(5,1)$ has a canonical dimension which is strictly lower than that of all other operators in the same representation, made out of other fields. Thus, by power counting, the polynomials in the scalar fields can only mix between themselves under renormalization. That is, for any homogeneous polynomial $\mathcal{P}_A$ of degree n in the traceless symmetric representation, one has

$$\mathcal{C}\big[\mathcal{P}_A\big(\omega(\varphi)\big) \cdot \Gamma\big] = \sum_B \gamma_A{}^B \big[\mathcal{P}_B\big(\omega(\varphi)\big) \cdot \Gamma\big] \tag{51}$$

Then, the Slavnov–Taylor identities imply

$$\mathcal{C}\big[\Delta_A\big(c,\omega(\varphi)\big)\cdot\Gamma\big] = \sum_B \gamma_A{}^B\big[\Delta_B\big(c,\omega(\varphi)\big)\cdot\Gamma\big] + \cdots \tag{52}$$

where the dots stand for possible $\mathcal{S}_{(Q)|\Gamma}$-invariant corrections. However, in the shadow-Landau gauge, $\Delta_A(c,\omega(\varphi))$ cannot appear in the right hand side because such term would break the ghost Ward identities. One thus gets the result that $\gamma_A{}^B = 0$

$$\mathcal{C}\big[\mathcal{P}_A\big(\omega(\varphi)\big)\cdot\Gamma\big] = 0 \tag{53}$$

Upon decomposition of this equation in function of the five independent supersymmetry parameters, one then gets the finiteness proof for each invariant polynomial $\mathcal{P}(\phi) \equiv \mathcal{P}(\phi^i,\phi^j,\phi^k,\cdots)$ in the traceless symmetric representation of the R-symmetry group, namely

$$\mathcal{C}\big[\mathcal{P}(\phi)\cdot\Gamma\big] = 0 \tag{54}$$

Having proved that all 1/2 BPS primary operators have zero anomalous dimension, the Q-symmetry implies that all the operators generated from them, by applying $\mathcal{N}=4$ super-Poincaré generators, have also vanishing anomalous dimensions. It follows that all the operators of the 1/2 BPS multiplets are protected operators.

It is worth considering as an example the simplest case of Tr $\omega(\varphi)^2$. One has

$$\begin{gathered} Q\text{Tr }\Big(\omega(\varphi)c - \frac{1}{3}c^3\Big) = \text{Tr }\omega(\varphi)^2 \qquad s\,Q\text{Tr }\Big(\omega(\varphi)c - \frac{1}{3}c^3\Big) = 0 \\ s\text{ Tr }\Big(\omega(\varphi)c - \frac{1}{3}c^3\Big) = \text{Tr }\Big(\mu\big(\omega(\varphi) - c^2\big) - [\Omega,\omega(\varphi)]c\Big) \end{gathered} \tag{55}$$

Following [10], one couples these operators to the theory by adding source terms to the effective action Σ

$$u\text{Tr }\Big(\omega(\varphi)c - \frac{1}{3}c^3\Big) + u^{(s)}\text{Tr }\Big(\mu\big(\omega(\varphi) - c^2\big) - [\Omega,\omega(\varphi)]c\Big) + u^{(Q)}\text{Tr }\omega(\varphi)^2 \tag{56}$$

This action satisfies the Slavnov–Taylor identities associated to the s and Q symmetries, provided that the sources $u^\bullet$ transform as follows

$$\begin{aligned} s\,u^{(Q)} &= 0 \qquad & Qu^{(Q)} &= u \\ s\,u^{(s)} &= u \qquad & Qu^{(s)} &= 0 \\ s\,u\ &= 0 \qquad & Qu\ &= 0 \end{aligned} \tag{57}$$

It is easy to check by inspection that the introduction of these new sources cannot introduce any potential anomaly for the Slavnov–Taylor identities. In the shadow-Landau gauge, the ghost Ward identities remain valid, with an additional dependence in the sources $u^{\bullet}$ [3]

$$\int\Big(\frac{\delta^L\Gamma}{\delta\mu}-\Big[\bar\mu,\frac{\delta^L\Gamma}{\delta b}\Big]+u^{(s)}\frac{\delta^L\Gamma}{\delta c^{(Q)}}-(-1)^a[\varphi^{(Q_s)}{}_a,\varphi^a]+[\Omega^{(Q_s)},\Omega]+[\mu^{(Q)},c]\Big)=0$$
$$\int\Big(\frac{\delta^L\Gamma}{\delta c}+\Big[\bar c,\frac{\delta^L\Gamma}{\delta b}\Big]-\Big[\bar\mu,\frac{\delta^L\Gamma}{\delta\bar\Omega}\Big]+(-1)^a\Big[\varphi^{(Q_s)}{}_a,\frac{\delta^L\Gamma}{\delta\varphi^{(s)}{}_a}\Big]-\Big[\Omega^{(Q_s)},\frac{\delta^L\Gamma}{\delta\Omega^{(s)}}\Big]$$
$$+u\frac{\delta^L\Gamma}{\delta c^{(Q)}}+u^{(s)}\frac{\delta^L\Gamma}{\delta\mu^{(Q)}}+[\varphi^{(Q)}{}_a,\varphi^a]+[\Omega^{(Q)},\Omega]+[c^{(Q)},c]+[\mu^{(Q)},\mu]\Big)=0$$
$$\int\Big(\frac{\delta^L\Gamma}{\delta\Omega}-\Big[\bar\Omega,\frac{\delta^L\Gamma}{\delta b}\Big]+\Big[\bar\mu,\frac{\delta^L\Gamma}{\delta\bar c}\Big]-\Big[c,\frac{\delta^L\Gamma}{\delta\mu}\Big]-(-1)^a\Big[\varphi^{(Q_s)}{}_a,\frac{\delta^L\Gamma}{\delta\varphi^{(Q)}{}_a}\Big] \qquad (58)$$
$$+\Big[\Omega^{(Q_s)},\frac{\delta^L\Gamma}{\delta\Omega^{(Q)}}\Big]+\Big[\mu^{(Q)},\frac{\delta^L\Gamma}{\delta c^{(Q)}}\Big]+[\varphi^{(s)}{}_a,\varphi^a]+[\Omega^{(s)},\Omega]\Big)=0$$

The most general $u^{\bullet}$ dependent counter term which satisfies both Slavnov–Taylor identities is

$$u^{(Q)}\mathcal{S}_{(Q)|\Sigma}\Delta^{(0,3)}_{[\frac{3}{2}]}+u\Delta^{(0,3)}_{[\frac{3}{2}]}+u^{(s)}\mathcal{S}_{(s)|\Sigma}\Delta^{(0,3)}_{[\frac{3}{2}]} \qquad (59)$$

where $\Delta^{(0,3)}_{[\frac{3}{2}]}$ must be a local functional of ghost and shadow number $(0,3)$, canonical dimension $\frac{3}{2}$, which verifies

$$\mathcal{S}_{(s)|\Sigma}\mathcal{S}_{(Q)|\Sigma}\Delta^{(0,3)}_{[\frac{3}{2}]}=0 \qquad (60)$$

$\Delta^{(0,3)}_{[\frac{3}{2}]}$ is also a scalar under the action of the symmetry group $SU(2)_+\times SU(2)\times U(1)$.

These constraints imply that $\Delta^{(0,3)}_{[\frac{3}{2}]}$ is proportional to $\mathrm{Tr}\,\big(\omega(\varphi)c-\frac{1}{3}c^3\big)$. Thus the three insertions that we have introduced can only be multiplicatively renormalized, having the same anomalous dimension. Moreover, the ghost Ward identities forbid the introduction of any invariant counter term including the shadow field c, if it is not trough a derivative term dc or particular combinations of c and the other fields that do not appear in the insertion $\mathrm{Tr}\,\big(\omega(\varphi)c-\frac{1}{3}c^3\big)$. This gives the result that

$$\mathcal{C}\left[\mathrm{Tr}\;\omega(\varphi)^2\;\cdot\Gamma\right]=0 \qquad (61)$$

Finally, owing to the factorization property we obtain that all the 20 operators that constitute the traceless symmetric tensor representation of rank two in $SO(5,1)$ are

[3] For invariant polynomials $\mathcal{P}$ of rank higher than 2, one has to introduce further sources in order to restore the ghost Ward identities [7]. But we can always carry out this with a finite number of such sources.

protected operators

$$\mathrm{Tr}\,\big(\Phi^2\big)\,,\ \mathrm{Tr}\,\big(\Phi L\big)\,,\ \mathrm{Tr}\,\big(\Phi\bar\Phi+\frac{1}{2}L^2\big)\,,\ \mathrm{Tr}\,\big(\bar\Phi L\big)\,,\ \mathrm{Tr}\,\big(\bar\Phi^2\big)\,,$$
$$\mathrm{Tr}\,\big(\Phi h_I\big)\,,\ \mathrm{Tr}\,\big(L h_I\big)\,,\ \mathrm{Tr}\,\big(\bar\Phi h_I\big)\,,\ \mathrm{Tr}\,\big(\delta_{IJ}\Phi\bar\Phi+\frac{1}{2}h_I h_J\big) \tag{62}$$

This constitutes the simplest example of (54), for $\mathcal{P}(\phi)\equiv \mathrm{Tr}\,\big(\phi^i\phi_j-\frac{1}{6}\delta^i_j\phi_k\phi^k\big)$.

6 Cancellation of the β function

We now give an improved version of the proof given in [11], that the β function is zero to all order in the perturbative $\mathcal{N}=4$ super-Yang–Mills theory.

The proof of the cancellation of the β function is a corollary of the three following propositions

- The β function is zero at first order (63)
- $$\mathcal{C}\,\Big[\int\mathcal{L}^0_4\cdot\Gamma\Big]=\mathcal{S}_{(s)|\Gamma}\big[\mathbf{\Psi}^{(1)}\cdot\Gamma\big] \tag{64}$$
- $$\frac{\partial\Gamma}{\partial g}+\frac{2}{g^3}a(g)\,\Big[\int\mathcal{L}^0_4\cdot\Gamma\Big]=\mathcal{S}_{(s)|\Gamma}\big[\mathbf{\Psi}^{(2)}\cdot\Gamma\big] \tag{65}$$

where $\mathbf{\Psi}^{(\xi)}$ are integrated insertions of ghost number -1 and shadow number 0. Moreover,

$$a(g)=1+\sum_{\mathbb{N}*}a_n\,g^{2n} \tag{66}$$

is a function of the coupling constant g that accounts for the possible radiative corrections to the classical equation

$$\frac{\partial\Sigma}{\partial g}+\frac{2}{g^3}\int\mathcal{L}^0_4=0 \tag{67}$$

The key part of of the proof follows from the equation

$$\Big[\mathcal{C},\frac{\partial}{\partial g}\Big]\mathscr{F}=-\frac{\partial\beta}{\partial g}\frac{\partial\mathscr{F}}{\partial g}-\mathcal{S}_{(s)|\mathscr{F}}\mathcal{S}_{(Q)|\mathscr{F}}\int\Big(\sum_a\frac{\partial\gamma^a}{\partial g}\,\varphi^a\varphi^{(Q_s)}{}_a+\frac{\partial\gamma^{(A)}}{\partial g}\mathrm{Tr}\,\bar\mu\frac{\delta^L\mathscr{F}}{\delta b}\Big) \tag{68}$$

If one equates $\mathscr{F}$ to the 1PI generating functional Γ, one obtains, as a direct consequence of (27), (64) and (65), that

$$\frac{\partial}{\partial g}\Big(\beta\,\frac{2}{g^3}a(g)\Big)\,\Big[\int\mathcal{L}^0_4\cdot\Gamma\Big]=\mathcal{S}_{(s)|\Gamma}\big[\mathbf{\Psi}^{(3)}\cdot\Gamma\big] \tag{69}$$

Since $\left[\int \mathcal{L}_4^0 \cdot \Gamma\right] \neq 0$ belongs to the cohomology of $\mathcal{S}_{(s)|\Gamma}$, the right-hand side and left-hand side of this equation must be zero. This implies

$$\frac{\partial}{\partial g}\left(\beta\,\frac{2}{g^3}a(g)\right) = 0 \tag{70}$$

This equation can be expanded in power of g, $\beta = \sum_{n\in\mathbb{N}} \beta_{(n)}\, g^{2n+1}$. It gives

$$\beta_{(n)} = -\sum_{p=1}^{n-1} a_{n-p}\,\beta_{(p)} \;\propto \beta_{(1)}, \qquad n \geqslant 2 \tag{71}$$

Using then the proposition (63), that is $\beta_{(1)} = 0$, one obtains that the β function is zero at all orders in perturbation theory.

Let us now demonstrate the basic ingredients (64,65) of the proof that the β function vanishes to all orders.

The cancellation of the one-loop β function (63) is a well-established result in perturbation theory.

The proposition (65) is a straightforward consequence of the property that the $\mathcal{N} = 4$ super-Yang–Mills action has only one physical parameter, namely the coupling constant g. The Slavnov–Taylor operators commute with the derivation with respect to g, and thus

$$\mathcal{S}_{(s)|\Gamma}\frac{\partial\Gamma}{\partial g} = \mathcal{S}_{(Q)|\Gamma}\frac{\partial\Gamma}{\partial g} = 0 \tag{72}$$

Starting from the classical equation (67), the quantum action principle implies that differentiation of the 1PI generating functional with respect to the coupling constant g amounts to the insertion of an integrated local functional of ghost and shadow number zero, which satisfies all the global linear symmetries of the theory, and which is invariant under the action of the two linearized Slavnov–Taylor operators $\mathcal{S}_{(s)|\Sigma}$ and $\mathcal{S}_{(Q)|\Sigma}$ (see [11] for more details). The only such functional in the cohomology of $\mathcal{S}_{(s)|\Sigma}$ is the classical action $\int \mathcal{L}_4^0$, what establishes the result (65).

The only non-trivial point for proving the vanishing of the β function is thus the demonstration of (64), which we now show, in the shadow-Landau gauge.

6.1 Cocycles and descent equations for the lagrangian density

To prove (64), we will use the fact that the lagrangian density is uniquely linked to a protected operator by descent equations, involving the equivariant part of the Q symmetry.

Because $\mathcal{L}_4^0$ and Ch_4^0 (13,14) are supersymmetric invariant only modulo a boundary term, the algebraic Poincaré lemma predicts series of cocycles, which are linked to $\mathcal{L}_4^0$ and Ch_4^0 by descent equations, as follows:

$$\begin{aligned}
\delta^{Susy}\mathcal{L}_4^0 + d\mathcal{L}_3^1 &= 0 & \delta^{Susy}Ch_4^0 + dCh_3^1 &= 0 \\
\delta^{Susy}\mathcal{L}_3^1 + d\mathcal{L}_2^2 &= \varpi i_\varepsilon \mathcal{L}_4^0 & \delta^{Susy}Ch_3^1 + dCh_2^2 &= \varpi i_\varepsilon Ch_4^0 \\
\delta^{Susy}\mathcal{L}_2^2 + d\mathcal{L}_1^3 &= \varpi i_\varepsilon \mathcal{L}_3^1 & \delta^{Susy}Ch_2^2 + dCh_1^3 &= \varpi i_\varepsilon Ch_3^1 \\
\delta^{Susy}\mathcal{L}_1^3 + d\mathcal{L}_0^4 &= \varpi i_\varepsilon \mathcal{L}_2^2 & \delta^{Susy}Ch_1^3 + dCh_0^4 &= \varpi i_\varepsilon Ch_2^2 \\
\delta^{Susy}\mathcal{L}_0^4 &= \varpi i_\varepsilon \mathcal{L}_1^3 & \delta^{Susy}Ch_0^4 &= \varpi i_\varepsilon Ch_1^3
\end{aligned} \tag{73}$$

Using the grading properties of the shadow number and the form degree, we conveniently define

$$\begin{aligned}
\mathcal{L} &\equiv \mathcal{L}_4^0 + \mathcal{L}_3^1 + \mathcal{L}_2^2 + \mathcal{L}_1^3 + \mathcal{L}_0^4 \\
Ch &\equiv Ch_4^0 + Ch_3^1 + Ch_2^2 + Ch_1^3 + Ch_0^4
\end{aligned} \tag{74}$$

The descent equations can then be written in a unified way

$$(d + \delta^{Susy} - \varpi i_\varepsilon)\mathcal{L} = 0 \qquad (d + \delta^{Susy} - \varpi i_\varepsilon)Ch = 0 \tag{75}$$

Note that on gauge-invariant polynomials in the physical fields, δ^{Susy} can be identified to $s + Q$, in such way that the differential $(d + \delta^{Susy} - \varpi i_\varepsilon)$ is nilpotent on them. Since $\mathcal{L}_4^0$ and Ch_4^0 are the unique solutions of the first equation in (73), one obtains that $\mathcal{L}$ and Ch are the only non-trivial solutions of the descent equations, that is, the only ones that cannot be written as $(d + \delta^{Susy} - \varpi i_\varepsilon)\,\Xi$. In fact Ch_4^0 and $\mathcal{L}_4^0$ are the unique non-trivial solutions of Eq. (73), even when δ^{Susy} is restricted to six supersymmetry parameters ($\upsilon_I = 0$).

The expression of the cocycles Ch_{4-s}^s can be simply obtained, by changing F into the extended curvature (2.1) in the topological term $\frac{1}{2}\mathrm{Tr}\, FF$, owing to the horizontality equation that expresses s and Q. Therefore:

$$Ch = \frac{1}{2}\mathrm{Tr}\,\Big(F + \varpi\Psi + \omega\bar{\Psi} + g(\varepsilon)\eta + g(J_I\varepsilon)\chi^I + \varpi^2\Phi + \varpi\omega L + (\omega^2 + |\varepsilon|^2)\bar{\Phi}\Big)^2 \tag{76}$$

As for determining the explicit form of $\mathcal{L}_{4-s}^s$ for $s \geqslant 1$, we found no other way than doing a brute force computation, starting from $\mathcal{L}_4^0$ in Eq. (13). In this way, one gets, in a step by step computation

$$\mathcal{L}^1_3 = \text{Tr}\ \Big(\varpi\big(\Psi_\wedge F + J_I(\bar\chi^I T - \bar\Psi[\Phi, h^I])\big)
+ \omega\big(\bar\Psi_\wedge F + J_I(-\eta d_A h^I + \chi_I T + \Psi[\bar\Phi, h^I] - \bar\Psi[L, h^I])\big)
+ J_{I\wedge} i_\varepsilon F\chi^I + i_\varepsilon \star \chi_I H^I + \varepsilon_{IJK} g(\varepsilon) J^I \chi^J H^K
+ (i_{J^I\varepsilon} \star \eta - i_\varepsilon \star \chi^I)\Big(\frac{1}{2}\varepsilon_{IJK}[h^J, h^K] + [L, h_I]\Big)
+ \star\bar\Psi i_\varepsilon T - i_\varepsilon(\bar\Psi \star d_A L) + i_\varepsilon(J_{I\wedge}\bar\Psi)_\wedge d_A h^I
+ i_\varepsilon \star \bar\eta[\bar\Phi, L] - i_{J_I\varepsilon} \star \bar\eta[\bar\Phi, h^I] + i_\varepsilon \star \eta[\Phi, \bar\Phi] + i_\varepsilon(\Psi \star d_A\bar\Phi) - g(\varepsilon)\Psi d_A \bar\Phi\Big) \quad (77)$$

$$\mathcal{L}^2_2 = \text{Tr}\ \Big(\varpi^2\big(\Phi F + \frac{1}{2}\Psi\Psi\big) + \varpi\omega\big(LF + \Psi\bar\Psi + J_I(h^I[\Phi, \bar\Phi] - \eta\bar\chi^I)\big)
+ \omega^2\big(\bar\Phi F + \frac{1}{2}\bar\Psi\bar\Psi + J_I(L[\bar\Phi, h^I] - \eta\chi^I)\big) + \varpi\big(g(J_I\varepsilon)\Psi\chi^I + i_\varepsilon J_{I\wedge}\bar\Psi\bar\chi^I - g(\varepsilon)\Phi d_A\bar\Phi\big)
+ \omega\big(-g(J_I\varepsilon)\bar\Phi d_A h^I + J_I i_\varepsilon \bar\Psi\chi^I - 2\eta(g(\varepsilon)\bar\Psi)^- - g(\varepsilon) L d_A \bar\Phi\big)
+ \frac{1}{2} g(J_I\varepsilon) g(J_J\varepsilon)\bar\Phi[h^I, h^J] + \star g(\varepsilon) g(J_I\varepsilon) L[\bar\Phi, h^I] + \frac{1}{2}|\varepsilon|^2\bar\Psi\bar\Psi - J_I i_\varepsilon \bar\Psi i_{J^I\varepsilon}\bar\Psi\Big) \quad (78)$$

$$\mathcal{L}^3_1 = \text{Tr}\ \Big(\varpi^3\Phi\Psi + \varpi^2\omega(L\Psi + \Phi\bar\Psi) + \varpi\omega^2(L\bar\Psi + \bar\Phi\Psi) + \omega^3\bar\Phi\bar\Psi
+ \varpi^2 g(J_I\varepsilon)\Phi\chi^I + \varpi\omega g(J_I\varepsilon)(L\chi^I - \bar\Phi\bar\chi^I)
+ \omega^2 g(\varepsilon)\bar\Phi\eta + \omega\bar\Phi g(J_I\varepsilon) i_{J^I\varepsilon}\bar\Psi\Big) \quad (79)$$

$$\mathcal{L}^4_0 = \frac{1}{2}\text{Tr}\ \Big(\big(\varpi^2\Phi + \varpi\omega L + \omega^2\bar\Phi\big)^2 + \varpi^2|\varepsilon|^2\bar\Phi^2\Big) \quad (80)$$

6.2 Finiteness property for the classical action $\int \mathcal{L}^0_4$

The last cocycle $\mathcal{L}^4_0$ is a linear combination of protected operators associated to the second Chern class Tr $\omega(\varphi)^2$. Therefore, its anomalous dimension is zero. We now show that this implies that its ascendant $\mathcal{L}^0_4$ is also protected modulo a d variation, which is the non-trivial condition (64) for proving the vanishing of the β function, which we now rewrite

$$\mathcal{C}\ \Big[\int \mathcal{L}^0_4 \cdot \Gamma\Big] = \mathcal{S}_{(s)|\Gamma}\Big[\int \Upsilon^{(-1,0)}_4 \cdot \Gamma\Big] \quad (81)$$

It is worth going into the details of the proof of this identity.

To define one insertion of the lagrangian density $\mathcal{L}_4^0$ and its descendants $\mathcal{L}_{4-p}^p$ in a way that preserves the Slavnov–Taylor identities, one introduces sources for each term $\mathcal{L}_{4-p}^p$ and defines the effective action

$$\Sigma \to \Sigma[u] \equiv \Sigma + \int \sum_p u_p^{-p} \mathcal{L}_{4-p}^p \tag{82}$$

This effective action verifies the Slavnov–Taylor identities, provided that the sources u_p^{-p} are s-invariant and

$$Q u_p^{-p} = -d u_{p-1}^{1-p} + \varpi i_\varepsilon u_{p+1}^{-1-p} \tag{83}$$

Using the extended form $u \equiv \sum_p u_p^{-p}$, one has

$$(d + Q - \varpi i_\varepsilon) u = 0 \tag{84}$$

Let us define as Δ_{4-p}^p the local counterterms that might occur for the renormalization of the operators $\mathcal{L}_{4-p}^p$. The question is that of determining the most general invariant counterterm for the effective action $\int \sum_p u_p^{-p} \Delta_{4-p}^p$, which is linear in the u_p^{-p}. It ought to be invariant under $SU(2)_+ \times SU(2) \times U(1)$ and to obey the Slavnov–Taylor identities:

$$\mathcal{S}_{(s)|\Sigma} \int \sum_p u_p^{-p} \Delta_{4-p}^p = 0 \qquad \mathcal{S}_{(Q)|\Sigma} \int \sum_p u_p^{-p} \Delta_{4-p}^p = 0 \tag{85}$$

If we call $\Delta \equiv \sum_p \Delta_{4-p}^p$, one must have

$$(d + \mathcal{S}_{(s)|\Sigma} + \mathcal{S}_{(Q)|\Sigma} - \varpi i_\varepsilon) \Delta = 0 \tag{86}$$

Since the cohomology of $\mathcal{S}_{(s)|\Sigma}$ modulo d can be identified with the gauge-invariant polynomials in the physical fields, Δ must be the sum of a gauge-invariant polynomial in the physical fields and of a $\mathcal{S}_{(s)|\Sigma}$-exact term $\mathcal{S}_{(s)|\Sigma}\Upsilon$, where Υ is an arbitrary extended form in the fields and the sources of ghost number -1. Because $\mathcal{L}$ and Ch generate the only non-trivial elements of the cohomology of $d + \delta^{Susy} - \varpi i_\varepsilon$ in the set of gauge-invariant polynomials in the physical fields, Δ must be of the form

$$\Delta = z_1 \mathcal{L} + z_2 Ch + (d + \delta^{Susy} - \varpi i_\varepsilon)\, \Xi + \mathcal{S}_{(s)|\Sigma} \Upsilon \tag{87}$$

Ξ is an arbitrary gauge-invariant extended form in the physical fields of total degree 3 and canonical dimension $\frac{7}{2}$.

We have seen in the previous section that $\mathcal{L}_0^4$ is a protected operator, and therefore, $\mathcal{C}\left[\mathcal{L}_0^4 \cdot \Gamma\right] = 0$. Thus, all invariant counterterms that might be generated in perturbation theory have to be such that

$$\Delta_0^4 = 0 \tag{88}$$

and therefore

$$z_1 \mathcal{L}_0^4 + z_2 Ch_0^4 + \delta^{Susy}\, \Xi_0^3 - \varpi i_\varepsilon\, \Xi_1^2 = 0 \tag{89}$$

Each term in this expansion must separately vanish. Indeed, $\mathcal{L}_0^4$ and Ch_0^4 are not δ^{Susy}-exact and

$$Ch_0^4 - \mathcal{L}_0^4 = i_\varepsilon\, g(\varepsilon)\bar{\Phi}\big(\varpi^2\Phi + \varpi\omega L + \frac{1}{2}(\omega^2 + |\varepsilon|^2)\bar{\Phi}\big) \tag{90}$$

cannot be written as a contraction with respect to the vector $\varpi\varepsilon$ of a 1-form that is analytic in ϖ.

It follows that the most general functional (87) which has vanishing component of shadow number four, must have a component of zero shadow number of the following form

$$\Delta_4^0 = d\,\Xi_3^0 + \mathcal{S}_{(s)|\Sigma}\Upsilon_4^{(-1,0)} \tag{91}$$

This is precisely the result (81) that we wanted to prove.

7 Conclusion

A great improvement due to the introduction of the shadow fields is that one has two separated and consistent Slavnov–Taylor identities corresponding respectively to gauge and supersymmetry invariance. This enables one to establish the cancellation of the anomalous dimension of some operators, considering them as insertions in any Green functions of physical observables, which are not restricted to be supersymmetric scalars. As for the physics, it is now defined to be the cohomology of $\mathcal{S}_{(s)|\Gamma}$ rather than the cohomology of $\mathcal{S}_{(Q)|\Gamma}$, and its supersymmetry covariance is easy to check.

Aspects of the superconformal invariance of the $\mathcal{N} = 4$ theory can be checked at any given finite order in perturbation theory, for any type of ultraviolet regularization. In this paper we have proved the cancellation of the β function and the finiteness of the 1/2 BPS operators. The method can certainly be extended to other features of superconformal invariance.

References

[1] L. Baulieu, G. Bossard and S. P. Sorella, "Shadow fields and local supersymmetric gauges," [arXiv:hep-th/0603248].

[2] S. Mandelstam, "Light Cone Superspace And The Ultraviolet Finiteness Of The $\mathcal{N} = 4$ Model," Nucl. Phys. B **213** (1983) 149; P. S. Howe, K. S. Stelle and P. K. Townsend, Nucl. Phys. B **236** (1984) 125; L. Brink, O. Lindgren and B. E. W. Nilsson, "The Ultraviolet Finiteness Of The $\mathcal{N} = 4$ Yang-Mills Theory," Phys. Lett. B **123**, 323 (1983).

[3] M. F. Sohnius and P. C. West, "Conformal Invariance In $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory," Phys. Lett. B **100** (1981) 245; P. L. White, "Analysis of the superconformal cohomology structure of $\mathcal{N} = 4$ superYang-Mills," Class. Quant. Grav. **9** (1992) 413.

[4] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].

[5] L. Baulieu, G. Bossard and A. Tanzini, "Topological Vector Symmetry of BRSTQFT and Construction of Maximal Supersymmetry," JHEP **0508** (2005) 037 [arXiv:hep-th/0504224].

[6] L. Baulieu and G. Bossard, "New results on $\mathcal{N} = 4$ superYang–Mills theory," Phys. Lett. B **632** (2006) 131 [arXiv:hep-th/0507003].

[7] N. Maggiore and A. Tanzini, "Protected operators in $\mathcal{N} = 2, 4$ supersymmetric theories," Nucl. Phys. B **613** (2001) 34 [arXiv:hep-th/0105005].

[8] L. Baulieu, H. Kanno and I. M. Singer, "Special quantum field theories in eight and other dimensions," Commun. Math. Phys. **194** (1998) 149 [arXiv:hep-th/9704167].

[9] O. Piguet and S. P. Sorella, "Algebraic renormalization: Perturbative renormalization, symmetries and anomalies," Lect. Notes Phys. **M28**, 1 (1995).

[10] A. Blasi, V. E. R. Lemes, N. Maggiore, S. P. Sorella, A. Tanzini, O. S. Ventura and L. C. Q. Vilar, "Perturbative beta function of $\mathcal{N} = 2$ super-Yang–Mills theories," JHEP **0005** (2000) 039 [arXiv:hep-th/0004048].

[11] V. E. R. Lemes, M. S. Sarandy, S. P. Sorella, O. S. Ventura and L. C. Q. Vilar, "An algebraic criterion for the ultraviolet finiteness of quantum field theories," J. Phys. A **34** (2001) 9485 [arXiv:hep-th/0103110].

[12] L. Baulieu and G. Bossard, in preparation.