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Comments on noncommutative gravity

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Abstract

We study the possibility of obtaining noncommutative gravity dynamics from string theory in the Seiberg-Witten limit. We find that the resulting low-energy theory contains more interaction terms than those proposed in noncommutative deformations of gravity. The rôle of twisted diffeomorphisms in string theory is studied and it is found that they are not standard physical symmetries. It is argued that this might be the reason why twisted diffeomorphisms are not preserved by string theory in the low energy limit. Twisted gauge transformations are also discussed.

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1 Introduction

The construction of consistent noncommutative deformations of Einstein gravity has been a subject of interest for some time (for an incomplete list of references see [1, 2, 3]). Following the standard procedure to construct noncommutative deformations of gauge and scalar theories [4], noncommutative versions of the Einstein-Hilbert action have been obtained by replacing the ordinary product by the noncommutative Moyal product

$$f(x) \star g(x) = f(x) e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} g(x). \tag{1.1}$$

There are many ways in which these deformations have been implemented. Generically, noncommutative deformations of gravity lead to a complexification of the metric as well as of the local Lorentz invariance of the theory which is no longer SO(1,3) but U(1,3) or larger [2]. This results in theories with hermitian metrics reminiscent of those studied long ago by Einstein and Straus [5]. These theories are known to contain ghost states [6] (for some proposal to overcome these problems see [3]).

An unsatisfactory feature of many of the approaches developed so far is that they are physically *ad hoc*, since the deformation of general relativity is not based on any general dynamical principle. Recently, however, a new approach has been proposed in which a deformation of the Einstein-Hilbert action is constructed based on a deformation of the group of diffeomorphisms [7, 8] (see [9] for a review). The idea behind it is to replace the diffeomorphism invariance of general relativity by a twisted version of this symmetry [10], i.e. deforming the Hopf algebra structure of the universal enveloping algebra of the Lie algebra of vector fields by twisting the coproduct in the appropriate way (see also [11]). Roughly speaking this amounts to keeping the action of the diffeomorphisms on the physical "primary" fields unchanged while deforming the Leibniz rule when taking the action of diffeomorphisms on the product of two fields. This change have the effect that diffeomorphisms now act "covariantly" on the star-product of two fields.

Once the deformation of the diffeomorphisms has been introduced a gravity action can be written with the requirement of invariance under the action of the twisted transformations. As with other proposals for noncommutative gravity, the deformed Einstein-Hilbert action can be written in terms of star-products. Nevertheless, unlike other proposals, this construction has the obvious advantage of being based on an underlying symmetry principle.

One may wonder whether string theory can reproduce the deformed gravity action proposed in [7, 8] in some limit. It is known that the strict Seiberg-Witten [12] limit results in a complete decoupling of gravity. This is because in order to retain nontrivial gauge dynamics on the brane one has to scale the closed string coupling constant to zero in the low-energy limit. As a result, closed string states decouple at low energies and the resulting theory is not coupled to gravity.

By looking at the next-to-leading order in the Seiberg-Witten limit it is possible to study the dynamics of closed strings in the presence of a constant B-field. In this paper we will study the gravitational action induced by the bosonic string theory on a space-filling D-brane with a constant magnetic field in the low energy limit. We find that the induced terms for the interaction vertex of three gravitons on the brane contain terms which cannot be derived from an action expressed solely in terms of star-products and therefore are not accounted for in the noncommutative gravity action proposed in [7, 8]. Moreover, these new terms scale in the same way in the Seiberg-Witten limit as the ones associated with star-products and we have found no physical argument that allows their consistent elimination to reproduce an induced gravity action that can be expressed in terms of star-products alone.

We will see that this inability to reproduce a gravity action invariant under deformed diffeomorphisms can be traced back to the fact that the star-product is not "inert" under the action of space-time diffeomorphisms. As it will be shown below, the twisted Leibniz rule can be interpreted as resulting from applying the ordinary Leibniz rule but taking into account the transformation of the star-product. Hence the invariance of the deformed gravity action under twisted diffeomorphisms cannot be considered a physical symmetry in the standard sense, since the transformation involves not only the physical fields but also the star-products. Thus, string theory cannot provide the deformed diffeomorphisms at low energies since these transformations do not yield a physical symmetry of the theory.

The present paper is organized as follows: in Section 2 we will review the construction of the induced gravity action on D-branes in the standard case. After that, in Section 3, we will summarize the relevant aspects of the construction of the noncommutative gravity action presented in [7, 8]. We move on in Section 4 to study the gravity action induced on the brane in the presence of a B-field to the next-to-leading order in the Seiberg-Witten limit. As advertised, we will find extra terms not present in the field theory construction of the noncommutative action. Finally, in Section 5 we will try to understand in physical terms the mismatch between the string and field theory constructions, before summarizing our results in Section 6. In order to make the presentation more self-contained, Appendix A contains a summary of results and definitions concerning Hopf algebras.

2 Brane induced gravity from string theory

We begin by computing the gravitational action induced on the brane in the absence of a B-field. For simplicity we work here with the bosonic string in the presence of a space-filling D-brane. The analysis can be extended to superstrings and/or lower-dimensional D-branes [14].

We carry out the computation by evaluating the correlation functions of three graviton vertex operators on the disk. These correlation functions induce three-graviton interaction terms in the gravitational action on the brane. It is important to stress at this point that we are computing the gravitational action *induced* on the brane. It is well known that the graviton amplitudes on the disk contain divergences associated with the coincidence limit of two or more vertex operator insertions that are usually handled by considering the different factorization limits including the contributions of Riemann surfaces without boundaries [16]. In our case, however, we eliminate these divergences by adding appropriate counterterms in the induced action on the brane. Then the divergences in the string amplitudes will be absorbed in a renormalization of the coefficients multiplying the different terms in the induced gravitational action. From this point of view the origin of the terms in the induced action is similar to the way in which $\lambda\phi^3$ and $\lambda\phi^4$ interaction terms are induced in the Yukawa theory.

We have to calculate the disk correlation function of three graviton vertex operators [17, 18]

$$V_p = \frac{g_s}{\alpha'}\varepsilon_{\mu\nu}(p)\int d^2z\partial X^\mu\overline{\partial}X^\nu e^{ip\cdot X}(z,\overline{z}), \qquad \varepsilon_{[\mu\nu]}(p) = 0 = \eta^{\mu\nu}\varepsilon_{\mu\nu}(p), \tag{2.1}$$

where the symmetric-traceless polarization tensor has to satisfy the transversality conditions $p^\mu\varepsilon_{\mu\nu}(p)$ and the momentum has to be on-shell, $p^2 = 0$, for the vertex operator to be a primary field with conformal weight (1,1). Naively, the coupling between three on-shell gravitons (or gauge fields) vanishes for kinematical reasons. In our case we remain on-shell while continuing the momenta to complex value, in such a way that the resulting amplitudes give a nonzero result from which the induced couplings can be read off.

The relevant quantity to evaluate is the correlation function of three graviton vertex operators

$$\langle V_{p_1} V_{p_2} V_{p_3} \rangle_D = \frac{g_s^2}{(\alpha')^3} \int_D \prod_{i=1}^{3} d^2 z_i \langle\!\langle \prod_{k=1}^{3} \partial X^\mu \overline{\partial} X^\nu e^{ip\cdot X}(z_k, \overline{z}_k) \rangle\!\rangle_D, \tag{2.2}$$

where $\langle\!\langle \ldots \rangle\!\rangle_D$ indicates the correlation function of the corresponding operator on the disk. In order to carry out the calculation, it is very convenient to rewrite the polarization tensors as $\varepsilon_{\mu\nu} \equiv \zeta_\mu \overline{\zeta}_\nu$ so that the graviton vertex operator can be expressed as

$$V_p = g_s \int d^2 z\, e^{i\mathcal{P}\cdot X} \Big|_{\zeta\overline{\zeta}} \tag{2.3}$$

where by the subscript we indicate that only the part linear in $\zeta_\mu \overline{\zeta}_\nu$ has to be kept, and $\mathcal{P}_\mu$ is defined by

$$\mathcal{P}_\mu \equiv p_\mu - \frac{i}{\sqrt{\alpha'}} \zeta_\mu \partial - \frac{i}{\sqrt{\alpha'}} \overline{\zeta}_\mu \overline{\partial}. \tag{2.4}$$

Proceeding in this way the correlation function of three gravitons can be written in a way similar to the three tachyons amplitude with the momenta p_μ replaced by $\mathcal{P}_\mu$,

$$\langle V_{p_1} V_{p_2} V_{p_3} \rangle_D = \frac{g_s^2}{(\alpha')^{13}} (2\pi)^{26} \delta(p_1 + p_2 + p_3) \int_D \prod_{i=1}^{3} d^2 z_i \prod_{k<\ell}^{3} e^{-\mathcal{P}_k \cdot \mathcal{P}_\ell G(z_k, z_\ell)} \Big|_{(\zeta\overline{\zeta})^3}, \tag{2.5}$$

where the delta function and the powers of α' arise from the integration over the zero modes of $X^\mu(z, \overline{z})$ and $G(z, w)$ is the propagator of the Laplacian on the disk

$$G(z, w) = -\alpha' \Big(\log|z - w| - \log|z - \overline{w}| \Big). \tag{2.6}$$

If we use now the momentum conservation together with the on-shell condition for the momenta, the exponent inside the integral in Eq. (2.5) simplifies to

$$\begin{aligned} \sum_{k<\ell}^{3} \mathcal{P}_k \cdot \mathcal{P}_\ell G(z_k, z_\ell) &= -\frac{i}{\sqrt{\alpha'}} \sum_{k\neq\ell}^{3} \left[(p_k \cdot \zeta_\ell) \partial_\ell + (p_k \cdot \overline{\zeta}_\ell) \overline{\partial}_\ell \right] G(z_k, z_\ell) \\ &- \frac{1}{2\alpha'} \sum_{k\neq\ell}^{3} \left[(\zeta_k \cdot \zeta_\ell) \partial_k \partial_\ell + (\overline{\zeta}_k \cdot \overline{\zeta}_\ell) \overline{\partial}_k \overline{\partial}_\ell \right] G(z_k, z_\ell) \\ &- \frac{1}{\alpha'} \sum_{i\neq j}^{3} (\zeta_i \cdot \overline{\zeta}_j) \partial_i \overline{\partial}_j G(z_i, z_j). \end{aligned} \tag{2.7}$$

In order to obtain the terms in the effective action we have to expand the exponential in Eq. (2.5), $\exp\left[-\sum_{k<\ell}^{3}\mathcal{P}_k\cdot\mathcal{P}_\ell G(z_k,z_\ell)\right]$ keeping the terms linear in $\zeta_\mu\overline{\zeta}_\nu$. Doing this we obtain terms with two, four and six momenta that induce interactions in the action that are associated respectively with terms linear, quadratic and cubic in the Riemann tensor [16] (the term without derivatives is associated to a "cosmological" term proportional to $\sqrt{-g}$).

The only terms that contributes to the Einstein-Hilbert term are those with two momenta given by

$$
\begin{aligned}
e^{-\sum_{k<\ell}^{3}\mathcal{P}_k\cdot\mathcal{P}_\ell G(z_k,z_\ell)}\Big|_{(\zeta\overline{\zeta})^3} &= -\frac{1}{(\alpha')^3}\sum_{i,\ell=1}^{3}\Bigg[\sum_{(j,a,b)=1}^{3}\sum_{(m,c,d)=1}^{3}(p_i\cdot\zeta_j)(p_\ell\cdot\overline{\zeta}_m)(\zeta_a\cdot\zeta_b)(\overline{\zeta}_c\cdot\overline{\zeta}_d) \\
&\quad\times\partial_j G_{\rm reg}(z_i,z_j)\overline{\partial}_m G_{\rm reg}(z_\ell,z_m)\partial_a\partial_b G_{\rm reg}(z_a,z_b)\overline{\partial}_c\overline{\partial}_d G_{\rm reg}(z_c,z_d) \\
&+\sum_{(j,a,c)=1}^{3}\sum_{[m,b,d]=1}^{3}(p_i\cdot\zeta_j)(p_\ell\cdot\overline{\zeta}_m)(\zeta_a\cdot\overline{\zeta}_b)(\zeta_c\cdot\overline{\zeta}_d) \\
&\quad\times\partial_j G_{\rm reg}(z_i,z_j)\overline{\partial}_m G_{\rm reg}(z_\ell,z_m)\partial_a\overline{\partial}_b G_{\rm reg}(z_a,z_b)\partial_c\overline{\partial}_d G_{\rm reg}(z_c,z_d) \\
&+\sum_{(a,j,m)=1}^{3}\sum_{\{a,b,c\}=1}^{3}(p_i\cdot\zeta_j)(p_\ell\cdot\zeta_m)(\zeta_a\cdot\overline{\zeta}_b)(\overline{\zeta}_c\cdot\overline{\zeta}_d) \qquad (2.8)\\
&\quad\times\partial_j G_{\rm reg}(z_i,z_j)\partial_m G_{\rm reg}(z_\ell,z_m)\partial_a\overline{\partial}_b G_{\rm reg}(z_a,z_b)\overline{\partial}_c\overline{\partial}_d G_{\rm reg}(z_c,z_d) \\
&+\sum_{(a,j,m)=1}^{3}\sum_{\{m,b,c\}=1}^{3}(p_i\cdot\overline{\zeta}_j)(p_\ell\cdot\overline{\zeta}_m)(\zeta_a\cdot\zeta_b)(\overline{\zeta}_c\cdot\zeta_d) \\
&\quad\times\overline{\partial}_j G_{\rm reg}(z_i,z_j)\overline{\partial}_m G_{\rm reg}(z_\ell,z_m)\partial_a\partial_b G_{\rm reg}(z_a,z_b)\overline{\partial}_c\partial_d G_{\rm reg}(z_c,z_d)\Bigg],
\end{aligned}
$$

where we have used the notation (a,b,c) to indicate a sum where all the three indices are different and $b<c$, $[a,b,c]$ to denote that all three indices are different and in addition $b>d$ and $\{a,b,c\}$ that the three indices are different without any further constraint. We have also indicated that the propagators have to be regularized in the coincidence limit for the expression to be finite.

A calculation of the corresponding integrals gives the following expression for the term with two momenta [16]

$$
\begin{aligned}
\langle V_1V_2V_3\rangle_{p^2} &= \mathcal{A}\Big[(p_3\cdot\varepsilon_1\cdot p_3)(\varepsilon_2\cdot\varepsilon_3)+(p_1\cdot\varepsilon_3\cdot p_1)(\varepsilon_1\cdot\varepsilon_2)+(p_2\cdot\varepsilon_1\cdot p_2)(\varepsilon_2\cdot\varepsilon_3) \\
&+ 2(p_1\cdot\varepsilon_3\cdot\varepsilon_2\cdot\varepsilon_1\cdot p_2)+2(p_2\cdot\varepsilon_1\cdot\varepsilon_3\cdot\varepsilon_2\cdot p_3)+2(p_3\cdot\varepsilon_2\cdot\varepsilon_1\cdot\varepsilon_3\cdot p_1)\Big], \qquad (2.9)
\end{aligned}
$$

where the coefficient $\mathcal{A}$ is given by[4]

$$\mathcal{A} = -\frac{g_s^2}{(\alpha')^3}\int_D \prod_{i=1}^{3} d^2 z_i \Big|(\partial_2 G_{12} - \partial_2 G_{32})\partial_1\partial_3 G_{13}\Big|^2. \tag{2.10}$$

This integral has divergences associated with the coincidence limits of two or more insertions which, by conformal invariance, are related to the factorization limits containing closed string amplitudes. These divergences can be regularized in a way compatible with both conformal invariance and target-space general covariance [16]. In our case, as explained above, these divergences will be reabsorbed in a renormalization of the coupling constant.

In constructing the induced graviton interactions on the brane we have to be careful in normalizing the graviton field properly. In particular, because the disk amplitude with two graviton vertex operators is nonvanishing [16] and scales as g_s, we have to reabsorb powers of the string coupling constant in the definition of the graviton field in such a way that the quadratic term in the effective action is independent of g_s. Therefore we take the following correspondence between the graviton wave function $h_{\mu\nu}(x)$ and the polarization tensor $\varepsilon_{\mu\nu}(p)$

$$g_s^{\frac{1}{2}}(\alpha')^7 \int \frac{d^{26}p}{(2\pi)^{26}} \varepsilon_{\mu\nu}(p) e^{ip\cdot x} \longrightarrow h_{\mu\nu}(x). \tag{2.11}$$

With this normalization, it is easy to see that the perturbative expansion can be written as an expansion not in powers of the closed string coupling, g_s, but of the so-called open string coupling constant, $g_o = g_s^{\frac{1}{2}}$. Then, the gravitational constant κ scales as

$$\kappa \sim g_s^{\frac{1}{2}}(\alpha')^6 = g_o(\alpha')^6. \tag{2.12}$$

Notice that this scaling with the string coupling constant is different from the one that emerges in the string low energy effective action, $\kappa \sim g_s(\alpha')^6$. The reason behind is of course that, strictly speaking, here we are not computing the low energy effective action but the gravity action induced on the brane at low energies.

Taking into account the previous discussion, the corresponding term in the effective action induced by the part of the amplitude represented by (2.9) can be computed from

$$\Delta S_2 = (\alpha')^{26} \int \frac{d^{26}p_1}{(2\pi)^{26}} \frac{d^{26}p_2}{(2\pi)^{26}} \frac{d^{26}p_3}{(2\pi)^{26}} (2\pi)^{26}\delta(p_1+p_2+p_3)\, \langle V_{p_1} V_{p_2} V_{p_3}\rangle_{p^2}, \tag{2.13}$$

[4]To avoid cumbersome expressions we have factored out explicitly the momentum conservation delta function $(\alpha')^{-13}(2\pi)^{26}\delta(p_1+p_2+p_3)$.

leading to the following induced term in the Lagrangian

$$\Delta\mathcal{L}_2 = -2\kappa\Big(2h^{\sigma\mu}\partial_\mu h^{\alpha\beta}\partial_\beta h_{\alpha\sigma} + h^{\mu\sigma}h_{\alpha\beta}\partial_\sigma\partial_\mu h^{\alpha\beta}\Big). \qquad (2.14)$$

This term can be obtained from the Einstein-Hilbert term $\mathcal{L}_{\rm EH} = \frac{1}{2\kappa^2}\sqrt{-g}R$ in the weak field expansion near Minkowski space-time, $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}$.

The terms with four and six momenta induce contributions to the action containing higher powers of the curvature tensor. With four momenta we find the following tensor structure

$$\begin{aligned}\langle V_1V_2V_3\rangle_{p^4} &= \mathcal{B}\Big[(p_1\cdot\varepsilon_2\cdot\varepsilon_3\cdot p_1)(p_2\cdot\varepsilon_1\cdot p_3) + (p_2\cdot\varepsilon_3\cdot\varepsilon_1\cdot p_2)(p_3\cdot\varepsilon_2\cdot p_1)\\ &+ (p_3\cdot\varepsilon_1\cdot\varepsilon_2\cdot p_3)(p_1\cdot\varepsilon_3\cdot p_2)\Big]\end{aligned} \qquad (2.15)$$

where the coefficient $\mathcal{B}$ is given by

$$\begin{aligned}\mathcal{B} &= -2\frac{g_s^2}{(\alpha')^3}\mathrm{Re}\int_D\prod_{i=1}^{3}d^2z_i\Big|(\partial_2G_{12}-\partial_2G_{32})\Big|^2\Big[(\partial_1G_{12}-\partial_1G_{31})(\partial_2G_{13}-\partial_3G_{23})\overline{\partial}_2\overline{\partial}_3G_{23}\\ &+ (\overline{\partial}_1G_{12}-\overline{\partial_1}G_{31})(\partial_2G_{13}-\partial_3G_{23})\partial_2\overline{\partial}_3G_{23}\Big]\end{aligned} \qquad (2.16)$$

and we assume that the divergences arising from the coincidence limits have been properly regularized. As in the previous case, this amplitude induces a term in the effective action that can be computed as in Eq. (2.13) with the result

$$\Delta\mathcal{L}_4 = -8a_4\alpha'\kappa h^{\mu\nu}\partial_\mu\partial_\sigma h^{\alpha\beta}\partial_\alpha\partial_\beta h_\nu{}^\sigma, \qquad (2.17)$$

where a_4 is a numerical constant. As explained in [16], a term of this type arises from the weak field expansion of a piece in the effective action quadratic in the curvature tensor, $\frac{\alpha' a_4}{2\kappa^2}\sqrt{-g}R_{\mu\nu,\sigma\lambda}R^{\mu\nu,\sigma\lambda}$.

Finally we analyze the term with six momenta. From the calculation of the disk amplitude we get

$$\langle V_1V_2V_3\rangle_{p^6} = \mathcal{C}(p_1\cdot\varepsilon_2\cdot p_1)(p_2\cdot\varepsilon_3\cdot p_2)(p_3\cdot\varepsilon_1\cdot p_3), \qquad (2.18)$$

with

$$\mathcal{C} = -\frac{g_s^2}{(\alpha')^3}\int_D\prod_{i=1}^{3}d^2z_i\Big|(\partial_2G_{12}-\partial_2G_{32})\Big|^2\Big|(\partial_3G_{23}-\partial_3G_{13})\Big|^2\Big|(\partial_1G_{31}-\partial_1G_{21})\Big|^2. \qquad (2.19)$$

Following the same procedure as above, it can be seen that this part of the three-graviton amplitude induces the following term in the effective action

$$\Delta\mathcal{L}_6 = 8a_6(\alpha')^2\kappa\partial_\mu\partial_\nu h^{\alpha\beta}\partial_\alpha\partial_\sigma h^{\lambda\nu}\partial_\lambda\partial_\beta h^{\mu\sigma}. \tag{2.20}$$

Again a_6 is a new dimensionless coupling. This piece of the induced action is the leading term of $\frac{a_6(\alpha')^2}{2\kappa^2}\sqrt{-g}R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\sigma\lambda}R^{\sigma\lambda}{}_{\mu\nu}$ in the expansion around flat space-time.

Putting together the previous results (2.14), (2.17) and (2.20) we conclude that open strings induce a three-point graviton action which reproduces the weak-field expansion of the induced action

$$S_{\text{ind}} = \frac{1}{2\kappa^2}\int d^{26}x\sqrt{-g}\left[\frac{a_0}{\alpha'} + R + a_4\alpha' R_{\mu\nu,\sigma\lambda}R^{\mu\nu,\sigma\lambda} + a_6(\alpha')^2 R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\sigma\lambda}R^{\sigma\lambda}{}_{\mu\nu} + \dots\right] \tag{2.21}$$

Terms containing higher powers of the curvature tensor do not contribute to the three-graviton amplitude. In the spirit of induced gravity, the divergences contained in the integrals in Eqs. (2.16) and (2.19) are absorbed into the renormalized couplings a_0, κ, a_4 and a_6.

3 Noncommutative gravity

Einstein's General Relativity, including the cosmological constant term, can be derived by requiring general covariance and that the action contains only up to two derivatives of the metric. In the case of noncommutative gravity a similar approach has been advanced in [7, 8] where local diffeomorphisms are twisted. In the following we review some basic aspects of this approach and study the corrections to the standard gravitational action in the weak field expansion around flat space-time. The extra couplings between gravitons depending on the noncommutativity parameter $\theta^{\mu\nu}$ are the ones we will try to obtain from string theory in Section 4.

3.1 Deformed diffeomorphisms

As already explained above, the starting point in the approach of [7, 8] is the deformation of diffeomorphisms by twisting. In General Relativity, infinitesimal diffeomorphisms are generated by vector fields $\xi(x) = \xi^\sigma\partial_\sigma$. Their action on the physical tensor fields $T^{(p)}_{(q)}(x)$ of type (p, q) is given by

$$T'^{(p)}_{(q)}(x+\xi) = T^{(p)}_{(q)}(x), \tag{3.1}$$

where $T'^{(p)}_{(q)}(x)$ represents the transformed tensor field. By expanding to linear order in ξ the transformation of $T^{(p)}_{(q)}(x)$ can be written in terms or its Lie derivative as

$$\delta_\xi T^{(p)}_{(q)}(x) \equiv T^{(p)}_{(q)}(x+\xi) - T^{(p)}_{(q)}(x) = -\mathcal{L}_\xi T^{(p)}_{(q)}(x), \tag{3.2}$$

or in components

$$\begin{aligned} \delta_\xi T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}(x) &= -\xi^\alpha(x)\partial_\alpha T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}(x) + \partial_\alpha\xi^{\mu_1}(x)T^{\alpha\ldots\mu_p}_{\nu_1\ldots\nu_p}(x) + \ldots + \partial_\alpha\xi^{\mu_n}(x)T^{\mu_1\ldots\alpha}_{\nu_1\ldots\nu_q}(x) \\ &- \partial_{\nu_1}\xi^\beta(x)T^{\mu_1\ldots\mu_p}_{\beta\ldots\nu_q}(x) - \ldots - \partial_{\nu_q}\xi^\beta(x)T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\beta}(x). \end{aligned} \tag{3.3}$$

In addition, the infinitesimal diffeomorphism generated by ξ acts on the product of two tensor fields $T_1{}^{(p_1)}_{(q_1)}(x)$, $T_2{}^{(p_2)}_{(q_2)}(x)$ via the Leibniz rule

$$\delta_\xi\left[T_1{}^{(p_1)}_{(q_1)}(x)T_2{}^{(p_2)}_{(q_2)}(x)\right] = \delta_\xi T_1{}^{(p_1)}_{(q_1)}(x)\,T_2{}^{(p_2)}_{(q_2)}(x) + T_1{}^{(p_1)}_{(q_1)}(x)\;\delta_\xi T_2{}^{(p_2)}_{(q_2)}(x). \tag{3.4}$$

In a theory deformed by replacing standard products by the Moyal star-products (1.1) the application of the Leibniz rule (3.4) shows that the product of two tensor fields does not transform covariantly, i.e. unlike in the case analyzed above $T_1{}^{(p_1)}_{(q_1)}(x) \star T_2{}^{(p_2)}_{(q_2)}(x)$ does not transform as a tensor of type (p_1+p_2, q_1+q_2). This means that a noncommutative gravity theory based on replacing standard (commutative) products by star-products will fail to be invariant under diffeomorphisms.

This fact, however, does not necessarily mean that there are no other transformations leaving invariant the action of the deformed theory and that, in the limit $\theta^{\mu\nu} \to 0$, reduce to the standard diffeomorphism invariance. There are in principle two possible ways to find the deformed transformations. The first one is to deform the action of infinitesimal diffeomorphisms (3.2) on the fields in such a way that, using the Leibniz rule, the deformed action remains invariant. This is analogous to how gauge invariance gets deformed to star-gauge invariance in gauge theories on noncommutative spaces.

A second alternative is to keep the transformations (3.2) intact but to deform the way it acts on products of fields. In [7, 8] a twist of the standard diffeomorphisms has been constructed to achieve precisely this. The idea behind it is the realization that the universal enveloping algebra of vector fields has a Hopf algebra structure (some basic facts about Hopf algebras, as well as the notation used here, are summarized in Appendix A). In particular, for an infinitesimal diffeomorphism generated by $\xi \neq \mathbf{1}$, the coproduct $\Delta(\xi)$ can be defined to be[5]

$$\Delta(\xi) = \xi \otimes \mathbf{1} + \mathbf{1} \otimes \xi, \tag{3.5}$$

while $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$.

In fact, the choice of the coproduct determines the way the algebra of diffeomorphisms acts on the products of fields. For a given vector field ξ, the action on the product of two tensor fields $T_1{}^{(p_1)}_{(q_1)}(x)$, $T_2{}^{(p_2)}_{(q_2)}(x)$ is given by [cf. Eq. (A.16)]

$$\delta_\xi [T_1{}^{(p_1)}_{(q_1)}(x) T_2{}^{(p_2)}_{(q_2)}(x)] \equiv \mu \left\{ \Delta(\xi) \left[T_1{}^{(p_1)}_{(q_1)}(x) \otimes T_2{}^{(p_2)}_{(q_2)}(x) \right] \right\}. \tag{3.6}$$

Using the coproduct (3.5) the standard Leibniz rule (3.4) is retrieved.

Equation (3.5) is not the only possible choice for a coproduct to define a Hopf algebra structure in the algebra of infinitesimal diffeomorphisms. In particular it is possible to introduce the twist operator

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}, \tag{3.7}$$

in terms of which a new twisted coproduct is defined as

$$\Delta(\xi)_{\mathcal{F}} = \mathcal{F} \left(\xi \otimes \mathbf{1} + \mathbf{1} \otimes \xi \right) \mathcal{F}^{-1}. \tag{3.8}$$

The twist operator (3.7) enters also in the definition of the star-product of two fields

$$T_1{}^{(p_1)}_{(q_1)}(x) \star T_2{}^{(p_2)}_{(q_2)}(x) = \mu \left\{ \mathcal{F}^{-1} \left[T_1{}^{(p_1)}_{(q_1)}(x) \otimes T_2{}^{(p_2)}_{(q_2)}(x) \right] \right\} \equiv \mu_\star \left[T_1{}^{(p_1)}_{(q_1)}(x) \otimes T_2{}^{(p_2)}_{(q_2)}(x) \right], \tag{3.9}$$

which is just a more sophisticated way of writing Eq. (1.1).

It is important to stress that the twist does not change the action of infinitesimal diffeomorphisms on the fields, that it is still given by (3.2). The effect of the twist is to change the coproduct and consequently the action of diffeomorphisms on the product. By using Hadamard's formula

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} [\underbrace{A, [A, \ldots [A}_{n}, B] \ldots]], \tag{3.10}$$

[5]In the case of infinitesimal diffeomorphisms, the identity **1** correspond to the diffeomorphism generated by a vanishing vector field

the twisted coproduct (3.8) can be written in powers of $\theta^{\mu\nu}$ as

$$\begin{aligned}\Delta(\xi)_{\mathcal{F}} &= \xi\otimes\mathbf{1}+\mathbf{1}\otimes\xi \\ &+ \sum_{n=1}^{\infty}\frac{(-i/2)^n}{n!}\theta^{\mu_1\nu_1}\ldots\theta^{\mu_n\nu_n}\Big\{[\partial_{\mu_1},[\partial_{\mu_2},\ldots[\partial_{\mu_n},\xi]\ldots]]\otimes\partial_{\nu_1}\partial_{\nu_2}\ldots\partial_{\nu_n} \\ &\quad+\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_n}\otimes[\partial_{\nu_1},[\partial_{\nu_2},\ldots[\partial_{\nu_n},\xi]\ldots]]\Big\}.\end{aligned} \tag{3.11}$$

This twisted coproduct defines the action of the Lie algebra of vector fields on the star-product of two fields as

$$\begin{aligned}\delta_\xi\Big[T_1{}^{(p_1)}_{(q_1)}(x)\star T_2{}^{(p_2)}_{(q_2)}(x)\Big] &\equiv \mu_\star\Big\{\Delta(\xi)_{\mathcal{F}}\Big[T_1{}^{(p_1)}_{(q_1)}(x)\otimes T_2{}^{(p_2)}_{(q_2)}(x)\Big]\Big\} \\ &= \delta_\xi T_1{}^{(p_1)}_{(q_1)}(x)\star T_2{}^{(p_2)}_{(q_2)}(x)+T_1{}^{(p_1)}_{(q_1)}(x)\star\delta_\xi T_2{}^{(p_2)}_{(q_2)}(x) \\ &+ \sum_{n=1}^{\infty}\frac{(-i/2)}{n!}\theta^{\mu_1\nu_1}\ldots\theta^{\mu_n\nu_n}\Big\{[\partial_{\mu_1}[\partial_{\mu_2},\ldots[\partial_{\mu_n},\delta_\xi]\ldots]]T_1{}^{(p_1)}_{(q_1)}(x)\star\partial_{\nu_1}\partial_{\nu_2}\ldots\partial_{\nu_n}T_2{}^{(p_2)}_{(q_2)}(x) \\ &+\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_n}T_1{}^{(p_1)}_{(q_1)}(x)\star[\partial_{\nu_1},[\partial_{\nu_2},\ldots[\partial_{\nu_n},\delta_\xi]\ldots]]T_2{}^{(p_2)}_{(q_2)}(x)\Big\}.\end{aligned} \tag{3.12}$$

The interesting thing of this choice of the coproduct is that unlike the standard Leibniz rule, Eq. (3.12) guarantees that the star-product of two tensor fields of type (p_1, q_1) and (p_2, q_2) is a tensor of type (p_1+p_2, q_1+q_2). Therefore the star-product transforms "covariantly" with respect to twisted diffeomorphisms.

3.2 The deformed gravity action and its weak field expansion

Once the deformation of diffeomorphisms has been carried out, it is possible to proceed with the construction of the deformed gravity action. In [7, 8] a gravity action is presented and shown to be invariant under deformed diffeomorphisms. In D-dimensions this action reads[6]

$$S_{\star\mathrm{EH}} = \frac{1}{2\kappa^2}\int d^Dx\Big[(\det{}_\star e_\mu{}^a)\hat{G}^{\mu\nu}\star R_{\mu\nu}+\text{c.c.}\Big]. \tag{3.13}$$

Here $\hat{G}^{\mu\nu}$ denotes the star-inverse of $\hat{G}_{\mu\nu}$, i.e. $\hat{G}_{\sigma\nu}\star\hat{G}^{\mu\sigma}=\delta^\mu{}_\nu$. In terms of the vielbein $e_\mu{}^a$ the deformed metric is given by

$$\hat{G}_{\mu\nu}=\frac{1}{2}\Big(e_\mu{}^a\star e_\nu{}^b+e_\nu{}^a\star e_\mu{}^b\Big)\eta_{ab}=g_{\mu\nu}-\frac{1}{8}\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}(\partial_{\alpha_1}\partial_{\alpha_2}e_\mu{}^a)(\partial_{\beta_1}\partial_{\beta_2}e_\nu{}^b)+\ldots, \tag{3.14}$$

[6]Following the notation of Refs. [7, 8] we denote by $\hat{G}_{\mu\nu}$ the deformed metric and by $g_{\mu\nu}$ its commutative limit $\theta^{\mu\nu}\to 0$. For other quantities, we denote the deformed ones with a hat.

and the star-determinant of the vielbein is defined by

$$\det{}_\star e_\mu{}^a = \frac{1}{D!}\epsilon^{\mu_1\dots\mu_D}\epsilon_{a_1\dots a_D} e_{\mu_1}{}^{a_1} \star \dots \star e_{\mu_D}{}^{a_D}. \tag{3.15}$$

The other ingredient of the deformed Einstein-Hilbert action (3.13), the Ricci tensor, is defined in terms of the deformed Riemann tensor

$$\hat{R}_{\mu\nu,\sigma}{}^{\lambda} \equiv \partial_\nu \hat{\Gamma}_{\mu\sigma}{}^{\lambda} - \partial_\mu \hat{\Gamma}_{\nu\sigma}{}^{\lambda} + \hat{\Gamma}_{\nu\sigma}{}^{\alpha} \star \hat{\Gamma}_{\mu\alpha}{}^{\lambda} - \hat{\Gamma}_{\mu\sigma}{}^{\alpha} \star \hat{\Gamma}_{\nu\alpha}{}^{\lambda} \tag{3.16}$$

by $\hat{R}_{\mu\nu} \equiv \hat{R}_{\mu\sigma,\nu}{}^{\sigma}$. In turn, the deformed Christoffel symbols $\hat{\Gamma}_{\mu\nu}{}^{\sigma}$ can be computed from the metric $\hat{G}_{\mu\nu}$ and its star-inverse by

$$\hat{\Gamma}_{\mu\nu}{}^{\sigma} = \frac{1}{2}\Big(\partial_\mu G_{\nu\alpha} + \partial_\nu G_{\mu\alpha} - \partial_\alpha G_{\mu\nu}\Big) \star \hat{G}^{\alpha\sigma}. \tag{3.17}$$

As shown in [7, 8] the deformed action (3.13) is invariant under the deformed algebra of diffeomorphisms, $\delta_\xi S_{\star EH} = 0$, acting on the fields in the way defined in the previous subsection.

Our main goal is to decide whether the deformed Einstein-Hilbert action (3.13) follows somehow from string theory, and the way in which we will do it is by comparing (3.13) to the terms in the gravitational action induced on the brane in the presence of a constant B-field. Since the string theory calculation proceeds by evaluating scattering amplitudes of gravitons, it is convenient to write the deformed gravity action in a weak-field expansion around flat space-time by writing

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}. \tag{3.18}$$

In the case of the vielbeins the weak field expansion is implemented by[7]

$$e_\mu{}^a = \delta_\mu{}^a + 2\kappa \tau_\mu{}^a, \tag{3.19}$$

where $\tau_\mu{}^a$ is related to the graviton field by

$$\tau_\mu{}^a = \frac{1}{2}\eta^{ab} h_{\mu\nu}\delta^\nu{}_a. \tag{3.20}$$

As above, we fix the gauge invariance by taking the transverse-traceless gauge, $h^\mu_\mu = 0$, $\partial_\mu h^{\mu\nu} = 0$.

[7] In the Euclidean case this relation can be obtained by noticing that in matrix notation $\mathbf{g} = \mathbf{e}\mathbf{e}^T$. In addition, the real matrix $\mathbf{e}$ can be written as the product of an orthogonal and a symmetric matrix, $\mathbf{e} = \mathbf{S}\mathbf{O}$. Therefore $\mathbf{g} = \mathbf{S}^2$ and $\mathbf{S}$ is given by the square root of the metric. Then in the weak field expansion $\mathbf{S} = \mathbf{1} + \frac{1}{2}(2\kappa\mathbf{h})$ and, fixing the gauge freedom to $\mathbf{O} = \mathbf{1}$, we find that $\mathbf{S} = \mathbf{e} \equiv \mathbf{1} + (2\kappa)\boldsymbol{\tau}$.

Moreover, in order to eventually compare with the string theory calculation presented in Section 4 in the weak field expansion (3.18) we will consider only the leading terms in the expansion in the noncommutativity parameter $\theta^{\mu\nu}$. Using (3.14) and (3.19) we find for the deformed metric

$$\hat{G}_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu} - \frac{\kappa^2}{32}\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}\partial_{\alpha_1}\partial_{\alpha_2}h_\mu{}^\sigma\partial_{\beta_1\beta_2}h_{\nu\sigma} + \mathcal{O}(\kappa^3,\theta^4), \tag{3.21}$$

whereas for its inverse there is also a linear term in $\theta^{\mu\nu}$

$$\begin{aligned}\hat{G}^{\mu\nu} &= \eta^{\mu\nu} - 2\kappa h^{\mu\nu} + \kappa^2\Big[4h^{\mu\sigma}h_\sigma{}^\nu - 2i\theta^{\alpha\beta}\partial_\alpha h^{\mu\sigma}\partial_\beta h^\mu{}_\sigma \\ &- \frac{3}{8}\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}(\partial_{\alpha_1}\partial_{\alpha_2}h^{\mu\sigma})(\partial_{\beta_1}\partial_{\beta_2}h^\nu{}_\sigma)\Big] + \mathcal{O}(\kappa^3,\theta^3).\end{aligned} \tag{3.22}$$

In the case of the star-determinant of the vielbein $\det{}_\star e_\mu{}^a$, as it also happens in the case in Einstein gravity, the only term contributing to the three-graviton amplitude is the leading one, $\det{}_\star e_\mu{}^a = 1 + \mathcal{O}(\kappa,\theta)$.

With these ingredients we can proceed to compute the leading θ-dependent terms in the three-graviton amplitude. Because the action (3.13) is real the first correction to the Einstein-Hilbert action has to contain two powers of the noncommutativity parameter. Hence, for dimensional reasons, we find that the first nontrivial contribution to the three-graviton vertex contains two powers of θ and six derivatives. Collecting all the terms of this type we find that the leading correction to the three-graviton vertex is

$$\Delta_\theta\mathcal{L}_{\rm EH} = \frac{1}{2\kappa^2}\Big[\hat{G}^{\mu\nu(2)}R_{\nu\mu} + g^{\mu\nu}\hat{R}^{(2)}_{\nu\mu} + \hat{G}^{\mu\nu(1)}\hat{R}^{(1)}_{\nu\mu}\Big]\Big|_{h^3} + \text{total derivatives},$$

where by the superindex (k) we denote the terms with k powers of $\theta^{\mu\nu}$ and the subindex h^3 indicates that we are keeping only the terms with three graviton fields. Using the expressions given above for the different terms and after a long but straightforward calculation we find the sought term to be

$$\begin{aligned}\Delta_\theta\mathcal{L}_{\rm EH} &= \frac{1}{2}\kappa\theta^{\nu\gamma}\theta^{\eta\rho}\Big(2h^{\sigma\mu}\partial_\mu\partial_\nu\partial_\eta h^{\alpha\beta}\partial_\beta\partial_\gamma\partial_\rho h_{\alpha\sigma} + \partial_\gamma\partial_\rho h^{\mu\sigma}h_{\alpha\beta}\partial_\sigma\partial_\mu\partial_\nu\partial_\eta h^{\alpha\beta}\Big) \\ &+ \text{terms vanishing on-shell.}\end{aligned} \tag{3.23}$$

In our analysis we are going to ignore those terms in the action that are zero by applying the equations of motion. The reason is that ultimately we want to compare this result with the induced gravity action obtained from the string theory amplitudes. Since string theory only allows the computation of on-shell scattering amplitudes, any term in the effective action vanishing on-shell cannot be accounted for. In order to compare with later results, we can rewrite this term in momentum space as

$$\begin{aligned}\Delta_\theta \mathcal{L}_{\rm EH} &= -\frac{\kappa}{12}(p_{2\,\mu}\theta^{\mu\nu}p_{3\,\nu})^2\Big[2p_{3\,\sigma}p_{2\,\lambda}h^{\sigma\beta}(p_1)h_{\beta\eta}(p_2)h^{\eta\lambda}(p_3) + h^{\sigma\lambda}(p_1)p_{2\,\alpha}p_{2\,\beta}h_{\sigma\lambda}(p_2)h^{\alpha\beta}(p_3)\Big] \\ &+ \mathcal{O}(p_i\cdot p_j). \end{aligned} \tag{3.24}$$

We notice that the term in brackets has the same tensor structure as the standard Einstein-Hilbert terms for three gravitons, Eq. (2.9). The reason is simple. Since we are expanding around flat space, the kinetic term in the graviton action $\mathcal{L}_{\rm kin} = \frac{1}{2}h_{\mu\nu}(-\nabla^2)h^{\mu\nu}$ is invariant under Lorentz transformations. In the absence of noncommutativity this symmetry would be preserved by all the terms in the weak-field expansion of the Einstein-Hilbert Lagrangian, unlike in our case where the presence of $\theta^{\mu\nu}$ breaks that symmetry. However, in the terms computed we find that the only source of Lorentz violation is the overall factor $(p_{2\,\mu}\theta^{\mu\nu}p_{3\,\nu})^2$ while the rest of the expression, not containing any power of $\theta^{\mu\nu}$, has to preserve Lorentz invariance. As we know, there is only one term with this property that contains two momenta and three graviton fields, and this is precisely the three-graviton vertex of the Einstein action given in Eq. (2.9).

4 Induced noncommutative gravity and the Seiberg-Witten limit

We now turn to the question of whether the term (3.23) can be obtained in some low energy limit of string theory. As in Section 2 we restrict our attention to the case of the bosonic string on a D25-brane. Graviton interactions on lower-dimensional branes have been considered in [13]. Applications of the Seiberg-Witten map to the study of induced gravity on noncommutative spaces have been studied in [23].

In the Seiberg-Witten limit [12] the low-energy limit $\alpha' \to 0$ is taken while keeping fixed the open string metric $G_{\mu\nu}$, the noncommutativity parameter $\theta^{\mu\nu}$ and the gauge coupling constant $g_{\rm YM}$. This limit can be implemented by introducing a control parameter $\epsilon \to 0$ an scaling α', the closed string metric $g_{\mu\nu}$ and the closed string coupling constant g_s according to

$$\epsilon^{\frac{1}{2}}\alpha', \qquad \epsilon\, g_{\mu\nu} \quad \text{and} \quad \epsilon^{\frac{1}{2}} g_s, \tag{4.1}$$

where we have assumed that the B-field has maximal rank $r = 24$.

From the scaling of g_s we see how closed string states decouple in the Seiberg-Witten limit. The resulting low-energy field theory is not coupled to gravity and no gravity action can be obtained in this low-energy limit. Nevertheless, the gravitational couplings can be studied by considering terms which are subleading in the Seiberg-Witten limit, i.e. terms in the action which scale with positive powers of ϵ in the limit $\epsilon \to 0$. Since we are interested in reproducing the θ-dependent terms in the three-graviton interaction vertex, the relevant string amplitude to compute is the disk with three graviton vertex operators insertions. The most important change with respect to the standard calculation presented in [16] is that now, due to the presence of the B-field, the disk propagator has the form

$$\begin{aligned} \langle X^\mu(z,\overline{z}) X^\nu(w,\overline{w})\rangle_D &= -\alpha' \left[\frac{1}{\sqrt{\epsilon}} g^{\mu\nu} \Big(\log|z-w| - \log|z-\overline{w}| \Big) \right. \\ &+ \left. \sqrt{\epsilon}(-\Theta^2)^{\mu\nu} \log|z-\overline{w}|^2 + \Theta^{\mu\nu} \log\left(\frac{z-\overline{w}}{\overline{z}-w}\right) \right], \end{aligned} \tag{4.2}$$

where we have introduced the dimensionless noncommutativity parameter $\Theta^{\mu\nu} = \frac{1}{2\pi\alpha'}\theta^{\mu\nu}$.

Once the disk propagator is known, the three-graviton amplitude can be computed using the same techniques applied in the calculation presented in Section 2, namely

$$\begin{aligned} \langle V_{p_1} V_{p_2} V_{p_3}\rangle_D &= \epsilon^{-\frac{11}{2}} \frac{g_s^2}{(\alpha')^{13}} (2\pi)^{26} \delta(p_1+p_2+p_3) \\ &\times \int_D \prod_{i=1}^{3} d^2 z_i \exp\left\{ -\sum_{k<\ell}^{3} \left[\frac{1}{\sqrt{\epsilon}} \mathcal{P}_k \cdot \mathcal{P}_\ell G(z_k,z_\ell) \right.\right. \\ &+ \left.\left.\left. \sqrt{\epsilon} \mathcal{P}_k \bullet \mathcal{P}_\ell H(z_k,z_\ell) + \mathcal{P}_k \wedge \mathcal{P}_\ell K(z_k,z_\ell) \right] - \frac{1}{2}\sqrt{\epsilon} \sum_{k=1}^{3} \mathcal{P}_k \bullet \mathcal{P}_k H(z_k,z_k) \right\}\right|_{(\zeta\overline{\zeta})^3}, \end{aligned} \tag{4.3}$$

where $G(z,w)$ is the propagator on the disk,

$$\begin{aligned} H(z,w) &= -\alpha' \log|z-\overline{w}|^2 \\ K(z,w) &= -\alpha' \log\left(\frac{z-\overline{w}}{\overline{z}-w}\right) \end{aligned} \tag{4.4}$$

and we have introduced the notation

$$a \bullet b \equiv a_\mu (-\Theta^2)^{\mu\nu} b_\nu, \qquad a \wedge b \equiv a_\mu \Theta^{\mu\nu} b_\nu. \tag{4.5}$$

It is also important to keep in mind that the $\bullet$- and $\wedge$-products contain all the dependence on the noncommutativity parameter. Notice also that we have factored out all powers of ϵ so that we have a good control on the Seiberg-Witten limit. This means that $\mathcal{P}_\mu$ is now given by

$$\mathcal{P}_\mu \equiv p_\mu - \frac{i}{\epsilon^{\frac{1}{4}}\sqrt{\alpha'}}\zeta_\mu\partial - \frac{i}{\epsilon^{\frac{1}{4}}\sqrt{\alpha'}}\overline{\zeta}_\mu\overline{\partial}. \tag{4.6}$$

In order to reproduce the first θ-dependent correction in the noncommutative gravity action (3.13) we need terms containing two θ's and six momenta. There are multiple ways of obtaining this term from Eq. (4.3). In particular, bringing down all the terms of the form $(p_i \wedge p_j)(p_k \wedge p_\ell)K(z_i, z_j)K(z_k, z_\ell)$ from the exponent, we find the following contribution to the amplitude[8]

$$\epsilon^{-\frac{11}{2}}\frac{g_s^2}{(\alpha')^{13}}(2\pi)^{26}\delta(p_1+p_2+p_3)(p_1\wedge p_2)^2\int_D\prod_{i=1}^{3}d^2z_i\,K(z_1,z_2)^2\prod_{k<\ell}^{3}e^{-\frac{1}{\sqrt{\epsilon}}\mathcal{P}_k\cdot\mathcal{P}_\ell G(z_k,z_\ell)}\Bigg|_{(\zeta\overline{\zeta})^3}, \tag{4.7}$$

where in the exponential we have to keep the terms with two momenta and three polarization tensors. Since all the dependence on the noncommutativity parameter is already factored out, we can use our results of Section 2 to write the amplitude as

$$\widehat{\mathcal{A}}\,(p_1\wedge p_2)^2\Big[(p_3\cdot\varepsilon_1\cdot p_3)(\varepsilon_2\cdot\varepsilon_3)+2(p_1\cdot\varepsilon_3\cdot\varepsilon_2\cdot\varepsilon_1\cdot p_2)+\text{permutations}\Big], \tag{4.8}$$

with

$$\widehat{\mathcal{A}} = -\epsilon^{-\frac{5}{2}}\frac{g_s^2}{(\alpha')^3}\int_D\prod_{i=1}^{3}d^2z_i\,K(z_1,z_2)^2\Big|(\partial_2G_{12}-\partial_2G_{32})\partial_1\partial_3G_{13}\Big|^2 \tag{4.9}$$

and we have not included explicitly the overall factor implementing momentum conservation, $(\sqrt{\epsilon}\alpha')^{-13}(2\pi)^{26}\delta(p_1+p_2+p_3)$. We proceed now as in Section 2 by rescaling the graviton wave function according to (2.11). Using Eq. (2.13) the corresponding induced term in the action can be computed to be

$$\epsilon^{\frac{21}{2}}b_2\kappa\theta^{\nu\gamma}\theta^{\eta\rho}\Big(2h^{\sigma\mu}\partial_\mu\partial_\nu\partial_\eta h^{\alpha\beta}\partial_\beta\partial_\gamma\partial_\rho h_{\alpha\sigma}+\partial_\gamma\partial_\rho h^{\mu\sigma}h_{\alpha\beta}\partial_\sigma\partial_\mu\partial_\nu\partial_\eta h^{\alpha\beta}\Big), \tag{4.10}$$

with b_2 the dimensionless coupling. This term is precisely of the form found in Eq. (3.24) for the first θ-dependent correction to the Einstein-Hilbert action. We notice that, as expected, the induced term is suppressed by an overall positive power of ϵ.

[8] At this stage one should not worry about the overall negative power of ϵ, since this is only due to the normalization chosen for the correlation function. As we see below the properly normalized induced term in the effective action is suppressed by a positive power of ϵ.

Unfortunately together with the wanted term (4.8) string theory gives us a whole plethora of other terms which are not found in the noncommutative gravity action (3.13) and that, moreover, scale the same way as (4.8) in the Seiberg-Witten limit. In particular we can look at the term obtained from (4.3) by taking down all the terms proportional to $p_k \bullet p_\ell$ ($k \neq \ell$) from the exponential

$$\epsilon^{-5}\, g_s^2 (2\pi)^{26} \delta(p_1+p_2+p_3)(p_1 \bullet p_2 + p_1 \bullet p_3 + p_2 \bullet p_3) \\ \times \int_D \prod_{i=1}^{3} d^2 z_i\, H(z_1,z_2) \prod_{k<\ell}^{3} e^{-\frac{1}{\sqrt{\epsilon}}\mathcal{P}_k \cdot \mathcal{P}_\ell G(z_k,z_\ell)}\bigg|_{(\zeta\bar{\zeta})^3} . \tag{4.11}$$

Since we have factored out two momenta and two powers of the noncommutative parameters we need to keep the terms in the expansion of the exponential that contain three polarization tensors and four momenta. Again we can go back to Section 2 to write

$$\widehat{\mathcal{B}}\ (p_1 \bullet p_2 + p_1 \bullet p_3 + p_2 \bullet p_3)\Big[(p_1 \cdot \varepsilon_2 \cdot \varepsilon_3 \cdot p_1)(p_2 \cdot \varepsilon_1 \cdot p_3) + (p_2 \cdot \varepsilon_3 \cdot \varepsilon_1 \cdot p_2)(p_3 \cdot \varepsilon_2 \cdot p_1) \\ + (p_3 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot p_3)(p_1 \cdot \varepsilon_3 \cdot p_2)\Big], \tag{4.12}$$

where now after factoring out $(\alpha'\sqrt{\epsilon})^{-13}(2\pi)^{26}\delta(p_1+p_2+p_3)$ we have

$$\widehat{\mathcal{B}} \;=\; -\epsilon^{-\frac{5}{2}} \frac{2g_s^2}{(\alpha')^3} \int_D \prod_{i=1}^{3} d^2 z_i \Big|\partial_2 G_{12} - \partial_2 G_{32}\Big|^2 H(z_1,z_2)\Big[(\partial_1 G_{12} - \partial_1 G_{31}) \tag{4.13}$$

$$\times\;\; (\partial_2 G_{13} - \partial_3 G_{23})\overline{\partial}_2\overline{\partial}_3 G_{23} + (\overline{\partial}_1 G_{12} - \overline{\partial_1} G_{31})(\partial_2 G_{13} - \partial_3 G_{23})\partial_2\overline{\partial}_3 G_{23} + \text{c.c.}\Big].$$

This amplitude induces a term in the action is not present in the weak-field expansion of the deformed action (3.13) and that furthermore cannot be written in terms of star-products. The problem however lies in that the new induced term has exactly the same scaling in the Seiberg-Witten limit, $\epsilon^{\frac{21}{2}}$, as the one in Eq. (4.8), so both terms in the effective action are equally important in the low energy limit.

Actually this is not the end of the story, since there are many other terms which contain other θ-dependent couplings and that are of the same order in the low-energy limit. These, for example, include in principle terms proportional to

$$(p_i \wedge \varepsilon_k \wedge p_i) \equiv p_{i\,\mu}\Theta^{\mu\nu}\varepsilon_{k\,\nu\sigma}\Theta^{\sigma\lambda}p_{i\,\lambda} \quad \text{and} \quad (p_i \bullet \varepsilon_k \cdot p_i) \equiv p_{i\,\mu}(-\Theta^2)^{\mu\nu}\varepsilon_{k\,\nu\sigma}p_i^{\sigma}. \tag{4.14}$$

Again, these terms scale in the Seiberg-Witten limit with the same power of ϵ as the others we have kept[9]. As in the case of the other term discussed above the induced interactions in the action cannot be expressed in terms of star-products of the fields.

The result of our calculation is that in taking the low-energy limit of gravitons interacting with open strings there are terms in the induced effective action that cannot be written only in terms of star-products. In the end this should not be a surprise, since this is also known to happen when studying the coupling of open string states to closed strings [21, 22]. In particular, couplings of the type (4.14) are found in Ref. [21] when computing the coupling of gauge fields to closed string tachyons.

The bottom line is that the brane-induced low-energy dynamics of closed string theory in the presence of a B-field is much richer than the one contained in the deformed action proposed in [7, 8] (as, for that matter, in any other noncommutative deformation of gravity based only on star-products). In the following we will try to shed some light on the physical reason of why string theory does not yield the noncommutative deformation of gravity presented in Section 3.

5 Star-deformed symmetries versus twisted symmetries

In the last section we have seen how string theory is unable to account for any noncommutative theory of gravity based on star-products, in particular the one proposed in [7, 8]. In spite of all the caveats to be kept in mind while trying to derive noncommutative gravity from the Seiberg-Witten limit, it is quite surprising that the situation is so different from the one arising in gauge theories, where the Seiberg-Witten limit leads to a well-defined noncommutative gauge theory with gauge invariance deformed appropriately.

In this section we attempt to give an explanation of why string theory does not reproduce noncommutative gravity. In order to do that we are going to propose an interpretation of the twisted diffeomorphisms that allows, in our view, a better understanding of the rôle of this twisted symmetry in field theory.

[9]In the philosophy of induced gravity it is possible to reabsorb powers of ϵ in the graviton wavefunction in order that, for example, the two-graviton induced term scales like ϵ^0. This changes the overall scaling of the three-graviton interaction terms containing two powers of $\theta^{\mu\nu}$ from $\epsilon^{\frac{21}{2}}$ to ϵ^3 but does not change the relative scaling between the different terms.

5.1 Diffeomorphisms

Let us consider an arbitrary diffeomorphism generated by a vector field $\xi(x) = \xi^\mu(x)\partial_\mu$. We consider two fields Φ_1, Φ_2 transforming under a finite diffeomorphism generated by a vector field ξ in two different representations $\mathcal{R}_1$, $\mathcal{R}_2$

$$\Phi_1' = \mathcal{D}_1(\xi)\Phi_1, \qquad \Phi_2' = \mathcal{D}_2(\xi)\Phi_2, \qquad \mathcal{D}_1(\xi) \in \mathcal{R}_1, \quad \mathcal{D}_2(\xi) \in \mathcal{R}_2. \tag{5.1}$$

If the star-product of the two fields $\Phi_1 \star \Phi_2$ transforms in the product representation $\mathcal{R}_1 \otimes \mathcal{R}_2$ then $\mathcal{F}^{-1}\Phi_1 \otimes \Phi_2$ has to transform as

$$(\mathcal{F}^{-1}\Phi_1 \otimes \Phi_2)' \equiv [\mathcal{D}_1(\xi) \otimes \mathcal{D}_2(\xi)](\mathcal{F}^{-1}\Phi_1 \otimes \Phi_2). \tag{5.2}$$

Inserting now the identity $\mathbf{1} = [\mathcal{D}_1(\xi)^{-1} \otimes \mathcal{D}_2(\xi)^{-1}][\mathcal{D}_1(\xi) \otimes \mathcal{D}_2(\xi)]$ we find

$$\begin{aligned}(\mathcal{F}^{-1}\Phi_1 \otimes \Phi_2)' &= \Big\{[\mathcal{D}_1(\xi) \otimes \mathcal{D}_2(\xi)]\mathcal{F}^{-1}[\mathcal{D}_1(\xi)^{-1} \otimes \mathcal{D}_2(\xi)^{-1}]\Big\}\Big\{[\mathcal{D}_1(\xi) \otimes \mathcal{D}_2(\xi)](\Phi_1 \otimes \Phi_2)\Big\} \\ &= \mathcal{F}'^{-1}\Phi_1' \otimes \Phi_2', \end{aligned} \tag{5.3}$$

where we have introduced the transformed operator

$$\mathcal{F}'^{-1} = [\mathcal{D}_1(\xi) \otimes \mathcal{D}_2(\xi)]\mathcal{F}^{-1}[\mathcal{D}_1(\xi)^{-1} \otimes \mathcal{D}_2(\xi)^{-1}]. \tag{5.4}$$

Actually, by expanding $\mathcal{F}^{-1}$ in powers of the noncommutativity parameter $\theta^{\mu\nu}$ and inserting the identity, the transformed twist operator can be written as

$$\mathcal{F}'^{-1} = e^{\frac{i}{2}\theta^{\mu\nu}[\mathcal{D}_1(\xi)\partial_\mu\mathcal{D}_1(\xi)^{-1}]\otimes[\mathcal{D}_2(\xi)\partial_\mu\mathcal{D}_2(\xi)^{-1}]}. \tag{5.5}$$

Let us now focus on infinitesimal diffeomorphisms. By writing

$$\mathcal{D}_1(\xi) = e^{\delta_{\xi,1}}, \qquad \mathcal{D}_2(\xi) = e^{\delta_{\xi,2}} \tag{5.6}$$

we find, at first order in the generators $\delta_{\xi,1}$, $\delta_{\xi,2}$

$$[\mathcal{D}_1(\xi)\partial_\mu\mathcal{D}_1(\xi)^{-1}] \otimes [\mathcal{D}_2(\xi)\partial_\mu\mathcal{D}_2(\xi)^{-1}] = \partial_\mu \otimes \partial_\nu - [\partial_\mu, \delta_{\xi,1}] \otimes \partial_\nu - \partial_\mu \otimes [\partial_\nu, \delta_{\xi,2}], \tag{5.7}$$

so the variation of the twist operator $\delta_\xi\mathcal{F}^{-1} = \mathcal{F}'^{-1} - \mathcal{F}^{-1}$ is given by

$$\delta_\xi\mathcal{F}^{-1} = e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu - \frac{i}{2}\theta^{\mu\nu}[\partial_\mu,\delta_{\xi,1}]\otimes\partial_\nu - \frac{i}{2}\partial_\mu\otimes[\partial_\nu,\delta_{\xi,2}]}\Big|_\delta, \tag{5.8}$$

where the subscript δ indicates that we should only keep the terms linear in $\delta_{\xi,1}$ and $\delta_{\xi,2}$.

In order to work out the expression (5.8) we make use of the relation

$$e^{A+\delta A}\big|_{\delta A} = \int_0^1 ds\, e^{sA}\delta A e^{(1-s)A} = e^A \int_0^1 dt\, e^{-tA}\delta A e^{tA}, \tag{5.9}$$

which together with Hadamard's formula (3.10) leads to the following expression

$$e^{A+\delta A}\big|_{\delta A} = e^A \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}[\underbrace{A,[A,\ldots[A}_{n},\delta A]\ldots]], \tag{5.10}$$

so $\delta_\xi \mathcal{F}^{-1}$ can be written as the following formal series linear in $\delta_{1,\xi}$, $\delta_{2,\xi}$

$$\begin{aligned} \delta_\xi \mathcal{F}^{-1} &= \mathcal{F}^{-1}\sum_{n=1}^{\infty}\frac{(-i/2)^n}{n!}\theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}\ldots\theta^{\mu_n\nu_n}\Big\{[\partial_{\mu_1},[\partial_{\mu_2},\ldots[\partial_{\mu_n},\delta_{\xi,1}]\ldots]]\otimes\partial_{\nu_1}\partial_{\nu_2}\ldots\partial_{\nu_n} \\ &+ \partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_n}\otimes[\partial_{\nu_1},[\partial_{\nu_2},\ldots[\partial_{\nu_n},\delta_{\xi,2}]\ldots]]\Big\}. \end{aligned} \tag{5.11}$$

Therefore, the transformation of the star-product of Φ_1 and Φ_2 is given by

$$\begin{aligned} \delta_\xi[\Phi_1(x)\star\Phi_2(x)] &\equiv \mu[\mathcal{F}'^{-1}\Phi_1'(x)\otimes\Phi_2'(x)] - \mu[\mathcal{F}^{-1}\Phi_1(x)\otimes\Phi_2(x)] \\ &= \mu[\mathcal{F}^{-1}\delta_{\xi,1}\Phi_1(x)\otimes\Phi_2(x)] + \mu[\mathcal{F}^{-1}\Phi_1(x)\otimes\delta_{\xi,2}\Phi_2(x)] \\ &+ \mu[(\delta_\xi\mathcal{F}^{-1})\Phi_1(x)\otimes\Phi_2(x)] \\ &= \delta_{\xi,1}\Phi_1(x)\star\Phi_2(x) + \Phi_1(x)\star\delta_{\xi,2}\Phi_2(x) + \Phi_1(x)(\delta_\xi\star)\Phi_2(x), \end{aligned} \tag{5.12}$$

where we have introduced the notation

$$\Phi_1(x)(\delta_\xi\star)\Phi_2(x) \equiv \mu[(\delta_\xi\mathcal{F}^{-1})\Phi_1(x)\otimes\Phi_2(x)]. \tag{5.13}$$

Using Eq. (5.8) we can write this extra term explicitly as

$$\begin{aligned} \Phi_1(x)(\delta_\xi\star)\Phi_2(x) &= \sum_{n=1}^{\infty}\frac{(-i/2)^n}{n!}\theta^{\mu_1\nu_1}\ldots\theta^{\mu_n\nu_n}\Big\{[\partial_{\mu_1},\ldots[\partial_{\mu_n},\delta_{\xi,1}]\ldots]\Phi_1(x)\star\partial_{\nu_1}\ldots\partial_{\nu_n}\Phi_2(x) \\ &+ \partial_{\mu_1}\ldots\partial_{\mu_n}\Phi_1(x)\star[\partial_{\nu_1},\ldots[\partial_{\nu_n},\delta_{\xi,2}]\ldots]\Phi_2(x)\Big\}. \end{aligned} \tag{5.14}$$

In this calculation we have retrieved the twisted Leibniz rule that was obtained by twisting the Hopf algebra coproduct by the twist operator $\mathcal{F}$. However, our analysis lead us to a different interpretation of the twisted Leibniz rule. Instead of thinking in term of a twisted symmetry we can interpret the deformed diffeomorphisms as the ordinary ones but with the additional condition that these act not only on the fields but on the star-product as well, according to Eq. (5.14). In other words, the twisted Leibniz rule emerges from the application of the *standard* one to $\Phi_1 \star \Phi_2$ and taking into account the transformation of the star-product itself

$$\delta_\xi(\Phi_1 \star \Phi_2) = (\delta_\xi \Phi) \star \Phi_2 + \Phi_1 \star (\delta_\xi \Phi_2) + \Phi_1(\delta_\xi \star)\Phi_2, \tag{5.15}$$

i.e., in transforming a star-product of operators we have consider the star-product as a differential operator with its own transformation properties. Notice that the transformation of the star-product depends on the representation of the fields we multiply.

This interpretation of the deformed diffeomorphisms actually allows a better understanding of the results found so far. In particular we see that the deformed diffeomorphisms, although leaving the deformed Einstein-Hilbert action invariant, are not *bona fide* physical symmetries, since they do not act just on the fields, but on the star-products as well. This prevents the application of the standard Noether procedure to obtain conserved currents. The same can be said with respect to Ward identities in the quantum theory.

It is interesting to particularize our analysis to the case of linear affine transformations, where we can recover the results of Ref. [24]. Considering, for simplicity, the product of two scalar fields $\Phi_1(x)$, $\Phi_2(x)$ the linear affine coordinate transformation

$$\delta x^\mu = B^\mu{}_\nu x^\nu + a^\mu. \tag{5.16}$$

induce the following tranformation for the scalar fields

$$\delta\Phi(x) = -B^\mu{}_\nu x^\nu \partial_\mu \Phi(x) - a^\mu \partial_\mu \Phi(x) \equiv \left(-B^\mu{}_\nu x^\nu \partial_\mu - a^\mu \partial_\mu\right) \Phi(x). \tag{5.17}$$

This implies that

$$[\partial_\mu, \delta] = -B^\alpha{}_\mu \partial_\alpha \tag{5.18}$$

and, as a result, only the term with $n = 1$ in Eq. (5.14) survives so we find

$$\begin{aligned} \Phi_1(x)(\delta\star)\Phi_2(x) &= \frac{i}{2}\theta^{\mu\nu}\left[B^\alpha{}_\mu \partial_\alpha \Phi_1 \star \partial_\nu \Phi_2(x) + B^\alpha{}_\nu \partial_\mu \Phi_1 \star \partial_\alpha \Phi_2(x)\right] \\ &= \frac{i}{2}\left(B^\alpha{}_\mu \theta^{\mu\nu} + \theta^{\alpha\sigma} B^\nu{}_\sigma\right) \partial_\alpha \Phi_1(x) \star \partial_\nu \Phi_2(x) \end{aligned} \tag{5.19}$$

The interesting thing about the case of linear affine transformation is that this extra term in the Leibniz rule can actually be reabsorbed by a simultaneous transformation of the noncommutativity parameter [24]

$$\delta\theta^{\mu\nu} = -\left(B^{\mu}{}_{\alpha}\theta^{\alpha\nu} + \theta^{\mu\sigma}B^{\nu}{}_{\sigma}\right), \tag{5.20}$$

since in this case the transformation of $\Phi_1(x) \star_\theta \Phi_2(x)$ picks up an extra term associated to the transformation of $\theta^{\mu\nu}$ itself given by[10]

$$\begin{aligned} \delta_\theta[\Phi_1(x) \star_\theta \Phi_2(x)] &\equiv \Phi_1(x) \star_{\theta+\delta\theta} \Phi_2(x) - \Phi_1(x) \star_\theta \Phi_2(x) \\ &= \frac{i}{2}\delta\theta^{\mu\nu}\partial_\mu\Phi_1(x) \star_\theta \partial_\nu\Phi_2(x). \end{aligned} \tag{5.21}$$

This cancels exactly the extra term (5.19) in the twisted Leibniz rule and one is left with

$$\begin{aligned} \delta_{\rm total}[\Phi_1(x) \star_\theta \Phi_2(x)] &\equiv \delta[\Phi_1(x) \star_\theta \Phi_2(x)] + \delta_\theta[\Phi_1(x) \star_\theta \Phi_2(x)] \\ &= \delta\Phi_1(x) \star_\theta \Phi_2 + \Phi_1(x) \star_\theta \delta\Phi_2(x), \end{aligned} \tag{5.22}$$

where δ indicates the variation of at constant $\theta^{\mu\nu}$ given by Eq. (5.15). Hence we have recovered the result of [24] that the star-product is covariant under affine linear transformations provided the noncommutativity parameter is also transformed. In particular, if one takes affine linear transformations belonging to the "little group", i.e. those leaving $\theta^{\mu\nu}$ invariant, the extra term in the Leibniz rule (5.19) vanishes.

It is important to keep in mind that for this to work it is crucial that in the case of affine linear transformation the noncommutativity parameter does not pick up any dependence on the space-time coordinates. Of course this is not the case for general nonlinear transformations of the coordinates, in which case there is no transformation of the (constant) noncommutativity parameter that allows a covariantization of the Moyal star-product.

5.2 Gauge theories

The situation in gauge theories is somewhat different. It is crucial that in this case we have in principle the Seiberg-Witten map relating ordinary gauge transformations to their star-deformation in a well defined series in $\theta^{\mu\nu}$, in such a way that at each stage the effective

[10]For the sake of clarity, here and in the remain of this subsection we have indicated by $\star_\theta$ the explicit dependence of the star-product on $\theta^{\mu\nu}$.

action is invariant in the ordinary sense. Gauge transformations are deformed in such a way that the star-product does not transform under a star-gauge transformation. This is clear in the case of an adjoint field Φ on which a finite star-gauge transformations act by

$$\Phi(x) \longrightarrow \mathcal{U}(x)_\star \star \Phi(x) \star \mathcal{U}(x)_\star^{-1}, \tag{5.23}$$

where the inverse is defined in the sense of the star-product

$$\mathcal{U}_\star(x) \star \mathcal{U}_\star(x)^{-1} = \mathcal{U}_\star(x)^{-1} \star \mathcal{U}_\star(x) = \mathbf{1}. \tag{5.24}$$

This identity guarantees the covariance of the star-product under star-gauge transformations. Indeed, given two adjoint fields $\Phi_1(x)$, $\Phi_2(x)$, one finds

$$\begin{aligned} \Phi_i'(x) \star \Phi_2'(x) &= [\mathcal{U}_\star(x) \star \Phi_1(x) \star \mathcal{U}_\star(x)^{-1}] \star [\mathcal{U}_\star(x) \star \Phi_1(x) \star \mathcal{U}_\star(x)^{-1}] \\ &= \mathcal{U}_\star(x) \star \Phi_1(x) \star [\mathcal{U}_\star(x)^{-1} \star \mathcal{U}_\star(x)] \star \Phi_2 \star \mathcal{U}_\star(x)^{-1} \\ &= \mathcal{U}_\star(x) \star [\Phi_1(x) \star \Phi_2(x)] \star \mathcal{U}_\star(x)^{-1} = [\Phi_1(x) \star \Phi_2(x)]', \end{aligned} \tag{5.25}$$

where the prime indicates the gauge transformed field. At the infinitesimal level, this means that the star-gauge variation of the star-product of two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ can be computed using the standard Leibniz rule

$$\delta_{\text{gauge}}(\mathcal{O}_1 \star \mathcal{O}_2) = (\delta_{\text{gauge}}\mathcal{O}_1) \star \mathcal{O}_2 + \mathcal{O}_1 \star (\delta_{\text{gauge}}\mathcal{O}_2). \tag{5.26}$$

Using the language introduced above we can consider that the star-product is invariant under star-gauge transformations, i.e. $\delta_{\text{gauge}}\star = 0$. A very important consequence of this deformation of gauge invariance is that it implies a restriction on the possible gauge groups which are reduced to U(N).

A second alternative [19, 20] consists of keeping the gauge transformations undeformed $\delta_\omega \Phi = iT_\omega \Phi$ and then twist the coproduct as in Eq. (3.8)

$$\begin{aligned} \Delta(T_\omega)_\mathcal{F} &\equiv \mathcal{F}(T_\omega \otimes \mathbf{1} + \mathbf{1} \otimes T_\omega)\mathcal{F}^{-1} = T_\omega \otimes \mathbf{1} + \mathbf{1} \otimes T_\omega \\ &+ \sum_{n=1}^{\infty} \frac{(-i/2)^n}{n!} \theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2} \ldots \theta^{\mu_n\nu_n} \Big\{ [\partial_{\mu_1}, [\partial_{\mu_2}, \ldots [\partial_{\mu_n}, iT_\omega] \ldots]] \otimes \partial_{\nu_1}\partial_{\nu_2} \ldots \partial_{\nu_n} \\ &\quad + \partial_{\mu_1}\partial_{\mu_2} \ldots \partial_{\mu_n} \otimes [\partial_{\nu_1}, [\partial_{\nu_2}, \ldots [\partial_{\nu_n}, iT_\omega] \ldots]] \Big\}. \end{aligned} \tag{5.27}$$

This results in a deformation of the Leibniz rule. In the case of adjoint fields[11],

$$\Phi(x) \longrightarrow \mathcal{U}(x)\Phi(x)\mathcal{U}(x)^{-1}, \tag{5.28}$$

the extra terms in the twisted Leibniz rule reflect the fact that

$$\begin{aligned} \Phi_i'(x) \star \Phi_2'(x) &= [\mathcal{U}(x)\Phi_1(x)\mathcal{U}(x)^{-1}] \star [\mathcal{U}(x)\Phi_1(x)\mathcal{U}(x)^{-1}] \\ &\neq \mathcal{U}(x)[\Phi_1(x) \star \Phi_2(x)]\mathcal{U}(x)^{-1}. \end{aligned} \tag{5.29}$$

Interestingly, unlike the case of star-gauge transformations, there is no restriction on the possible gauge groups that can be twisted.

Actually we can be more general by repeating the analysis performed above for the diffeomorphisms this time applied to gauge transformations. The covariance of the star-product of two fields Φ_1, Φ_2 transforming respectively under finite gauge transformations in two representations $\mathcal{R}_1$, $\mathcal{R}_2$ as

$$\Phi_1' = \mathcal{U}_1\Phi_1, \qquad \Phi_2' = \mathcal{U}_2\Phi_2, \qquad \mathcal{U}_1 \in \mathcal{R}_1, \quad \mathcal{U}_2 \in \mathcal{R}_2 \tag{5.30}$$

leads to a transformation of $\mathcal{F}^{-1}$ to $\mathcal{F}'^{-1}$ given by

$$\mathcal{F}'^{-1} = e^{\frac{i}{2}\theta^{\mu\nu}(\mathcal{U}_1\partial_\mu\mathcal{U}_1^{-1})\otimes(\mathcal{U}_2\partial_\nu\mathcal{U}_2^{-1})}. \tag{5.31}$$

Writing now $\mathcal{U}_1 = e^{iT^{(1)}_\omega}$, $\mathcal{U}_2 = e^{iT^{(2)}_\omega}$ we can write the variation of $\mathcal{F}^{-1}$ under a gauge transformation as

$$\begin{aligned} \delta_\epsilon\mathcal{F}^{-1} &= \mathcal{F}^{-1}\sum_{n=1}^{\infty}\frac{(-i/2)^n}{n!}\theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}\ldots\theta^{\mu_n\nu_n}\Big\{[\partial_{\mu_1},[\partial_{\mu_2},\ldots[\partial_{\mu_n},T^{(1)}_\omega]\ldots]]\otimes\partial_{\nu_1}\partial_{\nu_2}\ldots\partial_{\nu_n} \\ &\quad + \partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_n}\otimes[\partial_{\nu_1},[\partial_{\nu_2},\ldots[\partial_{\nu_n},iT^{(2)}_\omega]\ldots]]\Big\}. \end{aligned} \tag{5.32}$$

This expression produces again the extra terms in the twisted Leibniz rule (5.27). As in the case of diffeomorphisms discussed above, the twisted Leibniz rule can be thought of again as resulting from the noninvariance of the star-product $\delta_\omega\star \neq 0$ under standard, i.e. non-star, gauge transformations

$$\delta_\omega(\Phi_1\star\Phi_2) = [iT^{(1)}_\omega\Phi_1]\star\Phi_2 + \Phi_1\star[iT^{(2)}_\omega\Phi_2] + \Phi_1(\delta_\omega\star)\Phi_2, \tag{5.33}$$

where the product $\Phi_1\star\Phi_2$ transforms now in the product representation $\mathcal{R}_1\otimes\mathcal{R}_2$. The star-product is invariant only in the case of global transformations, for which $[\partial_\mu, iT_\omega] = 0$. It is in this case that the standard Leibniz rule is retrieved.

[11]Now we consider the standard inverse $\mathcal{U}(x)^{-1}\mathcal{U}(x) = \mathcal{U}(x)^{-1}\mathcal{U}(x) = \mathbf{1}$.

5.3 Discussion

Our analysis of gauge transformations in noncommutative theories shows that, in extending gauge symmetries to the noncommutative realm one is faced with a choice. Either gauge transformations are deformed in such a way that the standard Leibniz rule is satisfied or one keeps the gauge transformations as in the commutative case at the price of giving up the Leibniz rule. In the latter case the new rule to compute the gauge variation of the star-products of fields can be seen as resulting from a twist in the Hopf algebra structure of the universal enveloping algebra of the Lie algebra of the gauge group extended by translations.

The obvious advantage of deforming gauge transformations into star-gauge transformations is that gauge symmetries acts then only on the fields in a similar way as in the commutative theories. In this sense star-gauge symmetry is a *bona fide* physical symmetry, it can be implemented in the quantum case leading to Ward identities.

On the other hand, if ordinary gauge transformations are retained and a twisted Leibniz rule is introduced, the situation is not so clear. In this case the noncovariant nature of the star-product under "undeformed" gauge transformations has to be taken into account in order for the action to be invariant, which amounts to replacing the ordinary Leibniz rule with the twisted one. This means that, unlike the previous case, now the transformations do not act only on the fields. As a consequence this is not a physical symmetry in the usual sense and it is not clear whether Noether charges and Ward identities can be derived.

Let us consider the case of U(N) noncommutative gauge theories. What makes these theories special is the fact that the same action

$$S = -\frac{1}{4g^2}\int d^4x \,\mathrm{tr}\,[F_{\mu\nu}\star F^{\mu\nu}], \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star \tag{5.34}$$

is invariant both under U(N) star-gauge transformations and U(N) twisted gauge transformations. In Ref. [19] a conserved charge was found, given by

$$j^\mu = i[F^{\mu\nu}, A_\nu]_\star, \qquad \partial_\mu j^\mu = 0. \tag{5.35}$$

The fact that the action (5.34) has both types of invariances makes the physical interpretation of this current unclear. Indeed, the current (5.35) can be obtained from (5.34) using the standard Noether procedure with respect to the "global" transformation $\delta A_\mu = i[\omega(x), A_\mu]_\star$ and setting $\omega(x)$ to a constant at the end of the calculation. The resulting conservation law is consistent with star-gauge invariance. Namely,

$$\delta_{\star\omega} j^\mu = i[F^{\mu\nu}, \partial_\nu \omega]_\star + [[F^{\mu\nu}, A_\nu]_\star, \omega]_\star \tag{5.36}$$

which applying the equations of motion, $\partial_\nu F^{\mu\nu} = -i[F^{\mu\nu}, A_\nu]_\star$, gives

$$\delta_{\star\omega} j^\mu = i\partial_\nu [F^{\mu\nu}, \omega]_\star, \qquad \partial_\mu(\delta_{\star\omega} j^\mu) = 0. \tag{5.37}$$

In the case of twisted gauge transformations, in the absence of a Noether procedure valid for twisted symmetries, the only way to obtain the current (5.35) is as an integrability condition for the equations of motion. In the U(1) case j^μ is nevertheless invariant under twisted gauge transformations, $\delta_\omega j^\mu = 0$, while for U(N) the transformation of the current is given by

$$\delta_\omega j^\mu = i\partial_\nu [F^{\mu\nu}, \omega] + i[\omega, \partial_\nu F^{\mu\nu} + i[F^{\mu\nu}, A_\nu]_\star], \tag{5.38}$$

also compatible with current conservation after applying the equations of motion, $\partial_\mu(\delta_\omega j^\mu) = 0$. As discussed above, the action of the twisted gauge transformations can be seen as the action of ordinary gauge transformations acting not only on the fields but on the star-products as well.

Because of the simultaneous presence of both types of symmetries in the action (5.34) it is not easy to decide whether the origin of the conserved current in noncommutative gauge theories is star-gauge invariance or twisted gauge transformations. One possibility is that in this case star-gauge invariance plays the rôle of a custodial standard symmetry that forces not only that the low energy action is expressed exclusively in terms of star-products but also the existence of conserved currents and Ward identities. If this is the case twisted gauge transformations might play only an accidental rôle in the dynamics of noncommutative gauge theories. Of course, our discussion only applies to U(N) noncommutative gauge theories. In the case of other gauge groups we are left only with twisted gauge transformations as the invariance of the theory since star-gauge transformations cannot be implemented.

In the case of noncommutative gravity the apparent absence of a star-deformed version of diffeomorphism invariance might be at the heart of the difficulties in obtaining a noncommutative gravity action from string theory. Comparing with the case of noncommutative gauge theories it seems that in the case of gravity there is no symmetry of the standard type that plays the custodial rôle that star-gauge symmetry might be playing for gauge theories. Apparently twisted symmetries by themselves are not handled by string theory so well as standard symmetries acting only on the fields.

6 Conclusions and outlook

In this paper we have studied the possibility of obtaining noncommutative gravitational dynamics from string theory by studying the Seiberg-Witten limit of the graviton interactions induced by a space-filling brane in bosonic string theory. In particular our main interest is to investigate whether string theory can provide some ultraviolet completion of recently proposed noncommutative deformations of gravity based on the invariance under twisted diffeomorphisms [7, 8].

The conclusion of our work is that, in the case of gravitational interactions, string theory contains much richer dynamics than those codified in terms of star-products. We have found that the gravitational action induced on the brane in the presence of a constant B-field in the Seiberg-Witten limit cannot be expressed in terms of star-products alone, unlike the action for noncommutative gravity proposed in [7, 8].

The consequences of this result are still to be fully understood. In particular it would be very interesting to clarify the rôle played by twisted symmetries in the context of string theory. In the case of noncommutative gauge theories string theory provides in the Seiberg-Witten limit a theory which in addition to star-gauge symmetry also has a twisted invariance. So far it has not been possible to decide whether this twisted invariance plays a fundamental dynamical rôle in string theory or whether it should be regarded as an accidental symmetry additional to star-gauge invariance which would play a custodial role ensuring the existence of conserved currents and Ward identities.

There are several ways in which one can expect to find twisted symmetries in the context of open strings in the Seiberg-Witten limit. In particular in this limit strings become rigid rods with variable length and their gauge theory Fock space description is similar to that of open strings, in the sense that it is the product of two copies associated with the Hilbert space at each endpoint, $\mathcal{H}_1 \otimes \mathcal{H}_2$. Since while propagating these rods sweep the background magnetic flux, it can be expected that gauge transformations are twisted due to the presence of the background field. Open string field theory in the presence of constant B-fields might be specially suited to study this issue in detail [25].

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Appendix A. Hopf algebras: a summary of useful formulae and definitions

Our aim in this Appendix is to summarize basic facts about Hopf algebras and to introduce the notation used in the paper. A more detailed introduction to the subject of Hopf algebras and quantum groups can be found in standard reviews (for example, [15]).

Coalgebras. The concept of a coalgebra is in a sense "dual" to that of an algebra. If an algebra A is endowed with an associative product, a coalgebra C is a vector space over a field $\mathbb{K}$ together with a coproduct Δ which is a bilinear map

$$\Delta : C \longrightarrow C \otimes C. \tag{A.1}$$

In general, for any element a of the algebra the action of the coproduct Δ can always be written as

$$\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}, \tag{A.2}$$

where the superscript indicates the "copy" of C in $C \otimes C$ to which the element belongs. In addition, the coproduct is required to be coassociative, i.e. for all $a \in C$

$$(\Delta \otimes I)\Delta(a) = (I \otimes \Delta)\Delta(a). \tag{A.3}$$

In more concrete terms, this means that if $\Delta(a)$ is given by Eq. (A.2)

$$\sum_i \Delta(a_i^{(1)}) \otimes a_i^{(2)} = \sum_i a_i^{(1)} \otimes \Delta(a_i^{(2)}). \tag{A.4}$$

In the same way that an algebra can contain a unit element e, a coalgebra might include a counit. This is a map $\overline{e} : C \longrightarrow \mathbb{K}$ that satisfies

$$(I \otimes \overline{e}) \circ \Delta = I = (\overline{e} \otimes I) \circ \Delta. \tag{A.5}$$

Bialgebras. A bialgebra B is a vector space over a field $\mathbb{K}$ that is at the same time an algebra and a coalgebra. In addition the product and the coproduct must be compatible. This means that for all elements a, $b \in B$

$$\Delta(a \cdot b) = \sum_i \left(a_i^{(1)} \cdot b_i^{(1)}\right) \otimes \left(a_i^{(2)} \cdot b_i^{(2)}\right) = \Delta(a) \cdot \Delta(b), \tag{A.6}$$

where $\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$ and $\Delta(b) = \sum_i b_i^{(1)} \otimes b_i^{(2)}$.

Hopf algebras. A Hopf algebra is a bialgebra A together with a linear map $S : A \longrightarrow A$ called the antipode which for all $a \in A$ with $\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$ satisfies

$$\sum_i a_i^{(1)} \cdot S\left(a_i^{(2)}\right) = \sum_i S\left(a_i^{(1)}\right) \cdot a_i^{(2)} = (e \circ \overline{e})(a). \tag{A.7}$$

Examples. The simplest one is the Hopf algebra associated with a group $\mathcal{G}$. Given this group we can always define the group algebra $\mathbb{K}\mathcal{G}$ over a field $\mathbb{K}$ as the algebra of lineal combinations of elements of $\mathcal{G}$ with coefficients $\lambda \in \mathbb{K}$ and with the product

$$(\lambda_1 g_1) \cdot (\lambda_2 g_2) = (\lambda_1 \lambda_2)(g_1 g_2), \qquad \forall \lambda_1, \lambda_2 \in \mathbb{K} \quad g_1, g_2 \in \mathcal{G}, \tag{A.8}$$

where on the right hand side of the equation we use the product of $\mathbb{K}$ and $\mathcal{G}$. This algebra has the unit element $e = I$, where I the identity element of $\mathcal{G}$.

Actually, the algebra $\mathbb{K}\mathcal{G}$ has also a coalgebra structure given by the coproduct defined by

$$\Delta(g) = g \otimes g, \qquad g \in \mathcal{G}. \tag{A.9}$$

It is straightforward to show that this coproduct is coassociative. In addition the counit $\overline{e}$ is defined by $\overline{e}(g) = 1$ where 1 is the identity of the field $\mathbb{K}$. Bilinearity of the coproduct and the linearity of the counit determines the map for any element of $\mathbb{K}\mathcal{G}$. Moreover, the coproduct (A.9) is actually compatible with the product, so $\mathbb{K}\mathcal{G}$ is in fact a bialgebra. This structure can be extended to a Hopf algebra by the antipode map S defined by

$$S(g) = g^{-1}, \qquad \forall g \in \mathcal{G}, \tag{A.10}$$

which satisfies indeed the property (A.7).

The second instance of Hopf algebras that we are going to study is the universal enveloping algebra of a Lie algebra. Let $\mathcal{L}$ be a Lie algebra over a field $\mathbb{K}$ with generators ξ_i ($i = 1, \ldots, \dim \mathcal{L}$). The universal enveloping algebra $\mathcal{U}(\mathcal{L})$ associated with $\mathcal{L}$ is the algebra generated by ξ_i with the identification

$$\xi_i \cdot \xi_j - \xi_j \cdot \xi_i \sim [\xi_i, \xi_j], \qquad i, j = 1, \ldots, \dim \mathcal{L}, \tag{A.11}$$

where $[a, b]$ denotes the commutator operation in the Lie algebra.

Given $\xi \in \mathcal{U}(\mathcal{L})$ the mapping

$$\Delta(\xi) = \begin{cases} \xi \otimes e + e \otimes \xi & \xi \neq e \\ e \otimes e & \xi = e \end{cases} \tag{A.12}$$

defines a coassociative coproduct compatible with the algebra product. A counit $\overline{e}$ is defined by

$$\overline{e}(\xi) = \begin{cases} 0 & \xi \neq e \\ 1 & \xi = e \end{cases} . \tag{A.13}$$

This shows that $\mathcal{U}(\mathcal{L})$ is endowed with a bialgebra structure. This is actually extended to a Hopf algebra by the antipode map

$$S(\xi) = \begin{cases} -\xi & \xi \neq e \\ e & \xi = e \end{cases} , \tag{A.14}$$

which can be easily seen to satisfy the condition (A.7).

The action of a Hopf algebra on an algebra. For the applications of Hopf algebras as twisted symmetries we need to define its action on an algebra A. Roughly speaking we want to define a map $\alpha : H \otimes A \longrightarrow A$ with the property that for every $\xi, \zeta \in H$, $a \in A$

$$\alpha(\xi \cdot \zeta \otimes a) = \alpha[\xi \otimes \alpha(\zeta \otimes a)], \qquad \alpha(e \otimes a) = a, \tag{A.15}$$

where $e \in H$ is the unit element.

If H was just an algebra this would be the end of the story. However H has a Hopf algebra structure as well, so additional conditions are imposed involving the coproduct and the counit. If we denote now by μ the product map on the algebra A, $\mu(a \otimes b) = ab$ for $a, b \in A$ these conditions are

$$\alpha(\xi \otimes ab) = \mu \circ \alpha[\Delta(\xi) \otimes (a \otimes b)], \qquad \alpha(\xi \otimes \mathbf{1}) = \mathbf{1} \circ \overline{e}(\xi), \tag{A.16}$$

with $\mathbf{1}$ the unity element of the algebra A and the action of the map α is extended to

$$\alpha[(\xi \otimes \zeta) \otimes (a \otimes b)] = \alpha(\xi \otimes a) \otimes \alpha(\zeta \otimes b). \tag{A.17}$$

As an example we can consider the action of the universal enveloping algebra of a Lie algebra, $\mathcal{U}(\mathcal{L})$ on an algebra A. Since the coproduct Δ is given by Eq. (A.12), we have for $a, b \in A$, $\xi \in \mathcal{U}(\xi)$

$$\alpha[\Delta(\xi) \otimes (a \otimes b)] \;\; = \;\; \alpha(\xi \otimes a) \otimes b + a \otimes \alpha(\xi \otimes b).$$

Therefore Eq. (A.16) reads

$$\alpha(\xi \otimes ab) = \mu \circ [\alpha(\xi \otimes a) \otimes b] + \mu \circ [a \otimes \alpha(\xi \otimes b)]. \tag{A.18}$$

This identity is nothing but Leibniz' rule giving the action of an element of the Hopf algebra ξ on the product of two algebra elements a and b.

Twisting. Of course the coproduct given in Eq. (A.12) is just one among all the possible choices for a coproduct satisfying the appropriate conditions. Other possible coproduct can be obtained from this one by *twisting*. This means that given a invertible element $\mathcal{F} \in \mathcal{U}(\mathcal{L}) \otimes \mathcal{U}(\mathcal{L})$, a new Hopf algebra structure can be defined using the twisted coproduct

$$\Delta_{\mathcal{F}}(\xi) \equiv \mathcal{F}\Delta(\xi)\mathcal{F}^{-1} = \mathcal{F}(\xi \otimes e + e \otimes \xi)\mathcal{F}^{-1}, \tag{A.19}$$

and the twisted antipode

$$S_{\mathcal{F}}(\xi) \equiv u S(\xi) u^{-1}, \tag{A.20}$$

where u is defined by

$$u \equiv \mu[(I \otimes S)\mathcal{F}] = \sum_i f_i^{(1)} S(f_i^{(2)}). \tag{A.21}$$

In writing this last expression we have used the decomposition

$$\mathcal{F} = \sum_i f_i^{(1)} \otimes f_i^{(2)}. \tag{A.22}$$

Of course not every twist operator $\mathcal{F}$ gives rise to a well defined twisted Hopf algebra. In particular one should make sure that the new coproduct $\Delta_{\mathcal{F}}$ preserves coasociativity (A.3) and is compatible with the counit $\overline{e}$ (A.5). This respectively is guaranteed if $\mathcal{F}$ satisfies the following conditions

$$\begin{aligned}(e \otimes \mathcal{F})(I \otimes \Delta)\mathcal{F} &= (\mathcal{F} \otimes e)(\Delta \otimes I)\mathcal{F} \\ (I \otimes \overline{e})\mathcal{F} &= (\overline{e} \otimes I)\mathcal{F} = e. \end{aligned} \tag{A.23}$$

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