EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

ON HIGH-ENERGY BOUNDS FOR THE SCATTERING AMPLITUDE

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Paper to be published in the
Proceedings of the Sienna International Conference
on Elementary Particles, Sienna, Italy,
30 Sept. - 5 Oct. 1963.

7915/TH. 394
6 January 1964

1. Consider the elastic scattering of two spin zero particles with equal mass m. Let s, k and ϑ be the square of the energy, the momentum and the scattering angle in the centre-of-mass system of these two particles. Expand the elastic scattering amplitude f in a Legendre series as follows :

$$f(s,z) = (\sqrt{s}/k) \sum_{n=0}^{\infty} (2n+1) a_n(s) P_n(z) \qquad (1)$$

where $z = \cos\vartheta$. The unitarity condition reads

$$\mathrm{Im}\, a_n \geq |a_n|^2 \qquad (2)$$

for $n = 0,1,\ldots$.

If one assumes that :

(α) for every fixed s, real and bigger than $4m^2$, $f(s,z)$ is holomorphic for z in the plane cut along the real axis from $-\infty$ to $-(1+2m^2/k^2)$ and from $(1+2m^2/k^2)$ to ∞;

(β) the discontinuity of f along the cuts is a tempered distribution in s and z;

one can prove [1] using (2) that the elastic differential cross-section satisfies for sufficiently big s the inequality

$$d\sigma/d\Omega < C_1 (\ln s)^3 s^{-1} \qquad (3)$$

for fixed $\vartheta \neq 0$ or π, where C_1 is independent of s. This inequality is stronger than that given by Froissart in Ref. [2] [see also Greenberg and Low [3], and Martin [4]] which was

$$d\sigma/d\Omega < C (\ln s)^3 s^{1/2}$$

for fixed $\vartheta \neq 0$ or π.

In contrast, another result by Froissart [2], namely that the total cross-section satisfies

$$\sigma_{tot} < C_2 (\ln s)^2 \qquad (4)$$

(C_2 independent of s) cannot be strengthened by the combined use of (2) and of hypotheses (α) and (β). This is shown by two examples we will briefly explain in this note.

2. From assumptions (α) and (β), it follows that there exist two functions A and B with the following properties :

i) at fixed s, real, $> 4m^2$, $A(s,n)$ and $B(s,n)$ are defined and holomorphic for $\text{Re}\, n > N$, where N real is independent of s;

ii) $A(s,n) = a_n(s)$ for even integer $n > N$, $B(s,n) = a_n(s)$ for odd integer $n > N$;

iii) $|A(s,n)|$ and $|B(s,n)| < s^M |n|^P \exp(-C\,\text{Re}\, n/\sqrt{s})$ where M,P,C are real positive constants.

Conversely, given A and B, if there exist constants N,M,P,C such that conditions i) and iii) are fulfilled, and if $\text{Im}\, A \geq |A|^2$ for even integer n, and $\text{Im}\, B \geq |B|^2$ for odd integer n, then the amplitude defined as

$$f(s,z) = (\sqrt{s}/k)\left[\sum_{n\ \text{even}} (2n+1)A(s,n)P_n(z) + \sum_{n\ \text{odd}} (2n+1)B(s,n)P_n(z)\right] \quad (n > N)$$

satisfies (α) and (β), as well as the unitarity condition (2).

Consequently if $F(s,n)$:

a) is holomorphic at fixed s (real positive) for $\text{Re}\, n > 0$;

b) satisfies $|F(s,n)| \leq s^M \exp(-\text{Re}\, n/\sqrt{s})$, where M is a positive constant;

c) is real and satisfies $0 \leq F(s,n) \leq 1$ for n and s real positive;

d) is such that

$$\sum_{n=1}^{\infty} (2n+1)F(s,n) \sim s(\ln s)^2$$

for $s \to \infty$;

then $A = B = iF$ gives an example of an f which satisfies condition (2) as well as (α) and (β), and which gives a total cross-section

$$\sigma_{tot} \sim (\ln s)^2$$

for $s \to \infty$.

We found two such functions F. It would be meaningless to write them down just now. In order to make their features understandable, we will first sketch the paths we followed to arrive at them.

3. Supposons le problème résolu : assume we got a function F with a), b), and c). Introduce the variable

$$x = n/M\sqrt{s}\ \ln s \qquad (5)$$

and define (for $s > 1$)

$$g(s,x) = s^{-M} \exp(xM \ln s) F(s, xM\sqrt{s} \ln s) \ . \qquad (6)$$

Because of a), b), and c), we have then :

a') $g(s,x)$ is holomorphic for $\mathrm{Re}\, x > 0$ (at fixed $s > 1$);

b') $\ln|g(s,x)| \leq 0$;

c') $\ln|g(s,x)| \leq (x-1)M \ln s$ for x real, positive.

We use now the fact that $\ln|g(s,x)|$ is a subharmonic function in $\mathrm{Re}\, x > 0$ (at fixed s) [this follows from a'); see [5]]. It can be shown that among all real functions of x and s which are subharmonic for $\mathrm{Re}\, x > 0$ (at fixed s), and which satisfy inequalities b') and c'), there is a biggest one, call it $V(s,x)$. We have then

$$\ln|g(s,x)| \leq V(s,x) \ .$$

We have been lucky enough to find the explicit expression of $V(s,x)$. We found :

$$V(s,x) = V_0(x) M \ln s , \qquad V_0(x) = \mathrm{Re}\, U_0(x) ,$$

$$U_0(x) = x+iy+(i/\pi) \ln[(y-2/\pi)/(y+2/\pi)]$$

with $y = i[x^2-(2/\pi)^2]^{1/2}$ (determination for which $y/x \to i$ as $|x| \to \infty$) (determination of $\ln$: that which goes to 0 when $|y| \to \infty$). (The x plane is cut along the segment between $-2/\pi$ and $2/\pi$.)

If it would be possible to find a function g which satisfies both a') and $\ln|g| = V$, this function would give with (5) and (6) a function F which would have all four properties a) to d). That this is in fact impossible is easily seen : namely, if $\ln|g| = V$, one would have $\Delta \ln|g| = \Delta V$ ($x = u+iv$, $\Delta = \partial_u^2+\partial_v^2$; the derivatives are taken in the distribution theoretical sense); but one finds

$$\Delta V(s,x) = (2/u)[(2/\pi)^2-u^2]^{1/2}\, \vartheta(2/\pi - u)\delta(v) M \ln s \qquad (7)$$

and, using a')

$$\Delta \ln|g(s,x)| = 2\pi \sum_{\text{zeros}} r_i\, \delta(u-u_i)\delta(v-v_i) \qquad (8)$$

(x_i : zeros of g with $u_i > 0$; r_i : their respective multiplicities); so the mentioned impossibility is now evident.

But in order that F satisfies property d), it is needed that $\ln|g|$ becomes as close as possible to V along the real axis, especially for values of x situated on the segment between 0 and $2/\pi$.

So one is led to search for functions g for which $\ln|g|$ behave "asymptotically" (loosely speaking) like $V(s,x)$ for big values of s. We followed two different lines of thought to find such functions. This we will expose now.

4. First example

Consider the differential equation

$$\{(d/dx)^2 - [h\,p(u)]^2\}\ w(x) = 0$$

where h is a positive parameter, and p an analytic function. Then, the W.K.B. approximation method tells us that there exists a fundamental system of solutions $w_\pm$ which behave asymptotically, for $h \to \infty$, like

$$[p(x)]^{-1/2} \exp[\pm h \int_{x_0}^{x} p(u)\,du]$$

so that

$$|w_\pm| \sim |p(x)|^{-1/2} \exp[\pm h \,\mathrm{Re} \int_{x_0}^{x} p(u)\,du] \quad .$$

Identifying h with $M \ln s$, x_0 with $2/\pi$,

$$\int_{x_0}^{x} p(u)\,du$$

with $U_0(x) - x$ (we subtract x from U_0 in order to get a simpler expression for p), we are led to associate with our subharmonic bound $V = M \ln s \,\mathrm{Re}\, U_0$ the following Bessel equation :

$$\{(d/dx)^2 + (M \ln s)^2 [(2/\pi x)^2 - 1]\}\ w(x) = 0 \quad .$$

$\sqrt{x}\,H_t^{(1)}(ixM \ln s)$ and $\sqrt{x}\,H_t^{(2)}(ixM \ln s)$, with $t(t+1) = -(2M \ln s/\pi)^2$ (so that $t \sim i\,2M \ln s/\pi$) form a fundamental system of solutions showing the characteristic W.K.B. behaviour.

The considerations just presented have heuristic character only. But the precise study of the H-functions for large values of their parameters, as made by Martin in a different context [6], shows that indeed

$$F(s,n) = -C_0 \ln s(n/M\sqrt{s}\ln s - 2/\pi)^2 (n/M\sqrt{s}\ln s + 2/\pi)^{-1} [H^{(1)}_{i\,2M\ln s/\pi}(in/\sqrt{s})]^2$$

with suitable choice of C_0 (real positive constant) satisfies the four properties a) to d).

5. Second example

$V(s,x)$ is (at each fixed s) the solution of the following problem : find (in the domain $\mathrm{Re}\,x > 0$) the solution of (7) which is equal to zero in $\mathrm{Re}\,x = 0$, and which goes to zero when $|x| \to \infty$. This solution can be expressed in terms of the Green function $(2\pi)^{-1} \ln|(x-x')/(x+x')|$ associated with the boundary conditions, and of the right-hand member of (7) as follows :

$$V(s,x) = M \ln s \int_0^{2/\pi} \rho(u) \ln|(x-u)/(x+u)|\, du$$

where $\rho(u) = [(2/\pi)^2 - u^2]^{1/2}/\pi u$. Introducing the function

$$y(u) = \int_u^{2/\pi} \rho(v)\, dv \quad ,$$

we obtain

$$V(s,x) = M \ln s \int_0^{\infty} \ln|[x-u(y)]/[x+u(y)]|\, dy$$

where $u(y)$ is the inverse function of $y(u)$.

In order to find $g(s,x)$, holomorphic in $\mathrm{Re}\,x > 0$, such that $V(s,x)$ is an asymptotic expression for $\ln|g|$ when $s \to \infty$, we try to replace the right-hand side of (7) by an expression of the form (8); i.e., we try to approximate $V(s,x)$ by a sum like

$$r \sum \ln|(x-x_i)/(x+x_i)|$$

(r : positive integer), (replacement of an integral by a sum !) which is equal to $\ln|g|$ if

$$g = \prod [(x-x_i)/(x+x_i)]^r$$

In fact, if we take $x_i = u(y_i)$, where $y_i = (2i-1)\, r/2M \ln s$ $(i = 1,2,..)$, it is possible to show that there exists for each positive integer r a positive constant C_3 such that

$$g = C_3 \prod_{i=1}^{\infty} [(x-x_i)/(x+x_i)]^r$$

satisfies a'), b'), and c'). We then choose r even, in order that g be positive for real positive x. So the corresponding function F [see Eqs. (5) and (6)] satisfies conditions a), b), and c). Furthermore, one can show that condition d) is also satisfied.

6. Remark 1

We have no example of a function f satisfying (α), (β) and (2), and for which

$$d\sigma/d\Omega \sim s^{-1}(\ln s)^3$$

at fixed $\vartheta \neq 0$ or π.

Remark 2

Assumptions (α) and (β) are fulfilled if f satisfies Mandelstam representation. The converse is evidently false : (α) and (β) impose no analytic properties for f as a function of s.

The examples we found have analytic properties in s. But they do not seem to obey Mandelstam representation. It is not clear if they can be modified in order to do so.

Furthermore, we have used unitarity in one channel only.

Can inequality (4) be strengthened by a more complete use of the Mandelstam analyticity hypothesis, and of the unitarity condition in the three channels ? This is an open question.

REFERENCES

1) T. Kinoshita, J.J. Loeffel and A. Martin, Phys. Rev. Letters, 10, 460 (1963).

2) M. Froissart, Phys. Rev., 123, 1053 (1961).

3) O.W. Greenberg and F.E. Low, Phys. Rev., 124, 2047 (1961).

4) A. Martin, Phys. Rev., 129, 1432 (1963); also "Proc. Internat. Confer. on High Energy Nucl. Phys. Geneva 1962." (CERN Scientific Information Service, Geneva 1962).

5) See for example: T. Rado, "Subharmonic Functions", (Chelsea Publishing Company, New York 1949) p.23.

6) A. Martin, "On the behaviour of the partial wave amplitudes for large angular momenta in potential scattering", (CERN preprint TH 361, July 1963) (submitted to the Nuovo Cimento).