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SOME GENERAL ASPECTS OF MULTIPERIPHERAL DYNAMICS \*)

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A B S T R A C T

We review some general aspects of the multiperipheral formalism at vanishing momentum transfer. A special attention is devoted to the concept of operator  $Q$ -factorized representation of the production amplitudes and to the general condition which ensures the existence of a representation of this kind. A special attention is also devoted to the  $O(3,1)$  partial wave projection of the Chew-de Tar multiperipheral integral equation, using a promising and rigorous approach based on some representations of a semi-group  $S$  contained in  $SL(2C)$ .

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## 1. Introduction

The first multiperipheral models have been developed ten years ago by ABFST and others <sup>1) - 3)</sup>. Some general predictions of these models, as Regge behaviour and Feynman scaling, are, at least qualitatively, in accord with the most recent results of accelerator and ISR experiments. Other predictions, as the weak increase of correlations with energy, will be tested soon at ISR, permitting possibly a choice between the multiperipheral ideas and other possibilities, as the one or two fireball models and the diffractive excitation models, which predict a strong increase of correlations with energy.

Other general features of the simplest multiperipheral models, as the dominance of an isolated Pomeron pole and its consequences (factorization and short range of inclusive correlations) are expected to hold only approximately and do not follow necessarily from multiperipheral models of a more general kind.

More detailed features, as the shape and the energy dependence of the inclusive distributions, have been successfully interpreted by means of particular multiperipheral models, making use of specific phenomenological inputs.

In this situation, in order to facilitate the interpretation of the forthcoming experimental data, it is useful to re-analyse in detail the assumptions, the structure and the mathematical formalism of multiperipheral dynamics. In these lectures, I shall summarize some partial contributions in this direction, contained in some recent papers <sup>4) - 6)</sup>, adding when necessary a short exposition of previously known results, in order to get, as far as possible, a self-contained treatment.

I shall deal only with a particular aspect of multiperipheral physics, which can be summarized as follows: make suitable assumptions on the production amplitudes and derive, by integration over the momenta and sum over the multiplicity, the total cross section and the inclusive distributions. We shall see that this problem is not so simple as it could seem.

The new contributions which will be treated with more details are essentially a proposed characterization of the multiperipheral amplitudes by means of a factorized upper bound and some improvements in the  $O(3,1)$  projection of the multiperipheral integral equation based on the properties of a semi-group.

## 2. A Characterization of Multiperipheral Production Amplitudes

The mathematical formalism and the general results of multiperipherism are based on some general assumptions on the production amplitudes

$M_n(P_A, P_B, P_{n+1}, \dots, P_0)$  for the processes

$$A + B \rightarrow (n+1) + (n) + \dots + (1) + (0), \quad m = 0, 1, 2, \dots \quad (2.1)$$

We are assuming for simplicity that all the particles are identical and spinless, so that the functions  $M_n$  are symmetrical with respect to the permutations of the  $(n+2)$  final particles.

We introduce, as usual, the four-momentum transfers

$$Q_i = P_A - \sum_{j=0}^i P_j, \quad (2.2)$$

the corresponding invariants

$$t_i = (Q_i)^2, \quad i = 0, 1, \dots, n, \quad (2.3)$$

which, in the equal mass case we are considering, are necessarily negative, and the subenergies

$$s_i = (P_{i+1} + P_i)^2 = (Q_{i-1} - Q_{i+1})^2, \quad i = 0, 1, \dots, n. \quad (2.4)$$

In order to get a complete set of  $(3n+2)$  invariants, we may consider also the subenergies

$$\sigma_i = (P_{i+1} + P_{i-1})^2, \quad i = 1, 2, \dots, n. \quad (2.5)$$

We remark that in the definition of these invariants a special ordering of the final momenta has been assumed.

A first characteristic feature of the multiperipheral amplitudes is that they can be large only in the kinematical configurations in which, after a suitable permutation of the final particles, all the momentum transfers  $t_0, \dots, t_n$  are small (in absolute value). This requirement can be stated more precisely by means of the inequality

$$|M_m(P_A, P_B, P_{m+1}, \dots, P_0)| \leq \sup_{\pi} f_m(P_A, P_B, P_{\pi(m+1)}, \dots, P_{\pi(0)}) , \quad (2.6)$$

where we have indicated by  $\pi$  a permutation of the  $(n+2)$  final particles and  $f_m$  is a function which decreases in a suitable way when some momentum transfer  $t_i$  increases (in absolute value).

It is convenient to indicate by  $x$  a set of  $(3n+2)$  kinematical invariants and by  $P_{\pi} x$  the set of invariants obtained from  $x$  by means of the permutation  $\pi$  of the final particles. Then Eq. (2.6) takes the form

$$|M_m(x)| \leq \sup_{\pi} f_m(P_{\pi} x) . \quad (2.7)$$

This condition is discussed in detail in Ref. <sup>4)</sup>. For our present purposes, it is sufficient to consider the weaker condition

$$|M_m(x)|^2 \leq \sum_{\pi} (f_m(P_{\pi} x))^2 . \quad (2.8)$$

We remark that, if this condition holds, it is always possible to decompose the modulus square of the amplitude as follows

$$|M_m(x)|^2 = \sum_{\pi} \mathcal{F}_m(P_{\pi} x) , \quad (2.9)$$

where

$$0 \leq \mathcal{F}_m(x) \leq (f_m(x))^2 . \quad (2.10)$$

A possible choice of the function  $\mathcal{F}_n(x)$ , which is not necessarily the most convenient one, is

$$\mathcal{F}_m(x) = |M_m(x)|^2 (f_m(x))^2 \left[ \sum_{\pi} (f_m(P_{\pi}x))^2 \right]^{-1}. \quad (2.11)$$

In general, multiperipheral models are based on an explicit expression for the functions  $\mathcal{F}_n(x)$ , which have a peculiar factorization property, which was studied in the most general case by Chew, Goldberger and Low<sup>7)</sup>, who called it "short range correlation". In order to avoid confusion between this exclusive short range correlation and the inclusive short range correlation, which is a completely different concept, we call this property "Q-factorization". A sequence of functions  $\mathcal{F}_n(P_A, P_B, P_{n+1}, \dots, P_0)$  is called Q-factorized of order k if all these functions can be written in the form

$$\begin{aligned} \mathcal{F}_n(x) = & \mathcal{B}(P_B, Q_m, \dots, Q_{m-k+2}) \mathcal{H}(Q_m, \dots, Q_{m-k+1}) \dots \\ & \dots \mathcal{H}(Q_{k-1}, \dots, Q_0) \mathcal{A}(Q_{k-2}, \dots, Q_0, P_A). \end{aligned} \quad (2.12)$$

Remark that the functions  $\mathcal{B}$ ,  $\mathcal{H}$  and  $\mathcal{A}$  do not depend on n, so that the whole sequence of functions  $\mathcal{F}_n$  is described in terms of these three functions. In general, it is more convenient to express  $\mathcal{B}$ ,  $\mathcal{H}$  and  $\mathcal{A}$  in terms of invariants than in terms of four-vectors.

Many physically important contributions to the amplitude, as multiple pion exchange or multiple Reggeon exchange, are Q-factorized. However, one can show that any Q-factorized representation of the amplitudes is necessarily approximated.

For this reason, we prefer to found the multiperipheral formalism on a weaker assumption, which has some chance of being really true. Our assumption is that the functions  $f_n$ , which appear in the upper bounds (2.7), (2.8) and (2.10) are Q-factorized. For general purposes, it is sufficient to assume a simple Q-factorized upper bound, which takes into account only the decrease in

the momentum transfers  $t_i$  and the polynomial boundedness in the subenergies  $s_i$ . We shall use the explicit assumption

$$f_m(x) = c \prod_{i=0}^m \left[ d(t_i) \left( \frac{s_i}{4m^2} \right)^d \right], \quad (2.13)$$

where  $d(t)$  is a suitably decreasing function of  $|t|$  and the exponent  $d$  is one or slightly larger in order to allow for logarithmic factors.

More complicated and restrictive upper bounds can be useful in order to take into account more detailed properties of the production amplitudes, but they are not necessary for the construction of a multiperipheral formalism.

In conclusion, the formalism that we shall develop is completely based on the assumptions (2.8) and (2.13) plus some condition on the function  $d(t)$ . The more restrictive assumption (2.7) would be useful in the treatment of other problems, as the study of the elastic diffraction peak based on unitarity.

### 3. Total Cross Section and Multiplicity Distribution

Substituting Eq. (2.9) into the cross section formula, we see that the total cross section for a process with  $(n+2)$  final particles is given by

$$\begin{aligned} \sigma_m(s) &= \frac{1}{2} (2\pi)^{-3n-2} [s(s-4m^2)]^{-\frac{1}{2}} \cdot \\ &\cdot \int \mathcal{F}_m(p_A, p_B, p_{n+1}, \dots, p_0) \delta^4(p_A + p_B - \sum_{i=0}^{n+1} p_i) \cdot \\ &\cdot (2p_{n+1}^0)^{-1} d^3 \underline{p}_{n+1} \dots (2p_0^0)^{-1} d^3 \underline{p}_0 \cdot \end{aligned} \quad (3.1)$$

Substituting into this formula the inequality (2.10) and Eq. (2.13), after suitable majorizations we get (see Ref. 5)

$$\sigma_n(s) \leq \pi H c^2 [\lambda(\lambda - 4m^2)]^{-1} \left(\frac{s}{m^2}\right)^{2d} \cdot \frac{1}{n!} \left[ H \log \frac{s}{m^2} \right]^n, \quad (3.2)$$

where

$$H = (4\pi)^{-2} 4^{-d} \int_{-\infty}^0 (d(t))^2 \left(1 + \frac{\sqrt{-t}}{m}\right)^{8d} dt, \quad (3.3)$$

(we assume that this integral converges).

The inequality (3.2) contains a very important and characteristic feature of multiperipheral physics: due to the factor  $(n!)^{-1}$ , the cross section  $\sigma_n(s)$  is a very fast decreasing function of  $n$  in the region  $n \gg \log s$ . Of course,  $\sigma_n(s) = 0$  for  $n > s^{\frac{1}{2}} m^{-1}$ , but Eq. (3.2) implies that  $\sigma_n(s)$  is negligible for much smaller values of  $n$  (at high energy).

Starting from Eq. (3.2) and from the very weak assumption that the total cross section  $\sigma(s)$  does not decrease faster than any negative power of  $s$ , one can derive the inequalities

$$\langle n^p \rangle = (\sigma(s))^{-1} \sum_n n^p \sigma_n(s) \leq c_p \left(\log \frac{s}{m^2}\right)^p. \quad (3.4)$$

The detailed proof for  $p = 1$  can be found in Ref. <sup>5)</sup> and the generalization to arbitrary  $p$  is straightforward.

#### 4. The Bali-Chew-Pignotti Variables

For the further developments of the formalism, it is convenient to describe the production amplitudes in terms of Bali-Chew-Pignotti (BCP) variables <sup>8)</sup>. In order to introduce our notations, we give here a short account of this argument.

If  $a$  is an element of the group  $SL(2C)$ , namely a  $2 \times 2$  complex matrix with determinant equal to one, we indicate by  $L(a)$  the  $4 \times 4$  Lorentz transformation matrix defined in such a way that, if  $V$  and  $V'$  are four vectors, the relation

$$V' = L(a) V \quad (4.1)$$

is equivalent to the relation

$$\begin{pmatrix} V'_0 + V'_3 & V'_1 - iV'_2 \\ V'_1 + iV'_2 & V'_0 - V'_3 \end{pmatrix} = a \begin{pmatrix} V_0 + V_3 & V_1 - iV_2 \\ V_1 + iV_2 & V_0 - V_3 \end{pmatrix} a^\dagger \quad (4.2)$$

As well known,  $L(a)$  spans the proper orthochronous Lorentz group. We indicate respectively by  $u_z(\mu)$  and  $a_z(\chi)$  the elements of  $SL(2C)$  which correspond to a rotation of an angle  $\mu$  around the  $z$  axis and to a boost of rapidity  $\chi$  along the  $z$  axis. We use also similar notations with  $z$  replaced by  $x$  or  $y$ .

For each final particle, we introduce a frame of reference connected with a given arbitrary frame by means of the Lorentz transformation  $L(a_i)$ . The simplest procedure is to define the group elements  $a_i$  implicitly by means of the formulae

$$\begin{cases} P_B = L(a_{m+1}) (m, 0, 0, 0), \\ Q_i = L(a_i) (0, 0, 0, \sqrt{-t_i}), \\ Q_i = L(a_{i+1} a_z(\chi_{i+1})) (0, 0, 0, \sqrt{-t_i}), \quad i = 0, 1, \dots, m, \\ P_A = L(a_0 a_z(\chi_0)) (m, 0, 0, 0). \end{cases} \quad (4.3)$$

The condition that the outgoing particles have mass  $m$  permits to compute the quantities  $\chi_i$ . In this way we get



$$\begin{cases} \sinh \chi_{m+1} = (2m)^{-1} \sqrt{-t_m} , \\ \sinh \chi_0 = (2m)^{-1} \sqrt{-t_0} , \\ \chi_i = \chi(t_i, t_{i-1}) , \quad i = 1, 2, \dots, m , \end{cases} \quad (4.4)$$

where

$$\cosh \chi(t, t') = (m^2 - t - t')(4tt')^{-\frac{1}{2}} , \quad \chi(t, t') > 0 . \quad (4.5)$$

If we put

$$g_i = a_z(\chi_{i+1}) a_{i+1}^{-1} a_i , \quad i = 0, 1, \dots, m , \quad (4.6)$$

from Eq. (4.3) we see immediately that  $L(g_i)$  does not act on the  $z$  component of a four vector. It follows that  $g_i$  belongs to the subgroup  $SU(1,1)$  of  $SL(2C)$  and can be parametrized as follows

$$\begin{aligned} g_i &= \mu_z(\mu_i) a_x(\xi_i) \mu_z(\nu_i) , \\ 0 \leq \mu_i < 4\pi , \quad \xi_i > 0 , \quad 0 \leq \nu_i < 2\pi . \end{aligned} \quad (4.7)$$

We see also that the Eqs. (4.3) are not affected by the substitution

$$a_k \rightarrow a_k \mu_z(\gamma) , \quad (4.8)$$

which, in terms of the elements  $g_i$  takes the form

$$\begin{cases} g_k \rightarrow g_k \mu_z(\gamma) , \\ g_{k-1} \rightarrow \mu_z(-\gamma) g_{k-1} . \end{cases} \quad (4.9)$$

The BCP variables  $t_i, g_i$  ( $i = 0, \dots, n$ ) determine univocally all the kinematical invariants and therefore the amplitudes can be expressed in terms of them. In the spinless case we are considering, the amplitudes are invariant under transformations of the kind (4.9).

From Eq. (4.3), we get immediately the following expression for the subenergies

$$\Delta_i = t_{i-1} + t_{i+1} - \frac{1}{2t_i} \left[ \left( T(m^2, t_{i-1}, t_i) T(m^2, t_i, t_{i+1}) \right)^{\frac{1}{2}} \cdot \cosh \xi_i + (m^2 - t_{i-1} - t_i)(m^2 - t_i - t_{i+1}) \right], \quad (4.10)$$

where

$$t_{-1} = t_{n+1} = m^2, \quad (4.11)$$

$$T(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ac. \quad (4.12)$$

It is convenient in the following to express Q-factorized functions in terms of BCP variables. For instance, we shall replace the Q-factorized upper bound (2.13) by the following expression which is Q-factorized in a different way

$$f_m(x) = b(\xi_m, t_m) k(t_m, \xi_{m-1}, t_{m-1}) \dots \dots k(t_1, \xi_0, t_0) a(t_0), \quad t_i < 0, \quad \xi_i \geq 0. \quad (4.13)$$

If we put, for instance,

$$\left\{ \begin{aligned} b(\xi_m, t_m) &= \left(\frac{t_m}{m^2}\right)^{\frac{1}{4}} c [d(t_m)]^{\frac{1}{2}} \left(\frac{4m^2 - t_m}{8m^2}\right)^{\frac{d}{2}} (1 + \cosh \xi_m)^d, \\ k(t_{i+1}, \xi_i, t_i) &= \left(\frac{t_i}{t_{i+1}}\right)^{\frac{1}{4}} [d(t_i) d(t_{i+1})]^{\frac{1}{2}} \cdot \\ &\cdot \left(\frac{T(m^2, t_{i+1}, t_i)}{8m^2 \sqrt{t_{i+1} t_i}}\right)^d (1 + \cosh \xi_i)^d, \\ a(t_0) &= \left(\frac{m^2}{t_0}\right)^{\frac{1}{4}} [d(t_0)]^{\frac{1}{2}} \left(\frac{4m^2 - t_0}{8m^2}\right)^{\frac{d}{2}}. \end{aligned} \right. \quad (4.14)$$

We see, using Eq. (4.10) that the expression (4.13) is greater than the expression (2.13). In the following we shall use the less restrictive upper bound defined by Eqs. (4.13) and (4.14).

## 5. Operator Q-Factorization

The Q-factorization property (2.12) is essential in the mathematical treatment of the multiperipheral models, as it permits an iterative procedure which reduces the integration over the final momenta and the sum over the multiplicity to the solution of an integral equation. The group-theoretical analysis of this multiperipheral equation provides a natural approach to complex angular momentum.

It could seem that if one starts from the weaker assumption of a Q-factorized upper bound, all these formal developments have to be abandoned. Fortunately, it is not so, because one can show that, if a sequence of amplitudes satisfies a Q-factorized upper bound, these amplitudes possess an exact Q-factorized representation of the kind (2.12) with the new complication that the quantity  $\mathcal{R}$  is an operator valued function, the quantities  $\mathcal{B}$ ,  $\mathcal{A}$  are vector valued functions and the right-hand side of Eq. (2.12) has to be interpreted as the matrix element of a product of operators.

The same result holds if we express our amplitudes in terms of invariants or of BCP variables. The special form of this result which we shall use

and which is proven in Ref. <sup>4)</sup> is the following:

Proposition: If the sequence of continuous functions  $\mathcal{F}_n(x)$  satisfies the inequality (2.10) with  $f_n(x)$  given by Eq. (4.13), for every  $n = 0, 1, \dots$  we can write the operator Q-factorized representation

$$\begin{aligned} \mathcal{F}_n(q_m, t_m, \dots, q_0, t_0) &= (B(q_m, t_m), \mathcal{K}(t_m, q_{m-1}, t_{m-1}) \dots \\ &\dots \mathcal{K}(t_1, q_0, t_0) a(t_0)) , \end{aligned} \quad (5.1)$$

where  $a$  is a vector belonging to a suitable Banach space,  $\mathcal{K}$  is an operator in this space and  $B$  is a vector of the dual space. The norms of these quantities satisfy the conditions

$$\left\{ \begin{aligned} \|a(t_0)\| &\leq (a(t_0))^2 , \\ \|\mathcal{K}(t_{i+1}, q_i, t_i)\| &\leq (k(t_{i+1}, \xi_i, t_i))^2 , \\ \|B(q_m, t_m)\| &\leq (b(\xi_m, t_m))^2 . \end{aligned} \right. \quad (5.2)$$

For a more exact formulation and a complete proof, see Ref. <sup>4)</sup>. Here, in order to clarify the structure of the proof, I consider a simpler case in which the functions  $\mathcal{F}_n$  depend on  $n + 1$  real variables and satisfy the inequalities

$$|\mathcal{F}_n(z_m, \dots, z_0)| \leq k(z_m) \dots k(z_0) , \quad n = 1, 2, \dots \quad (5.3)$$

Considering the ratios between the left- and the right-hand sides, we can reduce the problem to the simpler case in which

$$|\mathcal{F}_n(z_m, \dots, z_0)| \leq 1 . \quad (5.4)$$

We consider the Banach space  $\mathcal{M}$  of the sequences of bounded complex measures

$$\varphi_0(z_0) , \varphi_1(z_1, z_0) , \varphi_2(z_2, z_1, z_0) , \dots , \quad (5.5)$$

with the norm

$$\|\varphi\| = \sum_{i=0}^{\infty} \int |\varphi_i(z_i, \dots, z_0)| dz_i \dots dz_0 \quad . \quad (5.6)$$

Then we define the bilinear functional

$$\begin{aligned} \Phi(\psi, \varphi) &= \sum_{i,k=0}^{\infty} \int \mathcal{F}_{i+k+1}(z'_0, \dots, z'_k, z_i, \dots, z_0) \psi(z'_0, \dots, z'_k) \cdot \\ &\varphi(z_i, \dots, z_0) dz'_0 \dots dz'_k dz_i \dots dz_0 \quad . \end{aligned} \quad (5.7)$$

From Eq. (5.4) we get

$$|\Phi(\psi, \varphi)| \leq \|\psi\| \|\varphi\| \quad . \quad (5.8)$$

We consider also the operators  $H(z)$  and  $\tilde{H}(z)$  defined by

$$\begin{cases} [H(z)\varphi]_0(z_0) = 0, \\ [H(z)\varphi]_i(z_i, \dots, z_0) = \delta(z-z_i) \varphi_{i-1}(z_{i-1}, \dots, z_0), \quad i > 0, \end{cases} \quad (5.9)$$

$$\begin{cases} [\tilde{H}(z)\varphi]_0(z_0) = 0, \\ [\tilde{H}(z)\varphi]_i(z_i, \dots, z_0) = \varphi_{i-1}(z_i, \dots, z_1) \delta(z-z_0), \quad i > 0. \end{cases} \quad (5.10)$$

One can easily show that

$$\Phi(\psi, H(z)\varphi) = \Phi(\tilde{H}(z)\psi, \varphi) \quad . \quad (5.11)$$

If we define also the vectors  $\alpha(z) \in \mathcal{M}$  given by

$$\begin{cases} [\alpha(z)]_0(z_0) = \delta(z-z_0), \\ [\alpha(z)]_i(z_i, \dots, z_0) = 0, \quad i > 0, \end{cases} \quad (5.12)$$

we easily see that

$$\mathcal{F}_m(z_m, \dots, z_0) = \Phi(\alpha(z_m), H(z_{m-1}) \dots H(z_1) \alpha(z_0)) \quad (5.13)$$

This formula already looks like an operator factorized representation, but it is not economical, as the space  $\mathcal{M}$  is in general "too large". In order to eliminate this redundancy, we consider the closed subspace  $\mathcal{O}$  of  $\mathcal{M}$  containing the vectors  $\varphi$  such that

$$\Phi(\psi, \varphi) = 0 \quad (5.14)$$

for any  $\psi \in \mathcal{M}$ . Then we consider the quotient space

$$\mathcal{N} = \mathcal{M} / \mathcal{O} \quad (5.15)$$

Using Eq. (5.11), we see that the operators  $H(z)$  transform  $\mathcal{O}$  into itself and therefore they define the new operators  $\mathcal{H}(z)$  in the quotient space  $\mathcal{N}$ . We see also that the mapping

$$\varphi \rightarrow \Phi(\alpha(z), \varphi) \quad (5.16)$$

is a linear functional on  $\mathcal{M}$  which vanishes on  $\mathcal{O}$  and therefore defines an element  $\mathcal{B}(z)$  of the dual space  $\mathcal{N}'$ . If we indicate by  $\mathcal{A}(z)$  the element of  $\mathcal{N}$  which corresponds to  $\alpha(z)$ , from Eq. (5.13), by successive passages to the quotient, we get

$$\mathcal{F}_m(z_m, \dots, z_0) = (\mathcal{B}(z_m), \mathcal{H}(z_{m-1}) \dots \mathcal{H}(z_1) \mathcal{A}(z_0)) \quad (5.17)$$

which is just the required operator factorized representation.

It is interesting to remark that if the functions  $\mathcal{F}_n$  are numerically factorized, the space  $\mathcal{N}$  turns out to be one-dimensional and Eq. (5.17) is indeed a numerical factorized representation.

## 6. The Chew-de Tar Multiperipheral Integral Equation

Now we use the Q-factorized representation (5.1) in order to derive the multiperipheral integral equation, following essentially a very elegant procedure introduced by Chew and de Tar<sup>9)</sup>. For simplicity, we consider the case of numerical Q-factorization, but it is easy to see that exactly the same procedure holds for operator Q-factorized amplitudes.

We start from the total cross section formula (3.1). The analogous treatment for  $r$  particle inclusive distributions can be found in Ref.<sup>4)</sup>. First we introduce as new integration variables the four-momenta  $Q_i$ . Then we introduce the fictitious integration variables  $Q'_i$ ,  $P'_A$  and  $P'_B$  together with suitable  $\delta$  functions which identify them with the corresponding unprimed variables. We get in this way the formula

$$\begin{aligned} \sigma_m(s) &= \frac{1}{2} (2\pi)^{-3n-2} [s(s-4m^2)]^{-\frac{1}{2}} \int \mathcal{F}_m(x) \cdot \\ &\cdot \delta^4(P_B - P'_B) \delta((P'_B + Q'_m)^2 - m^2) \theta(P'_B{}^0 + Q'_m{}^0) d^4 P'_B d^4 Q'_m \cdot \\ &\cdot \delta^4(Q'_m - Q_m) \delta((Q'_{m-1} - Q_m)^2 - m^2) \theta(Q'_{m-1}{}^0 - Q_m{}^0) d^4 Q_m d^4 Q'_{m-1} \cdots \\ &\cdots \delta^4(Q'_0 - Q_0) \delta((P'_A - Q_0)^2 - m^2) \theta(P'_A{}^0 - Q_0{}^0) d^4 Q_0 d^4 P'_A \cdot \\ &\cdot \delta^4(P'_A - P_A) \cdot \end{aligned} \tag{6.1}$$

In agreement with Eq. (4.3), we put

$$\begin{cases} P_B = L(l_B)(m, 0, 0, 0) , \\ P'_B = L(a_{m+1})(m', 0, 0, 0) , \end{cases} \quad (6.2)$$

$$\begin{cases} Q_i = L(a_i)(0, 0, 0, \sqrt{-t_i}) , \\ Q'_i = L(a_{i+1} a_z(\chi_{i+1}))(0, 0, 0, \sqrt{-t'_i}) , \end{cases} \quad (6.3)$$

$$\begin{cases} P'_A = L(a_0 a_z(\chi_0))(m'', 0, 0, 0) , \\ P_A = L(l_A)(m, 0, 0, 0) . \end{cases} \quad (6.4)$$

In Ref. <sup>4)</sup>, the following two lemmas have been proven:

Lemma 1 : For every value of the variables  $t_1$  and  $t_2$  we choose two four vectors  $\hat{Q}_1$  and  $\hat{Q}_2$  with the properties

$$(\hat{Q}_1)^2 = t_1 , (\hat{Q}_2)^2 = t_2 , (\hat{Q}_1 - \hat{Q}_2)^2 = m^2 , \hat{Q}_1^0 - \hat{Q}_2^0 > 0 , \quad (6.5)$$

and we put

$$Q_1 = L(a) \hat{Q}_1 , \quad Q_2 = L(a) \hat{Q}_2 . \quad (6.6)$$

Then we have the identity

$$\begin{aligned} & \int f(Q_1, Q_2) \delta((Q_1 - Q_2)^2 - m^2) \theta(Q_1^0 - Q_2^0) d^4 Q_1 d^4 Q_2 = (4\pi)^{-1} \cdot \\ & \cdot \int \tilde{f}(t_1, t_2, a) [T(t_1, t_2, m^2)]^{\frac{1}{2}} \theta[T(t_1, t_2, m^2)] dt_1 dt_2 d^6 a , \end{aligned} \quad (6.7)$$



where

$$\tilde{f}(t_1, t_2, a) = f(Q_1, Q_2), \quad (6.8)$$

and  $d^6 a$  is a properly normalized invariant measure on  $SL(2C)$ .

Lemma 2 : If we put

$$\begin{cases} Q_t = (\sqrt{t}, 0, 0, 0), & t > 0, \\ Q_t = (0, 0, 0, \sqrt{-t}), & t < 0, \end{cases} \quad (6.9)$$

for  $t \neq 0$  we have the identity

$$\delta^4(L(a)Q_t - L(a')Q_{t'}) = \frac{2\pi^2}{|t|} \delta(t-t') \delta_{\pm}^3(a^{-1}a'), \quad (6.10)$$

where  $\pm$  is the sign of  $t$  and  $\delta_{\pm}^3(a)$  are measures on  $SL(2C)$  defined by

$$\int_{SL(2C)} f(a) \delta_{\pm}^3(a) d^6 a = \int_{H_{\pm}} f(h) d^3 h. \quad (6.11)$$

We have indicated by  $H_+$  the group  $SU(2)$ , by  $H_-$  the group  $SU(1,1)$  and by  $d^3 h$  the invariant measure on these groups.

After a repeated use of these two lemmas, the Eq. (6.1) takes the form

$$\begin{aligned} \sigma_m(\lambda) &= \frac{1}{2} (2\pi)^{-3n-2} [\lambda(\lambda-4m^2)]^{-\frac{1}{2}} \int \mathcal{F}_m(x) \cdot \\ &\cdot \frac{2\pi^2}{m^2} \delta_+^3(b_B^{-1} a_{m+1}) (4\pi)^{-1} [T(m^2, m^2, t_m)]^{\frac{1}{2}} d^6 a_{m+1} \cdot \\ &\cdot \frac{2\pi^2}{|t_m|} \delta_-^3(q_m) (4\pi)^{-1} [T(m^2, t_m, t_{m-1})]^{\frac{1}{2}} dt_m d^6 a_m \dots \\ &\dots \frac{2\pi^2}{|t_0|} \delta_-^3(q_0) (4\pi)^{-1} [T(m^2, t_0, m^2)]^{\frac{1}{2}} dt_0 d^6 a_0 \cdot \\ &\cdot \frac{2\pi^2}{m^2} \delta_+^3(a_z(-x_0) a_0^{-1} b_A). \end{aligned} \quad (6.12)$$

where  $g_i$  is given by Eq. (4.6). If we introduce the Q-factorized expression for  $\mathcal{F}_n$  and we put

$$\underline{B}(a, t_m) = \frac{1}{8\pi |t_m|} [T(m^2, m^2, t_m)]^{\frac{1}{2}} \delta_-^3(a_z(-\chi_{m+1})a) \cdot \\ \cdot B(a_z(-\chi_{m+1})a, t_m), \quad (6.13)$$

$$\underline{\mathcal{F}k}(t', a, t) = \frac{1}{16\pi^2 |t|} [T(m^2, t', t)]^{\frac{1}{2}} \delta_-^3(a_z(-\chi(t', t))a) \cdot \\ \cdot \mathcal{F}k(t', a_z(-\chi(t', t))a, t), \quad (6.14)$$

$$\underline{a}(t_0) = \frac{1}{4\pi} [T(m^2, t_0, m^2)]^{\frac{1}{2}} a(t_0), \quad (6.15)$$

the Eq. (6.12) takes the form

$$\sigma_m(s) = \frac{1}{2} [\lambda(\lambda - 4m^2)]^{-\frac{1}{2}} \int \frac{2\pi^2}{m^2} \delta_+^3(b_B^{-1}a_{m+1}) \cdot \\ \cdot \underline{B}(a_{m+1}^{-1}a_m, t_m) \underline{\mathcal{F}k}(t_m, a_m^{-1}a_{m-1}, t_{m-1}) \dots \\ \dots \underline{\mathcal{F}k}(t_1, a_1^{-1}a_0, t_0) \underline{a}(t_0) \frac{2\pi^2}{m^2} \delta_+^3(a_z(-\chi_0)a_0^{-1}b_A) \cdot \\ \cdot d^6a_{m+1} \dots d^6a_0 dt_m \dots dt_0. \quad (6.16)$$

We remark that the integrals over the group elements  $a_{n+1}, \dots, a_0$  are of the convolution type. Summing over  $n$  the expressions (6.16), we get the total cross section  $\sigma(s)$ . It is clear that it can be expressed in terms of the quantity

$$\underline{R}(t', a, t) = \underline{\mathcal{F}k}(t', a, t) + \int \underline{\mathcal{F}k}(t', a', t'') \cdot \\ \cdot \underline{\mathcal{F}k}(t'', a'^{-1}a, t) d^6a' dt'' + \dots \quad (6.17)$$

As shown in Ref. <sup>4)</sup>, also the  $r$  particle inclusive distributions can be written as finite sums of integrals containing the expression (6.17). In conclusion, we see that the most difficult step in the calculation of the total cross section and of the inclusive distributions is the treatment of the series (6.17). It already contains, as we shall see, the complex angular momentum singularities which determine the asymptotic behaviour of the measurable quantities.

We see easily that the series (6.17) is, if it converges, the perturbative solution of the multiperipheral integral equation

$$\underline{R}(t', a, t) = \underline{\mathcal{R}}(t', a, t) + \int \underline{\mathcal{R}}(t', a', t'') \cdot \underline{R}(t'', a'^{-1}a, t) d'a' dt'' . \quad (6.18)$$

One has to keep in mind that  $\underline{R}$  is uniquely defined by Eq. (6.17), while the equation (6.18) could also have different solutions.

## 7. Diagonalization of Convolution Products on SL(2C)

In the Eqs. (6.17) and (6.18) we find integrations over the "radial" variables  $t, t' \dots$  and over the "angular" variables which are represented by the group elements  $a, a' \dots$ . The integrations over the angular variables can be interpreted as convolution products. We remember that the convolution of two functions defined on a group (e.g. SL(2C)) is defined by

$$[F_1 * F_2](a) = \int_{SL(2C)} F_1(a') F_2(a'^{-1}a) d'a' . \quad (7.1)$$

If, as in our case, the quantities  $F_i$  are measures <sup>\*)</sup> or distributions, Eq. (7.1) has to be interpreted as

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\*) We indicate a measure on SL(2C) by the notation  $F(a) d^6 a$ , where  $F$  is a generalized function, which for the sake of brevity will also be called a measure.

$$\int [F_1 * F_2](a) \varphi(a) d^6 a = \int F_1(a') F_2(a'') \varphi(a'a'') d^6 a' d^6 a'', \quad (7.2)$$

where  $\varphi(a)$  is a test function.

As well known, the Fourier or Laplace transformations transform the convolution product into an ordinary product. If  $\mathbb{D}(a)$  is a continuous representation of  $SL(2C)$  by means of bounded operators in a Hilbert or Banach space, if we define the projection integrals

$$\mathbb{D}(F) = \int_{SL(2C)} F(a) \mathbb{D}(a) d^6 a, \quad (7.3)$$

from Eq. (7.2) and from the representation property

$$\mathbb{D}(aa') = \mathbb{D}(a) \mathbb{D}(a'), \quad (7.4)$$

we have

$$\mathbb{D}(F_1 * F_2) = \mathbb{D}(F_1) \mathbb{D}(F_2). \quad (7.5)$$

Applying this procedure to the equations (6.17) or (6.18), we get the corresponding projected partial wave equations, which are in general simpler and have interesting properties. This procedure has been discussed in detail in Refs. 9) - 11).

The critical point of this treatment is the convergence of the integral (7.3). A sufficient condition for the existence of an integral of an operator valued function of this kind is

$$\int \|F(a)\| \|\mathbb{D}(a)\| d^6 a < \infty. \quad (7.6)$$

In order to discuss this condition, we have to remember some properties of the irreducible representations of  $SL(2C)$  and of the corresponding norms. A complete account can be found in Refs. 12) - 14).

We choose as representation space the space of the measurable functions of the complex variable  $z$  such that the expression

$$\|f\|_p = \left[ \int |f(z)|^p (1+|z|^2)^{-2-p(\operatorname{Re}\lambda-1)} d^2z \right]^{\frac{1}{p}}, \quad p > 1, \quad (7.7)$$

$$d^2z = d \operatorname{Re} z \, d \operatorname{Im} z, \quad (7.8)$$

is finite. With respect to this norm, the representation space is a Banach space, and even a Hilbert space if  $p = 2$ . The different representations are labelled by the complex parameter  $\lambda$  and by the integral or half integral parameter  $M$ . The representation operators are defined by

$$\begin{aligned} [D^{M\lambda}(a)f](z) &= (a_{12}z + a_{22})^{\lambda-M-1} \cdot \\ &\cdot (\bar{a}_{12}\bar{z} + \bar{a}_{22})^{\lambda+M-1} f(z_a), \end{aligned} \quad (7.9)$$

where

$$z_a = \frac{a_{11}z + a_{21}}{a_{12}z + a_{22}}. \quad (7.10)$$

From Eq. (7.10) we have

$$d^2z = |a_{11} - a_{12}z_a|^{-4} d^2z_a, \quad (7.11)$$

and from Eqs. (7.7) and (7.9) we get

$$\| \mathbb{D}^{M\lambda}(a) f \|_p = \left[ \int |f(z_a)|^p \cdot (|a_{11} - a_{12} z_a|^2 + |a_{22} z_a - a_{21}|^2)^{-2 - p(\operatorname{Re} \lambda - 1)} d^2 z_a \right]^{\frac{1}{p}} . \quad (7.12)$$

From the definition of the norm of an operator, we have finally

$$\| \mathbb{D}^{M\lambda}(a) \|_p = \sup_z \left( \frac{1 + |z|^2}{|a_{11} - a_{12} z|^2 + |a_{22} z - a_{21}|^2} \right)^v , \quad (7.13)$$

where

$$v = \operatorname{Re} \lambda - 1 + 2 p^{-1} . \quad (7.14)$$

If we use the parametrization

$$a = \mu_1 a_z(\xi) \mu_2 , \quad \mu_1, \mu_2 \in SU(2) , \quad (7.15)$$

we obtain from Eq. (7.13) after some calculation

$$\| \mathbb{D}^{M\lambda}(a) \|_p = \exp |v \xi| \geq 1 . \quad (7.16)$$

We see that the condition (7.6) can be satisfied only if  $F$  represents a bounded measure. If we apply this condition to the kernel (6.14) at fixed values of  $t$  and  $t'$ , we see, using Eq. (6.11), that the function  $\mathcal{K}(t', g, t)$  must be integrable on  $SU(1,1)$ . This is ensured by the bound (5.2) with  $k$  given by Eq. (4.14), only if  $d < -1/2$ , which is a condition too restrictive for physical purposes.

Of course, one has to remember that the condition (7.6) is not necessary for the existence of the integral (7.3). Nevertheless, one can see that there is no hope to project a physically interesting kernel on the representations  $\mathbb{D}^{M\lambda}$ .

Fortunately, the analysis given in Refs. <sup>10), 11)</sup> of the Chew-de Tar equation (6.18) and the treatment given in Refs. <sup>15) - 17)</sup> of the simpler multi-peripheral equations of the ABFST type suggest that there is some way to avoid these difficulties.

In the following we summarize a systematic approach developed in Ref. <sup>6)</sup>. It is based on the remark that the existence of the projection integral (7.3) is strictly related with the existence of the convolution integral (7.1). In fact, if the main property of a Laplace transformation is to transform a convolution product into an ordinary product, it is natural to expect that this Laplace transform is well defined only for functions which satisfy some conditions which ensure the existence of the convolution products. More exactly, the domain of definition of a Laplace transform should be a convolution algebra, i.e. a linear space of functions or measures which is closed under the convolution product. Different "natural" Laplace transforms should correspond to different convolution algebras. The bounded measures form a convolution algebra and the projection on the representations  $\mathcal{O}^{M,\lambda}$ , with  $p$  chosen in such a way that  $v = 0$ , is the corresponding Laplace transform.

Now we note that the convolutions which appear in Eq. (6.17) are well defined for physical reasons even if the kernel  $\mathcal{K}$  is not a bounded measure. As we shall see in the next section, the existence of these convolutions is due to the fact that the kernel  $\mathcal{K}$ , due to the positivity of the mass and the energy of physical states, vanishes outside a given subset of  $SL(2C)$ . Analysing this property, we shall find a convolution algebra to which the kernel belongs and the corresponding natural Laplace transform.

#### 8. A Semi-Group Contained in $SL(2C)$

From Eq. (6.14) we see that the kernel  $\mathcal{K}$  does not vanish only if its argument  $a$  is an element of  $SL(2C)$  of the form

$$a = a_z(\chi) q, \quad \chi > 0, \quad q \in SU(1,1). \quad (8.1)$$

It is easy to show that the elements of the form (8.1) have the properties

$$\left\{ \begin{array}{l} L_{33}(a) = \frac{1}{2} (|a_{11}|^2 + |a_{22}|^2 - |a_{12}|^2 - |a_{21}|^2) \gg 1, \\ L_{03}(a) = \frac{1}{2} (|a_{11}|^2 - |a_{22}|^2 - |a_{12}|^2 + |a_{21}|^2) \gg 0, \\ L_{30}(a) = \frac{1}{2} (|a_{11}|^2 - |a_{22}|^2 + |a_{12}|^2 - |a_{21}|^2) \gg 0. \end{array} \right. \quad (8.2)$$

We indicate by  $S$  the set of all the elements of  $SL(2C)$  which satisfy the inequalities (8.2). We want to show that  $S$  is a semi-group, namely that if  $a$  and  $b$  belong to  $S$ , also their product  $ab$  belongs to  $S$ . If  $L$  is any matrix of the orthochronous Lorentz group, we have

$$\left\{ \begin{array}{l} L_{00} \gg 1, \\ L_{00}^2 - L_{01}^2 - L_{02}^2 - L_{03}^2 = 1, \\ L_{00}^2 - L_{10}^2 - L_{20}^2 - L_{30}^2 = 1, \\ L_{30}^2 - L_{31}^2 - L_{32}^2 - L_{33}^2 = -1, \\ L_{03}^2 - L_{13}^2 - L_{23}^2 - L_{33}^2 = -1. \end{array} \right. \quad (8.3)$$

If  $a$  and  $b$  belong to  $S$ , using these equations and the Schwarz inequality, we have

$$\begin{aligned} L_{33}(ab) &= L_{30}(a)L_{03}(b) + L_{31}(a)L_{13}(b) + L_{32}(a)L_{23}(b) + \\ &+ L_{33}(a)L_{33}(b) \gg L_{30}(a)L_{03}(b) + L_{33}(a)L_{33}(b) - \\ &- [1 + (L_{30}(a))^2 - (L_{33}(a))^2]^{\frac{1}{2}} [1 + (L_{03}(b))^2 - (L_{33}(b))^2]^{\frac{1}{2}} \gg \\ &\gg L_{33}(a)L_{33}(b) \gg 1, \end{aligned} \quad (8.4)$$



$$\begin{aligned}
 L_{30}(ab) &= L_{30}(a)L_{00}(b) + L_{31}(a)L_{10}(b) + L_{32}(a)L_{20}(b) + \\
 &+ L_{33}(a)L_{30}(b) \gg L_{30}(a)L_{00}(b) + L_{33}(a)L_{30}(b) - \\
 &- [1 + (L_{30}(a))^2 - (L_{33}(a))^2]^{\frac{1}{2}} [(L_{00}(b))^2 - (L_{30}(b))^2 - 1]^{\frac{1}{2}} \gg \\
 &\gg L_{33}(a)L_{30}(b) \gg 0.
 \end{aligned} \tag{8.5}$$

In a similar way we get also

$$L_{03}(ab) \gg L_{03}(a)L_{33}(b) \gg 0. \tag{8.6}$$

From these inequalities we see that  $ab$  belongs to  $S$ .

We see also that the set  $S^{\circ}$  defined by the inequalities

$$L_{33}(a) > 1, \quad L_{30}(a) > 0, \quad L_{03}(a) > 0, \tag{8.7}$$

is a semi-group. One can show that  $S^{\circ}$  is the interior of  $S$  and that  $S$  is the closure of  $S^{\circ}$ . The elements of the kind (8.1) belong to  $S^{\circ}$ . Also the elements of the more general form

$$a = g a_z(x) g', \quad x > 0, \quad g, g' \in SU(1,1), \tag{8.8}$$

belong to  $S^{\circ}$ . We want to show that any element of  $S^{\circ}$  can be written in the form (8.8).

From the Eqs. (8.3) and (8.7) we see that the Minkowski three-vectors

$$\begin{cases} (L_{03}(a), L_{13}(a), L_{23}(a)) , \\ (L_{30}(a), L_{31}(a), L_{32}(a)) , \end{cases} \tag{8.9}$$

are time-like. Therefore, we can find two elements  $h, h'$  of  $SU(1,1)$  such that

$$L(hah') = \begin{pmatrix} \Lambda_{00} & \Lambda_{01} & \Lambda_{02} & \sinh \chi \\ \Lambda_{10} & \Lambda_{11} & \Lambda_{12} & 0 \\ \Lambda_{20} & \Lambda_{21} & \Lambda_{22} & 0 \\ \sinh \chi & 0 & 0 & \cosh \chi \end{pmatrix}, \quad \chi > 0. \quad (8.10)$$

From the properties (8.3) and

$$L_{00}L_{03} - L_{10}L_{13} - L_{20}L_{23} - L_{30}L_{33} = 0 \quad (8.11)$$

of the Lorentz matrices we get

$$\Lambda_{00} = \cosh \chi, \quad \Lambda_{01} = \Lambda_{02} = \Lambda_{10} = \Lambda_{20} = 0, \quad (8.12)$$

and therefore from Eq. (8.10) we obtain

$$hah' = a_z(\chi) u_z(\varphi) \quad (8.13)$$

and the representation (8.8) follows immediately.

### Linear Representations of the Semi-Group S

In this section we define and study an important class of Banach space representations of  $S$ . First of all, we consider the transformation (7.10) and we show that if  $a \in S$  and  $|z| \geq 1$ , we have  $|z_a| \geq 1$ . This is clear in the special case

$$a = a_z(\chi) = \begin{pmatrix} \exp \frac{1}{2} \chi & 0 \\ 0 & \exp(-\frac{1}{2} \chi) \end{pmatrix}, \quad \chi > 0. \quad (9.1)$$

In the case

$$a = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(1,1) , \quad (9.2)$$

we see immediately that the transformation (7.10) maps the circle  $|z| = 1$  onto itself and has therefore the required property. It is easy to extend the proof to any element of  $S^0$ , as it can be decomposed according to Eq. (8.8). The general result follows from a continuity argument.

From the property proven above, it follows that if  $a \in S$  the operator  $\mathcal{O}^{M\lambda}(a)$  defined by Eq. (7.9) transforms a function  $f(z)$  which vanishes outside the circle  $|z| < 1$  into a function which has the same property. Therefore we can define the operator  $\mathcal{B}^{M\lambda}(a)$  which is the restriction of  $\mathcal{O}^{M\lambda}(a)$  to a space of functions which vanish for  $|z| \geq 1$ . It can also be defined directly by means of the formula

$$\begin{aligned} [\mathcal{B}^{M\lambda}(a)f](z) &= (a_{12}z + a_{22})^{\lambda-M-1} \\ &\cdot (\bar{a}_{12}\bar{z} + \bar{a}_{22})^{\lambda+M-1} f(za) , \quad a \in S . \end{aligned} \quad (9.3)$$

In this case it is useful to introduce, instead of the norms (7.7), the norms

$$\begin{aligned} \|f\|_{p, \nu} &= \left[ \int_{|z| < 1} |f(z)|^p (1-|z|^2)^{-\nu p} d^2z \right]^{\frac{1}{p}} , \\ p &\geq 1 , \quad \nu \geq 0 . \end{aligned} \quad (9.4)$$

Proceeding as in the derivation of Eq. (7.13), we get the operator norms

$$\begin{aligned} \|\mathcal{B}^{M\lambda}(a)\|_{p, \nu} &= \sup_{|z| \leq 1} \left[ \left( \frac{1-|z|^2}{|a_{11}-a_{12}z|^2 - |a_{22}z - a_{21}|^2} \right)^{\nu} \right. \\ &\cdot \left. |a_{11}-a_{12}z|^{2(\nu-1)} \right] = \rho_{\nu}(a) , \quad a \in S . \end{aligned} \quad (9.5)$$

where  $v$  is given by Eq. (7.14). It is easy to show that this expression is finite.

We have obtained in this way a class of representations of  $S$  by means of bounded operators in Banach spaces. For  $p = 2$  we get Hilbert space representations. Representations which are labelled by the same values of  $M$  and  $\lambda$  but correspond to different values of  $p$  or  $r$  are not essentially different, as they coincide on some dense subspaces of the representation spaces.

From the representation property it follows that the norms (9.5) satisfy the inequality

$$\rho_{rv}(ab) \leq \rho_{rv}(a) \rho_{rv}(b) . \quad (9.6)$$

The explicit calculation of these norms is very difficult, but it is easy to get the following partial results

$$\rho_{rv}(u_z(\mu)) = 1 , \quad (9.7)$$

$$\rho_{rv}(a_z(\chi)) = \exp(-v\chi) , \quad \chi \geq 0 , \quad (9.8)$$

$$\rho_{rv}(a_x(\xi)) = \exp|(\nu-r)\xi| . \quad (9.9)$$

From Eqs. (9.6) - (9.9) we get

$$\rho_{rv}(a) = \exp(-v\chi) , \quad a \in S^0 , \quad (9.10)$$

where  $a$  is parametrized as in Eq. (8.8). In particular we have

$$\sup_{|z| \leq 1} \left( \frac{1 - |z|^2}{|a_{11} - a_{12}z|^2 - |a_{22}z - a_{21}|^2} \right) = \exp(-\chi) . \quad (9.11)$$

and introducing this result into Eq. (9.5), we obtain the inequality

$$\rho_{n+c, v+c}(a) \leq \exp(-\chi c) \rho_{nv}(a), \quad c \gg 0. \quad (9.12)$$

We are mainly interested in group elements of the kind (8.1) and we shall only need the following result which can be obtained from Eq. (9.5) after complicated calculations.

$$\rho_{nv}(a_z(\chi)g) \leq \exp[(v-n)\xi] \Phi_{nv}(\chi), \quad v \leq n \leq 2v, \quad (9.13)$$

where  $g$  is parametrized as in Eq. (4.7) and

$$\Phi_{nv}(\chi) = O(\chi^{2(v-n)}) \quad , \quad \chi \rightarrow 0, \quad (9.14)$$

$$\Phi_{nv}(\chi) = O(e^{-v\chi}) \quad , \quad \chi \rightarrow \infty. \quad (9.15)$$

## 10. Convolution Algebras and Laplace Transform on $S$

Now we consider a measure  $F(a)$  with support in  $S$ . A sufficient condition for the existence of the projection integral

$$B^{M\lambda}(F) = \int_S B^{M\lambda}(a) F(a) d^6 a \quad (10.1)$$

is that the quantity

$$\|F\|_{nv} = \int_S |F(a)| \rho_{nv}(a) d^6 a \quad (10.2)$$

is finite.

The measures with support in  $S$  such that the norm (10.2) is finite form a Banach space  $M_{rv}$ . They form also a convolution algebra and we have the following inequality

$$\begin{aligned} \|F_1 * F_2\|_{rv} &= \int_S \left| \int_S F_1(a') F_2(a'^{-1}a) d^6 a' \right| \varrho_{rv}(a) d^6 a \leq \\ &\leq \int_S \int_S |F_1(a')| |F_2(a'')| \varrho_{rv}(a') \varrho_{rv}(a'') d^6 a' d^6 a'' = \\ &= \|F_1\|_{rv} \|F_2\|_{rv} . \end{aligned} \tag{10.3}$$

We have used the inequality (9.6). The inequality (10.3) shows that  $M_{rv}$  is a Banach algebra with respect to convolution.

The measures belonging to  $M_{rv}$  have the fundamental property

$$\mathcal{B}^{M\lambda}(F_1 * F_2) = \mathcal{B}^{M\lambda}(F_1) \mathcal{B}^{M\lambda}(F_2) . \tag{10.4}$$

In order to show that this formalism is suitable for the treatment of multiperipheral equations, we have to show that the kernel (6.14) for fixed values of  $t, t'$  belongs to the algebra  $M_{rv}$  with some choice of the parameters  $r$  and  $v$ . From the inequalities (5.2) and (9.13) we get

$$\begin{aligned} \|\underline{\mathcal{K}}(t', t)\|_{rv} &\leq \frac{1}{16\pi^2 |t|} [T(m^2, t', t)]^{\frac{1}{2}} . \\ &\cdot \int_0^\infty (k(t', \xi, t))^2 \exp[(v-r)\xi] \Phi_{rv}(\chi(t', t)) \frac{1}{2} \sinh \xi d\xi \leq \\ &\leq 2^{-6} \pi^{-2} (2m)^{-4d} (tt')^{-d-\frac{1}{2}} [T(m^2, t', t)]^{\frac{1}{2}+2d} . \\ &\cdot d(t) d(t') \Phi_{rv}(\chi(t', t)) (r-v-2d-1)^{-1} , \end{aligned}$$

(10.5)

if the parameters  $r$  and  $v$  satisfy the inequalities

$$0 < 2d+1 < r-v < v . \quad (10.6)$$

Comparing with Eq. (7.14), we see that, with a proper choice of the parameters  $p$  and  $r$ , the projected kernel exists in the half plane

$$\operatorname{Re} \lambda > 2d . \quad (10.7)$$

This is just the result expected from previous treatments (7), (9) - (11), (18). Moreover, the results proven above show that all the convolutions which appear in Eq. (6.17) exist and belong to  $M_{rv}$  if  $r$  and  $v$  satisfy the condition (10.6).

#### 11. Discussion of the Multiperipheral Equation

In order to complete our analysis, we have to discuss the integration over the radial variables and the convergence of the series which appears in Eq. (6.17). For this purpose it is convenient to consider the kernel  $\underline{K}$  and the resolvent  $\underline{R}$  as functions of the variables  $t, t'$  with values in the convolution algebra  $M_{rv}$ . We always assume that these functions satisfy some measurability condition. Then the existence of the integrals over  $t'' \dots$  is ensured if the following sufficient condition is satisfied

$$N_{rv}(\underline{K}) = \left[ \int \int \|\underline{K}(t', t)\|_{rv}^2 dt dt' \right]^{\frac{1}{2}} < \infty . \quad (11.1)$$

This expression has all the properties of a norm and satisfies also the inequality

$$N_{rv}(\underline{K}_1 \underline{K}_2) \leq N_{rv}(\underline{K}_1) N_{rv}(\underline{K}_2) , \quad (11.2)$$

where we have used the notation

$$[\underline{\mathcal{J}R}_1, \underline{\mathcal{J}R}_2](t', t) = \int \underline{\mathcal{J}R}_1(t', t'') * \underline{\mathcal{J}R}_2(t'', t) dt'' . \quad (11.3)$$

It follows that if

$$N_{rv}(\underline{\mathcal{J}R}) < 1 , \quad (11.4)$$

the series (6.17) converges and we have

$$N_{rv}(\underline{R}) \leq N_{rv}(\underline{\mathcal{J}R}) \left(1 - N_{rv}(\underline{\mathcal{J}R})\right)^{-1} . \quad (11.5)$$

Moreover,  $\underline{R}$  satisfies the integral equation (6.18).

In order to see whether the condition (11.1) is satisfied, we have just to use Eqs. (10.5), (9.14) and (9.15). If the function  $d(t)$  decreases for large  $|t|$  faster than any negative power of  $|t|$ , one can easily show that the integral (11.1) converges if the condition (10.6) is satisfied. The case in which  $d(t)$  decreases as a power of  $|t|$  is treated in Ref. <sup>6)</sup>.

In order to prove Eq. (11.4), we remark that from Eqs. (6.14), (9.12) and (10.2) it follows

$$\|\underline{\mathcal{J}R}(t', t)\|_{n+c, v+c} \leq \exp[-c\chi(t', t)] \|\underline{\mathcal{J}R}(t', t)\|_{nv}, \quad c > 0, \quad (11.6)$$

and from Eq. (11.1)

$$\lim_{c \rightarrow \infty} N_{n+c, v+c}(\underline{\mathcal{J}R}) = 0 , \quad (11.7)$$

if  $r$  and  $v$  satisfy the condition (10.6).

It follows that, if  $v$  is sufficiently large and  $r$  is properly chosen,



the resolvent  $\underline{R}$  defined by Eq. (6.17) has the property

$$N_{rv}(\underline{R}) < \infty \quad (11.8)$$

and satisfies the Eq. (6.18). Moreover, if  $\text{Re } \lambda$  is sufficiently large and  $p, r$  are suitably chosen, the Laplace transform  $\mathcal{B}^{M\lambda}(\underline{R}(t, t'))$  exists and satisfies the "partial wave" equation

$$\begin{aligned} \mathcal{B}^{M\lambda}(\underline{R}(t', t)) &= \mathcal{B}^{M\lambda}(\underline{\mathcal{K}}(t', t)) + \\ &+ \int \mathcal{B}^{M\lambda}(\underline{\mathcal{K}}(t', t'')) \mathcal{B}^{M\lambda}(\underline{R}(t'', t)) dt'' . \end{aligned} \quad (11.9)$$

One can show that  $\mathcal{B}^{M\lambda}(a)$  are entire operator valued functions of  $\lambda$  and that the operators  $\mathcal{B}^{M\lambda}(\underline{\mathcal{K}}(t, t'))$  are analytic in  $\lambda$  in the half plane defined by Eq. (10.7). The operator  $\mathcal{B}^{M\lambda}(\underline{R}(t, t'))$  is analytic in the smaller half plane mentioned above, but, if one has a specific model, one can often use Eq. (11.9) to get its analytic continuation in a larger region and to study the nature of its singularities.

Summarizing, we have shown that very general assumptions are sufficient for a completely satisfactory definition of the multiperipheral integral equation and of its partial wave projection. In this procedure, the four-dimensional complex angular momentum  $\lambda$  and the other Lorentz quantum number  $M$  appear in a natural way.

In order to complete the treatment summarized above, one has to invert the Laplace transform  $\mathcal{B}^{M\lambda}(\underline{R}(t, t'))$  to get the resolvent  $\underline{R}$ , and to derive the explicit expressions of the asymptotic behaviours of the observable quantities in terms of the singularities in the complex  $\lambda$  plane.

A problem which is equivalent to the inversion of the transformation (10.1) has been treated in Ref. <sup>11)</sup>, while the analogous problem for the three-dimensional case is treated in Ref. <sup>19)</sup>. We think, however, that some work is still necessary in order to get a completely clear treatment.

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