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HOW IMPOSING ANALYTICITY ON A $\pi\pi$ PHASE SHIFT ANALYSIS CAN REVEAL NEW SOLUTIONS, EXPLORE EXPERIMENTAL STRUCTURES AND INVESTIGATE THE POSSIBILITY OF NEW RESONANCES

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A B S T R A C T

Experiments on $\pi\pi$ scattering determine only the modulus of the amplitude in the inelastic region. Fixed t and u analyticity in the form of conventional dispersion relations are used to determine the phase of the amplitude in this region. This analysis allows local structures in data to be isolated. We find alternative analytic unitary amplitudes exist equally well describing the CERN-Munich data on the $\pi^+\pi^-$ channel. These amplitudes have quite different physical properties with different daughter resonances, which data on other channels and/or e^+e^- annihilation will be able to distinguish in the very near future.

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1. Introduction

An understanding of strong interaction dynamics requires knowledge of the elements of the S-matrix. These elements alone determine experiment. The construction of these scattering amplitudes from experimental data is however fraught with ambiguity, even for a spinless particle process. That such ambiguities occur has nothing to do with experimental errors and would arise even if we had infinitely accurate data. In other words, while a given element of the S-matrix uniquely defines a set of experimental data, a given set of experimental data can equally well be described by a large number - in general, an infinity - of possible scattering amplitudes. Since each of these will have different physical properties, it is clear we must try to limit the range of possible ambiguities in translating experimental data into elements of the S-matrix if we are ever to reveal the basic properties of the strong interactions from data.

Now the elements of the S-matrix satisfy a trinity of fundamental properties, namely analyticity, crossing and unitarity, which it is clear an arbitrary function describing the data will not satisfy. By requiring that our scattering amplitudes should also satisfy these properties, we may hope to limit the range of possible candidates for elements of the S-matrix, if not to a unique choice, at least to just a few. The study of such a possibility in a practical situation is the purpose of this work.

Of these general properties, unitarity, or at least its single channel content, is the simplest to impose. It is the object of phase shift analysis to construct amplitudes describing the data and to find those solutions which are unitary and to discard the rest. However, and most importantly by beginning his analysis with a basic simplifying assumption, the phase shifter restricts his attention to a particular subclass of all possible scattering amplitudes describing the data and seeks unitary solutions only within this subclass. He obtains unitary amplitudes but what about analyticity?

At this point, the phase shifter makes a further step. He notes that his resulting phase shifts, being directly related to data, are subject to many fluctuations, most of them of a random statistical nature. Knowing that the true amplitude should not have such statistical fluctuations he smooths the phase shifts in some way, either before or after performing his analysis of the data. Such a smoothing procedure has no theoretical basis whatsoever except for a belief in a smooth amplitude. Indeed, such 'ad hoc' smoothing procedures obviously cannot distinguish dynamical structures in the data from random statistical fluctuations - all are smoothed away. Since our aim is to understand the dynamics of strong interactions it is clear we must find a way to remove (hopefully) most of the random statistical fluctuations and yet preserve local dynamical structures, which are the very substance of the dynamic S-matrix.

The phase shifter arrives at a set of solutions which are smooth and unitary; we wish to find those that are analytic and unitary. These are not the same. Indeed, we shall see that the analytic amplitudes do not lie in the restricted subclass of possible scattering amplitudes the phase shifter searches, but in the more general class of all possible amplitudes. However, since the phase shift analyst follows part of the path that we wish to, and how he, or she, proceeds is more familiar, it is convenient for us to base our discussion on the analysis of the phase shifter and only to be more general later.

Since our aim is to study the imposition of the general properties of analyticity and unitarity, not in the abstract, but in a practical situation we will consider a particular scattering process given by a particular set of experimental data. The physical process we study is $\pi\pi$ scattering, spinless and seemingly simple, the data for which is obtained from pion production experiments. The specific data we consider is that produced by the CERN-Munich group on both $\pi^{\mp}p \to \pi^{\mp}\pi^{+}n$ [1,2]. We do not concern ourselves here with any uncertainty in the extrapolation to the pion pole, but assume that these experiments provide us uniquely with data on $\pi^{\mp}\pi^{+} \to \pi^{\mp}\pi^{+}$ scattering. This is despite the fact that a recent polarized target experiment by this same group[3] has revealed a significant amount of non-π exchange, since preliminary analysis of these results indicates there to be little change in the $\pi\pi$ amplitude at least in the limited energy region so far investigated[4]. We take this as suggestive that the results of our extensive treatment of the original CERN-Munich data will survive further refinements. Moreover, in the energy region we study the data from the Omega spectrometer[5] are in good agreement with that of CERN-Munich. Our aim given these data is to construct the amplitude for $\pi\pi$ scattering. We will concentrate on the $\pi^{-}\pi^{+}$ channel.

In section 2 we define the ambiguities which arise in constructing an amplitude for such a channel from experimental data. In section 3 we explicitly form $\pi\pi$ amplitudes describing experiment which are analytic. We shall see that this implementation of analyticity provides us with two basic amplitudes both equally consistent. On partial wave projecting these we find, in section 4, that each produces a phase shift solution which is unitary. These two basic solutions, α and β, equally analytic, unitary and perfectly describing experiment in the $\pi^{-}\pi^{+}$ channel have quite different physical properties.

In these solutions, the leading partial waves are rather similar, with the D and F waves being dominated by the f and g resonances, respectively. Moreover, the G wave is completely consistent with expectations from the tail of the spin four h(2025) resonance, having an elasticity of 20 to 25 percent. It is however the structure of the daughter waves that distinguishes these solutions. Indeed, it is the unravelling of these structures in lower partial waves that is an outstanding question of meson spectroscopy. Solution α has a highly inelastic S wave resonating around 1350 MeV with a fairly structureless P wave showing no evidence for a ρ' coupling to $\pi\pi$ at either 1250 or 1600 MeV at the 2% level. In contrast, solution β has a much more elastic S wave, once again resonating in the f region with the P wave also showing a clearly resonant $\rho'(1600)$ having a 30% coupling to $\pi\pi$. The fact that a strong $\rho'(1600)$ signal exists in e^+e^- annihilation experiments(6) will shortly allow its coupling to $\pi\pi$ to be checked outside of $\pi\pi\to\pi\pi$ experiments.

For the first time evidence is presented for a significant structure in the D wave amplitude around 1550 MeV. In solution α, particularly, this structure is suggestive of a resonance. Since it is rather close to the limit of the energy region explored by the CERN-Munich experiment a conclusive discussion of this structure will require future data in the $\pi^+\pi^-$ and/or other channels. Nonetheless its appearance is unmistakable.

Though amplitudes α,β are those we explore in most detail, there exist other analytic amplitudes which we also investigate. In particular, close to solution β, we find another amplitude β', rather similar to β, but with different physical properties. This we also discuss in section 4.

This analysis has a number of rather unique features, which have not only revealed new phase shift solutions noticeably different from any seen before, but also, within each solution, uncovered structures, which only such an analytic phase shift analysis can provide. We compare our analysis with other related work in section 5. We close with our conclusions (section 6).

2. Defining the amplitude

(A) A question of phases

Data from pion production experiments are expressed in terms of moments of the final state $\pi\pi$ angular distribution. On extrapolating to the pion pole it is the M=0 spherical harmonic that contains all the information on physical mass $\pi\pi$ scattering. From these moments $A_L(s)$, with M=0, we can construct the square of the modulus of the $\pi\pi$ amplitude by

$$|F(s,\cos\theta)|^2 = \sum_L (2L+1) A_L(s)\; P_L(\cos\theta)\,. \tag{1}$$

With s and t the Mandelstam invariants for the four-pion process and θ the usual centre of mass scattering angle, we have $z \equiv \cos\theta = 1+2t/(s-4\mu^2)$, μ being the pion mass. The dipion mass $M_{\pi\pi}$ for the pion production process is then just $\sqrt{s}$.

From the CERN-Munich data[1] on the channel $\pi^- p \to \pi^- \pi^+ n$, we are given moments* on $\pi^-\pi^+ \to \pi^-\pi^+$ scattering in 20 MeV bins from 0.6 to 1.8 GeV. The dominant feature of the moments in this channel, to be seen particularly in the L=0 moment (which is essentially a reaction cross-section), are three peaks identified with the ρ, f and g resonances. That these have the expected spins, one, two and three, and form the states on the leading Regge trajectory is easily checked by looking at the higher $L \neq 0$ moments. Moreover their spin four recurrence, the h, is already known to couple to $\pi\pi$ from preliminary $\pi^o\pi^o$ data at higher momentum at 2.035 GeV[10]. The leading trajectory shows no surprise.

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*Since the CERN-Munich group[1] have not published moments extrapolated to the pion pole, we are forced to reconstruct these using the energy independent phase shift analyses[7,8,9] of their data. In this way we take account of, and hopefully remove, the non-π exchange effects in the observed moments.

However, it is what is happening at the daughter level on the Regge trajectories that is an outstanding question of meson spectroscopy. To understand this is much more complicated. It requires a detailed analysis of the moments; in fact, the construction of the $\pi\pi$ amplitudes themselves. To this end it is our aim, given the data, to determine the possible amplitudes for $\pi^+\pi^-\to\pi^+\pi^-$ scattering.

Now the experimental data on the moments, $A_L(s)$, only allow us to fix the modulus of the amplitude, eq. (1), and so the amplitude itself is given by

$$F(s,z) = |F(s,z)|\ e^{i\phi(s,z)}, \qquad (2)$$

where the phase ϕ as a function of both s and $z=\cos\theta$ is completely undetermined by experiment. We have an infinity of possible amplitudes corresponding to the infinite number of choices of ϕ, all describing exactly the same data. Each of these amplitudes will have a different daughter structure, though very similar ρ, f and g resonances - in terms of masses, widths and elasticities. It is clear that we must try to resolve these ambiguities if we are ever to have a hope of determining the daughter spectrum.

The situation is not quite so bad at all energies, since in the elastic region (as we shall discuss) unitarity uniquely defines the phase of the amplitude from its modulus, aside from some rather specific and perhaps unlikely ambiguities. Nonetheless, above the first inelastic threshold we have the complete continuum ambiguity of eq. (2) with $\phi(s,z)$ quite unknown.

Now it is found experimentally that in the $\pi\pi$ mass range $\sqrt{s}<1.8$ GeV of the CERN-Munich data[1)], those moments with $L>6$ are vanishingly small within errors. We shall take it as an exact statement of experimental fact that the moments with $L>6$ are zero. It is, at this point, that the phase shift analyst, whose objective is to construct scattering amplitudes describing the data too, makes his crucial, though quite unjustified, assumption. He assumes that since, at each energy, the modulus squared of the amplitude is a polynomial in $\cos\theta$ of order $L_{max}=6$, the amplitude itself must by a polynomial of order $\frac{1}{2}L_{max}=3$ and so have partial waves only up to $\ell_{max}=3$. This assumption greatly restricts the class of amplitudes studied since it is only for very specific choices of the phase $\phi(s,z)$ in eq. (2) that the amplitude will have partial waves up to $\ell_{max}=3$ and not an infinite number. The phase shifter finds just two ambiguous amplitudes to be contrasted with the infinite number produced by the continuum ambiguity, eq. (2).

With this in mind, let us see how we can translate the experimental information embodied in eq. (1) into something about the scattering amplitude to use more readily in eq. (2). It is clear that we want to factorize the right hand side of eq. (1), like the left hand side, into the product of a function and its complex conjugate. The simplest and physically intuitive way to proceed is to re-express the Legendre sum as a product of its zeros, as suggested by Gersten and by Barrelet[11]. Since for s real (strictly $s+i\varepsilon$) the sum is a polynomial of order six in $\cos\theta$, it will have six complex zeros. As it is a real function only three of these will be independent and the other three their complex conjugates. We then have with N=3

$$|F(s,z)|^2 = f(s) \prod_{i=1}^{N} (z-z_i(s))\ (z-z_i^*(s)). \qquad (3)$$

It is convenient to replace the proportionality constant f(s) by the modulus of the forward amplitude and write

$$|F(s,z)|^2 = |F(s,z=1)|^2 \prod_{i=1}^{N} \frac{(z-z_i(s))(z-z_i^*(s))}{(1-z_i(s))(1-z_i^*(s))} \qquad (4)$$

The truncation of the expansion of eq. (1) at L=6 means the modulus squared of the amplitude has at most three (complex conjugate) pairs of zeros near the physical region (i.e. with Imz small and $|\mathrm{Re}z|<1$) at the energies reached in the CERN-Munich experiment, so N=3. The remaining possibly infinite number of zeros are far away from the physical region in phenomenologically inaccessible domains. They effectively have their products $(z-z_i(s))(z-z_i^*(s))$ constant at each value of $z\varepsilon[-1,1]$ and hence only contribute to f(s), eq. (3). Eq. (4) is just a convenient representation of the data, where at each energy, the seven non-zero experimental quantities - the moments from L=0 to 6 - are translated into seven other quantities: $|F(s,z=1)|$ and the three complex zeros $z_i(s)$. Having re-expressed the data of eq. (1) in the form of eq. (4) so that we have the amplitude multiplied by its complex conjugate equal to a product of a function and its complex conjugate, we can factorize this expression into an equation for the amplitude. Then implementing the phase shifter's assumption of $\ell_{max}=3$ we have

$$F(s,z) = F(s,z=1) \prod_{i=1}^{N=3} \frac{z-z_i(s)}{1-z_i(s)} \tag{5}$$

which certainly satisfies eq. (4) and has partial waves up to $\ell_{max}=3$. However, this representation immediately exposes an ambiguity in constructing the amplitude. For it is clear in factorising eq. (4) that we do not know whether the zero at $z=z_i(s)$ belongs to F(s,z) or $F^*(s,z)$. Thus for each zero, while Re $z_i(s)$ and $|\mathrm{Im}z_i(s)|$ are determined, the sign of $\mathrm{Im}z_i$ is not. We cannot tell $z_i(s)$ from $z_i^*(s)$.

Now in principle this 2^N-fold ambiguity exists at every value of s, but since we believe physical quantities to be continuous in energy, this number is considerably reduced. The reason for this is that continuity only allows a doubling of the number of solutions at well-defined energies. When a zero enters the physical region - for $\pi^+\pi^-\to\pi^+\pi^-$ scattering this entrance is made through the forward direction only - its imaginary part is in general known. Only when its imaginary part can vanish (within experimental errors) is there any uncertainty as to whether $\mathrm{Im}z_i$ has changed sign or not. In the case of $\pi^+\pi^-\to\pi^+\pi^-$ scattering $\mathrm{Im}z_i(s)$ become vanishingly small twice for i=1 and once for i=2 in the energy range up to 1.8 GeV. So there is just an 8-fold ambiguity in this whole energy region. Indeed we first have a 2-fold ambiguity up to where $\mathrm{Im}z_1 \simeq 0$ (i.e. at $\sqrt{s}$ = 1.25 GeV), this produces a four-fold ambiguity until $\mathrm{Im}z_2 \simeq 0$ (at $\sqrt{s}$=1.45 GeV) when a further doubling occurs. So at this stage we have an eightfold multiplicity of possible scattering amplitudes describing one set of experimental data. Moreover, for each of these, their overall forward phase $\phi_o(s)$ is unknown since in eq. (5) we do not really know the forward amplitude, only its modulus. The phase shifter's amplitude is therefore

$$F_{PS}(s,z) = |F(s,z=1)|e^{i\phi_o(s)} \prod_{i=1}^{3} \frac{z-z_i(s)}{1-z_i(s)} \qquad (6)$$

with eight possible products of zeros each with $\phi_o(s)$ unknown. The situation would not normally be quite so bad since the phase $\phi_o(s)$ would usually be determined through the optical theorem from a measurement of the total cross-section. However, for $\pi\pi$ scattering no reliable measurements of σ_{tot} exist in the energy range below 2 GeV and so $\phi_o(s)$ is experimentally unknown. All we know is that $\phi_o \varepsilon [0,\pi]$ since the total cross-section must be positive. Thus each discrete solution has a continuous ambiguity as to its overall phase.

But where is the complete continuum ambiguity of eq. (2)? Eq. (4) is just a representation of the data, of the moments $A_L(s)$. The factorisation of this equation is only eq. (6) if we require $\ell_{max}=3$; in general, the correct factorisation in this equation of $|F(s,z=1)|^2$ is $|F(s,z=1)|$ times an arbitrary phase factor which is not just a function of s but also of $\cos\theta$. Once again conveniently separating out the forward phase we have

$$F(s,z) = |F(s,z=1)| \, e^{i\phi_o(s)} \, e^{i\chi(s,z)} \prod_{i=1}^{3} \frac{z-z_i(s)}{1-z_i(s)} \qquad (7)$$

where $\chi(s,z=1)=0$ defining $\phi_o(s)$ to still be the forward phase. This amplitude thus has an experimentally unknown phase not just at every energy but at every value of $\cos\theta$ too. Eq. (7) is in fact just eq. (2) in which our knowledge of the seven non-zero experimental moments has been re-expressed. Unlike the phase shifter's restricted amplitude of eq. (6), the true amplitude of eq. (7) has in general an infinite number of partial waves. This, of course, is perfectly allowed, since all we know is that $|F|^2$ has effective moments up to L=2N, which does not imply F must have all waves with $\ell>N$ vanishingly small, but only exponentially decreasing with ℓ so that the Legendre expansion converges in appropriate domains[12).]

As already argued, the phase shifter has a discrete number of products of zeros he can write down to construct his amplitudes in eq. (6). However, the true continuum ambiguity allows us to include all these simultaneously. This is because we can transform eq. (7) from one discrete amplitude of the phase shifter's to another by replacing $\chi(s,z)=0$ by

$$e^{i\chi(s,z)} = \frac{z-z_i^*(s)}{z-z_i(s)} \; \frac{1-z_i(s)}{1-z_i^*(s)} \, , \text{ for } i=1,2,3. \qquad (8)$$

This phase factor (it is only a phase for z real) takes us from one discrete solution with a zero at $z=z_i(s)$ to one with a zero at $z=z_i^*(s)$ both with $\ell_{max}=3$.

It is perhaps appropriate at this point to make a cautionary remark on the interpretation of these zeros. We have rather naturally called the complex parameters $z_i(s)$, obtained from real data on the real z axis from the polynomial of eq. (1), which appear in eqs. (6,7), the zeros of the amplitude. However, strictly speaking, the identification of the parameters $z_i(s)$, thus determined, with the positions of the complex zeros of the amplitude as a function of z complex, requires an analytic continuation away from the real z axis to be performed - a continuation known to be quite unstable except in rather special circumstances. This is of no concern here. The z_i, z_i^* are just parameters conveniently representing the data, which we colloquially call "zeros".*

The continuum phase $\chi(s,z)$ being non-zero will be seen (in section 3) to be crucial for constructing analytic amplitudes. However, $\chi(s,z)\neq 0$ does take us away from one of the phase-shifter's amplitudes with $\ell_{max}=3$. It is therefore not unnatural to believe that the larger the phase $\chi(s,z)$ the larger the higher partial waves with $\ell>3$. That this intuition may be quite misleading is provided by the example of eq. (8), which takes us from one amplitude with $\ell_{max}=3$ to another with $\ell_{max}=3$, despite the fact that the phase variation between forward and backward directions given by eq. (8) can be very large indeed, for example with i=1 χ varies from 0^o to 300^o in crossing the physical region from forward to backward direction at a typical energy of 1.5 GeV (see fig. 1 of ref. 13). Nonetheless, $\chi(s,z)$ is a measure of how far an amplitude is from a particular phase shift solution, of eqs. (6,7).

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*We are grateful to Dr. S. Ciulli for a discussion which prompted these remarks.

Despite the fact that the phase shifter's solutions of eq. (6) form a highly restricted subclass of the amplitudes of eq. (7) it is useful to follow the phase shifter one step further and check which of his eight possible amplitudes are unitary.

(B) Unitarity introduced, partial waves defined

Since unitarity is most simply expressed in terms of partial wave amplitudes, let us first deal with the partial wave projection of our $\pi^+\pi^-$ amplitudes. While the odd waves contribute only to isospin one as a result of Bose symmetry, the even partial waves of this amplitude involve both isospin zero and two:

$$F(s,t) = \frac{1}{3} F^o(s,t) + \frac{1}{2} F^1(s,t) + \frac{1}{6} F^2(s,t). \qquad (9)$$

In order to separate the I=0,2 amplitudes* we introduce the $\pi^+\pi^+$ data; which in principle determines the I=2 partial waves. We use the data of Hoogland et al.[2] and of Durusoy et al.[14] up to 1.8 GeV. We then define the partial waves with definite isospin, $f^I_\ell(s)$, by

$$F^I(s,z) = \sum_{\ell=0} (2\ell+1)\, f^I_\ell(s)\, P_\ell(z) \qquad (10)$$

where $f^I_\ell(s)=0$ if $(-1)^{I+\ell}=-1$. For the phase shifter's amplitudes of eq. (6) (that is $\chi=0$), this is in fact just a re-expression of the product of zeros of eq. (6). The f^I_ℓ's are then determines by the zeros $z_i(s)$ which have been fixed by the moments $A_L(s)$.

This separation into partial waves completed, we can easily impose the single-channel content of the unitary nature of the S-matrix by requiring

$$\mathrm{Im} f^I_\ell(s) \geq \sqrt{\frac{s-4\mu^2}{s}}\ \ |f^I_\ell(s)|^2 \qquad (11)$$

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*This separation is necessary as unitarity is diagonal in the partial wave amplitudes with definite isospin.

The most important fact about this relation is that it is an equality in the region of elastic unitarity. Rigorously, this region runs from 2π threshold to 4π threshold, but phenomenologically it extends much further. There is no evidence for any inelasticity below $K\bar{K}$ threshold, close to 1 GeV, so that this energy rather than $\sqrt{s}=4\mu$ marks the limit of the elastic region. In this domain, unitarity completely defines the amplitude knowing its modulus: there are no discrete ambiguities, no continuum ambiguity. The signs of $\mathrm{Im}z_i$ are determined, the forward phase known. The amplitude is unique.*

In the inelastic region, the single channel content of unitarity is much weaker being an inequality, eq. (11). It is in this region that we have the full continuum ambiguity to resolve. It is useful then to express each partial wave in terms of a phase shift $\delta_\ell^I(s)$ and an inelasticity $\eta_\ell^I(s)$ to simplify the unitarity condition, eq. (11), where

$$f_\ell^I(s) = \sqrt{\frac{s}{s-4\mu^2}} \; \frac{(\eta_\ell^I \, e^{2i\delta_\ell^I}-1)}{2i} \; . \tag{12}$$

Then unitarity simply requires $\eta_\ell^I=1$ in the elastic region and $0\leq\eta_\ell^I\leq1$ above the first inelastic threshold. By plotting the partial waves (actually $\sqrt{\frac{s-4\mu^2}{s}}\, f_\ell^I$) in their argand circles it is easy to check whether unitarity is satisfied.

In the elastic region the partial waves are guaranteed to be on their argand circles. This fixes the sign of the imaginary part of the only nearby zero in this region and so reduces the number of phase shifter's possible amplitudes. We have just one amplitude below 1.2 GeV; there

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*Aside from rather specific Crichton-type ambiguities, for which see the admirable work of ref. 15.

Imz_1 may change sign or perhaps not giving solutions A, B respectively. The second bifurcation occurs around 1.45 GeV where Imz_2 may change sign; if it does solutions A, B become C, D. However, these latter solutions appear to be in disagreement[16)] with preliminary $\pi^o\pi^o$ data from Serpukhov[10)] and so we will not consider them further. The phase shifter[9)] then has just two possible amplitudes A and B corresponding to whether we have z_1 or z_1^* in the product of eq. (6) for $\sqrt{s}>1.2$ GeV. But are both these amplitudes unitary?

In the inelastic region the phase $\phi_o(s)$ is not determined by single-channel unitarity and so the orientation of the partial waves (though not their lengths) in the argand plane is free. In order to check whether the partial waves lie within their appropriate unitarity circles in the inelastic region, the phase shifter must fix the phase $\phi_o(s)$ by some (hopefully) physically reasonable guess. The peaks seen in the zeroth moment $A_o(s)$ identified with the f and g are assumed to occur too in the imaginary part of the amplitude, as one would expect for resonances. Away from these positions, $\phi_o(s)$ is described by assuming the leading waves to be given by Breit-Wigner forms, i.e. defining $\phi_o(s) = \phi_{BW}(s)$. Though, of course, not exact, this is certainly a reasonable, physically motivated assumption. We too will take this as our forward phase at least at the beginning when we start to construct analytic amplitudes.

With this choice of $\phi_o(s)$ and with $\ell_{max}=3$, the phase shift analysts can check whether unitarity forbids one solution or other. However, both solutions A and B have partial waves lying within or close to their unitarity limits though the S wave in solution B does wander far out of its circle. But with such a crude choice of phase, we cannot be more definite.

However, the phase shifter cannot fail to notice that his phase shifts wiggle all over the place as a function of energy particularly for the S and P waves. This is because the moments being experimental quantities are subject to fluctuations mainly of a random statistical nature. These random fluctuations are passed on in turn to the zeros via eq. (3) and then to the phase shifts by translation from eq. (5) to eq. (10). Believing the scattering amplitude not to behave in a random fashion, the phase shifter finds this amplitude unacceptable and so he smooths his results either directly, or by returning to smooth the data from which he begins, or as done in ref. 9 to smooth the intermediate zeros $z_i(s)$ of eqs. (3-6). This particular smoothing of ref. 9 is clearly highly dangerous since several types of local dynamical structures are known to produce rapid variations of the zero trajectories, for example the S* effect. But other 'ad hoc' smoothing methods are little better. Nonetheless, this smoothing highlights the violation of unitarity of the S wave for solution B - for now this wave has $\eta \simeq 1.3$ for several hundred MeV around 1.4 GeV. Nevertheless, this solution cannot be ruled out since this wave has the largest uncertainty, the overall phase is only guessed, the partial wave sum artificially truncated at $\ell=3$ and the amplitude arbitrarily smoothed: the violation of unitarity is then not particularly significant.

It is important to note that we have only considered unitarity for the phase shifter's amplitude of eq. (6) with no continuum ambiguity, that is $\chi=0$. For the true amplitude with the complete continuum ambiguity we cannot even expand the amplitude in partial waves without knowing $\chi(s,z)$. The partial wave vectors then not only have an unknown overall orientation but unknown relative orientation and undetermined lengths too. That is why the phase shifter assumes $\chi=0$.

We have seen that to make progress in constructing unitary $\pi\pi$ amplitudes, the phase shifter has been more or less forced into making three critical assumptions: (i) that the maximum number of partial waves is half the number of non-zero moments, (ii) to guess the forward phase in the inelastic region, (iii) to smooth the resulting amplitude in some arbitrary fashion that cannot distinguish local dynamics from random fluctuations. All these problems we hope to resolve by requiring that the scattering amplitude describing the data should be an element of the analytic S-matrix.

3. Analyticity

(A) Its Implementation

In order to impose analyticity on our amplitudes describing the $\pi^+\pi^-$ data it is essential that we allow the full continuum ambiguity. As we shall see, the restricted class of amplitudes studied by the phase shifter, whether smoothed or not, are not analytic.

Our aim is to use analyticity to determine the experimentally unknown phase $\phi(s,z)$ of the amplitude, eq. (2). We will base our analysis on the representation of eq. (7) since this embodies our experimental knowledge of the moments eq. (1).

At values of t,u fixed in the range $-28\mu^2 \leq t,u \leq 4\mu^2$ it has been rigorously proved that the $\pi\pi$ amplitude is analytic in the cut s plane and satisfies twice subtracted dispersion relations[12]. It is these properties that we wish to use.

So as to make our treatment as straightforward as possible, we will use a simple Cauchy representation for the scattering amplitude at fixed values of t,u to impose the required cut s-plane analyticity. How can this be expected to determine the phase of the amplitude? Dispersion relations relate the real part of the amplitude to its imaginary part, and hence the phase to the modulus. However, they do this in a very specific global manner. They determine the real part of the amplitude at all values of the energy (at a fixed momentum transfer) provided one knows the imaginary part of the amplitude at all energies. They thus determine the phase if we know the modulus everywhere. Of course, we do not know the modulus everywhere; from experiment it is only known up to 1.8 GeV. Above this we must make some assumption. We will assume Regge behaviour in a sense to be defined below. The global manner in which analyticity relates the phase to the modulus is to be contrasted with the local way elastic unitarity determines the amplitude from its modulus. In the elastic region, eq. (11) is an equality at each energy separately for each partial wave. The fact that we know the elastic amplitude locally at each energy in the elastic region will be crucial in allowing us to use analyticity to determine the phases in the inelastic region.

To see how our method works let us consider a simple illustration. Let us assume that the amplitude only has a right hand cut and write a Cauchy representation for this amplitude in the cut s-plane. We choose the contour of integration to envelope the cut up to some finite but large energy where $s=s_o \pm i\varepsilon$ and then to close the contour by the almost complete circle $|s|=s_o$. So we have at fixed t (or equally fixed u)

$$F(s,t) = \frac{1}{\pi} \oint ds' \, \frac{F(s',t)}{s'-s} \quad . \qquad (13)$$

The contributions to this contour integral are divided into the integral along each side of the cut and that around the circle. We assume that $s=s_o$ is so large that Regge behaviour has set in for the amplitude in all complex directions. Then having only a right hand cut

$$F(s,t) = \frac{1}{\pi} \int_{4\mu^2}^{s_o} ds' \, \frac{\mathrm{Im}F(s',t)}{s'-s} + \frac{1}{2\pi i} \int_{|s'|=s_o} ds' \, \frac{F(s',t)}{s'-s} \qquad (14)$$

This equation determines the real part of the amplitude along the cut knowing its imaginary part, since it is an identity for the imaginary part. Let us now divide up the integration region along the cut into 3 parts:

(a) In the region of elastic unitarity from $\pi\pi$ threshold to $K\bar{K}$ threshold, both the real and imaginary parts of the amplitude are known. Strictly this is not true since the data on $\pi^+\pi^-$ scattering only goes down to 0.6 GeV and not to threshold, 2μ=0.28 GeV. However, we can reliably determine the amplitude in this region by combining other dispersive equations, namely the Roy equations[17)], with K_{e4} results[18)] and the data in the rest of the elastic region and so fix the amplitude down to threshold.

(b) The so-called inelastic region runs from $K\bar{K}$ threshold to where the data ends at 1.8 GeV. Here we only know $|F|$ at each 20 MeV data point. This is where eq. (14) is to be used to determine the unknown phase.

(c) In the region from where the data ends to where Regge behaviour sets in for the whole amplitude at $s=s_o$, we expect, from the success of finite energy sum rule duality in other processes, that on the average Regge behaviour has set in for the imaginary part. The real part is then whatever it is required to be by analyticity via eq. (14). By Regge behaviour we mean $\mathrm{Im}F(s,t) = \sum_i \beta_i(t)(\alpha' s)^{\alpha_i(t)}$, where we know which trajectories $\alpha_i(t)$ to include, but for which their residues have simple forms with parameters to be fitted. For example, for near forward $\pi^+\pi^-$ scattering, Regge behaviour is assumed given by pomeron, f and ρ exchanges with free residues to be fixed by the dispersion relations from the data, but with trajectories known within limits.

Armed with this information we now set about determining the phase of the amplitude in the inelastic region (1-1.8 GeV). Clearly, we need an imaginary part for the amplitude in this region to put into the dispersion relation. We guess this by assuming we know the correct phase in the inelastic region - let us call this $\psi_o(s,t)$. These are numbers, not a function, guessed at each chosen fixed value of t or u at each 20 MeV data point from 1 to 1.8 GeV. Then $\mathrm{Im}F=|F|\sin\psi_o$ in this region is to be substituted into the dispersion relation.

With this input, we first fix the few (n) Regge residue parameters by requiring that at some n points in the elastic region the output of the dispersion relation agrees with the known real part of the elastic amplitude. This fixes the Regge parameters. Then if the phases $\psi_o(s,t)$ are the true phases of an analytic amplitude two criteria must be satisfied:

(i) at other energies in the elastic region the output real part of eq. (14) should equal the known real part of the elastic amplitude with the (now) given Regge residues. This agreement checks that the global properties of the phases $\psi_o(s,t)$ above 1 GeV are correct.

(ii) At each 20 MeV data point in the inelastic region the output real part should reproduce the expected phase by being equal to $|F|\cos\psi_o$. This checks that the phases are locally self-consistent.

If both these criteria are satisfied the phases, $\psi_o(s,t)$, are the phases in the inelastic region of an analytic amplitude.

Of course, we cannot guess the phases $\psi_o(s,t)$ exactly, but let us assume we can at least approximately. Then the output in the elastic region will only roughly reproduce the known elastic amplitude and the output in the inelastic region will not replicate the input, but let us call this output* $|F| \cos \psi_1$. We assume, and it is just an assumption at this stage,

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*This requires that at any point where $|\mathrm{Re}F_{out}|>|F|$ we set $\psi_1=0$ or π as appropriate.

that the phases $\psi_1(s,t)$, so defined at each data point in the inelastic region, are closer to the true analytic phases than the ψ_o were. We then repeat the procedure, inputing the phases ψ_1 as our guess instead of ψ_o. We redetermine the Regge parameters, check how well the output real parts in the elastic and inelastic regions reproduce what they should be and define the new output real parts above 1 GeV to be $|F| \cos\psi_2$. Hopefully the ψ_2 are closer to the ψ_1 than ψ_1 and ψ_o were and the elastic amplitude better replicated. The cycle is continued until $\psi_k \simeq \psi_{k-1}$ at each data point and at each momentum transfer. We then have a self-consistent amplitude in the inelastic region. If it is to converge this will in practice take only 5 to 8 iterations. Then these self-consistent phases $\psi(s,t)$ will be the true phases of an analytic amplitude provided (criterion (ii) is already satisfied) the output of the dispersion relations in the elastic region does agree with the known elastic amplitude (criterion (i))*.

It is important to note that this iterative procedure is designed to preserve the experimentally known modulus in the inelastic region with its error bars at every step. As outlined, such a procedure will only converge if we have a good starting guess for the phase $\psi_o(s,t)$. However, the phase shifter feels he knows the allowed amplitudes, eq. (6), and so provides us with suitable starting points.

In practice the procedure is a little more complicated because the amplitude has both right hand and left hand cuts. This important fact takes

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*It is, of course, quite possible for the amplitude in the inelastic region to be self-reproducing without eq. (14) giving the elastic amplitude satisfactorily. This is not an analytic amplitude.

into account some of the requirements of crossing symmetry, namely the fact that both the s and t channels are $\pi^+\pi^-\to\pi^+\pi^-$. Then the input in the integration regions (a), (b), (c) of our simple example above are divided into domains (I,II), (III,IV), (V,VI), respectively, detailed below and displayed in fig. 1.

I) From threshold up to $M_{\pi\pi}$=0.6 GeV we use the solution of the Roy equations for the S, P, D, F and G waves[17] having an I=0 S wave scattering length a_0^0=0.3 (in pion mass units), since this matches on with the CERN-Munich data most easily and agrees with recent K_{e4} data of the Geneva-Saclay group[18]. We will discuss below why, in this threshold region, we include up to ℓ=4.

II,III) For 0.6<$M_{\pi\pi}$<1.8 GeV in the s and t channels we have $|F(s,t)|$ for $\pi^+\pi^-$ scattering constructed from the moments of refs. 1, 7, 9 by eq. (1) in 20 MeV bins. This region is divided in two (II,III):

II) In the elastic region below the $K\bar{K}$ threshold the amplitude is formed from the S, P, D and F wave phase shifts fixed by unitarity.[9]

III) Above this inelastic threshold only $|F|$ is known and it is here the representation of eq. (7) is conveniently used, with $\phi_0(s)$ and $\chi(s,z)$ to be determined.

IV) In the u-channel from 0.6-1.3 GeV we input the $\pi^+\pi^+$ data of ref. 2, and beyond that up to 1.8 GeV either a smooth interpolation to the Regge form is used or the data of ref. 14. Since the $\pi^+\pi^+$ amplitude only contributes to the fixed t dispersion relation and then only to the left hand cut, despite its large uncertainty, it thus gives only a very small contribution to the dispersive integral for $\sqrt{s}$>0.6 GeV.

V) For $M_{\pi\pi}$>1.8 GeV we assume that simple Regge pole behaviour has set in "on the average", as finite energy sum rule duality for other processes would suggest, for the imaginary part of the amplitude. Then at some

$M_{\pi\pi}$ finite, but as large as necessary, this same Regge pole behaviour is assumed to hold for both the real and imaginary parts in all complex directions (away from the cuts) and there the contour of integration is closed in both the upper and lower half planes by semicircles.

At fixed t, in region V, the Regge contribution is given by pomeron, f and ρ exchanges:

$$F(\nu,t) = \frac{\sigma^{\pi\pi}(\infty)e^{bt}}{32\pi\alpha'} \xi_+^P (\alpha'\nu)^{\alpha_P(t)} + \frac{1}{3}\beta_f \frac{\alpha_f(t)}{\alpha_f(0)} \xi_+^f (\alpha'\nu)^{\alpha_f(t)} + \frac{1}{2}\beta_\rho \frac{\alpha_\rho(t)}{\alpha_\rho(0)} \xi_-^\rho (\alpha'\nu)^{\alpha_\rho(t)} \quad (15)$$

with $\nu=(s-u)/2$, $\alpha'=0.9$ GeV^{-2}, and where the standard trajectories $\alpha_P(t) = 1 + 0.2t$, $\alpha_{\rho,f}(t) = \alpha_o + \alpha' t$ with t in GeV^2. We take $\alpha_o=\frac{1}{2}$ throughout the present discussion, but will comment on other choices later. The signature factors are normalized so that Imξ=1, with $\xi_+^R = i-\cot\frac{\pi}{2}\alpha_R(t)$ and $\xi_-^R = i+\tan\frac{\pi}{2}\alpha_R(t)$. The slope b is chosen in the range (1.5, 4.5) GeV^{-2}. The asymptotic cross section, $\sigma^{\pi\pi}(\infty)$, and β_f are free parameters to be determined by the fit. The ρ exchange having odd signature contributes to ReF much less than the even-signatured pomeron and f exchanges for s below (1.8 GeV)^2. Therefore, rather than let it go free we first of all fix its residue $\beta_\rho=0.75$ (with $\alpha_o=\frac{1}{2}$) to agree with duality and FESR expectations*, and only later consider other values. The factor $\alpha_\rho(t)/\alpha_\rho(0)$ in eq. (15) approximates the desired t dependence of the ρ Regge residue* in the near forward

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*see, for example, ref. 19, and references therein.

direction we consider. It should be remembered that between the end of the data and $\nu=\nu_o$ (some large value) just the imaginary part of eq. (15) is assumed to hold - the corresponding real part is just given by eq. (13).

VI) The high energy contribution for the fixed u dispersion relation is calculated using a Regge paramerisation for some effective "exotic" exchange,

$$F(\nu,u) = \gamma(1+\gamma' u)\xi_+^x \, (\alpha'\nu)^{\alpha_x(u)} \tag{16}$$

with $\nu = (s-t)/2$. We choose the exotic trajectory to have its intercept in the range (-1.5, -0.5) and canonical slope, α'. The residue parameters γ and γ' are to be determined by the fit. Our results depend only weakly on the exact choice of the exotic trajectory, though of course the fitted values of γ and γ' are strongly α_x dependent. Once again it is only the imaginary part of eq. (16) that is assumed to hold for $\nu<\nu_o$. Typically ν_o can be regarded as 20 GeV^2. The real part is given by eq. (13).

The regions in which the various contributions occur are sketched in Fig. 1. For some momentum transfers under heading I, the amplitude must be known in regions which are unphysical. Care must be taken in constructing the amplitude in such a region. The absorptive part can be formed from its partial wave series for $-28\mu^2<t<4\mu^2$, though an increasing number of waves contribute as $|t|$ increases. However, the real part, with which we check the output of the Cauchy representation, can only be evaluated from its partial waves outside the physical region for $|t|$ or $|u|<4\mu^2$. We avoid this problem by only evaluating the dispersion relation for $s\gtrsim -u$ for fixed t and $s\gtrsim -t$ for fixed u. Since the CERN-Munich data[1)] only reliably begins at 0.6 GeV, we avoid such unphysical regions by restricting t, $u\gtrsim -0.3$ GeV^2.

By keeping to the range $-15\mu^2<t,u<0$, rather than the full interval $(-28\mu^2, 4\mu^2)$ only waves up to $\ell=4$ will be needed to evaluate the absorptive part in region I.

(B) Analyticity Imposed

Having discussed the input we are going to use in our dispersion relations, we now proceed to the determination of the unknown phases $\phi_o(s)$ and $\chi(s,z)$ in our representation of the amplitude of eq. (7). Our aim will be to fix these phases, so as to construct analytic amplitudes explicitly, using the iterative method we outlined above for a simple example. Before actually performing this construction let us briefly answer the question of whether the phase shifter's rather arbitrary, highly restricted amplitudes A, B are by chance analytic in addition to being smooth and roughly unitary.

To answer this we proceed no further than the first step in our method as outlined above since no iterations are necessary. The phase shifter's amplitudes are just those of eq. (6) with $\phi_o(s)$ guessed to be $\phi_{BW}(s)$ defined in ref. 9 and discussed informally above. Now if the solutions A and B, as thus defined, were analytic they would simply satisfy the Cauchy representation of eq. (13). We would just input the imaginary part of each amplitude along the cuts and out would come the correct real parts to satisfy our criteria (i) and (ii) of section 3A with no iterations. This is not the case, so let us use forward dispersion relations to determine the forward phase for each discrete solution as an improvement on the educated guess $\phi_{BW}(s)$. The only, though nonetheless appreciable, difference between the two discrete solutions A and B is whether Im $z_1(s)$ has changed sign or not around 1250 MeV. Since this possible sign change occurs when the zero is near the backward direction, it is not surprising that forward dispersion relations, eq. (13), determine the forward phase $\phi_o(s)$ to be very similar for the two solutions. As we begin here with the smoothed amplitudes of Estabrooks and Martin[9] (discussed in the previous section) we do not need to associate a phase $\psi_o(s,t=o)$ independently at every data point in the inelastic region, but we can simply express $\phi_o(s)$ for $1<\sqrt{s}<1.8$ GeV in terms of a 3 or 4 parameter form added to the first guess $\phi_{BW}(s)$. Its determination is then quite straightforward.

While solutions A and B then have very similar forward amplitudes, they are quite different near the backward direction, close to where $z \simeq \text{Re}\ z_1(s)$, and so we can expect that fixed-u dispersion relations may prefer one rather than another or perhaps neither of these solutions. In ref. 13 we showed that having determined the forward phase from fixed-t dispersion relations, fixed-u dispersion relations overwhelmingly favoured solution B to solution A, the latter showing a strong disagreement with analyticity around 1300 to 1400 MeV in particular. The phase shifter's amplitude B is thus rather close to an analytic amplitude and A not. If ℓ_{max} was really 3 and not infinite, we would conclude, like the phase shifter, that amplitude B is the correct amplitude being smooth, roughly unitary and approximately analytic and A disfavoured.

Of course, $\chi(s,z)\neq 0$ and the amplitude is not arbitrarily smoothed, nonetheless we begin our construction of analytic amplitudes expecting there will be an analytic amplitude with a phase structure similar to that of solution B (viz. Im $z_1>0$) having a small continuum phase $\chi(s,z)$. Moreover, we do not expect to find an analytic amplitude with $\chi(s,z)$ small close to the phase shifter's amplitude A. Perhaps the only analytic amplitude will be that close to B, or perhaps some other analytic amplitudes exist describing the very same data. We shall see.

We will use the general iterative procedure we have already outlined. So as not to artificially smooth our amplitudes we will associate, as discussed above, a phase $\psi_o(s,t)$ with each data point and at each momentum transfer independently as numbers not functions. It is clear our method will certainly converge if we have a suitable starting point close to an analytic amplitude - we shall explore how close "close to" is below: it is in fact quite far. Since the phase shift analyst provides us with one amplitude (at least) which we believe is close to an analytic one, let us begin there. Our method will certainly converge if the difference between the phase of the true analytic amplitude and that of starting amplitude, $[\psi(s,z)-\psi_o(s,z)]$, is small. It is therefore convenient to define, as our iteration parameter, the phase difference between the result of the k^{th} iteration and our starting amplitude (initially a phase shift solution) to be

$$\Delta\psi_k(s,z) \equiv \psi_k(s,z)-\psi_o(s,z) = [\phi_o(s)-\phi_{BW}(s)] + \chi(s,z) \qquad (17)$$

so that on convergence $\Delta\psi_k = \Delta\psi_{k-1} = \Delta\psi$ when $\psi_k = \psi$, the true analytic phase. Rather than leave this phase difference $\Delta\psi(s,z)$ to be determined at all 40 data points from 1 to 1.8 GeV at once, we divide this range into N intervals where to remain tractable N is between 8 and 24. In each of these intervals at its central energy $s=s_i (i=1,\ldots,N)$ - always chosen to be a data point - we associate the phase angle $\psi_o(s_i,t)$ at each fixed value of t or u. We overlap these bins so as to be able to obtain this phase every 20 MeV at every data point after 2, or sometimes 3, runs. We will have some 320 phases to determine, clearly a more complicated task than the 3 or 4 numbers we had to find to check whether solutions A or B were themselves analytic. However, we do expect that there will be an analytic amplitude close to solution B with $\chi(s,z)\simeq 0$. It is with this expectation that we begin.

We take $\Delta\psi_o(s,z)=0$ as our starting point so that the initial amplitude is just the phase shifter's solution B with $\phi_o(s) = \phi_{BW}(s)$ and $\chi(s,z)=0$. We evaluate the real part of the amplitude as the output of the Cauchy integral, eq. (13), for t,u = 0, -0.05, -0.1, -0.15, -0.2, -0.3 GeV^2 for values of $\sqrt{s}$ in the range (0.6, 1.79) GeV. We then iterate to convergence determining $\Delta\psi_k(s,z)$ at each value of the energy from 1 to 1.8 GeV after 2 or 3 runs and at each value of momentum transfer. At each step the four Regge parameters, $\sigma^{\pi\pi}(\infty)$, β_f, γ, γ' of eqs. (15,16), are fitted by demanding the best possible agreement (in the sense of χ^2) between the input and output real parts at some hundred separate points for both the forward and backward Regge residues.

Starting with amplitude B convergence is complete after some 5 iterations with $\Delta\psi(s_i,z)$ rather small. This means the resulting amplitude is not too different from solution B - differences we explore below. This we call analytic amplitude β. In the forward hemisphere, for $0 \geq t \geq -0.3$ GeV^2, $\Delta\psi(s,z)$ is typically $\pm 4^o$, with 20^o being an extreme value. In the backward hemisphere the angles are somewhat larger typically $\pm 15^o$. The largest values occur in the energy range 1.29 to 1.41 MeV and for t,u=-0.3 GeV^2, where $|F|$ is smallest anyway.

Now if this amplitude β were to be the only analytic solution that exists, if we start from solution A of eq. (6) - which is about as extreme as we can get - analyticity should then force the $\Delta\psi(s,z)$ to be so large that the amplitude beginning with the 'non-analytic' phase of one discrete solution (A) should become like the other (B,β) - the $\Delta\psi(s_i,z)$ would then be like χ in eq. (8) with i=1. In principle, this can, of course, happen. In this search for analytic amplitudes the discrete solutions A and B just act as previously charted reference points to help us find our way. However, one may be concerned that, by its very nature, our particular iterative procedure can only cope with very small angles and so this hypothetical check could not work. So before proceeding from solution A, let us explore some details of the method we have used starting from B.

In eqs. (7,17) we used the phase ϕ_{BW} as our first "guess" at the forward phase. The introduction of this particular phase into the amplitude is not essential to the method. This particular phase has the advantage that it already contains the gross features of the forward phase and so its introduction shortens the minimisation and simplifies the determination of the several hundred values of $\Delta\psi(s_i,z)$. If we remove these gross features of the initial forward phase and replace it by a constant $\pi/2$, say, so now $\Delta\psi(s,z)=[\phi_o(s)-\pi/2] + \chi(s,z)$, then $\Delta\psi_1(s,t=o)$ is much larger $\sim\pm30^o$ for certain values of s and the overall convergence is slower. Nevertheless, we find that we do obtain just the same amplitude from such a starting point. That this does happen is a prerequisite for considering solution A for which we expect the angles, $\Delta\psi(s,z)$, to which we apply our iterative procedure to be similarly large.

Let us now begin with amplitude A (see eq. (6)) and repeat the procedure already discussed. At each step $\Delta\psi_o\to\Delta\psi_1\to\Delta\psi_2$ etc., we move further from the original solution, but the procedure quickly converges with $\Delta\psi_k\to\Delta\psi(s,z)$. In this way we obtain an amplitude as completely self-consistent as amplitude β. The additional phase, $\Delta\psi$, in the forward hemisphere is just $\pm$ 1 or 2^o, while in the backward region it is typically 25^o. In this case it is for u=0 that the phase $\Delta\psi$ is consistently large. Not surprisingly, this implies that this analytic amplitude is different from the non-analytic starting solution A. However, a quick glance at these backward phases indicates it is not amplitude β. It is a new amplitude α, distinct from A and far from B, revealed only by the imposition of analyticity.

In figures 2a,b the argand diagrams are displayed at t,u=0, -0.2 GeV^2 for these two amplitudes α,β in the energy range 1.15 to 1.71 GeV. It is seen that while the forward amplitudes are essentially identical, they are very different in the backward region. Before discussing these results in detail in section 4, let us consider some general points first (subsection 3C) and then elaborate on our technique in subsections 3D-G.

(C) <u>A first look at the analytic amplitudes</u>

Despite the fact that from 1 to 1.8 GeV, in region III, we have input the data on $|F|$ for the $\pi^+\pi^-$ channel in 20 MeV bins just as given by eq.(1) with all their random statistical fluctuations, our resultant analytic amplitudes of figs. 2a,b are remarkably smooth at each momentum transfer. This smoothness has come about solely from the demands of analyticity. It is seen that this "analytic" smoothing in energy has removed much of the statistical variations; yet hopefully the structures of dynamics remain. It is important to note that methods of implementing cut s-plane analyticity other than ours tend to yield ultra-smooth amplitudes, in which even local "dynamical" structure is washed out. We will comment on this "analytic" smoothing in energy, as well as smoothing in $\cos\theta$, again in later sections.

A striking feature of the argand plots, Figs. 2a,b is the tightly wound loops of the forward amplitude, while the backward amplitude hardly completes a single circuit of a large diameter loop between 1 and 1.8 GeV. This distinctive behaviour is to be contrasted with that expected from the Lovelace-Shapiro model[20]. In this model, when unitarised[21], the forward argand amplitude executes loops moving ever further from the origin along the imaginary axis, while the backward amplitude curls rapidly round the origin in ever decreasing circles.

In order to understand why the physical amplitude is so different it is necessary to partial wave project our amplitudes to see the interplay between the leading resonant waves and their daughter waves, which may, or may not, be resonating too. It is to this we turn in section 4 after considering some necessary technicalities.

(D) Technicalities: Regge Parameters and their Variation

The fact that the amplitude is known in the low energy region (since elastic unitarity completely defines the amplitude from its modulus) is crucial in determining the phases above the inelastic threshold. Moreover, the phases in the forward and backward hemispheres above $K\bar{K}$ threshold though determined separately are not really independent, but are correlated by the fact that in this same region of elastic unitarity, there is considerable overlap between the fixed t and fixed u bands we use [for example, at $\sqrt{s}=0.69$ GeV, $t=-0.2$ GeV^2 is $u=-0.2$ GeV^2],and the same known amplitude must be reproduced. This appears to impart a remarkable stability to our approach despite the fact that the phase has to be determined at several hundred points in the physical region.

All our fits with different numbers of energy bins, different values of t and u, different weights given to certain energy ranges all yield stable forward Regge parameters. These we find for both amplitudes α,β to be

$$\sigma^{\pi\pi}(\infty) = 8.3\text{mb}, \quad \beta_f = 1.0$$

with $\alpha_o=\frac{1}{2}$. The value of the f residue is to be compared with exchange degeneracy which gives $\beta_f = \frac{3}{2}\beta_\rho = 1.1$ (for $\alpha_o=\frac{1}{2}$). It is reassuring that these Regge parameters are in complete agreement with a recent FNAL experiment[22)] that gives $\sigma(\pi^+\pi^-) = 15\pm4$mb at $s=20$ GeV^2 and 13.5 ± 2.5 mb at $s=32$ GeV^2. (It is only at these two energies that we can believe π exchange controls their reaction $\pi^+ n\to pX$ because of kinematical limits on the mass of the exchanged particle). Most importantly, even though the values of these

Regge parameters are stable, our results are, within limits, rather insensitive to them. For example, doubling $\sigma^{\pi\pi}(\infty)$, as factorisation would suggest we should, does not essentially change our amplitudes. Yes, the fit is worse; β_f decreases to partially compensate for the increase in $\sigma^{\pi\pi}$ and the forward and near forward amplitudes are rotated, most in the forward direction, but by only 5° at 1.15 GeV and by less at higher energies. Similarly, varying the intercept of the ρ,f trajectory not only changes the energy dependence of the high energy amplitude but, from the form of the residue in eq. (15), changes the position of the nonsense wrong signature zero also. However, setting α_o=0.35 rather than 0.5 rotates the forward amplitude by at most 6° about the basic amplitudes α, β: again a remarkable resistance to change.

In contrast the situation for the parameters of the effective "exotic" Regge pole controlling backward scattering at high energies is rather different. The values obtained depend strongly on the errors assigned to the backward data points above 1.7 GeV and on the grid of energies we choose to fit. This dependence reflects itself in the larger uncertainties in the results of the backward hemisphere dispersion relations above 1.7 GeV than in the forward hemisphere results. This is why we do not plot our results beyond 1.71 GeV in figs. 2a,b. Typically, with $\alpha_x(0)$=-1, we find (see eq. (16))

$$\alpha : \gamma = -0.70, \qquad \gamma' = 3.9\ \text{GeV}^{-2}$$

$$\beta : \gamma = -0.17, \qquad \gamma' = 7.5\ \text{GeV}^{-2}.$$

The quality of the fits depend rather weakly on the choice of $\alpha_x(0)$ in the interval (-1.5, -0.5), though the fitted values of γ, γ' are strongly α_x dependent, as we would expect from eq. (16) if the amplitude below 1.7 GeV is to be fairly insensitive to the variation in such parameters.

Although Regge behaviour is not required of the real part of the amplitude until some high energy such as 4 or 5 GeV (the results being independent of the exact value of ν_o), the forward amplitude has on the average a comparatively small real part above 1.8 GeV, just as the early onset of roughly exchange degenerate Regge behaviour would suggest. This is not the case in the backward direction where the imaginary part is small above 1.8 GeV but where the real part is quite considerable till much higher energies are reached. In particular, note that solution α has a sizeable negative real part at 1.8 GeV, while solution β is moving closer to the origin and so closer to an early "average" onset of Regge behaviour. However, we have no reason to expect that Regge behaviour for the real part must set in till much higher energies. Nonetheless, this is an important difference between the analytic amplitudes α,β to be considered again later.

(E) Technicalities: Varying the S^* parameters

While discussing such points it is appropriate to mention that larger uncertainties not only arise above 1.7 GeV but also between 0.95 and 1.1 GeV - this is why we do not start to plot our results till 1.15 GeV, where they are quite reliable. How big the uncertainties are in the inelastic region between 1.15 to 1.71 GeV with respect to changes in the Regge parameters we have already touched on, with regard to the experimental errors will be discussed later in the next section.

The inaccuracies here close to the inelastic threshold arise because, although in principle elastic unitarity determines the amplitude completely below $K\bar{K}$ threshold the moments, eq. (1), are very small in this narrow energy range and large uncertainties occur. In particular, while the I=0 S-wave phase shift δ_o^o is rising very rapidly from 90^o through 180^o in this precise energy region, we are uncertain as to its exact value at the moment the $K\bar{K}$

channel opens up[9]. This sizeable uncertainty propagates up to 1.1 GeV in the determination of the phase of the inelastic amplitude. In resonance terms this manifests itself in an imprecise relation between the S^* parameters and the background. To pin this down more precisely requires a detailed analysis of not just $\pi^+\pi^- \to \pi^+\pi^-$ and $\pi^+\pi^+ \to \pi^+\pi^+$ data but also $\pi^+\pi^- \to K\bar{K}$ and $\pi^+\pi^- \to \pi^o\pi^o$. At present, some inconsistencies arise between data on the latter two channels[23,24] and those on the former two[1,2]. High statistics accurate data on the $\pi^o\pi^o$ channel, in particular, might be useful in this connection[25] when combined with data on the other channels so as to accurately fix both the phase shifts and inelasticities of not just the I=0 channel but also the I=2, for which $\eta_\ell^2=1$ is necessarily assumed[2] for want of any reliable information on inelastic channels. We shall return to discuss this energy range around $K\bar{K}$ threshold later.

(F) Technicalities: Logarithmic end-point singularities

Next one may worry that the division of the range 1 to 1.8 GeV into N intervals, each with a discrete phase $\Delta\psi(s_i,z)$, introduces logarithmic singularities at the edges of the subregions when substituted into eq. (13), cf eq. (14). This is, of course, the case, but their effect is negligible. This can be checked (i) by increasing N - we have done this from N=8 to 24 - and (ii) by changing the energies s_i defining the intervals - this is done several times - and repeating the analysis. The results remain unchanged. This is largely because the additional non-continuous phase $\Delta\psi(s,z)$ (for s in bin centred on s_i) has a smooth behaviour with s_i at each fixed value of t,u except perhaps at $\sqrt{s}$ = 1.79 GeV - the last data point.

The ρ Regge term contributes much less than the pomeron or f to the high energy integral for $\sqrt{s}$<1.8 GeV having an additional (ν/ν_1) factor as odd signature requires, where ν_1 corresponds to an energy of 1.8 GeV. Thus, while the high energy contribution is relatively insensitive to the residue

β_ρ, the ρ term is just as important as its even-signatured counterpart, the f, in the imaginary part of the amplitude along the right hand cut. Because of this, we can with hardly a change in the integral of the Regge contribution for $\sqrt{s}$=1.5 GeV, say, eliminate any logarithmic singularity at $\sqrt{s}$=1.8 GeV by defining the ρ residue so as to give continuity, i.e. define $\mathrm{ImF}_{data}(s,t) = \mathrm{ImF}_{Reg}(s,t)$ at $\sqrt{s}$=1.79 GeV and t=0 or $-0.2\ \mathrm{GeV}^2$ for example, where the Regge amplitude is given by eq. (15). This makes a negligible difference to our results and typically β_ρ becomes 0.9 instead of 0.75.

(G) Two Analytic Amplitudes Alone?

We have constructed the amplitudes α and β to have cut s-plane analyticity for a certain range of fixed momentum transfer by starting from the non-analytic solutions A and B of the phase shift analyst (e.g. ref. 9). To obtain amplitude α, in particular, we have had to search for parameters $\Delta\psi$ far from the starting point A. It is natural then to ask if we were to search further would we find yet other analytic amplitudes besides α,β describing this same data of the CERN-Munich group[1,2]. Indeed, if such exist they would most easily be found by beginning our iterative procedure at some other reference points than the phase shifter's solutions A and B by, for example, taking $\chi(s,z)$ to be non-zero initially.

Though far from attempting an exhaustive search of the 320 parameter space of $\Delta\psi$'s,we have found that other starting amplitudes may lead us to the very same amplitude α,β at least within errors. This is of course reassuring. However, others take us far away to yet other amplitudes equally analytic and equally well describing the data but with quite different properties.

Nonetheless, as we shall see, amplitudes α,β constitute the basic analytic amplitudes we have been able to construct,so in the next section we shall explore these in greater detail and chart their physical properties before even taking the most tentative step into the quagmire of a large parameter space. Even then we shall see that amplitudes α,β and their neighbourhoods are indeed special.

4. Phase Shift Analysis

(A) Partial wave projection

We have constructed our basic amplitudes α, β by requiring that they not only describe the experimental moments of eq. (1), but are analytic in s. This construction can be performed independently at each fixed value of momentum transfer. The only inter-relation between different momentum transfers comes from the fact that at low energies in the elastic region the dispersion relations must reproduce the same known amplitude with its specific momentum transfer dependence given by experiment and that in the high energy region we have chosen definite forms for the t and u dependence of the Regge residues - fewer parameters than the number of values of t and u used. Thus our phases at a given energy in the inelastic region are not completely independent from one momentum transfer to another, but weakly correlated by these low and high energy requirements. However, while analyticity in s at a given momentum transfer is seen to produce rather smooth amplitudes [figs. 2(a,b)] describing the data, no smoothness in $\cos\theta$ at a given energy is imposed except in this very weak way. It is to the examination of these almost independent amplitudes, analytic in s, as a function of momentum transfer that we now turn. This is necessary if we are to investigate the physical properties of these amplitudes in terms of resonances and structures, which are most easily seen in individual partial waves. Moreover, it is on partial wave analysing our amplitudes that we can answer the important question of whether unitarity is satisfied or not. This is by no means obvious.

Recall that our starting amplitude B has an S-wave which is highly non-unitary over several hundred MeV. This we argued away by recognising the arbitrariness of the phase shifter's decision on the maximum number of partial waves, choice of overall orientation of the partial wave vectors and method of smoothing. However, we have removed these three factors of arbitrariness. Thus, for our amplitude β, or for that matter α, unitarity, eq. (11), now poses a realistic constraint which can possibly rule out one analytic amplitude or another.

Our first step in examining amplitudes α,β for their physical properties is therefore to check whether they satisfy unitarity or not. For this, as discussed in section 2B, we must consider the amplitudes as functions of $\cos\theta$ and project out partial waves. Our amplitudes α,β will have an infinity of partial waves as defined. To be able to determine these would require the amplitudes to be known at an infinite number of points in $\cos\theta$ at each energy. However, fixed t,u dispersion relations can only determine the amplitude in the near forward and backward cones. The centre of the physical region is unexplored [for example, at $\sqrt{s}$=1.6 GeV, only $|z|>0.75$ is covered by the dispersion relations we use] and then to be tractable we only use a finite number of points in t and u. We therefore cannot use the full range of $\cos\theta$, $-1\leq z\leq 1$, to project out partial waves.

Armed with the amplitude at only a finite number of momentum transfer points (t,u=0, -0.05,... GeV^2) at each energy we are forced to approximate the amplitude by a finite number of partial waves. In principle, this can be a large number but in practice we are able to represent the amplitude adequately using five waves up to and including $\ell=4$. Using four waves, $\ell\leq 3$, is not sufficient.

Since a minimum χ^2 criterion is used to fit the dispersion relation amplitudes with partial waves, it is possible to obtain fits which while giving a good description of the amplitudes, in the sense of χ^2, no longer fit the original experimental moments quite so well. To avoid this we require that the partial waves not only fit the dispersion relation amplitudes, but equally the experimental moments from which they are derived. If appreciable higher waves were necessary, which we have presently set to zero, this would show up by 'predicting' higher moments $L \geq 7$ non-zero outside their experimental error bars.* This, in fact, will not happen and for $\sqrt{s}<1.65$ GeV up to $\ell=4$ is certainly sufficient. Thus we find the major effect of the continuum phase $\chi(s,z)$ is not so much the generation of many higher partial waves, but rather only a few, together with the mixing of lower waves, so as to produce an analytic amplitude.

Every 20 MeV, at each data point from 1.15 to 1.69 GeV, we perform this partial wave analysis. However, when a partial wave is very small, particularly some high wave, like the $\ell=3$ and 4 waves below 1.4 GeV, the uncertainties in their determination, as given by the experimental and dispersion relation errors are very large and the points somewhat scattered. In this situation we can obtain more meaningful results without a deterioration in χ^2 by fixing these waves to be the tails of the appropriate Breit-Wigner resonances. We do this for the $\ell=3$ wave below 1.44 GeV assumed to be controlled by the g-resonance and for the $\ell=4$ wave below 1.5 GeV controlled by the h resonance.** Above these energies these waves are allowed to go

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*Recall that the way we have constructed the amplitude, if we were to take the complete infinite number of waves, the amplitude would require all moments with $L>6$ to be vanishingly small within limits.

**We are aware that this cannot be exact near threshold since the $\ell=4$ scattering lengths, whether in the resonant I=0 or the supposedly non-resonant I=2 wave, are comparable. We are only using these resonance tails as convenient forms down to 1 GeV.

free, since then their magnitudes exceed the size of their error bars. Below we will give the explicit Breit-Wigner forms used.

In calculating the even partial waves both I=0 and 2 components contribute according to eq. (9). For the S-wave we take the I=2 phase shift to be a constant, elastic, -25^{o} [2,14]. With this input we present the (supposedly) pure I=0 S-wave. Nevertheless, the D and G waves will each include an I=2 component which is expected to be small from $\pi^+\pi^+$ analyses[2,14].

To test whether our resultant partial waves are unitary or not, eq. (11), we conveniently represent them in terms of phase shifts and inelasticities, eq. (12), when the η_ℓ^I must lie in the range $0\leqslant\eta\leqslant1$. In figs. 3(a,b),4 we show the partial waves obtained by fitting each of the dispersion relation amplitudes α and β respectively, together with the experimental moments, weighted so that the moments and the dispersive results contribute roughly an equal amount to χ^2. In figs. 3(a,b) the S_o, P, D and F waves are displayed in their relevant argand diagrams from 1.15 to 1.69 GeV, where the circles set the unitarity bound given by eq. (11)*. Since the G wave is very small it is not shown in such a diagram but rather its phase shift is given separately in fig. 4 for both solutions.

It is immediately seen (fig. 3) that neither solution is ruled out. Indeed all the partial waves of both our basic analytic amplitudes satisfy unitarity remarkably well. If it is recalled that solution B of ref. 9, to which our amplitude β is quite closely related, has its S-wave far outside

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*Remembering the "small" I=2 part has not been separated out in the D wave, the circle is not the exact unitarity limit. What is plotted is $D=D_o+\frac{1}{2}D_2$.

the unitarity circle in the energy range from 1.25 to 1.55 GeV, we see our procedure has, by implementing analyticity and so fixing the overall phase of the amplitude and the relative phase of each partial wave, restored unitarity to the amplitude.

Moreover, our resulting partial waves are not only unitary but rather smooth (fig. 3). Not as free from structure as the phase shifter's artificially smoothed results[9] discussed in sect. 2B, but far smoother than the phase shifter's raw partial waves (see figs. 4.14,15 of ref. 16). This smoothness results solely from the requirement of analyticity in energy and by the approximate analyticity in cosθ we now have. Despite this 'analytic' smoothness our resulting partial waves fit, not only the smooth dispersion relation amplitudes of figs. 2(a,b), but also the experimental moments exceedingly well. Remember that with only guesses for the phase relation between different partial waves, the phase shifter finds very very raggedy partial wave amplitudes fitting these same moments. In figs. 5(a,b) we show how well the partial waves of amplitude β fit the original moments[1] - the situation for solution α is even slightly better. Let us discuss these results in more detail.

Comparing figs. 3(a,b) showing the $\ell\leqslant3$ waves for amplitudes α,β respectively, we see the leading waves in the f and g regions are very similar. For example, reading from these graphs the parameters of the f-resonance are m_f=1.272 GeV, Γ_f=0.18 GeV, x_f=0.83 for α compared with 1.261 GeV, 0.17 GeV and 0.84, respectively, for β. These parameters are exactly the same within errors - errors we shall discuss separately below.

While the leading waves are very much the same, it is the lower daughter waves that produce the difference between our amplitudes α,β of figs. 2(a,b). Since solution β is most like previously discovered solutions it is more natural to discuss it first.

As already remarked solution β has a unitary S wave (unlike B), which is in fact highly elastic in the entire energy range from 1.29 to 1.69 GeV. It is of course somewhat difficult to believe that this wave really can be quite so elastic after such a strong onset of inelasticity at $K\bar{K}$ threshold produced by the S* resonance. Indeed, we can bring the S wave into the unitarity circle by hand with $\eta_0^0 \simeq 0.9$ hardly worsening the χ^2 for fitting the dispersive amplitudes and moments. This is achieved by a small compensating movement mainly in the D wave, for which very roughly $|\Delta D| \simeq \frac{1}{5}|\Delta S|$. This will be seen more fully in the error regions we explore below. Nonetheless we can say the S wave is resonant in the region 1.3-1.45 GeV. Indeed, we can describe not only this S wave with an 'elastic' ε resonance in this mass range together with an S* just below $K\bar{K}$ threshold, but also approximately the experimental $\pi\pi \to K\bar{K}$ cross-section[23)] below 1.5 GeV. The detailed couplings however depend on the detailed forms used: see, for example, ref. 26).

The behaviour of the P wave in solution β is unmistakably that of a resonant loop characteristic of a second sheet pole. The parameters of this ρ' are $m_{\rho'}$=1.575 GeV, $\Gamma(\rho' \to \pi\pi)$=0.105 GeV, $\Gamma(\rho'$ inelastic$)$=0.235 GeV assuming the inelastic channels to have $K\bar{K}$ kinematics. Thus the ρ' in this solution has a 30% coupling to ππ and so should be clearly seen in this decay mode. We return to this below.

The highest effectively non-zero wave, the G wave, is seen in fig. 4 for both solutions α,β. It is of the size of 1 to 3^o. To see whether this is a reasonable magnitude for this wave generated by analyticity (recall the phase shifter artificially sets this to zero) we compare these results for δ_4 with that suggested by the tail of a Breit-Wigner form for the h-resonance at 2.035 GeV. Taking the unitary partial wave to be given by

$$f_\ell(s) = \sqrt{\frac{s}{s-4\mu^2}} \quad \frac{xM\Gamma(s)}{M^2-s-iM\Gamma(s)} \tag{17}$$

where the energy dependence of the width is described by

$$\Gamma(q^2) = \Gamma_o \left(\frac{q}{q_R} \right)^{2\ell+1} \frac{D_\ell(q_R^2R^2)}{D_\ell(q^2R^2)} \tag{18}$$

with the radius R in the centrifugal barrier factors being set at 5 GeV^{-1} (1 fm), and

$$q^2 = \tfrac{1}{4}(s-4\mu^2), \quad q_R{}^2 = \tfrac{1}{4}(M^2-4\mu^2).$$

For the ℓ=4, $D_4(x)$=11025 + 1575x + $135x^2$ + $10x^3$ + x^4. We then find that with M=2.035 GeV, Γ_o = 0.2 GeV, the phase shifts δ_4 for amplitudes α,β are fitted with elasticities, x, of 0.2 and 0.25, respectively: both very reasonable values indeed. These give the curves shown in fig. 4. [Incidentally, a similar form to eqs. (17,18) is used for the ℓ=3 wave below 1.45 GeV, as already explained, with $D_3(x)$ = 225 + 45x + $6x^2$ + x^3 and g-parameters of M_g = 1.692 GeV, Γ_g = 0.24 GeV and x_g = 0.245.]

The overall smoothness of the free ℓ=4 wave of fig. 4 and its strong agreement with the assumed tail of the h-resonance leads us, in fact, to use this form explicitly up to 1.69 GeV when fitting the lower waves and this is what is actually plotted in figs. 3 (a,b). Leaving out this G wave would give a poorer representation of the dispersion relation amplitudes and the moments and produce a non-unitary S wave.

Before turning to our amplitude α let us remark on its G wave too. It is smaller than for β though very much the same: an elasticity of 20% as compared with 25%. This illustrates an important point. Amplitude α is obtained by a far larger phase change $\chi(s,z)$, of eq. (7), from the phase shifter's non-analytic solution A with its $\ell\geq 4$ waves zero, than β from B. We thus might have expected that α would have larger higher waves (e.g. ℓ=4) than β. In fact, the converse is true. The larger values of $\chi(s,z)$ in the backward hemisphere for α produce not larger higher waves but rather mix the lower ones among themselves so as to generate an analytic amplitude.

Let us now look at the other daughter waves of solution α, fig. 3a. The S wave is now quite inelastic. Though the wave does return briefly to the unitarity circle beyond the S* effect just as $K\bar{K}$ data indicates[23], before becoming inelastic again and remaining so. The S wave executes a relatively tight resonant loop once again suggesting a high mass ε resonance in addition to the S* at $K\bar{K}$ threshold.

The P wave is distinctly different from that of solution β, showing no $\rho'(1600)$ coupling to $\pi\pi$ at the few percent level. This is a major physical difference between our solutions. Although the P wave does go outside its unitarity circle for some 100 MeV, this is of no great significance when we take account of the errors on this wave (considered below). We have made no attempt to constrain any wave to be unitary, which could certainly be done without too great an increase in χ^2. Moreover while the D, F waves are rather similar to those of β when these are leading waves, the D wave above the f is quite different. Not only is the shape of this D wave different but it shows a distinct loop in the 1.5-1.6 GeV region, again characteristic of a resonance. This we call the f* having a mass in the region of 1.55 GeV, a width of 0.1 GeV and roughly 10% coupling to $\pi\pi$ if it is in the I=0 channel or 20% coupling in the I=2, remembering we have not separated the D wave into its isospin components. Exactly what this loop in solution α might be we will consider below, nevertheless its appearance is quite unmistakable. In this same energy range the D wave in solution β also has some structure which is however not particularly significant.

It is appropriate here to mention that this loop is separate from the

the sharp fall in the moments A_o, A_2 and A_4 seen between 1.45 and 1.47 GeV in the CERN-Munich data[1] of figs.5(a,b), which also show the fit given by solution β*. This sharp drop results in the rapid movement of the D wave in this 20 MeV range seen in both figs. 3a and b for amplitudes α and β. Though this is at the onset of the f* effect, its interpretation appears to be unrelated.**

(B) The Physical Differences between our Analytic Amplitudes

Since our two basic amplitudes α,β are both not only analytic but also unitary we discuss in this section how we can differentiate between these solutions, particularly remembering that we have only obtained the amplitude α by the imposition of analyticity. It is useful to summarise their different physical properties in the inelastic region from 1 to 1.8 GeV in a resonance shorthand

Wave	α	β
G	tail of h	tail of h
F	g	g
D	f f*	f no f*
P	very little ρ'	ρ'
S	ε becoming highly inelastic	ε highly elastic

Three differences are clearly apparent:

(i) The S wave in α is much more inelastic than in β.

(ii) In the P wave, solution α has essentially no ρ'(1575) coupling to ππ,

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*Incidentally, solution α fits these falls somewhat better.

**The confirmation of this sharp structure awaits the final analysis of both the Omega experiment of ref. 5 and completion of the new ACCMOR experiment, ref. 29.

while solution β has a 30% coupling* to the elastic channel.

(iii) In the D wave, solution α has a resonance-like effect, the f* in the 1.55 GeV region, while β has none.

How can we distinguish between these possibilities. It is sensible to remember that the features of α(f*, no ρ') and of β(ρ', no f*) come in inseparable 'packages' so that if we can prove or reject one feature the others follow.

Clearly these amplitudes α,β having different S,D waves ((i),(iii) above) they predict different $\pi^+\pi^- \to \pi^o\pi^o$ amplitudes which high statistics data on this channel could confirm or rule out. Unfortunately, no really high statistics data exist on this channel (eg. ref. 24), but some is promised from an analysis of the Serpukhov data[10], which has augured well with the discovery of the h-resonance.

Detailed information on inelastic channels, both $K\bar{K}$ and 4π, could in principle distinguish between the different S wave inelasticities in the range 1.4-1.7 GeV. Indeed, a far better knowledge of the S* resonance would allow us to determine the phase of the whole amplitude with greater certainty through this region, at the crucial onset of inelasticity, than we have been able to here (see discussion section 3E). Present intuition, based on knowledge, such as it is, of the inelastic S wave below 1.1 GeV[26], would tend to favour solution β, rather than α, as its S wave can be more easily fitted with a "conventional" almost elastic ε plus S* prescription[26]. However, because of the sizeable uncertainties in the S wave (see below) and with the lack of definitive data in this region, we are unable at the present time, to distinguish our solutions in this way. Since neither of the differences (i) and (iii) are therefore very easy to test, we turn

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*Once again the errors on this will be discussed below.

to the second point, namely the different P waves.

If it were only more easily accessible, data on the $\pi^{\pm}\pi^{o} \rightarrow \pi^{\pm}\pi^{o}$ channel[25]) having just isospin one and two, would clearly see the ρ' and allow its coupling to be determined. However, the discovery of the J/ψ family of particles has highlighted the power of $e^{+}e^{-}$ annihilation to detect vector mesons and to discover their decay modes. Evidence already exists for a ρ'(1575) in $e^{+}e^{-}$ [6b,27]), $\bar{p}p$[27b]), 2π and 4π decay channels[6,27]), but its branching ratio to $\pi\pi$ - the crucial distinction between solutions α,β - is not determined other than indications that the two pion decay mode is less favoured than the four pion mode - how much less favoured is the question. This will hopefully be resolved in two ways: (a) the Fermilab experiment on the photoproduction of vector mesons off beryllium sees evidence for a ρ' in both 2π and 4π charged modes with 4π predominating[6a]); when their analysis is complete they should be able to quote the corresponding branching ratio for the ρ'. (b) Similarly, the DCI experiment at Orsay[6b]), that discovered the ω'(1778), sees the ρ' in the collective multipion modes $e^{+}e^{-}\rightarrow 2n\pi$ in the 1.6 GeV region. They will eventually, on collecting enough statistics, be able to quote the branching ratio $(\rho'\rightarrow 2\pi)/(\rho'\rightarrow 4\pi)$. It is believed that just as the ρ' prefers to decay to four pions rather than two[27a]), the ω' prefers to couple to five pions rather than three[6b]) (just like the ψ in fact): the problem is how much more. Indeed, if the ratio $(\omega'\rightarrow 3\pi)/(\omega'\rightarrow 5\pi)$ is quite small[6b]), this might suggest that the analogous 2π to 4π ratio for the ρ' is similarly small. This is, of course, only a suggestion but one which would be a natural implication if the dominant decay of the ω' were to be $\rho'\pi$, but this, in fact, is only one possibility.

In any case we have the exciting prospect that $e^{+}e^{-}$ experiments will in the very near future tell us which of our solutions α,β is nearest the truth. If the decay mode $\rho'\rightarrow\pi\pi$ is a large fraction, amplitude β would clearly be favoured. However, if the coupling of the ρ' to $\pi\pi$ really were found to be very small, it would imply* solution α is the correct one.

Now this solution has the so-called f* effect in its D wave. This leads us to enquire what this might be. We shall therefore list the possibilities:

0) It is in the I=0 and not I=2 D wave. No evidence is seen in the $\pi^+\pi^+$ channel[2], but $\pi^o\pi^o$ or $\pi^\pm\pi^o$ data would be useful to check this completely. We will assume it is an I=0 effect.

1) The f* effect is the f': this requires the f' to have higher mass, and width even greater, than found in the ANL $K\bar{K}$ analysis[23b] (which give a mass of 1.506 GeV and width of 0.066 GeV) and in other recent experiments[28] and a larger coupling to $\pi\pi$ than previously seen (which is less than 2.5% from ref. (28)), and finally, what we have taken to be pure $\pi\pi\to\pi\pi$ data extracted from $\pi N\to\pi\pi N$ by one-pion-exchange must have a sizeable non-pion-exchange component (the A_1) not yet eliminated.

If this is ruled out as conflicting with all present evidence on the f' we have two further options based on assuming the f* seen here and the f' are distinct.

2) The f* is the first daughter of the g. This is an easy and seemingly uncontroversial suggestion to make. However, this may nonetheless be somewhat disfavoured by the fact that we have no evidence for other states on this same daughter trajectory: the $\varepsilon(750)$ and $\rho'(1250)$, both of which have been superceded by states seen on the second daughter trajectory: $\varepsilon(1250)$ and $\rho'(1600)$. So this possibility raises more problems than it resolves.

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*Assuming the CERN-Munich data on $\pi^+\pi^- \to \pi^+\pi^-$ is correct - an assumption we have naturally made all along.

3) If the f* is neither of these (1,2) then we have no room in the simple quark model for another tensor meson in this region made out of a quark and an antiquark. Instead, if such a new tensor meson exists, it is perhaps made not of quarks but of coloured gluons. Such a highly inelastic state would lie on the Regge trajectory corresponding to gluon exchange, viz the pomeron, on which it would fit quite nicely. We do not go into this further, since the speculations are endless, but add that there are those who would not be unhappy if such a state did exist.*

Detailed analysis of $K\bar{K}$ data might suggest that to explain the CERN-Munich results[23d)] on this channel, even f, f', A_2 interference found by the ANL group[23b)] is not sufficient and that either another state (the f*) is needed or a far broader f'[23d)]. Thus $K\bar{K}$ data is perhaps the key to testing the existence of such a state but because of the complicated interference effects this may in fact be rather difficult in practice.

It is clear that such difficulties in confirming resonances extracted from phase shift analyses arise for all daughter states except for those with $J^{PC} = 1^{--}$ which couple to the photon, and so are preferentially produced diffractively in photoproduction and easily seen in e^+e^- annihilation. The final analysis of presently or nearly completed experiments[5,6,10,29)] are eagerly awaited, as well as future results on new experiments.

(C) Other Analytic Amplitudes Investigated

Having explained in some detail the physical properties of our basic amplitudes α, β which are equally analytic, equally unitary and equally well

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*We are grateful to John Ellis, who sensing a whiff of glue, shared with us his vision of what a glueball should really look like.

describe experimental data, we return to the question raised in section 3G, namely are α,β the only analytic amplitudes that we can find? This will provide us with a suitable vehicle for studying the errors in our previously described analyses.

Amplitudes α,β have been found by starting at the phase shifter's solutions A,B and iterating to find the analytic values of $\chi(s,z)$ of eq. (7), i.e. iterating $\Delta\psi$ of eq. (17). As already emphasised the phase shifter's solutions A and B are just convenient reference points in our search for analytic amplitudes. It is clear we can start anywhere. If we do, we find more often than not, that an arbitrary choice for $\psi_o(s,z)$, of eq. (17), does not lead to convergence or, even if it does, the amplitude is not analytic; namely, the system converges according to criterion (ii) of section 3A, so that the amplitude is self-replicating in the inelastic region, but criterion (i), that the known elastic amplitude should be correctly reproduced, is not fulfilled. When either of these two situations arise we take this to imply that our starting phases are more than some 50^o away from the nearest analytic amplitude, which seems about the maximum range for which we have found convergence. Of course other starting points than A and B do lead to completely analytic amplitudes, some to amplitudes so close to α or β as to be indistinguishable from them (see, for example, sections 3B,D). However, others lead us to quite new analytic amplitudes. These we discuss here.

In all such cases we have been unable to move far away from the forward hemisphere amplitudes of solutions α, β, aside from small changes discussed in section 3D. This remarkable stability comes as no surprise, since we have already found that replacing $\phi_o = \phi_{BW}(s)$ by a constant leads to exactly the same result (section 3B). This stability occurs because the imaginary parts of the forward and near forward amplitudes are positive definite, and this, together with the fact that the contributions to the modulus of the amplitude given by the moments, eq. (1), all add positively too, leaves relatively little room for manoeuvre. This is especially so at t=0 and, of course, less so at t=-0.3 GeV^2, where the amplitude is much smaller anyway. In the backward hemisphere, which is only very weakly correlated with the forward amplitudes, the situation is quite different.

We have found several different analytic amplitudes starting with different initial phases, all having a more rapidly rotating backward amplitude than either α or β. This invariably leads to a violation of unitarity for these analytic amplitudes. In particular, the S and D waves are forced far outside their unitarity circles in the region 1.29-1.35 GeV*. We consider these no further, other than to remark that it is surprising enough that both our analytic amplitudes α,β are unitary without expecting all other analytic amplitudes to be so.

We have, however, found one other amplitude which is not only analytic, of course, but also unitary. This we obtained from a starting point which is neither A nor B, α nor β, in a seemingly distant part of the parameter space, but not as far as our other trial starts giving non-unitary amplitudes were. This converges to yet a further analytic amplitude with the same forward Regge parameters as for α,β. For the moment, we call this amplitude γ. It is, however, not dissimilar to our amplitude β even though its start is typically 30^o away (namely, $|\Delta\psi(s,z)|\simeq 30^o$ on convergence). On partial wave analysing this amplitude γ, as in section 4A, we find it to be satis-

*This arises in this region because the forward and backward amplitudes have maximum modulus there. Rotating the backward amplitude (relative to α,β into the left half of the argand circle) reduces $\mathrm{Im} f_\ell^I$ and eq.(11) is violated.

factorily unitary as shown in fig. 6. Its S wave is less smooth than for α,β, not at elastic as β nor as inelastic as α. The amplitude is, however, of a β-type in that it clearly has a $\rho'(1575)$ in its P wave with a sizeable coupling to $\pi\pi$ rather than none at all. This time the coupling is 15%. Its D wave is more elastic in the 1.45 GeV region than α or β, and shows the same rapid movement from 1.45 to 1.47 GeV seen in all solutions.

It is important to note that this amplitude γ is, at different momentum transfers, just as smooth as α,β. This smoothness follows from the imposition of analyticity in energy. However, this amplitude γ does not have partial waves quite as smooth as α,β. This is related to the cos θ dependence at a given energy for which we have imposed no arbitrary smoothing other than a finite ℓ cut-off. Fig. 6 shows the results for $\ell_{max}=4$. The larger G wave found here we take as collective of all higher waves; indeed a non-zero $\ell=5$ wave is needed at the higher energies below 1.79 GeV to give as good a description as for α,β. Fitting the G wave phase shift (i.e. just the real part) to the Breit-Wigner form (shown in fig. 4 for α,β) and using the same mass, width and radius R as in eqs.(17,18) we find the elasticity to be 60% rather than 20-25%. Now we ask is this amplitude γ, which has a smaller coupling ρ', a less elastic S wave and a bigger G wave than β really a different amplitude, or is it more or less the same. In other words, are the minima of χ^2, in which β,γ lie, separated by a steep mountain or just a small hillock so that β and γ are in the same shallow plane?

Such an exploration requires the determination of meaningful errors for the phase shifts, always somewhat complicated in such analyses. This is solved by taking the errors given by the MIGRAD subroutine of the CERN minimisation programme MINUIT once it has converged. These errors are meaningful in that they correspond on convergence to the change brought about by increasing χ^2 by one. For illustrative purposes we show the corresponding error ellipses (defined by η,δ for each wave) at just two convenient, typical energies: 1.39 and 1.59 GeV. From these it is seen that β and γ are in fact minima in a fairly shallow plain and not in separated and distinct valleys. We therefore regard amplitude γ (obtained quite differently from β) as an equally likely alternative to β and so rename it β'. Despite slightly different physical properties β' and β are both analytic, unitary amplitudes lying in the same continuum patch. Indeed, from now on when referring to amplitude β we mean equally β'.

The sizeable error ellipses on the S wave of fig. 6, being roughly three times larger than those on the D wave, highlights, as already intimated in the previous section (4B), how impossible it is to use the S wave inelasticities to distinguish our basic amplitudes α from β', β' from β - hence α from β. As we have already emphasised, the clear cut distinction between these physical amplitudes is the strength of their couplings of the $\rho'(1575)$ to $\pi\pi$; a strength e^+e^- annihilation can most easily reveal.

Most importantly, we have found, in as much as we have been able to explore a three hundred parameter space, no other* analytic and unitary amplitudes than α,β. This is why we have regarded these as our basic results and considered them in such great detail.

5. Discussion

The purpose of this section is not so much to summarise what we have achieved, as we have summarised as we have gone along, but rather to discuss our analysis in the light of other work in this area.

Let us return to the phase shifter's solutions A, B, C and D[9] with their arbitrary partial wave cut-off, articial smoothing and arbitrarily

*This assumes that the final $\pi^o\pi^o$ production data will continue to eliminate the amplitudes that are obtained starting from solutions C and D of ref. 9.

chosen overall phase. It is clear that after ruling out solutions C, D as inconsistent[16] with preliminary $\pi^o\pi^o$ data[10], it is a natural problem to try to see if analyticity can distinguish between the remaining solutions A, B. As already discussed, we previously found[13] that, as they stand, B is closer to an analytic amplitude than A, using straightforward dispersion relations in a very simple way.

Before this, Common[30] performed a different study of this same problem. Since amplitudes A, B have, by definition the same modulus but different phases above $K\bar{K}$ threshold, it is natural to consider phase-modulus dispersion relations which are essentially dispersion relations for log F, rather than those we have used for the amplitude itself, eq. (13). These relate the phase directly to the modulus and so one does not have to make a starting guess for the phase $\psi_o(s,z)$ as we do. However, they are only useful in the forward direction, since it is necessary to know the zeros of the amplitude in the complex s plane to use a relation for log F. In the forward direction these are completely under control. It should be clear from our analysis that using only the very forward direction cannot distinguish A from B, α from β or from any other analytic amplitude we have been able to reveal. Common[30] found however that if Regge behaviour for the full amplitude (both the phase and modulus) sets in at 3 GeV solution A is favoured, while if it sets in at 1.8 GeV B is chosen. If Regge behaviour is required only for the imaginary part of the amplitude above 1.8 GeV, that is the phase is not a priori preset above 1.8 GeV but given a posteriori by dispersion relations, we find no such difference.

In the earliest of all analyses Froggatt and Petersen[31] have like us, constructed analytic $\pi\pi$ amplitudes describing the experimental data on the moments: an analysis which has very recently been finalised. This they do in a quite different way from us with, in general, basically similar results, results which nonetheless have some crucial distinctions.

Froggatt and Petersen impose cut s-plane analyticity by expanding the amplitude in terms of functions which are explicitly analytic in s. In pursuing this technique, they follow Pietarinen[32] who has applied such methods to πN,KN scattering. However, there are a number of important differences between these meson-baryon processes and $\pi\pi$ scattering. Unlike πN, KN, we have in $\pi\pi$ no reliable total cross-section measurements in the phase shift region and very little higher energy information[22]. This lack of working high energy parametrisations is particularly critical for the $\pi\pi$ amplitude controlled by exotic exchange with I=2, for this, of course, cannot arise in πN scattering and for which we must therefore make hypotheses [for example eq. (16)], which, however physically reasonable, are neither theoretically nor phenomenologically well tested (e.g. ref. 19).

In order to expand the amplitude in a convergent series of explicitly analytic functions, Froggatt and Petersen[31]* (like Pietarinen[32]) transform the cut s-plane into a unit circle by a conformal mapping. At each fixed t (and fixed u) they parametrize the amplitude as a polynomial in the new mapped variable and fit this to the data for $|F|$, including a certain penalty function to ensure smoothness. A smoothness analyticity does not necessarily require. This method has the advantage that the amplitude is manifestly analytic with the need for principal value integration obviated. Though not essential to the method,FP choose to explicitly include the ρ, f and g resonance poles in the full amplitude (not just in the relevant partial waves) in order to leave a smoother function for polynomial parametrisation. It is clear that some care is necessary if such a procedure is not to bias

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*Hereafter referred to as FP.

the analysis in favour of amplitudes with daughter structures beneath each leading resonance, since it is only for one form of the residue of each pole that its daughters can be eliminated.

Moreover, the high energy behaviour of an amplitude expanded in terms of a large order polynomial in the unit circle is far from transparent and difficult to control. It does not in a natural way lead to Regge behaviour with moving rather than fixed, singularities used in ref.(31a), nor to a smooth onset of this behaviour before infinite energies. In order to bring this under complete control FP have performed a new analysis[31b)] contemporaneously with this present work. This treatment assumes the standard Regge ansatz, rather like our eqs. (15,16) but for the whole amplitude beyond 1.8 GeV and not just for the imaginary part as we do below 4-5 GeV.

FP perform their analysis at t,u=0,-0.2,...,-1.0 GeV^2. A somewhat different range from us, since we restrict t,u to the smaller interval (0,-0.3) GeV^2. This is because we do not have the courage to use our dispersion relations across the double spectral regions without having any foreknowledge of how strong double spectral effects might be. Moreover at an energy of 2 GeV, for example, and a momentum transfer of t or u=-1 GeV^2, to assume Regge behaved amplitudes is a far stronger assumption that we can possibly make - perhaps at such relatively large momentum transfers a constituent interchange model is more appropriate. It is true that our forms for the Regge residues, eqs.(15,16), do have a couple fewer parameters than FP use; this does give them slightly greater flexibility at larger t and u than our forms (of restricted usage) would allow if simply continued there.

With these requirements imposed, FP find one analytic amplitude, which is very similar to our amplitude β of fig. 2b. This is reassuring as our methods and assumptions are quite different. However, FP find just this amplitude alone. This they claim to be the unique analytic description of the CERN-Munich data[1,2]. This is, needless to say, at variance with our determination of the unitary, analytic solution α and our discussions of sections 3G and 4C. How does this arise? FP have made stronger requirements than we have. Resonance poles are explicitly built into their amplitudes at the ρ, f and g positions - positions set by the Particle Data Group[27a)] - regardless of spin. The amplitude with a strong coupling ρ', daughter of the g, results in refs.(31a,b). For us no excursions on to the second sheet are necessary but, of course, a principal value integration is.

It is important to stress that in our determination of the phases we do not constrain them by a third criterion (in addition to the two of section 3A) that the real part of the amplitude above 1.8 GeV (and below, say 5 GeV) should be Regge behaved and given by eqs. (15,16). This is particularly so for the effective, rather than absolute, Regge singularities we call the pomeron and that controlling backward scattering. These not necessarily being poles may have rather different real parts than a simple pole form gives, even asymptotically. Instead, as discussed in section 3A, our real parts in this intermediate energy region above the data are constrained only by imposing analyticity in the form of eq. (13). However, we have already noted that "average" Regge behaviour for the real parts does approximately happen, even though we do not constrain it to occur, for both amplitudes α,β in the very stable forward direction and for β, but not α, in the backward region. For FP Regge behaviour is required to have an early onset for both real and imaginary parts. It is then not too surprising that, along with a built-in daughter for the g, amplitude β only results[31b)]*.

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*In the earlier FP analysis[31a)], in which the precocious onset of Regge behaviour was not imposed and no claim of uniqueness made, essentially the same amplitude was found, viz. like our β. This happens because in this

(footnote cont'd)

In πN scattering it is well-known that the real parts of the amplitude are at low energies quite different, even 'on average', from Regge expectations, even when the imaginary part is 'on the average' Regge pole dominated. One consequence of the early onset of Regge behaviour for the backward $\pi^+\pi^-$ amplitude, together with the assumed low-lying nature of the responsible Regge trajectory, is that this amplitude should satisfy a superconvergence relation with a cut-off as low as 1.8 GeV**. It is seen that the amplitude of FP and our solution β approximately do fulfill this type of condition, while α does not.

The cut-off dependence of such relations has however been well-publicized in many papers[33]. Indeed if our amplitude β (like that of FP) were to continue to satisfy such a relation with a cut-off after the h, for example, just as after the g, odd spin daughters, having isospin one, would be needed to cancel the contribution of the h - or perhaps we have to wait till after the leading spin five resonance for the relation to happen to be satisfied again. As discussed in sect. 3D of ref. 19, it is clear that such speculations are, from a practical point of view, of little use. Indeed, theoretically, we may expect the leading "exotic" singularity to be the ρ-ρ and A_2-A_2 cuts, having $\alpha_{eff}(t)=2\alpha_\rho(\frac{1}{4}t)-1$ with "intercept" rather close to zero. From such discussions[33,19] it is therefore quite unclear whether we can believe the backward $\pi^+\pi^-$ amplitude satisfies an unsubtracted

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*(footnote cont'd)
analysis, along with built in daughter poles, the amplitude is assumed known throughout the S* region up to 1100 MeV despite inelasticity. As already indicated in sections 3E,4B, our analysis is rather sensitive to this region. Knowing the amplitude there fixes the solution. Indeed, our solution β is more easily compatible with current analyses of $K\bar{K}$ data[26].

**One of us (MRP) is grateful to J. L. Petersen for a most useful discussion on this subject.

dispersion relation, never mind a superconvergence relation with effective cut-off just where the CERN-Munich data happens to end. We have therefore chosen not to prejudice the issue of the exact position of this effective Regge singularity nor how its consequent Regge phase is approached. Thus we feel both amplitudes α and β are physically reasonable from the point of view of the high energy behaviour of the backward amplitude*.

Another source of difference between our analysis and that of Froggatt and Petersen[31)] is the question of smoothness. As set out in the introduction (section 1), our aim has been to avoid the artificial removal of irregularities in the data used by the phase shift analyst and to demand only that amount of smoothing theoretically justified by analyticity, and no other. In this way, we hope to expose all the dynamical, and hence analytic, structures 'hidden' in the data for a given process

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*It is perhaps useful to remark parenthetically that both we and FP find that changing the pomeron coupling, viz $\sigma_{tot}(\infty)$, from 8 mb (the value favoured by the fits) to 15 mb (the value suggested by factorisation) has a rather small effect on the forward amplitude [note: we plot our amplitudes for $\sigma_{tot}(\infty)$ = 8.3 mb, while FP set $\sigma_{tot}(\infty)$ = 15 mb for the plots of their final analysis]. However the variation we find, discussed in sect. 3D, is somewhat smaller and its energy dependence the opposite of that shown by FP in refs. (16,31). This difference arises because FP, in varying σ_{tot}, change the real and imaginary parts above 1.8 GeV together, while we allow the real part freely to follow the change in the imaginary part only as required by analyticity. The resultant changes below 1.8 GeV are smaller in our case and are least at the energies approaching 1.8 GeV and larger at lower energies. This is because the data in the elastic region prefers a smaller total cross-section, through eq. (13); so increasing this, worsens the agreement there, but it improves towards 1.8 GeV. Nonetheless, these differences are small and mainly of a technical nature.

from a given experiment just as they stand, and not to bias our analysis by believing some effects and not others. To be guided by experiment and basic theoretical tools requires us to keep as close to the data as possible. Nonetheless our resulting amplitudes, seen in figs. 2a,b, are remarkably smooth. These may of course not be quite smooth enough for nature and may still have wiggles which are unphysical - we cannot tell. In partial wave analysing our dispersive amplitudes we have been guided by the same principle. Figs. 3(a,b), 4,6 are our results unsmoothed except for the continuous Breit-Wigner forms used for the F wave below 1.44 GeV and for the G wave as previously discussed.

For Froggatt and Petersen[16,31], however, smoothness is of paramount importance. In constructing their analytic amplitude a penalty function is included to ensure smoothness in addition to analyticity. Then their phase shifts are wholly determined by just the t=0, u=0 analytic amplitudes and by the experimental moments*. The amplitude at other values of t,u, though determined out to $-1\ \mathrm{GeV}^2$, are used by FP, only in an iterative way, to smooth the resulting partial waves. Furthermore, a penalty factor is included in χ^2 to minimise the importance of all partial waves with $\ell>3$. This results in their $\ell=4$ and higher waves being negligible. As argued throughout, higher waves are absolutely essential for us to obtain analytic amplitudes if these are to describe the experimental moments with any precision. This is because, in our approach, the phase shifter's amplitudes of eq.(6) with $\chi(s,z)=0$ (c.f. eq.(7)) are not analytic. Judging by fig. 7 of FP's recent paper[31b] our fits to the dispersive amplitudes are decidedly better than theirs - this is hardly surprising since we have a free G wave, an $\ell=4$ wave

*We find the $t,u=-0.1,-0.2\ \mathrm{GeV}^2$ amplitudes are just as important for our partial wave analysis.

that is nonetheless physically most reasonable.

Moreover, the S wave of FP, like that of the phase shift solution B[9]) is non-unitary. The S wave of our analytic solution β satisfies unitarity as a direct consequence of our $\ell=4$ wave being non-zero. For FP, with $\ell\geq4$ waves forced to be negligible, a non-unitary S wave results. To solve this problem FP add to their χ^2 function for determining partial waves yet another contribution, viz $5(\eta_0^0)^4$, of a rather 'ad hoc' nature. This, by its very size, forces the S wave to have $\eta_0^0\sim0.7$. Since, as we show in fig. 6, the S wave is the least well determined partial wave, such a change would hardly worsen χ^2 for us too. However, we need no such constraint to obtain a unitary amplitude. Because our partial waves are simply those needed to represent the data on the moments and the dispersion relation amplitudes they may not be as smooth, or perhaps even as physical, but how they are obtained is quite straightforward.

The amplitude of FP, like our solution β, has a $\rho'\to2\pi$ coupling of some 30%. Since preliminary experimental results[6,27a]) suggest - but only suggest - this coupling may be much smaller and FP have no solution like our α, they try to reduce this coupling as much as possible by the addition of further weight factors to χ^2 in their partial wave analysis. They find interestingly enough 10% to be an extreme possibility but then the moments and their 'dispersive' amplitudes are poorly fitted. Since our approach is somewhat different, we only look for perfect fits and ask what coupling for $\rho'\to2\pi$ they give for each analytic amplitude. For instance, our amplitude β', in the continuum patch of β, has a coupling of the $\rho'\to2\pi$ half that of solution β but with a much larger $\ell=4$ wave. FP's approximation to this situation, having essentially no $\ell\geq4$ waves, can only considerably worsen the fit to the moments - just as they find.

In the introduction and in section 2A we have strongly argued that the phase shifter's restricted class of solutions (eq.(6)) does not contain the class of analytic amplitudes.[34] Our solutions α and β are explicit examples of this. For unlike solutions A,B both of these have $\chi(s,z)$ different from zero, particularly in the backward hemisphere. The final amplitude* of Froggatt and Petersen[31] constructed from summing the non-negligible, and hence $\ell \leqslant 3$, partial waves is however, in as much as it fits the experimental moments, just one of the phase shifter's amplitudes of eq.(6), in which the only unknown phase $\phi_o(s)$ has been fixed by analyticity and the zeros z_i as a function of s have been smoothed in a different way than done for, example, in ref. 9.

The purpose of comparing our analysis in such detail with that very recently completed by Froggatt and Petersen[31] is that such a comparison serves to summarise what we have attempted, what we believe we have achieved and to present the differences so that future analysts of other data (for other processes, even) can be aware of the assumptions, pitfalls, etc, in the different methods available.

6. Conclusions

We have set out to solve some of the problems inherent in present $\pi\pi$ phase shift analyses[7,8,9]. These are of three types: (i) the assumption that the maximum number of partial waves is half the number of non-zero moments, (ii) the forward phase in the inelastic region is guessed, (iii) the resulting amplitude is smoothed in some arbitrary fashion that cannot distinguish local dynamics from random fluctuations. We have attempted to solve all of these by the use of one basic theoretical requirement that

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*which is not exactly the same as their amplitude analytic in s.

the amplitude describing the data should be an element of the analytic S-matrix. This has required us to make assumptions about the experimentally unknown high energy behaviour of the $\pi\pi$ scattering amplitude. In this we have been guided by the success of the simple Regge pole model and considerations of duality.

We have found that the phase shifter's assumptions define a class of non-analytic solutions and that we must take account of the full continuum ambiguity in order to find amplitudes which are analytic[34]. However, both the phase shifter's non-analytic solutions A and B lead us to analytic, unitary solutions α and β respectively. Though our search for other analytic, unitary amplitudes not in the continuum patches of α,β cannot be considered exhaustive, we have in fact found no others.* Each of these is equally analytic; each equally well describes the CERN-Munich data upon which our analysis is wholly based. Nonetheless these amplitudes have quite distinct physical properties, which more experimental information could distinguish; for example, a detailed analysis in the S* region of both $\pi\pi$ and $K\bar{K}$ channels or a simultaneous analysis with high statistics $\pi^o\pi^o$ data, when it is available. Perhaps most excitingly our amplitudes have quite different P waves, one with a clearly resonant $\rho'(1575)$ coupling strongly to $\pi\pi$ and the other having none or very little coupling at all. These differences can be resolved by e^+e^- and/or photoproduction experiments[6], hopefully within the next few months. These results we eagerly await.

This analysis uses only the results of the high statistics $\pi^{\mp}\pi^+$ experiments of the CERN-Munich group[1,2]. It clearly can be applied to any set of data. We have restricted our attention to one only to show how an

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*This assumes that the final $\pi^o\pi^o$ production data will continue to eliminate the amplitudes that are obtained starting from solutions C and D of ref. 9.

analysis, which tries to limit the degree of unprescribed smoothing, can be used to isolate local structures in the data if these are real dynamical, and hence analytic, effects. Future experiments (e.g. ref. 29) and hopefully the final results of completed experiments (e.g. refs. 5,10) will, of course, show whether these details will remain in all future analytic amplitudes or disappear, or be replaced by new effects. In any case, our method of analysis provides a simple, straightforward way to delineate the relationship between experimental data and elements of the S-matrix, and thereby to explore and expose the features of strong interaction dynamics.

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Figure Captions

Fig. 1 The domains of the Mandelstam plot for which knowledge of the (absorptive part of) the amplitude is used in evaluating fixed t and fixed u dispersion relations as enumerated in section 3A. The s and t channels are $\pi^+\pi^- \to \pi^+\pi^-$ scattering and the u channel is $\pi^+\pi^+ \to \pi^+\pi^+$ scattering.

Figs. 2a,b Argand plots of the $\pi^+\pi^- \to \pi^+\pi^-$ amplitude, F(s,z), for t, u=0, -0.2 $(GeV/c)^2$ in the energy range $1.15 \leq \sqrt{s} \leq 1.71$ GeV. The points, at 20 MeV intervals, are obtained by using dispersion relations iteratively as described in the text. The amplitudes labelled α and β, shown in figs. a,b respectively, are obtained by starting the iterative procedure from solutions A and B of ref. 9.

Figs. 3a,b The Argand diagrams showing the $\pi\pi$ partial waves obtained by fitting, at 20 MeV intervals, the CERN-Munich moments (fig. 5) together with the values of the analytic amplitudes at t, u=0, -0.1, -0.2 GeV^2. The unitarity circles are for I=0 for S and D waves, and for I=1 for P and F waves. The I=2 S wave is assumed elastic with $\delta_0^2 = -25^\circ$, and the I=2 D wave is neglected. Solutions α and β, of figs. a,b respectively, correspond to using amplitudes α and β of figs. 2a,b. Below 1.44 GeV the $\ell=3$ wave is input as the tail of the Breit-Wigner form, eqs. (17,18), for the g resonance. The input for the $\ell=4$ wave is as described in fig. 4.

Fig. 4 The $\pi\pi$ $\ell=4$ wave obtained from the simultaneous partial wave analysis of the analytic amplitudes (fig. 2) and of the CERN-Munich moments. The results are displayed in terms of δ_4^o, obtained by fitting the real part of the $\ell=4$ wave. The parameter, δ_4^o, would be the phase shift if the $\ell=4$ wave were entirely I=0 and elastic. The curves correspond to the tail of a spin four I=0 (h) resonance (M=2.035 GeV, Γ_o=0.2 GeV, R=5 GeV^{-1}) of elasticity x = 0.2 and 0.25 respectively. These spin 4 resonance forms were input as the $\ell=4$ wave, for solutions α and β respectively, in the partial wave analysis of fig. 3.

Figs. 5a,b The Legendre moments, $A_L(s)$ of eq.(1), reconstructed (for $L\leq 6$) from an analysis[9] of the CERN-Munich data. If the $\ell=4$ is determined in the partial wave analysis then the L=7,8 moments (taken to be zero) are included in the fit. We show a typical error assumed for the L=7 moment. The continuous line corresponds to the fit of the partial wave solution β of figs. 3b, 4.

Fig. 6 The $\pi\pi$ partial waves obtained by the simultaneous analysis of the dispersion relation solution β' and the Legendre moments of fig. 5. The analysis is the same as those of fig. 3 except that the $\ell=3$ wave is determined by the fit at all energies and that the $\ell=4$ wave is determined for $M_{\pi\pi}$>1.32 GeV. The curves around the points at $M_{\pi\pi}$ = 1.39 and 1.59 GeV show the uncertainty as given by $\Delta\chi^2=1$.

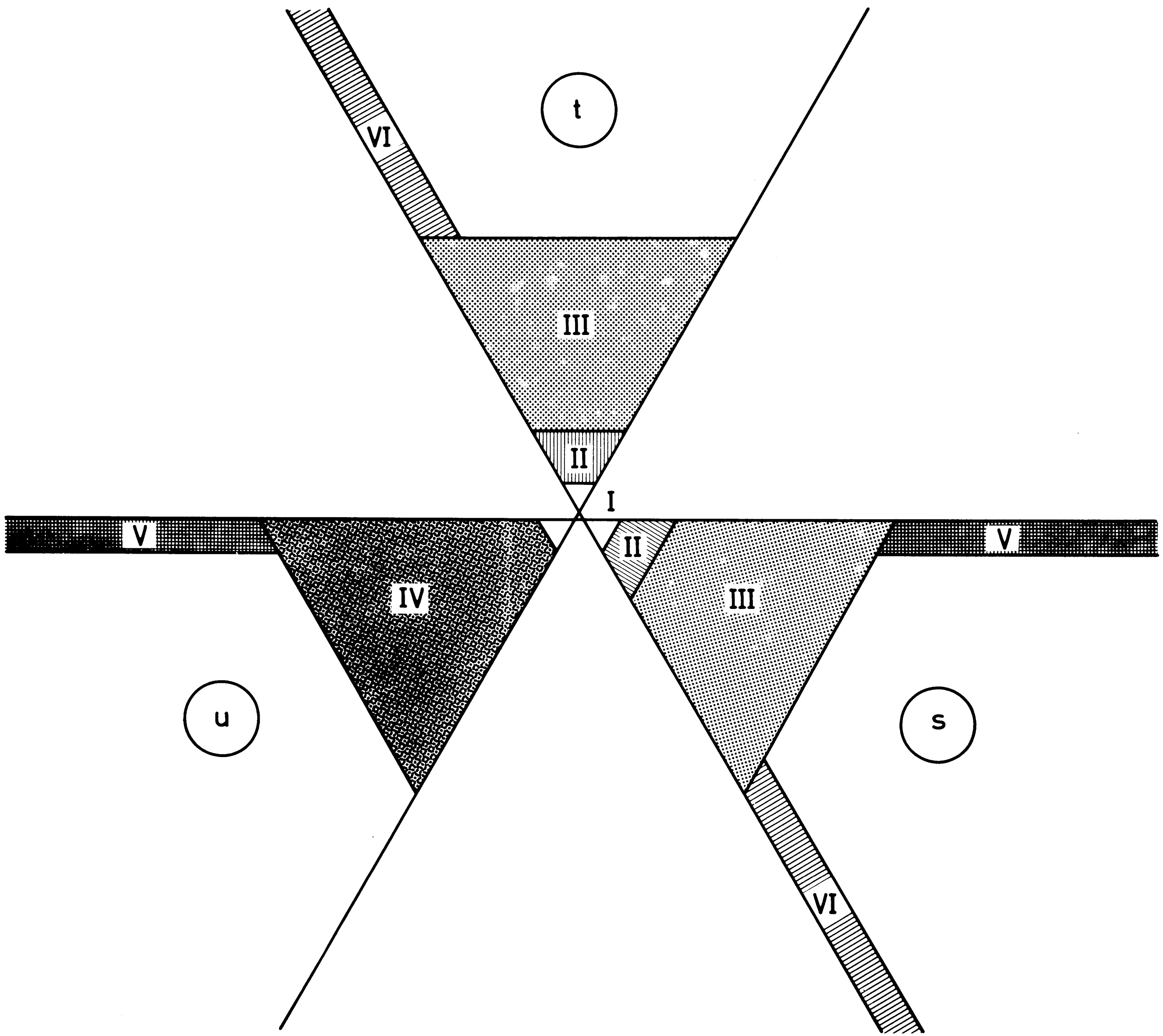


FIG.1

$\pi^- \pi^+$ Amplitude

Solution α

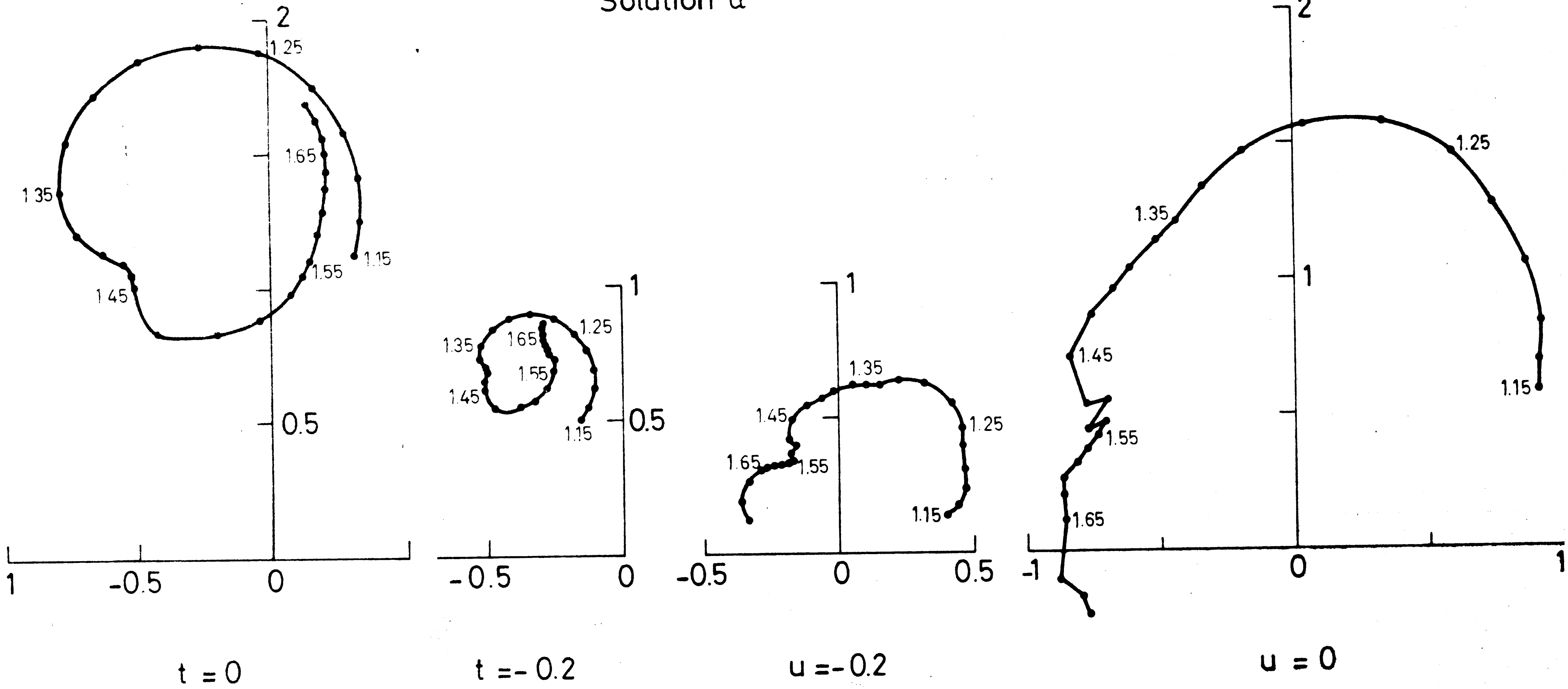
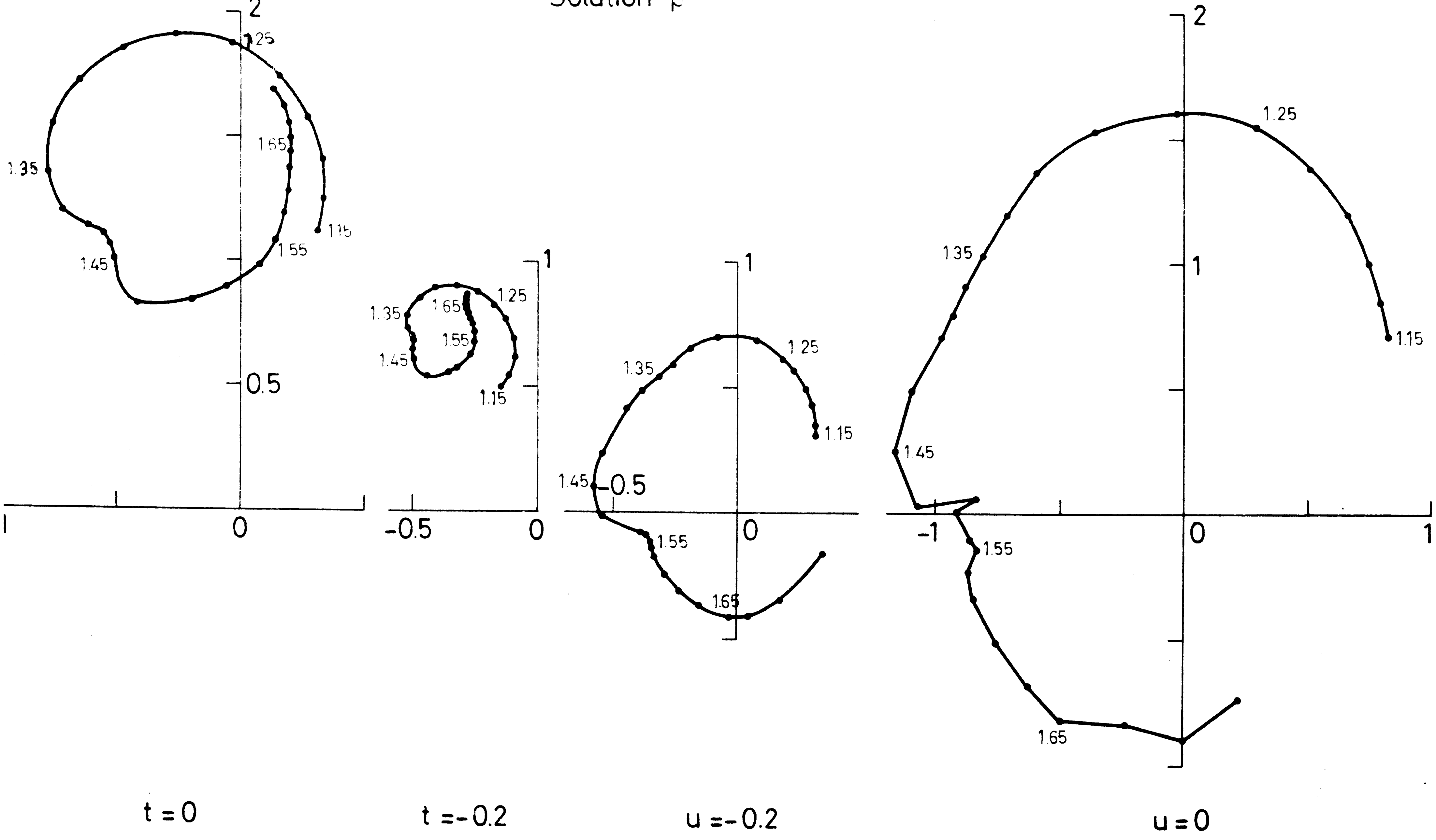


FIG. 2a

$\pi^- \pi^+$ Amplitude

Solution β



Solution α

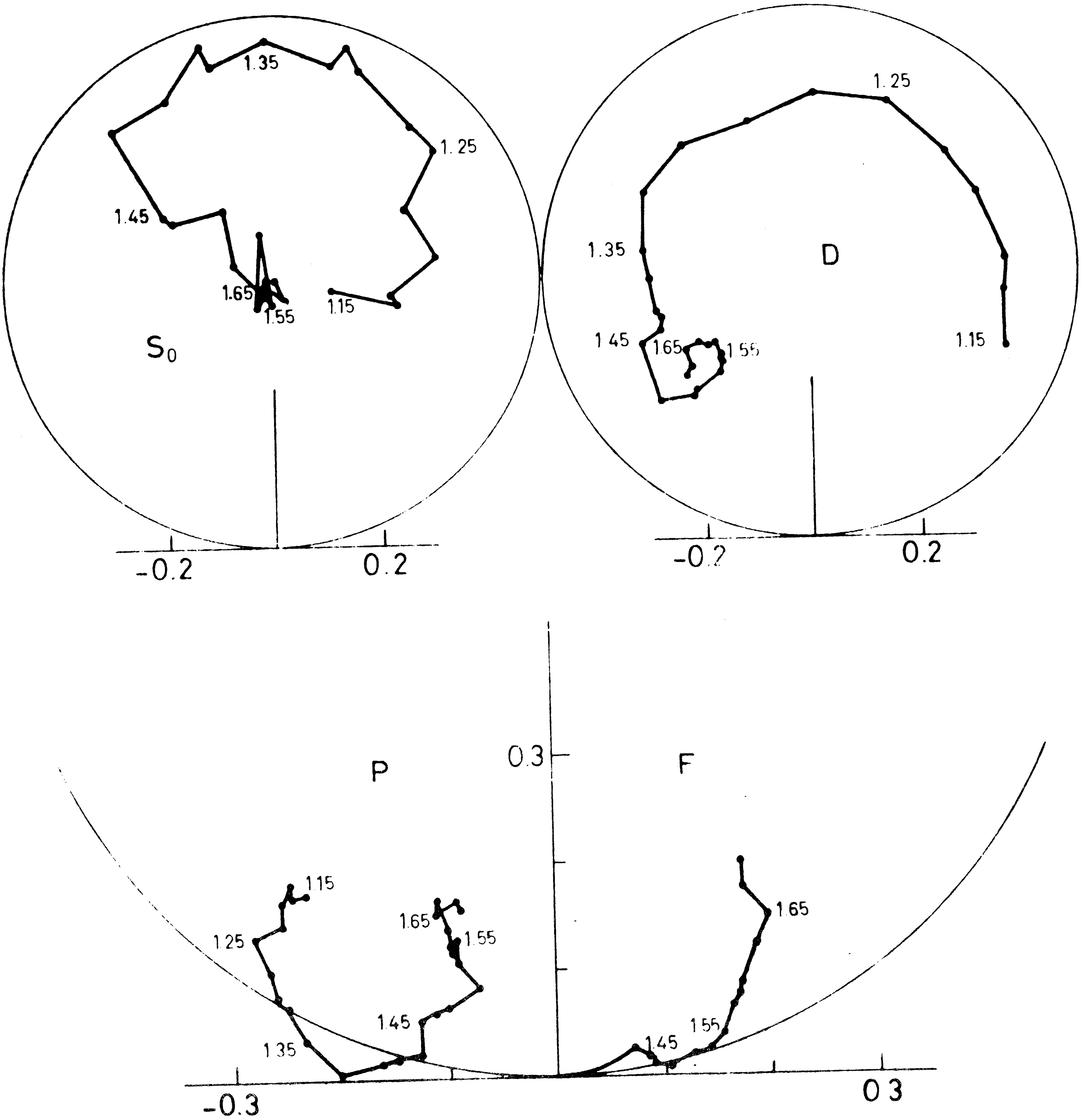


FIG.3a

Solution β

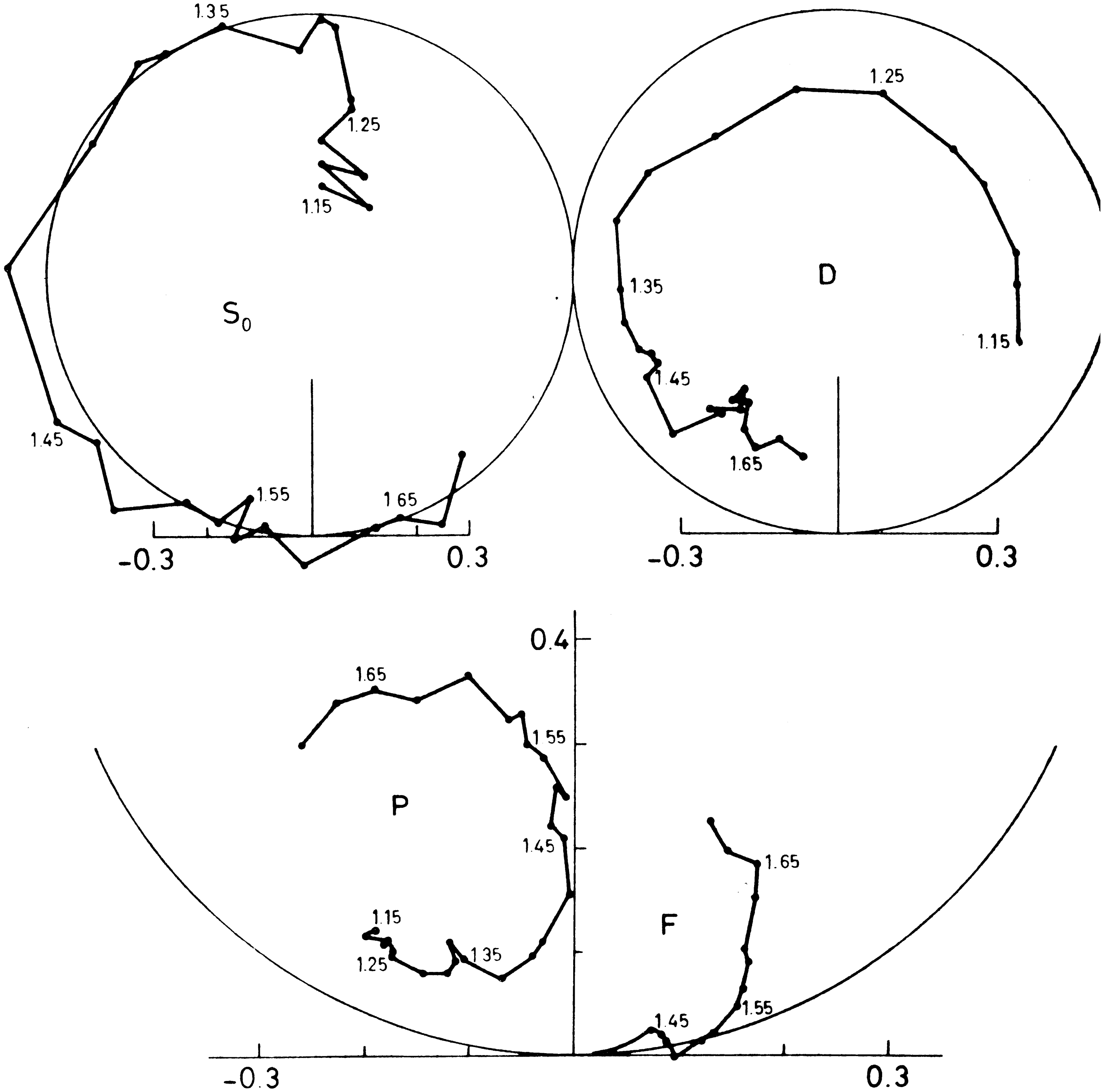


FIG. 3b

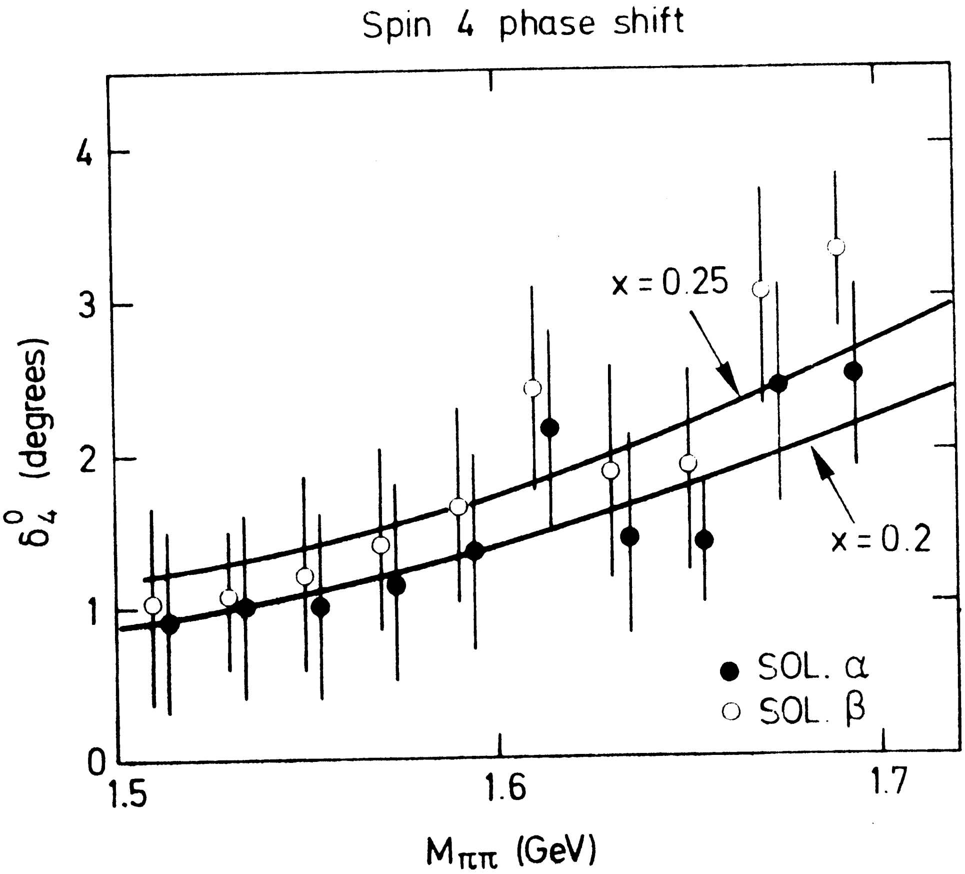

Spin 4 phase shift
δ_4^0 (degrees)
4
3
2
1
0
1.5
1.6
1.7
$M_{\pi\pi}$ (GeV)
x = 0.25
x = 0.2
● SOL. α
○ SOL. β

FIG. 4

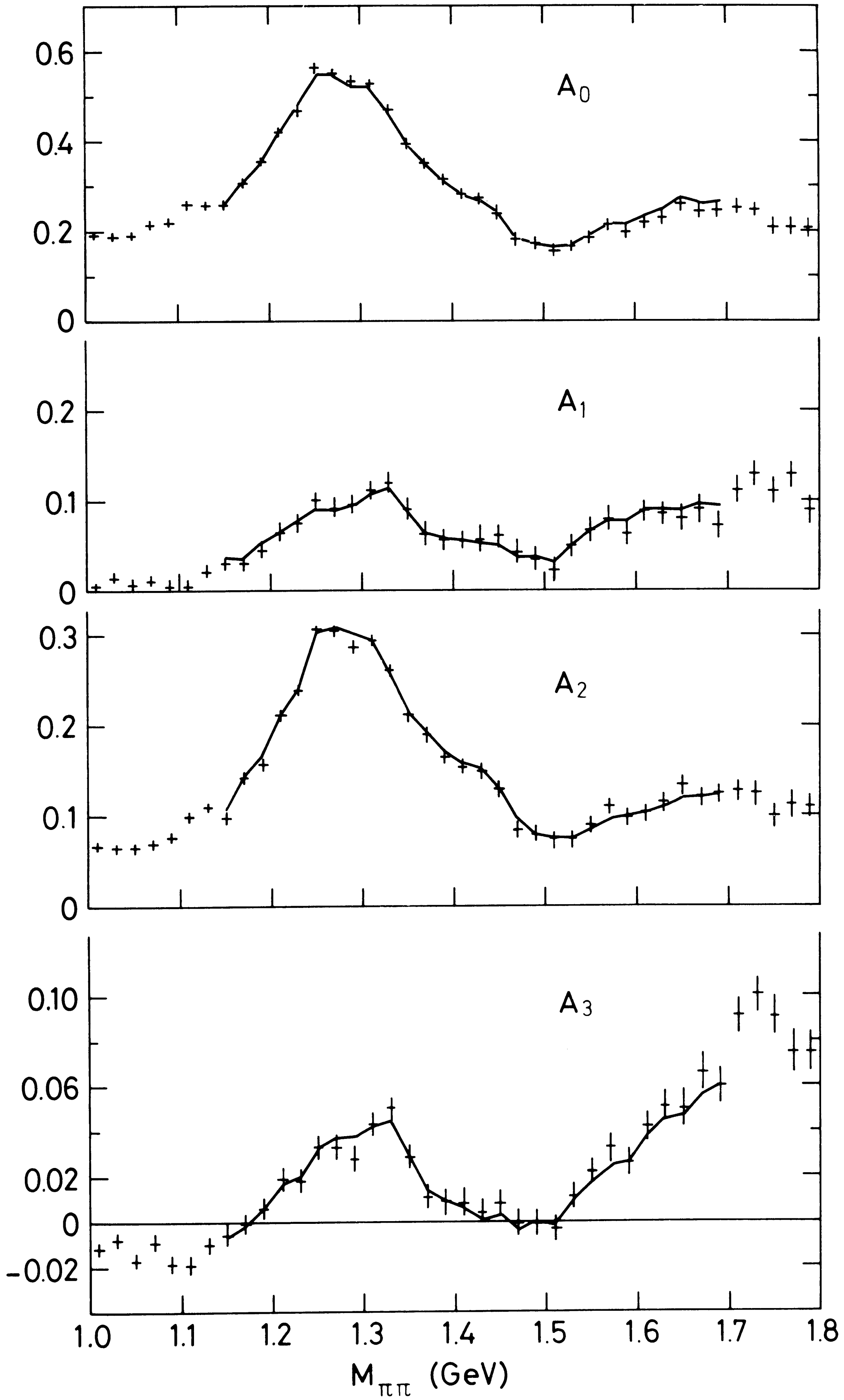
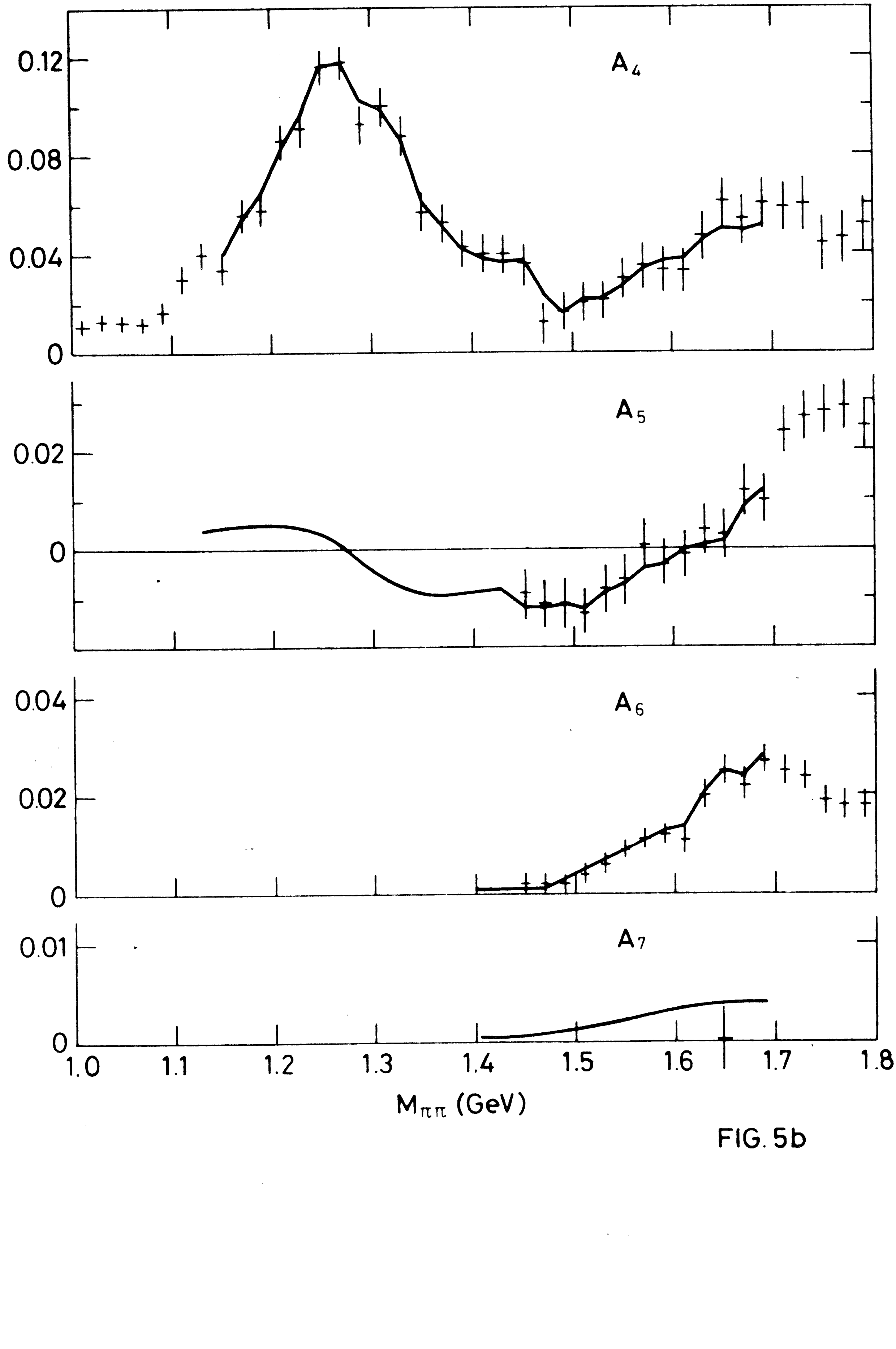

0.6
0.4
0.2
0
A0
0.2
0.1
0
A1
0.3
0.2
0.1
0
A2
0.10
0.06
0.02
0
-0.02
A3
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
Mππ (GeV)

FIG. 5a



A4
0.12
0.08
0.04
0
A5
0.02
0
A6
0.04
0.02
0
A7
0.01
0
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
Mππ (GeV)

FIG. 5b

Solution β'

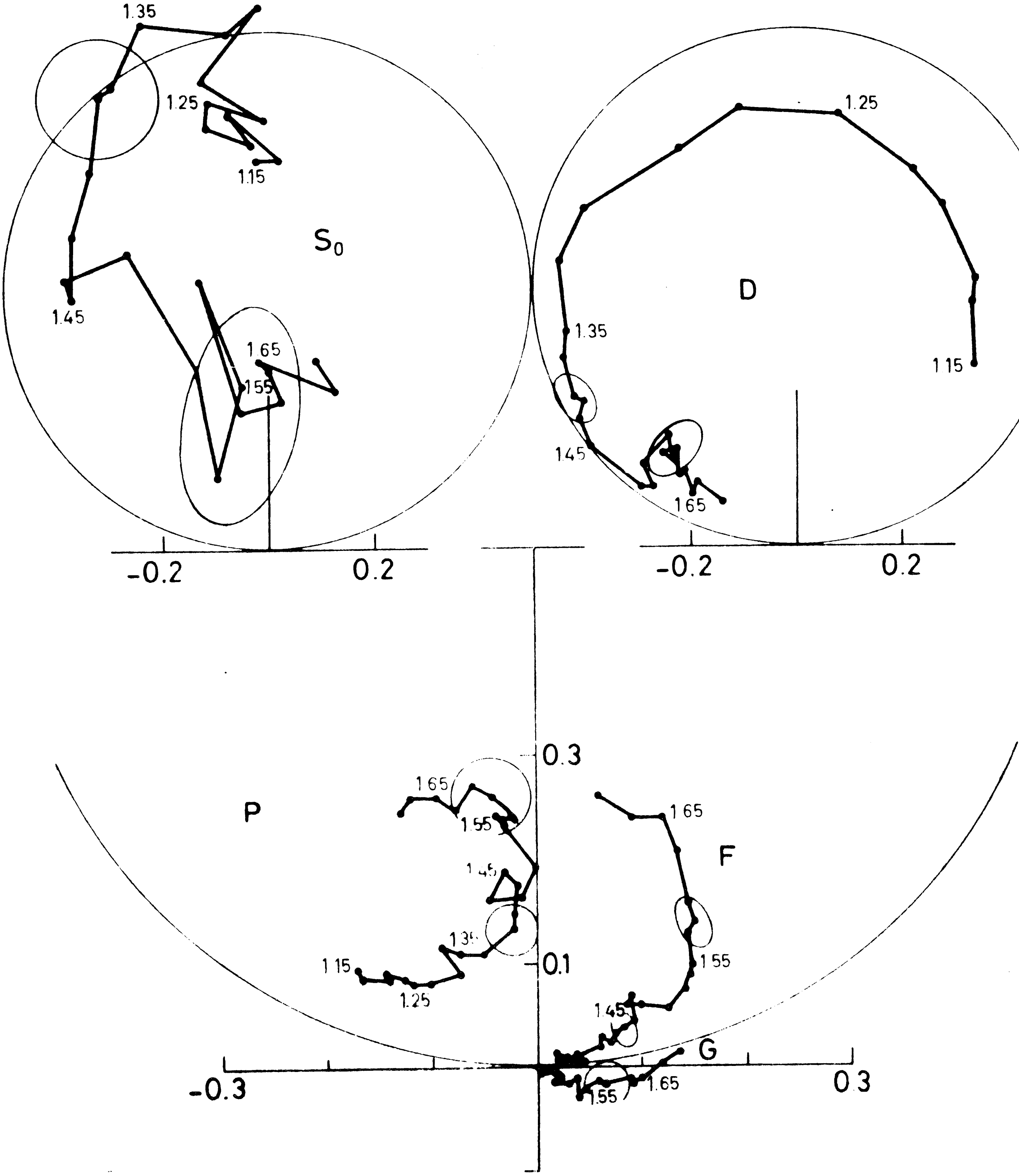


FIG. 6