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# GLAUBER THEORY, UNITARITY, AND THE AGK CANCELLATION

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## A B S T R A C T

It is shown that the Glauber theory for the scattering of a high-energy hadron on a nucleus becomes unitary, when all the possible inelastic intermediate states are included between successive scatterings. This result is used to prove that the n-th order multiple inelastic contributions satisfy the Abramovskii, Gribov and Kancheli cancellation, which is therefore a consequence of the multiple scattering structure of the theory.

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## 1. - INTRODUCTION

In a remarkable paper, Abramovskii, Gribov and Kancheli (1973) have proved, in the context of Reggeon calculus, the following theorem, known as the "AGK cancellation".

If one considers the single inclusive cross-section in the central region, one finds that it is given only by the contribution which results from cutting a single Pomeron exchange ; the contributions of all the other diagrams are cancelled by absorptive corrections. As the inclusive distribution corresponding to a single Pomeron exchange is a plateau in rapidity, the same result holds for the integrated inclusive distribution (excluding the fragmentation regions).

The detailed technical proof of the theorem is based upon the use of combinatorial techniques, and in particular of relations which closely recall the combinatorial expressions one finds in multiple scattering theories when dealing with the scattering of a projectile on a composite system. There the combinatorial factors come in when one expresses the S matrix for the scattering on the composite system, which is the product of the individual s matrices, in terms of the individual t matrices, or when one expresses a cross-section on the composite system in terms of the individual cross-sections.

It appears then likely that the AGK cancellation can be expressed in a simpler way through the use of the s matrices ; moreover one might wonder whether it is valid in multiple scattering theories wider than Reggeon calculus.

In fact, it is known (Chang and Yan, 1974 ; Capella et al., 1975 ; Andreev, 1975) that the AGK cancellation holds in eikonal models *).

In this paper we show that the AGK cancellation holds in any multiple scattering theory, which is unitary, and in which the individual scattering matrices are also unitary [for a closely related result in Reggeon field theory, see Ciafaloni and Marchesini, 1976].

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*) By eikonal models, we mean models in which only the incident particle is allowed to rescatter.

For this purpose, we shall use the generalized Glauber multiple scattering theory for the scattering and production in the high energy interaction of a hadron with a nuclear target containing A nucleons. Our proof will be along the following lines. First we prove that the Glauber theory is unitary when one includes between successive scattering all the possible elastic and inelastic intermediate states. The meaning of unitarity is here that the Glauber theory satisfies the optical theorem, namely that the total hadron-nucleus cross-section, computed by summing over all the possible final states, coincides with the expression given by the imaginary part of the forward coherent elastic scattering amplitude.

We then compute the total inelastic cross-section, as the difference between the total and total elastic (coherent plus incoherent) cross-sections. This inelastic cross-section is expressed as the sum of the partial inelastic cross-sections of order n, denoted by $\sigma_n$, which correspond to the process in which one has inelastic production on n nucleons, and in addition inelastic absorption on the remaining A-n nucleons. It is then found that the sum $\sum_{n=1}^{A} n\sigma_n$ equals $A\sigma_{in}$, namely is A times the projectile nucleon inelastic cross-section. This relation is just the AGK cancellation.

The technicalities of the proof go as follows : recalling the general expression (Gribov, 1970 ; Bertocchi, 1972, 1973a, 1973b) obtained from the Feynman diagram technique, of the coherent production amplitude for hadron-nucleus collisions, and which includes all the possible elastic and inelastic intermediate states, we construct the coherent and incoherent elastic and production cross-section, whose sum is shown to satisfy the optical theorem. To better understand the subtle points involved in the problem, the standard proof of the AGK cancellation is first rederived for the eikonal approximation. The next step consists in the proof of the AGK cancellation when all the inelastic intermediate states are present, but their longitudinal momentum transfers are neglected. Finally, the proof is given in the most general situation in which the longitudinal momentum transfers of the intermediate states are present.

## 2. - GLAUBER THEORY AND UNITARITY

The general amplitude for the coherent transition from the incident hadron state $\alpha$ to the final state $\beta$ on a nuclear target $|A>$ can be obtained from the Feynman diagram technique (Gribov, 1970 ; Bertocchi, 1972, 1973a, 1973b) which generalizes the Glauber theory (Glauber, 1967). If one neglects the constraints coming from the nuclear centre-of-mass motion, it can be written as *)

$$\langle A|\hat{F}_{\alpha\beta}|A\rangle = 4imp\sum_{n=1}^{A}\frac{A!(-1)^{n+1}}{(A-n)!}\langle A|\hat{G}^{(n)}_{\alpha\beta}|A\rangle \tag{1}$$

where the $n^{th}$ order operator $\hat{G}^{(n)}_{\alpha\beta}$ is given by

$$\begin{aligned}\hat{G}^{(n)}_{\alpha\beta}(\vec{x}_i) = \int d_2b\, e^{i\vec{q}\cdot\vec{b}} \sum_{\beta_1,\beta_2,\dots,\beta_{n-1}} & e^{-il_\alpha z_1}\Gamma_{\alpha,\beta_1}(\vec{b}_1-\vec{b})\cdot e^{il_{\beta_1}z_1}\cdot e^{-il_{\beta_1}z_2}\Gamma_{\beta_1,\beta_2}(\vec{b}_2-\vec{b})\cdot\\ & \cdot e^{il_{\beta_2}z_2}\cdots e^{-il_{\beta_{n-2}}z_{n-1}}\Gamma_{\beta_{n-2},\beta_{n-1}}(\vec{b}_{n-1}-\vec{b})e^{il_{\beta_{n-1}}z_{n-1}}\cdot e^{-il_{\beta_{n-1}}z_n}\cdot\Gamma_{\beta_{n-1},\beta}(\vec{b}_n-\vec{b})\cdot\\ & \cdot e^{il_\beta z_n}\cdot\theta(z_n-z_{n-1})\cdot\theta(z_{n-1}-z_{n-2})\cdots\theta(z_3-z_2)\cdot\theta(z_2-z_1)\end{aligned} \tag{2}$$

and $<A|\;|A>$ means $\prod_{i=1}^{A}\int d_3x_i\rho(\vec{x}_i)$ ; $\rho(\vec{x})$ is the one-particle nuclear density, normalized to $\int d_3x\,\rho(\vec{x}) = 1$.

In (1) and (2) the different symbols have the following meaning : $m$ is the nucleon mass ; $p$ the total impact parameter ; $\vec{b}_i$, $z_i$ the transverse and longitudinal co-ordinates of the $i^{th}$ target nucleons ; $l_{\beta_i} = m^2_{\beta_i}/2p$ the longitudinal momentum transfer on the $\beta_i$ state ; $\Gamma_{\gamma,\delta}(\vec{b}_i)$ the profile matrix for the transition $\gamma\to\delta$ on the $i^{th}$ nucleon, related to the $\gamma\to\delta$ production or scattering amplitude by

$$f_{\gamma\delta}(\vec{q}) = \frac{1}{2\pi ip}\int d_2b_i\, e^{-i\vec{q}\cdot\vec{b}_i}\,\Gamma_{\gamma,\delta}(\vec{b}_i)$$

The expression (2) represents the most general amplitude for the interaction with $n$ of the target nucleons (the remaining $A$-$n$ being "spectators"). The intermediate states $\beta_i$ are the most general

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*) The normalizations are the same as in Bertocchi (1972).

(multiparticle) intermediate states allowed by energy conservation. It is the presence of the inelastic intermediate states $\beta_i \neq \alpha$, for whom $\ell_{\beta_i} \neq \ell_\alpha$, which renders the theory much more complicated than the standard Glauber eikonal theory in which, for elastic scattering, one keeps only the elastic intermediate states $\beta_i = \alpha$.

We shall see, however, that it is still possible to express in a simple form the total cross-section and various integrated quantities, like the summed inelastic cross-section, even in the presence of the inelastic intermediate states.

Let us first recall the usual terminology of particle-nucleus interaction ; we shall call :

- <u>elastic</u>, a reaction in which $\alpha = \beta$ ;
- <u>inelastic</u>, the one in which $\alpha \neq \beta$ ;
- <u>coherent</u>, the reaction in which there is no change of the nuclear state $|A' > = |A >$ ;
- <u>incoherent</u>, the one in which there is a change of nuclear state $|A' > \neq |A >$ ;
- <u>summed</u>, the reaction in which one does not distinguish the final nuclear state, and therefore one sums over all the states $|A' >$.

We shall also use the word "integrated" to denote a reaction in which one has integrated over the angles a differential cross-section.

Starting from the expression (2), we can now immediately write down the amplitude for a reaction in which the nucleus makes a transition to a final state $|A' >$, which can be either equal or different from $|A >$, as

$$\langle A | \hat{F}_{\alpha\beta} | A' \rangle \qquad (3)$$

where $\hat{F}_{\alpha\beta}$ is again given by (2).

We rewrite now (2) introducing the co-ordinate space individual s matrices, denoted by $S^{(i)}_{\alpha\beta}(\vec{b}_i, z_i)$, and related to $\Gamma^{(i)}_{\alpha\beta}(\vec{b}_i)$ by

$$S^{(i)}_{\alpha\beta}(\vec{b}_i, z_i) = \delta_{\alpha\beta} - e^{-i\, l_\alpha z_i}\, \Gamma_{\alpha\beta}(\vec{b}_i)\, e^{i\, l_\beta z_i} \qquad (4)$$

The unitarity relations obeyed by $S^{(i)}$ and $\Gamma^{(i)}$ are

$$\sum_{\gamma} S^{(i)}_{\alpha\gamma} S^{(i)*}_{\beta\gamma} = \delta_{\alpha\beta} \quad ; \quad \Gamma^{(i)}_{\alpha\beta} + \Gamma^{(i)*}_{\beta\alpha} = \sum_{\gamma} \Gamma^{(i)}_{\alpha\gamma} \Gamma^{(i)*}_{\beta\gamma} \tag{5}$$

or, in operator form

$$\hat{S}^{(i)} \hat{S}^{(i)\dagger} = \hat{1} \quad ; \quad \hat{\Gamma}^{(i)} + \hat{\Gamma}^{(i)\dagger} = \hat{\Gamma}^{(i)} \hat{\Gamma}^{(i)\dagger} \tag{6}$$

The operator $\hat{F} = \hat{1} - \hat{S}$ in (1), (2), (3), can be easily rewritten as follows

$$\hat{F} = \hat{1} - \hat{S} = \hat{1} - \vec{Z}\left[\prod_{i=1}^{A} \hat{S}^{(i)}\right] \tag{7}$$

where the symbol $\vec{Z}$ has the meaning of taking all the possible permutations of the $z$ ordered products of the $\hat{S}^{(i)}$ operators ; the convention on the arrow is that the $z$ co-ordinate of an operator at the left is smaller than the $z$ co-ordinates of the operators at its right.

We are now going to compute the following quantities :

- the total $\alpha$ nucleus cross-section, using the optical theorem, which is given by :

$$\sigma^{\alpha A}_{tot} = 2\int d_2 b \prod_{i=1}^{A} \int d_3 x_i\, \rho(\vec{x}_i)\, \mathrm{Re}\left[1 - \left[\vec{Z}\left(\prod_{j=1}^{A} \hat{S}^{(j)}(\vec{b}_j - \vec{b}, z_j)\right)\right]_{\alpha\alpha}\right] \tag{8}$$

- the summed $(\sum_{A'})$, integrated $(\int d_2 q)$ cross-section for the reaction $\alpha \to \beta$

$$\sigma^{\alpha\to\beta}_{summed} = \int d_2 b \prod_{i=1}^{A} \int d_3 x_i\, \rho(\vec{x}_i) \left| \left[\hat{1} - \hat{S}\right]_{\alpha\beta} \right|^2 = \int d_2 b \prod_{i=1}^{A} \int d_3 x_i\, \rho(\vec{x}_i) \left[\hat{1} - \hat{S}\right]_{\alpha\beta} \cdot \left[\hat{1} - \hat{S}^{\dagger}\right]_{\beta\alpha} \tag{9}$$

To get this result, we have used the closure relation of the final nuclear states, $\sum_{A'} |A'\rangle\langle A'| = 1$, plus the orthogonality of the plane waves, which upon the angular integration diagonalizes the impact parameter. In particular, we have for the elastic summed integrated cross-section

$$\sigma^{\text{elastic}}_{\text{summed}} = \sigma^{\alpha\to\alpha}_{\text{summed}} = \int d_2 b \prod_{i=1}^{A} \int d_3 x_i\, \varrho(\vec{x}_i) \left| 1 - \hat{S}_{\alpha\alpha} \right|^2 \tag{10}$$

- the sum over all the final states $\beta$ of the summed integrated cross-section given by (9)

$$\sum_\beta \sigma^{\alpha\to\beta}_{\text{summed}} = \int d_2 b \prod_{i=1}^{A} \int d_3 x_i\, \varrho(\vec{x}_i) \sum_\beta \left[\hat{1} - \hat{S}\right]_{\alpha\beta} \cdot \left[\hat{1} - \hat{S}^\dagger\right]_{\beta\alpha} =$$

$$= \int d_2 b \prod_{i=1}^{A} \int d_3 x_i\, \varrho(\vec{x}_i) \left[ 1 - 2\,\mathrm{Re}\,\hat{S}_{\alpha\alpha} + (\hat{S}\hat{S}^\dagger)_{\alpha\alpha} \right] \tag{11}$$

Now, since

$$\hat{S} = \vec{Z}\left[\prod_{i=1}^{A} \hat{S}^{(i)}\right] \quad ; \quad \hat{S}^\dagger = \overleftarrow{Z}\left[\prod_{i=1}^{A} \hat{S}^{(i)\dagger}\right] \tag{12}$$

we have $\hat{S}\,\hat{S}^\dagger = \prod_{i=1}^{A} \hat{S}^{(i)}\hat{S}^{(i)\dagger} = \hat{1}$. In other words, the unitarity of the individual $\hat{S}^{(i)}$ operators implies the unitarity of the complete $\hat{S}$ operator.

Therefore, the expression (11) coincides with (8), and the optical theorem is verified.

This result is not trivial, as it might appear at first sight ; first of all, in order to have the optical theorem verified, we must compute the summed cross-sections, adding the coherent and incoherent cross-sections ; secondly, this result tells us that the standard eikonal form of the Glauber theory is non-unitary, and unitarity is restored only summing over all the possible elastic and inelastic intermediate states. In particular, the eikonal form for the total hadron-nucleus cross-section and the Kölbig-Margolis (1968) expressions for the coherent and incoherent nuclear production are not mutually consistent.

This result, however, does not necessarily imply the failure of the phenomenological applications of the standard Glauber-Margolis theory to total cross-sections or coherent production reactions ; the fact that this theory is actually in good agreement with the presently available experiments, with only small corrections coming from the inelastic intermediate states, implies the dominance of the eikonal approximation ; there are also theoretical arguments (Weis, 1976) to understand this dominance for the total cross-sections.

We end this section writing down the expression for the total inelastic cross-section in the $\alpha$ nucleus collision as the difference between the total cross-section given by (8) and the summed integrated elastic cross-section given by (10)

$$\sigma^{inel}_{summed} = \sigma^{\alpha A}_{Tot} - \sigma^{elastic}_{summed} = \int d_2 b \prod_{i=1}^{A} \int d_3 x_i \, \rho(x_i) \left[ 1 - \left| \hat{S}_{\alpha\alpha} \right|^2 \right] \tag{13}$$

## 3. - THE AGK CANCELLATION

We concentrate now on the total inelastic cross-section (13). As we have shown that Glauber theory is unitary, it is obvious that the inelastic cross-section must have this expression ; the complication given by the presence of inelastic intermediate states is, however, hidden in the matrix expression for $\hat{S}_{\alpha\alpha}$, which contains all the complications of the $z$ ordering.

Our aim is now to produce for $\sigma^{inel}_{summed}$ two different multiple scattering expansions.

To fix our ideas, we start with the simplified expression for the $\hat{S}$ operator (7), in which we neglect all the longitudinal momenta (this would be the situation at infinite energy, if all the intermediate states have finite mass). The $z$ ordering is then absent, since the $\hat{S}^{(i)}$ no longer depend upon $z_i$, and we simply have

$$\hat{S} = \prod_{i=1}^{A} \hat{S}^{(i)}(\vec{b}_i - \vec{b}) \quad ; \quad \hat{S}^{(i)} = \hat{1} - \hat{\Gamma}^{(i)}(\vec{b}_i - \vec{b}) \tag{14}$$

Let us also start with the eikonal approximation, in which only the elastic intermediate state is kept ; we have in this case

$$\hat{S}_{\alpha\alpha} = \prod_{i=1}^{A} S_{\alpha\alpha}^{(i)}(\vec{b}_i - \vec{b}) \quad ; \quad |\hat{S}_{\alpha\alpha}|^2 = \prod_{i=1}^{A} |S_{\alpha\alpha}^{(i)}(\vec{b}_i - \vec{b})|^2$$

$$\prod_{i=1}^{A} \int d_3 x_i\, \varrho(\vec{x}_i) \prod_{j=1}^{A} |S_{\alpha\alpha}^{(j)}(\vec{b}_j - \vec{b})|^2 = \left[ \int d_3 x_i\, \varrho(\vec{x}_i) |S_{\alpha\alpha}^{(i)}(\vec{b}_i - \vec{b})|^2 \right]^A \tag{15}$$

This expression can be simplified, using the wide nucleus approximation, valid for $A \gg 1$. We repeat here the details of the derivation, both for completeness and since this will be useful later. We have

$$\int d_3 x_i\, \varrho(\vec{x}_i) |S_{\alpha\alpha}^{(i)}(\vec{b}_i - \vec{b})|^2 = \int d_3 x_i\, \varrho(\vec{x}_i) \left[ 1 - 2\,\mathrm{Re}\left(\Gamma_{\alpha\alpha}^{(i)}(\vec{b}_i - \vec{b})\right) + |\Gamma_{\alpha\alpha}^{(i)}(\vec{b}_i - \vec{b})|^2 \right]$$

Now

$$2\,\mathrm{Re}\left(\int d_3 x_i\, \varrho(\vec{x}_i)\, \Gamma_{\alpha\alpha}(\vec{b}_i - \vec{b})\right) = \frac{1}{\pi p} \mathrm{Re}\left[ \frac{1}{i} \int d_3 x_i\, \varrho(\vec{x}_i) \int d_2 q' f(\vec{q}') e^{-i\vec{q}'\cdot(\vec{b}_i - \vec{b})} \right] \simeq$$

$$\underset{A \gg 1}{\simeq} \frac{1}{\pi p} \mathrm{Re}\left[ \frac{f(0)}{i} \int d_3 x_i\, \varrho(\vec{x}_i) \int d_2 q' e^{-i\vec{q}'\cdot(\vec{b}_i - \vec{b})} \right] = \sigma_T \cdot T(\vec{b})$$

where

$$T(\vec{b}) = \int_{-\infty}^{+\infty} dz\, \varrho(\vec{b}, z)$$

is the usual thickness function, and

$$\int d_3 x_i\, \varrho(\vec{x}_i) |\Gamma_{\alpha\alpha}(\vec{b}_i - \vec{b})|^2 = \frac{1}{4\pi^2 p^2} \int d_3 x_i\, \varrho(\vec{x}_i) \int d_2 q_1\, d_2 q_2\, f(\vec{q}_1) f^*(\vec{q}_2)\, e^{-i(\vec{q}_1 - \vec{q}_2)(\vec{b}_i - \vec{b})} =$$

$$= \frac{1}{4\pi^2 p^2} \int d_3 x_i\, \varrho(\vec{x}_i) \int d_2 q\, d_2 Q\, f(\vec{Q} + \tfrac{\vec{q}}{2}) f^*(\vec{Q} - \tfrac{\vec{q}}{2}) e^{-i\vec{q}\cdot(\vec{b}_i - \vec{b})} \simeq$$

$$\underset{A \gg 1}{\simeq} \int d_3 x_i\, \varrho(\vec{x}_i) \int \frac{d_2 q}{4\pi^2} e^{-i\vec{q}\cdot(\vec{b}_i - \vec{b})} \int \frac{d_2 Q}{p^2} |f(\vec{Q})|^2 = \sigma_{el}\, T(\vec{b})$$

Therefore

$$\int d_3x_i\, \rho(\vec{x}_i)\,|S^{(i)}_{\alpha\alpha}|^2 = 1 - \sigma_{in}\, T(\vec{b}) \tag{16}$$

where $\sigma_{in} = \sigma_T - \sigma_{el}$ is the total inelastic cross-section of the incident hadron with one single nucleon. One gets therefore the usual expression of the inelastic summed cross-section in the eikonal approximation

$$\sigma^{inel}_{summed} = \int d_2b \left[ 1 - \left(1 - \sigma_{in}\, T(\vec{b})\right)^A \right] \tag{17}$$

We remark here that even when only the elastic intermediate state is allowed, so that the total $\hat{S}$ operator becomes explicitly non-unitary, production comes in through the unitarity of the individual $\hat{S}^{(i)}$ operators. We write now (17) in the form of two different expansions

a) $$\sigma^{inel}_{summed} = \sum_{n=1}^{A} \tilde{\sigma}_n\,;$$

$$\tilde{\sigma}_n = (-1)^{n+1} \frac{A!}{(A-n)!\, n!} \int d_2b \left[ \sigma_{in}\, T(\vec{b}) \right]^n \tag{18a}$$

b) $$\sigma^{inel}_{summed} = \sum_{n=1}^{A} \sigma_n\,;$$

$$\sigma_n = \frac{A!}{(A-n)!\, n!} \int d_2b \left[ \sigma_{in}\, T(\vec{b}) \right]^n \cdot \left[ 1 - \sigma_{in}\, T(\vec{b}) \right]^{A-n} \tag{18b}$$

Expression a) is a power series expansion in terms of the individual inelastic cross-sections $\sigma_{in}$, while in b) $\sigma_n$ represents the cross-section of the physical process in which n nucleons have undertaken production, while the remaining A-n nucleons have only provided inelastic absorption. This interpretation of $\sigma_n$ is correct when the successive interactions with the different target nucleons are independent. We stress here three points :

i) the $\tilde{\sigma}_n$ are alternatively of positive and negative sign, while all the $\sigma_n$ are positive definite [approximation (15) is only meaningful when $[1-\sigma_n T(\vec{b})] \geq 0$, as it must give $|S^{(i)}_{\alpha\alpha}|^2$] ;

ii) in the $\sigma_n$, the absorption factor $[1-\sigma_n T(\vec{b})]^{A-n}$ [which becomes $\exp[-A\sigma_n T(\vec{b})]$ in the optical limit $A\to\infty$] contains the inelastic cross-section $\sigma_{in}$ and not the total individual cross-section $\sigma_T$ ; the origin of this fact is very well known ; as we look at an integrated quantity, only the truly inelastic events remove particles from the incident beam, while those removed by elastic scattering are found again at different angles ;

iii) $\tilde{\sigma}_n \neq \sigma_n$. In particular

$$\tilde{\sigma}_1 = A\,\sigma_{in}\int d_2 b\, T(\vec{b}) = A\,\sigma_{in} \tag{19}$$

If we compute now the sum

$$M = \sum_{n=1}^{A} n\,\sigma_n$$

where the partial cross-sections $\sigma_n$ are weighted with the factor $n$, we get

$$M = \sum_{n=1}^{A} n\,\sigma_n = \tilde{\sigma}_1 = A\,\sigma_{in} \tag{20}$$

which is equivalent to the AGK cancellation.

We will now prove that relation (20) is still valid, always neglecting (for the moment) the longitudinal momenta, when we do not restrict to the eikonal approximation, but we allow all the intermediate states.

Computing again $|\hat{S}_{\alpha\alpha}|^2$ we have

$$|\hat{S}_{\alpha\alpha}|^2 = \left(\hat{S}^{(1)}\cdot\hat{S}^{(2)}\cdots\hat{S}^{(A)}\right)_{\alpha\alpha}\cdot\left(\hat{S}^{(1)*}\,\hat{S}^{(2)*}\cdots\hat{S}^{(A)*}\right)_{\alpha\alpha} =$$

$$= \sum_{\substack{\beta_1,\beta_2,\cdots\beta_{A-1}\\ \gamma_1,\gamma_2,\cdots\gamma_{A-1}}} S^{(1)}_{\alpha\beta_1}\cdot S^{(2)}_{\beta_1\beta_2}\cdots S^{(A)}_{\beta_{A-1}\alpha}\cdot S^{(1)*}_{\alpha\gamma_1}\cdot S^{(2)*}_{\gamma_1\gamma_2}\cdots S^{(A)*}_{\gamma_{A-1}\alpha}$$

The term depending upon $\vec{b}_j$ has the general structure

$$S^{(j)}_{\mu\nu}(\vec{b}_j-\vec{b})\, S^{(j)*}_{\varrho\sigma}(\vec{b}_j-\vec{b}) = \left[\delta_{\mu\nu} - \Gamma^{(j)}_{\mu\nu}(\vec{b}_j-\vec{b})\right]\cdot\left[\delta_{\varrho\sigma} - \Gamma^{(j)*}_{\varrho\sigma}(\vec{b}_j-\vec{b})\right]$$

Working again in the wide nucleus approximation as in (16) we have

$$\int d_3x_j\, \varrho(\vec{x}_j)\, S^{(j)}_{\mu\nu}(\vec{b}_j-\vec{b})\cdot S^{(j)*}_{\varrho\sigma}(\vec{b}_j-\vec{b}) =$$

$$= \delta_{\mu\nu}\delta_{\varrho\sigma} - \left[\frac{2\pi}{ip} f_{\mu\nu}(0)\,\delta_{\varrho\sigma} - \frac{2\pi}{ip}\,\delta_{\mu\nu}\, f^*_{\varrho\sigma}(0)\right] T(\vec{b}) +$$

$$+ \frac{1}{p^2}\int d_2q\, f_{\mu\nu}(\vec{q})\, f^*_{\varrho\sigma}(\vec{q})\cdot T(\vec{b}) = \left[\hat{1} - \hat{G}(\vec{b})\right]^{\varrho\sigma}_{\mu\nu} \tag{21}$$

where

$$\hat{G}(\vec{b})^{\varrho\sigma}_{\mu\nu} = \left[\frac{2\pi}{ip} f_{\mu\nu}(0)\,\delta_{\varrho\sigma} - \frac{2\pi}{ip}\,\delta_{\mu\nu}\, f^*_{\varrho\sigma}(0) - \frac{1}{p^2}\int d_2q\, f_{\mu\nu}(\vec{q})\, f^*_{\varrho\sigma}(\vec{q})\right]\cdot T(\vec{b})$$

We get therefore for $\sigma^{\text{inel}}_{\text{summed}}$ the matrix expression

$$\sigma^{\text{inel}}_{\text{summed}} = \int d_2b\left[\hat{1} - \left[\hat{1} - \hat{G}(\vec{b})\right]^A\right]^{\alpha\alpha}_{\alpha\alpha} \tag{22}$$

As in (18), we expand $[\hat{1} - [\hat{1} - \hat{G}(\vec{b})]^A]$ to give

$$\sigma^{\text{inel}}_{\text{summed}} = \sum_{A=1}^{n} \tilde{\sigma}_n$$

$$\tilde{\sigma}_n = (-1)^{n+1}\frac{A!}{(A-n)!\,n!}\int d_2b\left[\left(\hat{G}(\vec{b})\right)^n\right]^{\alpha\alpha}_{\alpha\alpha} \tag{23a}$$

$$\sigma^{\text{inel}}_{\text{summed}} = \sum_{A=1}^{n} \sigma_n$$

$$\sigma_n = \frac{A!}{(A-n)!\,n!}\int d_2b\left[\left(\hat{G}(\vec{b})\right)^n\left(\hat{1} - \hat{G}(\vec{b})\right)^{A-n}\right]^{\alpha\alpha}_{\alpha\alpha} \tag{23b}$$

It is clear that expressions (23a) and (23b) for $\tilde{\sigma}_n$ and $\sigma_n$ are quite different from those given in (18a,b) ; however, it is still true that

$$M = \sum_{n=1}^{A} n\,\sigma_n = \tilde{\sigma}_1$$

and moreover

$$\tilde{\sigma}_1 = A\int d_2 b\; G(\vec{b})^{\alpha\alpha}_{\alpha\alpha} = A\left[\frac{4\pi}{p}\,\mathrm{Im}\,f_{\alpha\alpha}(0) - \frac{1}{p^2}\int d_2 q\,|f_{\alpha\alpha}(\vec{q})|^2\right]\cdot\int d_2 b\; T(\vec{b}) = A\,\sigma_{in}$$

so that the AGK cancellation is again valid.

We now turn to the most general case, where we retain the longitudinal momenta ; $\sigma^{inel}_{summed}$, $\hat{S}$ and $\hat{S}^+$ are given by (12) and (13). We remark that when we compute the product $\hat{S}_{\alpha\alpha}\cdot\hat{S}^*_{\alpha\alpha}$, we have only to keep in the product those terms corresponding to the same permutation of the z's in $S_{\alpha\alpha}$ and $S^*_{\alpha\alpha}$ ; the product of terms corresponding to different permutations vanishes. Take now, for example, one definite permutation, as the one given by 1,2,3,...,A. We shall have, as a partial term in the product, the expression

$$\left(\hat{S}_{\alpha\alpha}\cdot\hat{S}^*_{\alpha\alpha}\right)_p = \sum_{\substack{\beta_1,\beta_2,\cdots\beta_{A-1}\\ \gamma_1,\gamma_2,\cdots\gamma_{A-1}}} S^{(1)}_{\alpha\beta_1}\,S^{(2)}_{\beta_1\beta_2}\cdots S^{(A)}_{\beta_{A-1},\alpha}\cdot S^{(A)\dagger}_{\gamma_{A-1},\alpha}\cdots S^{(2)\dagger}_{\gamma_2\gamma_1}\cdot S^{(1)\dagger}_{\gamma_1\alpha}$$

where the index $p$ means the contribution of that definite permutation. The term depending upon $\vec{b}_j$, $z_j$ will have the structure

$$S^{(j)}_{\mu\nu}(\vec{b}_j-\vec{b},z_j)\,S^{(j)*}_{\varrho\sigma}(\vec{b}_j-\vec{b},z_j) = \delta_{\mu\nu}\,\delta_{\varrho\sigma} - \tilde{T}^{\varrho\sigma}_{\mu\nu}(\vec{b}_j-\vec{b},z_j)$$

Even if we do not specify the operator $\hat{\tilde{T}}$ *), we can always write, for the contribution of a given permutation to $\sigma^{inel}_{summed}$, the two expansions

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*) As it can be realized, the wide nucleus approximation introduced in (16) and (21) is not needed in order to get $M=\tilde{\sigma}_1$ ; this approximation is needed if moreover we want to have $\tilde{\sigma}_1 = A\sigma_{in}$. Without the wide nucleus approximation we would simply have $M = A.\int d_2 b\int d_3 x_j\,\rho(\vec{x}_j)\left[1-|S^{(j)}_{\alpha\alpha}|^2\right]$.

$$(\sigma^{inel}_{summed})_p = \sum_{n=1}^{A} (\tilde{\sigma}_n)_p$$

$$(\tilde{\sigma}_n)_p = \int d_2 b \prod_{i=1}^{A} \int d_3 x_i\, \rho(\vec{x}_i) \left[ (\hat{\tilde{P}}_n)_p \right]^{\alpha\alpha}_{\alpha\alpha} \tag{24a}$$

$$(\sigma^{inel}_{summed})_p = \sum_{n=1}^{A} (\sigma_n)_p$$

$$(\sigma_n)_p = \int d_2 b \prod_{i=1}^{A} \int d_3 x_i\, \rho(\vec{x}_i) \left[ (\hat{P}_n)_p \right]^{\alpha\alpha}_{\alpha\alpha} \tag{24b}$$

where $(\hat{\tilde{P}}_n)_p$ represents the sum of all the ordered products of order $n$ of $\tilde{T}$ matrices, while $(\hat{P}_n)_p$ represents the sum of all the ordered products which are of order $n$ in $\hat{\tilde{T}}$ and of order $A-n$ in $(\hat{1} - \hat{\tilde{T}})$.

It is then trivial to show again that

$$(M)_p = \sum_{n=1}^{A} n (\sigma_n)_p = (\tilde{\sigma}_1)_p \tag{25}$$

Now we have

$$(\tilde{\sigma}_1)_p = A \int d_2 b \prod_{i=1}^{A} \int d_3 x_i\, \rho(\vec{x}_i) \left[ (\hat{\tilde{P}}_1)_p \right]^{\alpha\alpha}_{\alpha\alpha} =$$

$$= A\, \sigma_{in} \int d_2 b\, T(\vec{b}) \cdot \left( \prod_{i=1}^{A} \int d_3 x_i\, \rho(\vec{x}_i) \right)_p \tag{26}$$

since $S^{(j)}_{\alpha\alpha}\, S^{(j)*}_{\alpha\alpha}$ is $z_j$ independent.

Summing (25) over all the permutations, and using

$$\sum_p \left( \prod_{i=1}^{A} \int d_3 x_i\, \rho(\vec{x}_i) \right)_p = 1$$

we get again the AGK cancellation

$$M = \sum_p M_p = A\, \sigma_{in}$$

We have therefore shown that the AGK cancellation is always valid in the unitary formulation of Glauber theory.

What is, in this context, the meaning of this relation ?

In the eikonal model, the $\sigma_n$ are the cross-sections to produce n "inelastic blocks" (Andreev, 1975) ; therefore $\sum_{n=1}^{A} n\sigma_n$ represents the sum of the different contributions to the (integrated) inclusive cross-section, and

$$\bar{\nu} = \frac{\sum_{n=1}^{A} n\,\sigma_n}{\sigma^{inel}_{summed}} \tag{27}$$

represents the ratio of the multiplicity of particles produced in hadron-nucleus collisions over the multiplicity of particles produced in hadron-nucleon collisions.

The relation we have obtained is therefore equivalent to the AGK calcellation whenever the $\sigma_n$'s defined in (18b) and (23b) have the meaning of the partial inelastic cross-sections contributed by n target nucleons, with the inclusion of inelastic absorption on all the remaining nucleons.

In this case we will have that the result (Gottfried, 1973)

$$\bar{\nu} = \frac{A\,\sigma_{in}}{\sigma^{inel}_{summed}} \tag{28}$$

is valid in the general case, and is not limited to the eikonal model. Therefore in a model which satisfies unitarity, the cascading of the secondary particles does not increase the total multiplicity, as compared to the eikonal case (with the caveat that, however, one must compute $\sigma^{inel}_{summed}$ including all the intermediate states).

We end with a number of relevant remark :

- as seen from (28), if one computes $\bar{\nu}$ from the measured values of $\sigma_{in}$ and of $\sigma^{inel}_{summed}$, in the latter one has to include all the particle productions, both coherent and incoherent ;
- as compared with Reggeon calculus, the relevant elementary parameter is the physical quantity $\sigma_{in}$, and not some perturbative approximation of it, as the cross-section corresponding to a single (bare) Reggeon exchange ; this is the virtue of the Glauber theory, which always involves measurable quantities at the individual level ;

- the general validity of the result is affected by energy conservation effects in the same way as the eikonal models (Andreev, 1975) or Reggeon calculus (Capella and Kaidalov, 1976), since energy conservation has been neglected in the present treatment insofar as limiting the number of produced particles (the only rôle of energy conservation is here to give the longitudinal momentum effects).

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